Note on the Uniqueness Holomorphic Function on the Unit Disk

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Abstract. Let \( f \) be an holomorphic function the unit disk to itself. We provide conditions on the local behavior of \( f \) along boundary near a finite set of the boundary points that requires \( f \) to be a finite Blaschke product.

1. Introduction

In 1994, Daniel M. Burns and Steven G. Krantz ([1]) proved that if the holomorphic function \( f : D \rightarrow D \) satisfies the condition
\[
f(z) = z + O((z - 1)^4) \quad z \rightarrow 1, \quad z \in D, \tag{1.1}
\]
then \( f(z) \equiv z \) on the unit disk.

The example
\[
f(z) = z + \frac{1}{10} (z - 1)^3
\]
shows that the exponent 4 in (1.1) can not be replaced by 3. In fact, the proof shows that \( O((z - 1)^4) \) can be replaced by \( o((z - 1)^3) \).

In 2001, Dov Chelst ([2]), in turn, established the following generalization of this result.

\textbf{Theorem 1.1.} Let \( f : D \rightarrow D \) be a holomorphic function from the disk to itself. In addition, let \( \phi : D \rightarrow D \) be a finite Blaschke product which equals \( \tau \in \partial D \) on a finite set \( A_f \subset \partial D \). If
\begin{enumerate}
\item[(i)] for a given \( \gamma_0 \in A_f \),
\[f(z) = \phi(z) + o((z - \gamma_0)^3), \quad \text{as } z \rightarrow \gamma_0,\]
\item[(ii)] for all \( \gamma \in A_f - \{\gamma_0\}, 
\[f(z) = \phi(z) + O((z - \gamma)^{k_\gamma}), \quad \text{for some } k_\gamma \geq 2 \text{ as } z \rightarrow \gamma,\]
\end{enumerate}
then \( f(z) \equiv \phi(z) \) on the disk.

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It was shown that the above condition \( k_\gamma \geq 2 \) can not be replaced by \( k_\gamma \geq 1 \).

In ([3]) and ([4]), this problem was generalized in the following aspects:

a) more general majorant was taken instead of the usual power majorant in (i) and (ii);

b) in (i) and (ii), the conditions \( z \to \gamma \), which usually stated approaching from inside of the disk before, were taken as the behavior of the function \( f \) along the boundary.

In 2015, M. Mateljević proved Theorem 1 in ([5]), where instead of Blaschke product was taken inner function and in (i) and (ii), the behavior of the function \( f \) along the boundary was considered.

Recently similar problems were investigated in ([6]) and ([7]). For more detail literature and the other types of the results, we refer to ([8]), ([9]), ([5]), ([10]) and references therein.

In the present study, we refined the results in ([4]). In particular, from our proofs it is followed that \( O(z - \gamma)^k \) in Theorem 1.1 can be replaced by \( o(z - \gamma) \).

We propose the following assertion for the proofs of our results.

(A) Let \( u = u(z) \) be a positive harmonic function on the open disk \( \mathbb{U}(z, r_0), r_0 > 0 \). Suppose that for \( \theta_0 \in [0, 2\pi) \), \( \lim_{r \to r_0} u(re^{i\theta_0}) = 0 \) is satisfied. Then

\[
\lim_{r \to r_0} \inf \frac{u(re^{i\theta_0})}{r_0 - r} > 0.
\]

This assertion follows from Harnack inequality. For more general results and related estimates, see also ([11, Theorem 1.1]), ([12]), ([13]).

(B) Let the function \( u \) be a subharmonic function in the unit disk, \( E \) is the finite subset of the unit circle \( \partial D \) such that

\[
\lim_{z \to \zeta, z \in \partial D} u(z) = 0, \quad \forall \zeta \in \partial D \setminus E,
\]

and

\[
u(z) = o(|z - z|^{-1}) \text{ as } z \to \zeta \text{ for each } \zeta \in E,
\]

then \( u(z) \leq 0 \) for all \( z \in D \).

The basic exposition for this version of Phragmen-Lindelöf Principle can be found in ([14, pp. 79-90]), ([15, pp. 176-186]) and ([16, Chapter 4, section 8 and Chapter 5, section 9]).

Let \( \mathfrak{N} \) be a class of functions \( \mu : (0, +\infty) \to (0, +\infty) \) for each of which \( \log \mu(x) \) is concave with respect to \( \log x \). For each function \( \mu \in \mathfrak{N} \) the limit

\[
\mu_0 = \lim_{x \to 0} \frac{\log \mu(x)}{\log x}
\]

exists, and \( -\infty < \mu_0 \leq +\infty \). Here, the function \( \mu \in \mathfrak{N} \) is called bilogarithmic concave majorant ([17]).

\( \mathfrak{N} \) be the class of sets with zero inner capacity ([18, p.210]).

2. Main Results

Let \( d(z, A) \) be the distance from the point \( z \) to the set \( A \).

**Theorem 2.1.** Let \( \phi : D \to D \) be a finite Blaschke product which equals \( \tau \in \partial D \) on a finite set \( A_\tau \subset \partial D \) and \( f : D \to D \) be a holomorphic function that is continuous on \( \overline{D} \cap \{ z : d(z, A_\tau) < \delta_0 \} \) for some \( \delta_0, \mu_1, \mu_2 \in \mathfrak{N}, \mu_0^1 > 3 \), \( \mu_0^2 > 1 \). Suppose that the following conditions are satisfied

(i) for a given \( \gamma_0 \in A_\tau \),

\[
f(z) = \phi(z) + O(\mu_1(|z - \gamma_0|)), z \in \partial D, z \to \gamma_0,
\]

(ii) \( \lim_{z \to \gamma_0} f(z) = 0 \).
Theorem 2.2. Let
\[ f(z) = \phi(z) + O(\mu^2(\lvert z - \gamma \rvert)), \quad z \in \partial D, \quad z \to \gamma. \]

Then \( f(z) \equiv \phi(z) \) on \( D \).

Following result is generalization of Theorem 2.1.

**Theorem 2.2.** Let \( \phi : D \to D \) be a finite Blaschke product which equals \( \tau \in \partial D \) on a finite set \( A_f \subset \partial D \) and \( f : D \to D \) be a holomorphic function, \( Q \in \mathfrak{M}, \mu^1, \mu^2 \in \mathfrak{M}, \mu^1_0 > 3, \mu^2_0 > 1 \). Let the following conditions are satisfied
\[(i) \text{ for a given } \gamma_0 \in A_f, \]
\[
\limsup_{z \to \zeta \in D} \left| f(z) - \phi(z) \right| = O(\mu^1(\lvert \zeta - \gamma_0 \rvert)), \quad \zeta \in \partial D \setminus Q, \quad \zeta \to \gamma_0, \tag{2.1}
\]
\[(ii) \text{ for all } \gamma \in A_f \setminus \{\gamma_0\}, \]
\[
\limsup_{z \to \zeta \in D} \left| f(z) - \phi(z) \right| = O(\mu^2(\lvert \zeta - \gamma \rvert)), \quad \zeta \in \partial D \setminus Q, \quad \zeta \to \gamma \tag{2.2}
\]

Then \( f(z) \equiv \phi(z) \) on \( D \).

**Proof.** Let the assumptions of Theorem 2.1 are satisfied. By the condition (2.1), there exist a number \( C_1 > 0 \) and \( \delta_0 \in (0, 1) \) such that
\[
\limsup_{z \to \zeta \in D} \left| f(z) - \phi(z) \right| = C_1 \mu^1(\lvert \zeta - \gamma_0 \rvert)), \quad \zeta \in \partial D \setminus Q, \quad \lvert \zeta - \gamma_0 \rvert \leq \delta_0.
\]

Let us denote \( k \) and \( C_2 \) as follows
\[
k := \sup_{\lvert z - \gamma_0 \rvert = \delta_0, z \in D} \left| f(z) - \phi(z) \right|,
\]
\[
C_2 := \max \left\{ \frac{k}{\mu^1(\delta_0)}, C_1 \right\}.
\]

It can be easily seen that for all points of the set \( \partial(D \cap U(\gamma_0, \delta_0)) \setminus Q \), the inequality
\[
\limsup_{z \to \zeta \in D} \left| f(z) - \phi(z) \right| = C_2 \mu^1(\lvert \zeta - \gamma_0 \rvert)
\]

is satisfied.

Applying Theorem 3 in ([17]) (see also ([19]), ([20])) to the set \( D \cap U(\gamma_0, \delta_0) \) and to the function \( f(z) - \phi(z) \), we get
\[
\left| f(z) - \phi(z) \right| \leq C_2 \mu^1(\lvert z - \gamma_0 \rvert)), \quad \forall z \in D \cap U(\gamma_0, \delta_0). \tag{2.3}
\]

From \( \mu^1_0 > 3 \) there are some positive constants \( \varepsilon \) and \( \sigma < \min(\delta_0, 1) \) such that
\[
\frac{\log \mu^1(x)}{\log x} \geq 3 + \varepsilon \quad \forall x \in (0, \sigma)
\]

and
\[
\log \mu^1(x) \leq (3 + \varepsilon) \log x, \quad \forall x \in (0, \sigma)
\]

In other words,
\[ \mu^2(x) \leq x^3 + c, \quad \forall x \in (0, a). \] (2.4)

From the inequalities (2.3) and (2.4) we take the inequality
\[ |f(z) - \phi(z)| \leq C_2 |z - \gamma_0|^{3 + \epsilon}, \quad \forall z \in D \cap U(\gamma_0, a). \] (2.5)

Similarly, for any point \( \gamma \in A_f \setminus \{\gamma_0\} \), from the condition \( \mu^2_0 > 1 \) and (2.2) we have
\[ |f(z) - \phi(z)| \leq C_3 |z - \gamma|^{1 + \epsilon}, \quad \forall z \in D \cap U(\gamma, \sigma_1) \] (2.6)

with some constants \( C_3 \) and \( \sigma_1 \).

Consider the following harmonic function in the unit disk
\[ \psi(z) = \Re \left( \frac{1 + f(z)}{1 - f(z)} \right) - \Re \left( \frac{1 + \phi(z)}{1 - \phi(z)} \right). \]

Since a finite Blaschke Product \( \phi \) is holomorphic on \( \overline{D} \) and and \( |\phi(z)| = 1 \) on \( \partial D \), we have the second term of \( \psi \) is zero on \( \partial D \setminus A_f \), and also the first term of \( \psi \) is nonnegative. Consequently, after taking limitinfs to any boundary point in \( (\partial D \setminus \Omega) \setminus A_f \), one always reaches the nonnegative value (infinity is also possible).

Now, let us examine the behaviour of the function \( \psi \) at points of set \( A_f \). Let us represent \( \psi(z) \) in the form
\[ \psi(z) = \Re \left( \frac{2 (f(z) - \phi(z))}{(1 - f(z))(1 - \phi(z))} \right). \]

Now, let us take any point \( \gamma \in A_f \setminus \{\gamma_0\} \). It can be easily seen that for any \( z, |z| = 1, |\phi'(z)| > 0 \). If \( \left| \phi'(\gamma) \right| = c_{\gamma} \), then there exists a constant \( \sigma_\gamma \in (0, a_1) \) such that
\[ |1 - \phi(z)| \geq \frac{c_\gamma}{2} |y - z|, \quad \forall z \in D \cap U(y, \sigma_\gamma). \] (2.7)

From (2.6)
\[ \lim_{z \to \gamma} \frac{1 - f(z)}{y - z} = c_\gamma \]

and there exists \( \sigma'_{\gamma} \in (0, \sigma_\gamma) \) such that
\[ |1 - f(z)| \geq \frac{c_\gamma}{2} |y - z|, \quad \forall z \in D \cap U(y, \sigma'_{\gamma}). \] (2.8)

Then, from (2.6), (2.7) and (2.8)
\[ \frac{2 (f(z) - \phi(z))}{(1 - f(z))(1 - \phi(z))} \leq \frac{8C_3}{c_\gamma^2} \frac{1}{|y - z|^{1 + \epsilon}} \quad \forall z \in D \cap U(y, \sigma'_{\gamma}). \]

Thus, the function \( \psi(z) \) satisfies the following relation
\[ \lim_{z \to \gamma} |z - \gamma| \psi(z) = 0 \] (2.9)
on every point $\gamma \in A_f \setminus \{\gamma_0\}$.

Similarly, for the point $\gamma_0$, using (2.5), we have

$$|\psi(z)| \leq C_4 |z - \gamma_0|^{1+\varepsilon} \quad \forall z \in D \cap U(\gamma_0, \sigma')$$

for some positive constants $C_4$ and $\sigma'$. In particular,

$$\lim_{z \to \gamma_0} \psi(z) = 0. \quad (2.11)$$

From also here

$$\lim_{z \to \gamma_0} |z - \gamma_0| \psi(z) = 0. \quad (2.12)$$

So, the function $\psi(z)$ satisfies the relation (2.9) on every point of finite set $A_f$. From the assertion (B) we have either $\psi(z) > 0$, $z \in D$ or $\psi(z) \equiv 0$. If $\psi(z) \equiv 0$, then the proof is finished. Assume that the relation $\psi(z) \equiv 0$ is not satisfied. If we take $z = r\gamma_0$ in (2.10), we obtain

$$\lim_{r \to 1} \frac{\psi(r\gamma_0)}{1 - r} = 0. \quad (2.12)$$

If $\psi$ is not constant, (2.11) and (2.12) contradict with assertion (A) statement. Hence, $\psi \equiv 0$. This implies that $f(z) = \phi(z)$ on the disk. \[ \square \]

Theorem 2.2 and Theorem 2.3 generalize the results in ([4]), where instead of the condition $\mu_0^2 > 1$ were taken $\mu_0^2 > 2$. Moreover, the part of the proof of Theorem 2.3 which is after (2.4) shows that $O(z - \gamma)^k$, $k \geq 2$ in Theorem 1.1 can be replaced by $o(z - \gamma)$.

References


