On an Inversion Formula for the Fourier Transform on Distributions by Means of Gaussian Functions

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Abstract. Gaussian functions are useful in order to establish inversion formulae for the classical Fourier transform. In this paper we show that they also are helpful in order to obtain a Fourier inversion formula for the distributional case.

1. Introduction

In a series of papers published by the authors, different aspects of the Fourier transform on the spaces of distributions denoted by $S_k'$ (duals of the spaces $S_k$ introduced by J. Horváth in [9]) were studied (see [3], [4], [5] and [6]).

These spaces can be identified with subspaces of the Schwartz space $S'$ and its members can be considered as tempered distributions. Moreover, the usual distributional Fourier transform of $f \in S_k'$ [12, Chap. VII, §6, p. 248] is the regular distribution generated by the function in $\mathbb{R}^n$ given by $(\mathcal{F} f)(y) = \left< f, e^{ixy} \right>$. In [4, Theorem 2.1] it was established that if $f \in S_k'$, $k \in \mathbb{Z}$, $k < 0$, then for all $\phi \in S$ the Parseval equality

$$\left< f, \mathcal{F} \phi \right> = \left< T_{<f,e^{ixy}>}, \phi(y) \right>$$

holds, where $T_{<f,e^{ixy}>}$ is the member of $S'$ given by

$$\left< T_{<f,e^{ixy}>}, \phi(y) \right> = \int_{\mathbb{R}^n} \left< f, e^{ixy} \right> \phi(y) dy,$$

and $\mathcal{F} \phi$ denotes the classical Fourier transform of $\phi$, namely

$$(\mathcal{F} \phi)(t) = \int_{\mathbb{R}^n} \phi(y) e^{i yt} dy, \quad t \in \mathbb{R}^n.$$

Moreover, in [4, Theorem 3.1] it was proved the following inversion formula:
Let \( f \in S_k \), \( k \in \mathbb{Z} \), \( k < 0 \), and set \((\mathcal{F}f)(y) = \langle f, e^{iy} \rangle\) for \( y \in \mathbb{R}^n \). Then for any \( \phi_1, \cdots, \phi_n \in \mathcal{D}(\mathbb{R}) \), \( t = (t_1, \ldots, t_n) \in \mathbb{R}^n \) and \( \phi(t) = \phi_1(t_1) \cdots \phi_n(t_n) \), one has
\[
\langle f, \phi \rangle = \lim_{Y \to +\infty} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{C(0,Y)} (\mathcal{F}f)(y)e^{-iy\phi(t)}dy\, dt,
\]
where \( C(0;Y) \) is the \( n \)-cube \([-Y, Y] \times \cdots \times [-Y, Y] \subset \mathbb{R}^n \), \( Y > 0 \).

Later, in [6, Theorem 1], this inversion formula was extended to functions \( \phi \in \mathcal{S} \) such that \( \phi(t) = \phi_1(t_1) \cdots \phi_n(t_n) \), \( t = (t_1, \ldots, t_n) \in \mathbb{R}^n \), where \( \phi_1, \cdots, \phi_n \in \mathcal{S}(\mathbb{R}) \).

The purpose of the present paper is to obtain a distributional Fourier inversion formula which be valid for any \( \phi \in \mathcal{S} \). For it we follow to Lang in [10, Theorem 4, p. 264] for obtaining an inversion formula for the classical Fourier transform by means of Gaussian functions.

As a consequence of this distributional inversion formula we get a representation over \( \mathcal{S} \) of the solution in \( S_k \) of convolution equations and, consequently, of linear partial differential equations with complex constant coefficients.

A representation of the Fourier transform on distributions was obtained in [1] (amongst others).

Gaussian functions have been useful in the context of integral transforms, as has been revealed in recent papers (see [7] and [13]). We also recall some interesting recent advances concerning to integral transforms [15].

Related differential equations have been solved in [16] by using the operational method.

We recall that the spaces \( S_k \), \( k \in \mathbb{Z} \) [9, p. 90], are defined as the vector spaces of all functions \( \phi \) on \( \mathbb{R}^n \) which possess continuous partial derivatives of all orders and which satisfy the condition that if \( p \in \mathbb{N}^n \) and \( \varepsilon > 0 \), then there exists \( A(\phi, p, \varepsilon) > 0 \) such that
\[
\| (1 + |x|^2)^k \partial^p \phi(x) \| \leq \varepsilon, \quad \text{for} \quad |x| > A(\phi, p, \varepsilon).
\]

For every \( p \in \mathbb{N}^n \), Hováth defines on \( S_k \) the seminorms
\[
q_{k,p}(\phi) = \max_{x \in \mathbb{R}^n} \left| (1 + |x|^2)^k \partial^p \phi(x) \right|.
\]

The spaces \( S_k \) equipped with the countable family of seminorms \( q_{k,p} \) are Fréchet spaces. The well known space of test functions \( \mathcal{D} \) is a dense subspace of \( S_k \) (see [9], p. 419). As it is usual, \( S_k' \) denotes the dual of the space \( S_k \).

In this paper we make use of the well known fact that
\[
(2\pi)^{n/2} \cdot \int_{-\infty}^{+\infty} \exp \left[ i v x - (x^2/2c) \right] dx = \exp(cv^2/2), \quad v \in \mathbb{C}, \quad c > 0.
\]

Throughout this paper we shall use the terminology and notation of [9].

2. The inversion formula

Firstly, we will establish the next assertion

**Lemma 2.1.** Let \( \phi \in \mathcal{S} \), \( k \in \mathbb{Z} \) and \( k < 0 \), then
\[
\frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} \phi(x + 2aw)e^{-2|aw|^2} dw \longrightarrow \phi(x),
\]
in \( \mathcal{S} \) for \( a \to 0^+ \).
Proof. First, we claim that for all $\phi \in S$ and all $a > 0$ one has

$$\frac{1}{\pi^2} \int_{\mathbb{R}^n} \phi(x + 2aw)e^{-\|x\|^2} \, dw \in S.$$ 

In fact, for any $p \in \mathbb{N}^n$ there exists a $M_{p,\phi} > 0$ such that $|\partial^p \phi(x)| \leq M_{p,\phi}$, for all $x \in \mathbb{R}^n$. Thus, for $\phi = (0, \ldots, 0)$, it is clear that

$$|\phi(x + 2aw)e^{-\|x\|^2}| \leq M_{0,\phi}e^{-\|x\|^2}.$$ 

Also, for $r(j) = (r_1(j), \ldots, r_n(j))$, where $r_m(j) = 0$ for $m \neq j$ and $r(j) = 1, \ j = 1, \ldots, n$, it follows that

$$|\partial_{x_j} \phi(x + 2aw)e^{-\|x\|^2}| \leq M_{r(j),\phi}e^{-\|x\|^2}, \ j = 1, \ldots, n \text{ and all } x \in \mathbb{R}^n.$$ 

Since $M_{0,\phi}e^{-\|x\|^2}$ and $M_{r(j),\phi}e^{-\|x\|^2}$, $j = 1, \ldots, n$, are integrable functions over $\mathbb{R}^n$, the use of [2, Theorem 5.9, p. 238] yields to

$$\frac{\partial}{\partial x_j} \int_{\mathbb{R}^n} \phi(x + 2aw)e^{-\|x\|^2} \, dw = \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j} \phi(x + 2aw)e^{-\|x\|^2} \, dw.$$ 

A similar argument allows us to prove that for all $p_j \in \mathbb{N}$,

$$\frac{\partial^{p_j}}{\partial x_j^{p_j}} \int_{\mathbb{R}^n} \phi(x + 2aw)e^{-\|x\|^2} \, dw = \int_{\mathbb{R}^n} \frac{\partial^{p_j}}{\partial x_j^{p_j}} \phi(x + 2aw)e^{-\|x\|^2} \, dw,$$ 

for all $j = 1, \ldots, n$. Now, since for $p = (p_1, \ldots, p_n) \in \mathbb{N}^n$, is $\partial^p = \frac{\partial^{p_1 + \ldots + p_n}}{\partial x_1^{p_1} \ldots \partial x_n^{p_n}}$, it follows that

$$\partial^p \int_{\mathbb{R}^n} \phi(x + 2aw)e^{-\|x\|^2} \, dw = \int_{\mathbb{R}^n} \partial^p \phi(x + 2aw)e^{-\|x\|^2} \, dw.$$ 

On the other hand, being

$$\frac{1}{\pi^2} \int_{\mathbb{R}^n} e^{-\|x\|^2} \, dw = 1,$$ 

we find that

$$\left(1 + |x|^2\right)^k \frac{1}{\pi^2} \int_{\mathbb{R}^n} \phi(x + 2aw)e^{-\|x\|^2} \, dw \leq \left(1 + |x|^2\right)^k M_{p,\phi} \frac{1}{\pi^2} \int_{\mathbb{R}^n} e^{-\|x\|^2} \, dw = \left(1 + |x|^2\right)^k \cdot M_{p,\phi},$$

(1)

from which, being $k < 0$, it follows that (1) tends to zero as $|x|$ tends to infinity.

Now, for all $p = (p_1, \ldots, p_n) \in \mathbb{N}^n$,

$$\max_{x \in \mathbb{R}^n} \left| \left(1 + |x|^2\right)^k \frac{1}{\pi^2} \partial^p \left( \int_{\mathbb{R}^n} \phi(x + 2aw)e^{-\|x\|^2} \, dw - \phi(x) \right) \right|$$

$$= \max_{x \in \mathbb{R}^n} \left| \left(1 + |x|^2\right)^k \frac{1}{\pi^2} \partial^p \left( \int_{\mathbb{R}^n} \left[ \phi(x + 2aw) - \phi(x) \right] e^{-\|x\|^2} \, dw \right) \right|,$$

(2)

which, applying again [2, Theorem 5.9, p. 238], we have that the last expression is equal to

$$\frac{1}{\pi^2} \max_{x \in \mathbb{R}^n} \left| \partial^p \phi(x + 2aw) - \partial^p \phi(x) \right| e^{-\|x\|^2} \, dw.$$
Thus, as a consequence of [4, Theorem 2.1], we have that (4) is equal to

\[ \phi \]

and by Fubini theorem it is equal to

\[ \phi \]

\[ \phi \]

Note that, since \( p(j) = (p_1, \ldots, p_j + 1, \ldots, p_n) \).

Also, using spherical coordinates in \( \mathbb{R}^n \) it is easily obtained that

\[ \phi \]

from which (2) is less than or equal to

\[ \phi \]

\[ \phi \]

and, thus, the Lemma holds.

\[ \square \]

We are now ready to prove the main result

**Theorem 2.2.** Let \( f \in \mathcal{S}', k \in \mathbb{Z}, k < 0, \) and \( (\mathcal{F} f)(y) = \langle f, e^{ixy} \rangle, \ y \in \mathbb{R}^n, \) then, for all \( \phi \in \mathcal{S} \) it follows

\[ \langle f, \phi \rangle = \lim_{\epsilon \to 0^+} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\mathcal{F} f)(y) e^{-\frac{yn}{\epsilon}} e^{-\frac{|y|}{\epsilon}} dy \phi(t) dt. \]  

(3)

**Proof.**

First, from [9, Proposition 2, p. 97], there exist a \( C > 0 \) and a nonnegative integer \( r \), both depending on \( f \), such that

\[ \|(\mathcal{F} f)(y)\| = \left| \langle f, e^{ixy} \rangle \right| \leq C \max_{|y| \leq r} \left| 1 + |x|^2 \right|^{\frac{k}{2}} \left| e^{ixy} \right| = C \max_{|y| \leq r} |y|^r. \]

Thus, for any \( \phi \in \mathcal{S} \), one has

\[ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\mathcal{F} f)(y) e^{-\frac{yn}{\epsilon}} e^{-\frac{|y|}{\epsilon}} dy \phi(t) dt \]

\[ = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle f, e^{ixy} \rangle e^{-\frac{yn}{\epsilon}} e^{-\frac{|y|}{\epsilon}} dy \phi(t) dt, \]

and by Fubini theorem it is equal to

\[ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \langle f, e^{ixy} \rangle e^{-\frac{yn}{\epsilon}} |y|^r \int_{\mathbb{R}^n} e^{-\frac{|y|}{\epsilon}} \phi(t) dt dy. \]

(4)

Note that, since \( \phi \in \mathcal{S} \) it follows that

\[ e^{-\frac{|y|}{\epsilon}} \int_{\mathbb{R}^n} e^{-\frac{yn}{\epsilon}} \phi(t) dt \in \mathcal{S}. \]

Thus, as a consequence of [4, Theorem 2.1], we have that (4) is equal to

\[ \langle f, \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ixy} e^{-\frac{yn}{\epsilon}} |y|^r \int_{\mathbb{R}^n} e^{-\frac{|y|}{\epsilon}} \phi(t) dt dy \rangle, \]
which, making use again of Fubini theorem, is equal to
\[ \langle f, \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x, y \rangle} e^{\frac{-y^2}{2a}} \, dy \phi(t) \, dt \rangle. \] (5)

Now, observe that by (1) we have
\[ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(x-t)y} e^{-\frac{y^2}{2a}} \, dy = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(x-y-\frac{t}{2})^2} \, dy \]
\[ = \frac{1}{2\sqrt{\pi a}} e^{-\frac{(x-t)^2}{2a}} = \frac{1}{2\sqrt{\pi a}} e^{-\frac{(x-t)^2}{2a}}, \]
and thus we get that
\[ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x, y \rangle} e^{-\frac{y^2}{2a}} \, dy = \frac{1}{2^n \pi^{n/2} a^n} e^{-\frac{x^2}{2a^2}}. \] (6)

Therefore, (5) is equal to
\[ \left\langle f, \frac{1}{2^n \pi^{n/2} a^n} \int_{\mathbb{R}^n} \phi(t) e^{-\frac{m^2}{2a^2}} \, dt \right\rangle. \] (7)

Now, performing the change of variables \( t = x + 2aw, \) (7) becomes
\[ \left\langle f, \frac{1}{\pi^2} \int_{\mathbb{R}^n} \phi(x + 2aw) e^{-\frac{\|\cdot\|^2}{2a^2}} \, dw \right\rangle, \] (8)
from which, since \( f \in S' \) by Lemma 2.1, the equality (3) follows. \( \square \)

As it is well known, the Dirac distribution \( \delta_u \) at \( u \in \mathbb{R}^n \) given by \( \langle \delta_u, \phi \rangle = \phi(u) \), for all \( \phi \in S_u \), is a member in \( S'_u \). As it is usual we denote \( \delta = \delta_0 \). Also, for all \( m \in \mathbb{N}^n \), \( \partial^m \delta_u \) at \( u \in \mathbb{R}^n \) given by
\[ \langle \partial^m \delta_u, \phi \rangle = \langle \delta_u, (-1)^m \partial^m \phi \rangle = (-1)^m \partial^m \phi(u), \] for all \( \phi \in S_u \), is a member in \( S'_u \).

Now, one obtains the next result

**Corollary 2.3.** For all \( \phi \in S, u \in \mathbb{R}^n \) and all \( m \in \mathbb{N}^n \), one has
\[ \langle \partial^m \delta_u, \phi \rangle = \lim_{a \to 0^+} \frac{(-1)^m}{2^n \pi^{n/2} a^n} \int_{\mathbb{R}^n} e^{-\frac{\|x-u\|^2}{2a^2}} \partial^m \phi(t) \, dt, \]
and
\[ \partial^m \phi(u) = \lim_{a \to 0^+} \frac{1}{2^n \pi^{n/2} a^n} \int_{\mathbb{R}^n} e^{-\frac{\|x-u\|^2}{2a^2}} \partial^m \phi(t) \, dt. \]

**Proof.**
Since \( \langle \delta_u, e^{iy} \rangle = e^{iy}, y \in \mathbb{R}^n \), and according to the above inversion formula, for any \( \phi \in S \), one has
\[ \langle \partial^m \delta_u, \phi \rangle = \lim_{a \to 0^+} \frac{(-1)^m}{2^n \pi^{n/2} a^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x, y \rangle} e^{-\frac{y^2}{2a}} \, dy \partial^m \phi(t) \, dt. \] (9)

Now, using (6), formula (9) becomes
\[ \langle \partial^m \delta_u, \phi \rangle = (-1)^m \partial^m \phi(u) = \lim_{a \to 0^+} \frac{(-1)^m}{2^n \pi^{n/2} a^n} \int_{\mathbb{R}^n} e^{-\frac{\|x-u\|^2}{2a^2}} \partial^m \phi(t) \, dt. \]

Also, using Theorem 2.2 above and [6, Theorem 2.1] one has
Corollary 2.4. Set $f \in S'_k, k \in \mathbb{Z}, k < 0$. Then
\[
\lim_{Y \to +\infty} \int_{\mathbb{R}^n} \int_{C(0,Y)} (\mathcal{F} f)(y) e^{-ity} dy \phi(t) dt
= \lim_{a \to 0+} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\mathcal{F} f)(y) e^{-ity} e^{-a\|y\|^2} dy \phi(t) dt,
\]
for all $\phi \in S$ such that $\phi(t) = \phi_1(t_1) \cdots \phi_n(t_n)$, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, where $\phi_1, \ldots, \phi_n \in S(\mathbb{R})$.

The next result is a variant of [5, Corollary 2.1] concerning the solution of convolution equations.

Corollary 2.5. Set $h, g \in S'_k, k \in \mathbb{Z}, k < 0$. Assume that $\mathcal{F} h$ has no zeros in $\mathbb{R}^n$, suppose that $\mathcal{F} h \in C^{-2k+2n}(\mathbb{R}^n)$ and there exists a polynomial $P$ such that
\[
\left| \partial^m \left( \frac{1}{(\mathcal{F} h)(y)} \right) \right| \leq P(|y|), \quad \forall y \in \mathbb{R}^n, \quad \forall m \in \mathbb{N}^n, \quad |m| \leq -2k + 2n.
\]
Then, the convolution equation
\[
h \ast f = g,
\]
has a unique solution $f \in S'_k$ and this solution has the next representation over members in $S$
\[
< f, \phi > = \lim_{a \to 0+} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\mathcal{F} g)(y)}{(\mathcal{F} h)(y)} e^{-ity} e^{-a\|y\|^2} dy \phi(t) dt, \quad \phi \in S.
\]

Proof.
In fact, from the hypothesis of this Corollary and using [5, Theorem 2.1] it follows that there exists an element $w \in S'_k$ such that $\mathcal{F} w = \frac{1}{\mathcal{F} h}$. Therefore, using [4, Proposition 4.1] one has
\[
\mathcal{F} [h \ast w] = \mathcal{F} h \cdot \frac{1}{\mathcal{F} h} = 1 = \mathcal{F} \delta.
\]
So, using [4, Corollary 3.1], it follows that $h \ast w = \delta$.
Now, the member of $S'_k$ given by $f = w \ast g$ is a solution of equation (10).
In fact,
\[
h \ast (w \ast g) = (h \ast w) \ast g = \delta \ast g = g.
\]
Note that if $f_1, f_2 \in S'_k$ satisfy $h \ast f_1 = g$ and $h \ast f_2 = g$ then $f_1 = f_2$. Indeed, taking Fourier transform it follows that
\[
\mathcal{F} f_1 = \mathcal{F} f_2 = \frac{\mathcal{F} g}{\mathcal{F} h},
\]
and, again by [5, Corollary 3.1], we have $f_1 = f_2$.
Also, since $\mathcal{F} [h \ast f] = \mathcal{F} g$ and using again [5, Proposition 4.1] one obtain that
\[
\mathcal{F} f = \frac{\mathcal{F} g}{\mathcal{F} h},
\]
which by Theorem 2.2 above allows us to the representation over $S$ given by (11). □

Remark (invertible elements of $S'_k$).
Observe that the distribution \( w = h^{-1} \) in \( S'_k, \, k \in \mathbb{Z}, \, k < 0 \), which satisfies the equation \( h \ast w = \delta \), is the inverse by convolution of the member \( h \in S'_k \). So, when the distributional Fourier transform of \( h \) has no zeros in \( \mathbb{R}^n \), with \( \mathcal{F} h \in C^{-2k+2n}(\mathbb{R}^n) \) and it satisfies the inequality

\[
\left| \partial^m \left( \frac{1}{(\mathcal{F} h)(y)} \right) \right| \leq P(|y|), \quad \forall y \in \mathbb{R}^n, \quad m \in \mathbb{N}, \quad |m| \leq -2k + 2n,
\]

for some polynomial \( P \), this distribution \( h^{-1} \) has the next representation over \( S \)

\[
<h^{-1}, \phi> = \lim_{a \to 0^+} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{(\mathcal{F} h)(y)} e^{-iy \cdot \phi} e^{-a \frac{||y||^2}{2}} dy \phi(t) dt, \quad \phi \in S.
\]

**Final observation**

As in [8] and [11], we consider linear partial differential equations with constant coefficients of the form

\[
P(\partial) u = v, \tag{1}
\]

where as it is usual \( P \) is a polynomial in \( \mathbb{R}^n \) (with complex coefficients) and \( P(\partial) \) denotes the corresponding polynomial differential operator given by

\[
\sum_{|\alpha| \leq m} a_{\alpha} \partial^\alpha, \quad \alpha \in \mathbb{N}^n, \quad a_{\alpha} \in \mathbb{C}, \quad m \in \mathbb{N},
\]

and \( v \) is an element of \( S'_k, \, k \in \mathbb{Z}, \, k < 0 \).

Note that, since

\[
P(\partial) u = (P(\partial) \delta) \ast u,
\]

equation (1) can be written as a convolution equation.

Having into account that

\[
(\mathcal{F}[P(\partial) \delta])(y) = P(-iy), \quad y \in \mathbb{R}^n,
\]

and using Corollary 2.5 above, one has that when \( P \) has no zeros of type \( \alpha i \), where \( \alpha \in \mathbb{R}^n \), then there exists a unique solution \( u \) in \( S'_k \) of (1).

Also, one obtains the next representation over \( S \) of the solution \( u \) of equation (1):

\[
<u, \phi> = \lim_{a \to 0^+} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\mathcal{F} v)(y)}{P(-iy)} e^{-iy \cdot \phi} e^{-a \frac{||y||^2}{2}} dy \phi(t) dt,
\]

for all \( \phi \in S \).

Furthermore, observe that if in (1) we set \( v = \delta \), then one obtains a representation over \( S \) of the fundamental solution \( E \) of equation (1). In fact, having into account that \( \mathcal{F} \delta = 1 \), then one has

\[
<E, \phi> = \lim_{a \to 0^+} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{P(-iy)} e^{-iy \cdot \phi} e^{-a \frac{||y||^2}{2}} dy \phi(t) dt,
\]

for all \( \phi \in S \).

Observe that this fundamental solution \( E \) is the inverse by convolution of the member \( h \) of \( S'_k \) given by \( h = P(\partial) \delta \).
References