Fredholm Generalized Composition Operators on Weighted Hardy Spaces

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Abstract. The main purpose of this paper is to study Fredholm generalized composition operators on weighted Hardy spaces.

1. Introduction

Let $f$ be an analytic function on the open unit disk $\Omega$ in a complex plane $\mathbb{C}$ given by $f(z) = \sum_{n=0}^{\infty} f_n z^n$, where $\{f_n\}_{n=0}^{\infty}$ is a sequence of complex numbers. Let $\{\beta_n\}$ be a sequence of positive real numbers with $\beta(0) = 1$. For $p \in [1, \infty)$, let $H^p(\beta) = \{f : f(z) = \sum_{n=0}^{\infty} f_n z^n, \sum_{n=0}^{\infty} |f_n|^p \beta^n < \infty\}$ be the space of formal series. Then $H^p(\beta)$ is a Banach space under the norm $\|f\|_p = \sum_{n=0}^{\infty} |f_n|^p \beta^n$. For $p = 2$, the space $H^2(\beta)$ is a Hilbert space under the inner product defined as $\langle f, g \rangle = \sum_{n=0}^{\infty} f_n \overline{g_n} \beta^n$, where $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and $g(z) = \sum_{n=0}^{\infty} g_n z^n$. The weighted Hardy space is denoted by $H^2(\beta)$. Let $\phi(z) = e^{zk}$ and $\hat{\phi}(z) = \frac{e^{zk}}{\beta^k}$, clearly $\{\hat{\phi}_k\}_{k=0}^{\infty}$ is an orthogonal basis for $H^2(\beta)$.

If $\phi : \Omega \to \Omega$ is a mapping such that the transformation $C_\phi : H^2(\beta) \to H^2(\beta)$ defined by $C_\phi f = f \circ \phi$, for every $f \in H^2(\beta)$, is continuous, we shall call it a composition operator induced by $\phi$. A generalized composition operator $C^d_\phi : H^2(\beta) \to H^2(\beta)$ is defined by $C^d_\phi f = f' \circ \phi_0$, where $f'$ is the derivative of $f$. By the anti-differential operator $D_a$ we shall mean the operator $D_a : H^2(\beta) \to H^2(\beta)$ defined by

$$D_a (\sum_{n=0}^{\infty} f_n z^n) = \sum_{n=0}^{\infty} \frac{f_n z^{n+1}}{n+1}$$
Also the Differential operator $D$ on $H^2(\beta)$ is defined by

$$D(\sum_{n=0}^{\infty} f_n z^n) = \sum_{n=0}^{\infty} n f_n z^{n-1}$$

Composition operators on the spaces of analytic functions were studied by Cowen[1], Ryff[4], Schwartz[5] and Singh[8]. Properties of generalized composition operators on weighted Hardy spaces were mentioned in the papers of Sharma[6]-[7], further Fredholm composition and weighted composition operators can be seen in the papers of Kumar[2], Maccluer[3] and Takagi[9]. In this paper we initiate the study of Fredholm generalized composition operators on weighted Hardy spaces. The symbol $B(H)$ denote the Banach algebra of all bounded linear operators on $H$ into itself and $N_0$ denote the set $\{0, 1, 2, 3, \ldots\}$.

2. Fredholm generalized composition operators on weighted Hardy spaces

The necessary and sufficient condition for generalized composition operators to be Fredholm is investigated in this section.

**Theorem 2.1.** Suppose $\phi : \Omega \to \Omega$ is a mapping such that $\{\phi^n : n \in N_0\}$ is an orthogonal family in $H^2(\beta)$. Then $ker C^d_\phi = span[\epsilon_0]$, where $\phi^n(z) = (\phi(z))^n$.

**Proof.** If $f = \alpha \epsilon_0$, then clearly $C^d_\phi f = 0$, therefore $f \in ker C^d_\phi$

Next, if $C^d_\phi f = 0$ then for $f = \sum_{n=0}^{\infty} f_n \epsilon_n$

We have

$$C^d_\phi f = \sum_{n=1}^{\infty} n f_n \phi^{n-1} = 0$$

this implies that

$$||C^d_\phi f||^2 = \sum_{n=1}^{\infty} |f_n|^2 \beta_n^2 ||\phi^{n-1}||^2 = 0$$

so that

$$|f_n| = 0 \quad \text{for every} \quad n \in N$$

Hence

$$f = \alpha \epsilon_0.$$ 

**Theorem 2.2.** Suppose $\phi : \Omega \to \Omega$ is a mapping such that $\{\phi^n : n \in N_0\}$ is an orthogonal family in $H^2(\beta)$. Then $C^d_\phi$ has closed range if and only if there exists $\epsilon > 0$ such that $n||\phi^{n-1}|| \geq \beta_n$ for all $n \in N$.

**Proof.** We first assume that $C^d_\phi$ has closed range. Then $C^d_\phi$ is bounded away from zero on $(ker C^d_\phi)^\perp$, therefore there exists $\epsilon > 0$ such that

$$||C^d_\phi \epsilon_n|| \geq \epsilon ||\epsilon_n|| \quad \text{for all} \quad n \in N$$

which implies that

$$n||\phi^{n-1}|| \geq \epsilon \beta_n \quad \text{for all} \quad n \in N$$
Conversely suppose that the conditions is true. Then for $f \in (\ker C^d_{\phi})^\perp$ we have

$$
\|C^d_{\phi} f\|^2 = \| \sum_{n=1}^{\infty} f_n C^d_{\phi} e_n \|^2 = \sum_{n=1}^{\infty} |f_n|^2 n^2 \|\phi^{n-1}\|^2 \geq c^2 \sum_{n=1}^{\infty} |f_n|^2 \beta_n^2 = c^2 \|f\|^2$$

for every $f \in (\ker C^d_{\phi})^\perp$.

Then $C^d_{\phi}$ is bounded away from zero on $(\ker C^d_{\phi})^\perp$. Consequently $C^d_{\phi}$ has closed range. \(\square\)

**Theorem 2.3.** Let $\phi : \Omega \to \Omega$ be such that $\{\phi^n : n \in \mathbb{N}_0\}$ is an orthogonal family in $H^2(\beta)$. Then $C^d_{\phi}$ is Fredholm if and only if there exists $\epsilon > 0$ such that

$$\frac{n\|\phi^{n-1}\|}{\beta_n} \geq \epsilon \text{ for every } n \in \mathbb{N}.$$ 

**Proof.** Suppose the condition is true. Then in view of the theorem (2.2) $C^d_{\phi}$ has closed range. Also in view of theorem (2.1), $\ker C^d_{\phi}$ is a finite dimensional.

We show that $\ker C^d_{\phi}$ is zero dimensional. Let $g \in \ker C^d_{\phi}$, then $C^d_{\phi} g = 0$.

Therefore, for $n \in \mathbb{N}_0$ we have

$$0 = \langle C^d_{\phi} g, e_n \rangle = \langle g, C^d_{\phi} e_n \rangle = n \langle g, \phi^{n-1} \rangle.$$ 

Hence $g = 0$, thus $\ker C^d_{\phi} = \{0\}$. Hence $C^d_{\phi}$ is Fredholm.

The converse is easy to prove in view of theorem (2.1) and theorem (2.2). \(\square\)

**Example 2.4.** Let $\phi : \Omega \to \Omega$ be defined by $\phi(z) = z$, let $\beta_n = n!$, then $\frac{n\|\phi^{n-1}\|}{\beta_n} = \frac{n\beta_{n-1}}{\beta_n} = 1$. Therefore $C^d_{\phi}$ has closed range. Now $\ker C^d_{\phi} = \text{span}\{e_0\}$ and $\ker C^e_{\phi} = \{0\}$.

Hence $C^d_{\phi}$ is Fredholm.

### 3. Fredholm Differential and Anti-Differential operators on weighted Hardy spaces

In this section we obtain adjoint of anti-differential operator on weighted Hardy spaces. The condition for anti-differential operator to be Fredholm is also investigated in this section.

**Theorem 3.1.** Let $f \in H^2(\beta)$. Then

$$D^*_a f = \sum_{n=0}^{\infty} \frac{f_{n+1} \beta_{n+1}}{(n+1) \beta_n} z^n$$

where $D^*_a$ is the adjoint of $D_a$.

**Proof.** For any $n \in \mathbb{N}_0$

Consider

$$\langle D^*_a e_{n+1}, f \rangle = \langle e_{n+1}, D_a f \rangle = \frac{1}{n+1} \left( \frac{\beta_{n+1}}{\beta_n} \right)^2 \langle e_n, f \rangle \text{ for every } f \in H^2(\beta).$$

Therefore,

$$D^*_a e_{n+1} = \frac{1}{n+1} \left( \frac{\beta_{n+1}}{\beta_n} \right)^2 e_n \text{ and } D^*_a e_0 = 0.$$
Now for \( f = \sum_{n=0}^{\infty} f_n e_n \)

\[
D_a^* f = \sum_{n=0}^{\infty} f_n D_a^* e_n = \sum_{n=0}^{\infty} f_{n+1} \frac{1}{n+1} \left( \frac{\beta_{n+1}}{\beta_n} \right)^2 e_n
\]

**Theorem 3.2.** Let \( D_a \in B(H^2(\beta)) \). Then \( D_a \) is Fredholm operator if and only if \( \frac{\beta_n}{\beta_{n-1}} \geq \epsilon \) for every \( n \geq 1 \).

**Proof.** Clearly, for \( n \geq 1 \), \( D_a^* e_n = \frac{1}{n} \left( \frac{\beta_n}{\beta_{n-1}} \right)^2 e_{n-1} \).

Since

\[
D_a^* e_0 = 0, \text{ so } e_0 \in \ker D_a^*.
\]

We shall show that \( \ker D_a^* = \text{span}\{e_0\} \).

Let \( f \in \ker D_a^* \), then

\[
D_a^* f = D_a^* \sum_{n=0}^{\infty} f_n e_n = \sum_{n=1}^{\infty} f_n \frac{1}{n} \left( \frac{\beta_n}{\beta_{n-1}} \right)^2 e_{n-1} = 0
\]

which implies that \( f_n = 0 \), \( \forall \ n \geq 1 \).

Hence \( f = f_0 e_0 \).

Thus \( \ker D_a^* = \text{span}\{e_0\} = M \).

Next we will see that \( D_a^* \) is bounded away from zero on \( (\ker D_a^*)^\perp \) if and only if \( \frac{\beta_n}{\beta_{n-1}} \geq \epsilon \) for every \( n \geq 1 \).

Let \( f \in (\ker D_a^*)^\perp = M^\perp \).

Consider

\[
||D_a^* f||^2 = \sum_{n=1}^{\infty} \left( \frac{\beta_n}{\beta_{n-1}} \right)^2 (f_n)^2 \geq \epsilon^2 \sum_{n=1}^{\infty} \beta_n^2 = \beta^2 ||f||^2
\]

This is true for every \( f \in (\ker D_a^*)^\perp \).

Hence \( D_a^* \) has closed range. Also \( \ker D_a = \{0\} \). For if we have \( D_a f = 0 \),

then \( \sum_{n=0}^{\infty} f_n D_a e_n = 0 \) implies that \( \sum_{n=0}^{\infty} f_{n+1} e_n = 0 \) or \( \frac{f_n}{n+1} = 0 \) for all \( n \in N_0 \).

This implies that \( f = 0 \).

Thus \( \ker D_a = \{0\} \). Hence \( D_a \) is Fredholm. The converse follows by reversing the arguments.

In the next theorem we characterize Fredholm differential operator.

**Theorem 3.3.** Let \( D \in B(H^2(\beta)) \). Then \( D \) is Fredholm operator if and only if \( \frac{\beta_{n+1}}{\beta_n} \geq \epsilon \) for every \( n \geq 1 \).

**Proof.** We first note that \( \ker D = \text{span}\{e_0\} \).

For if we suppose that \( D f = 0 \) for \( f \in H^2(\beta) \),

then for \( f = \sum_{n=0}^{\infty} f_n e_n \) we have

\[
D f = \sum_{n=1}^{\infty} f_{n+1} e_n = 0
\]
which implies that 
\[ \sum_{n=1}^{\infty} n^2|f_n|^2 \beta_{n-1}^2 = 0 \]
which further implies that \( f_n = 0 \) for all \( n = 1, 2, \ldots \).
Hence \( f = f_0e_0 \) so that \( f \in \text{span}\{e_0\} \).
Next we shall see that \( \ker D^* = \{0\} \). Suppose \( f \in \ker D^* \).
Then \( D^*f = 0 \) or
\[ D^*(\sum_{n=0}^{\infty} f_n e_n) = \sum_{n=0}^{\infty} f_n(n+1)(\frac{\beta_n}{\beta_{n+1}})^2 e_{n+1} = 0 \]
which implies that \( f_n = 0 \) for all \( n = 0, 1, \ldots \). Thus \( f = 0 \).
Finally we can show that if the given condition is satisfied, then \( D \) has closed range.
Let \( f \in (\ker D)^\perp \) and \( f = \sum_{n=1}^{\infty} f_n e_n \).
Then
\[ ||Df||^2 = ||\sum_{n=1}^{\infty} f_n n e_{n-1}||^2 = \sum_{n=0}^{\infty} |f_{n+1}|^2(n+1)^2 \beta_n^2 = \sum_{n=0}^{\infty} |f_{n+1}|^2(n+1)^2 \frac{\beta_n^2}{\beta_{n+1}^2} \beta_{n+1}^2 \geq \epsilon^2 \sum_{n=0}^{\infty} |f_{n+1}|^2 \beta_{n+1}^2 = \epsilon^2 ||f||^2 \]
Thus \( D \) is bounded away from zero on \( (\ker D)^\perp \) which proves that \( D \) has closed range. We can conclude that \( D \) is Fredholm.

Conversely suppose \( D \) is Fredholm. Then \( D \) has closed range. Therefore \( D \) is bounded away from zero on \( (\ker D)^\perp \).
We can find \( \epsilon > 0 \) such that
\[ ||De_n|| \geq \epsilon ||e_n|| \quad \forall \ n = 1, 2, \ldots \]
or
\[ \frac{n \beta_{n-1}}{\beta_n} \geq \epsilon \quad \forall \ n = 1, 2, \ldots \]
This complete the proof of the theorem. □

References