On the Jacobson and Simple Radicals of Semigroups

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Abstract. In this paper we study radicals of a semigroup which is the union of a family of its subsets, indexed by a nonempty set, such that the intersection of two distinct subsets is contained in the set of left zeroes of that semigroup.

1. Introduction

Radicals of semigroups have been investigated in many articles (see for instance [29]). They represent semigroup analogs of radicals of rings. Theory of radicals of rings is an active area of research (see [6]), and in particular, the theory of radicals of graded rings has been actively investigated and has various important applications (see [21, 22, 27, 28, 31]). Recently (see [11–14, 20, 21, 23]), some results are obtained regarding radicals and graded radicals of rings graded in the following sense.

Definition 1.1 ([21, 23]). Let $R$ be a ring, and $\Delta$ a partial groupoid, that is, a set with a partial binary operation. Also, let $\{R_\delta\}_{\delta \in \Delta}$ be a family of additive subgroups of $R$. We say that $R = \bigoplus_{\delta \in \Delta} R_\delta$ is $\Delta$-graded and $R$ induces $\Delta$ (or $R$ is a $\Delta$-graded ring inducing $\Delta$) if the following two conditions hold:

i) $R_\xi R_\eta \subseteq R_{\xi\eta}$ whenever $\xi\eta$ is defined;

ii) $R_\xi R_\eta \neq 0$ implies that the product $\xi \eta$ is defined.

$\Delta$-graded rings inducing $\Delta$ will be simply called graded rings in the rest of the paper. The set $\bigcup_{\delta \in \Delta} R_\delta$ is called the homogeneous part of a graded ring $R = \bigoplus_{\delta \in \Delta} R_\delta$, a notion equivalent to that in Definition 1.1 has been studied in [2, 9, 25] from a different point of view. If we restrict addition and multiplication to the homogeneous part of a graded ring, we obtain a partial structure called an anneid [2, 9, 25], and an origin of this point of view to grading goes back to [24]. For a survey on anneids, one may consult [26, 33]. The homogeneous part of a graded ring is obviously a semigroup with respect to multiplication. This motivates us to introduce the following type of a semigroup (see also the notion of a quasi-graded ring from [2, 25]). However, we decide to work with semigroups which are generally without zero, and if $S$ is a semigroup, by $O(S)$ we denote the set of all left zeroes of $S$, according to notation from [10].

Definition 1.2. Let $S$ be a semigroup, and $\Delta$ a partial groupoid, that is, a set with a partial binary operation. Also, let $\{S_\delta\}_{\delta \in \Delta}$ be a family of subsets, called components, of $S$. We say that $S = \bigcup_{\delta \in \Delta} S_\delta$ is homogeneous semigroup inducing $\Delta$ (shortly, homogeneous semigroup) if the following three conditions hold:

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Let $\mu$.

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A mapping $f : N \rightarrow M'$ between two homogeneous S-acts $M$ and $M'$ is called a homomorphism of homogeneous acts if it is a homomorphism between $M$ and $M'$ observed as S-acts and if $x \# y, x, y \in M,$ implies $f(x) \# f(y)$.

Remark 2.4. Let us notice that any trivial classical homomorphism is also a homomorphism in the sense of Definition 2.3.

Let $\mu$ be a congruence on a homogeneous $S$-act $M = \bigcup_{\Delta \in \Delta} M_{\Delta}$. By $[x]_{\mu}$, or just by $[x]$ when there is no threat of confusion, we denote the congruence class which contains the element $x \in M$ and $M/\mu$ denotes the set
of all congruence classes of $M$ with respect to $\mu$. It is easy to see that $M/\mu$ is itself a homogeneous $S$-act $\cup_{d \in D}(M/\mu)_d$ with $(M/\mu)_d = M_d/\mu \cap M_d \times M_d$ ($d \in D$) if we put $[x]_\mu a = [xa]_\mu$ for all $x \in M$ and $a \in S$, by analogy with the case of factor structures obtained from a graded ring [2, 9, 21, 25] (see also [19]). If we write $[x]_\mu \# [y]_\mu$, it means that representatives of these classes come from the same component of $M$.

We may consider a homogeneous semigroup $S = \cup_{\delta \in \Delta} S_\delta$ as a homogeneous $S$-act under right multiplication with $FS = O(S)$. A homogeneous semigroup $S$ is right regular if it is regular as a homogeneous $S$-act. It is clear how to define a left regular homogeneous semigroup, and a homogeneous semigroup is said to be regular if it is both left and right regular, that is, if $\Delta$ is a cancellative partial groupoid. Let us notice that if $\mu$ is a right congruence on $S$, then any classical homomorphism of $S/\mu$ is a homomorphism of $S/\mu$ in the sense of Definition 2.3.

We proceed with definitions of the regular Jacobson radical and the regular simple radical of a homogeneous semigroup by analogy with the notion of the graded Jacobson radical of a graded ring from [8, 9].

The notion of an irreducible homogeneous $S$-act coincides with the classical one. A homogeneous $S$-act $M$ is said to be totally irreducible if $MS \not\subseteq FM$ and if $M$ has no non-trivial homomorphisms in the sense of Definition 2.3. Let $M$ be a homogeneous $S$-act and let us define a congruence $\delta_M \leq S \times S$ by $(a_1, a_2) \in \delta_M$ if and only if for all $x \in M$ we have $xa_1 = xa_2$.

**Definition 2.5.** The congruence $J'(S) = \cap_{\delta \in \Delta} \delta_M$, where $I$ is the set of all irreducible regular homogeneous $S$-acts is called the regular Jacobson radical of $S$. The congruence $P'(S) = \cap_{\delta \in \Delta} \delta_M$, where $I$ is the set of all totally irreducible regular homogeneous $S$-acts is called the regular simple radical of $S$. In both cases, if $I = \emptyset$, for the radical we take the universal congruence.

**Remark 2.6.** If $S$ is a homogeneous semigroup, then let us notice that the classical Jacobson (classical simple) radical of $S$, denoted by $J(S)$ ($P(S)$), is contained in the regular Jacobson (regular simple) radical $J'(S)$ ($P'(S)$) of $S$. Also, the classical Jacobson and the classical simple radical of a homogeneous semigroup correspond to the notion of the large Jacobson radical of a graded ring from [8].

Let us recall (see for instance [29]), a right congruence $\mu$ on a semigroup $S$ is called modular if it is distinct from the universal congruence and if there exists an element $e \in S$ such that $[e]a = [a] \in S/\mu$ for all $a \in S$. Such an element $e$ is called a left unity modulo $\mu$.

From [8, 9] we know that every graded left unity modulo a proper homogeneous modular right ideal of a regular graded ring has the same degree, which is a nonzero idempotent element of the grading set. Here we have the following result.

**Lemma 2.7.** Let $S = \cup_{\delta \in \Delta} S_\delta$ be a regular homogeneous semigroup and $\mu$ a modular right congruence on $S$. Then all the left unities modulo $\mu$ have the same degree which is an idempotent element of $\Delta^*$. This degree is called the degree of a modular right congruence $\mu$.

**Proof.** Let $e$ and $f$ be left unities modulo $\mu$ of $S$. Then $[e]a = [f]a = [a] \in S/\mu$ for all $a \in S$. Since $\mu$ is distinct from the universal congruence, there exists $a \in S$ such that $[e]a \not\in F(S/\mu)$. Now, since $S$ is regular, we obtain $ef = f$. Hence, all the left unities modulo $\mu$ come from the same component of $S$, say $S_e$, and moreover $e^2 = e$ since $e^2 \# e$. Note that $e^2 \not\in O(S)$, again since $\mu$ is distinct from the universal congruence. \(\Box\)

Since classical homomorphism theorems obviously transfer to the case of homogeneous acts, it is easy to verify the following lemma.

**Lemma 2.8.** A homogeneous $S$-act $M$ is strictly cyclic (totally irreducible) if and only if $M \cong S/\mu$, where $\mu$ is a modular right congruence (maximal modular right congruence) on $S$.

Following the proof of the fact that the classical Jacobson (classical simple) radical of a semigroup $S$ coincides with the intersection of all modular right congruences $\mu$ on $S$ for which $S/\mu$ is irreducible (of all maximal modular right congruences on $S$) (Corollary 6.15 and Corollary 6.16 in [10]), one may easily verify the following theorem.
Theorem 2.9. Let $S$ be a homogeneous semigroup. Then the regular Jacobson (regular simple) radical of $S$ coincides with the intersection of all (maximal) modular right congruences $\mu$ on $S$ for which $S/\mu$ is an irreducible (totally irreducible) regular homogeneous $S$-act.

Corollary 2.10. If $S$ is a regular homogeneous semigroup, then the regular Jacobson (regular simple) radical of $S$ coincides with the classical Jacobson (classical simple) radical of $S$.

Let $R = \bigoplus_{s \in S} R_s$ be a graded ring for which $S$ is a cancellative partial groupoid (a regular graded ring), and let $e$ be a nonzero idempotent element from $S$. We know from [8, 9] that then there exists a one-to-one correspondence between the maximal homogeneous modular right ideals of $R$ of degree $e$ and the maximal modular right ideals of $R$, given by $I \mapsto I \cap R_e$ (the inverse mapping given by $I \mapsto \bigoplus_{s \in S} \{ x \in R_s \mid xR \cap R_e \subseteq I \}$). (For similar observations regarding 2-graded rings, see [32].) This corresponds implies that the classical Jacobson radical of the ring $R$ coincides with the intersection of the ring $R_e$ and the graded Jacobson radical of $R$ (see [8, 9]). The following theorems represent generalizations of the above mentioned results to homogeneous semigroups. In the proof of the first of them we as usual write $S^1$ to designate a semigroup which is obtained from a semigroup $S$ by adjoining an identity element $1$.

Theorem 2.11. Let $S = \bigcup_{s \in S} S_s$ be a regular homogeneous semigroup, $|O(S)| \leq 1$, and $e \in \Delta'$ an idempotent element. Then there exists a one-to-one correspondence between (maximal) regular modular right congruences on $S$ of degree $e$ and (maximal) modular right congruences on $S_e$.

Proof. Let $\mu$ be a modular right congruence on the semigroup $S_e$, and $e$ a left unity modulo $\mu$. Let us define a relation $\tilde{\mu} \subseteq S \times S$ by $(a_1, a_2) \in \tilde{\mu}$ if and only if $(a_1, a_2)^{S_1} \cap S_e \times S_e \subseteq \mu$, where $(a_1, a_2)^{S_1}$ denotes the set $\{(a_1s, a_2 s) \mid s \in S \}$. Clearly, $\tilde{\mu}$ is a right congruence on $S$. We claim that $\tilde{\mu} \cap S_e \times S_e = \mu$. Let $(a_1, a_2) \in \mu$ and let $s \in S$ be such that $(a_1s, a_2 s) \in S_e \times S_e$, and $(a_1s, a_2 s) \notin O(S) \times O(S)$. Then, since $S$ is regular, we obtain that $s \in S$. Therefore $(a_1s, a_2 s) \in \mu$. Obviously, $(a_1, a_2) \in \mu$ for every $(a_1, a_2) \in \mu$. Hence $\mu \subseteq \tilde{\mu} \cap S_e \times S_e$. Now, let $(a_1, a_2) \in \tilde{\mu} \cap S_e \times S_e$. Since $e$ is a left unity modulo $\mu$, we have $(ea_1, a_2) \in \mu$. Therefore $(ea_1, ea_2) \in \tilde{\mu}$ which implies $(ea_1, ea_2) \in \mu$ since $ea_1, ea_2 \in S_e$. Now, since $(ea_1, a_2) \in \mu$ and $(ea_2, a_2) \in \mu$, we have $(a_1, a_2) \in \mu$. Hence $\tilde{\mu} \cap S_e \times S_e \subseteq \mu$. Let us now check whether $e$ is a left unity modulo $\tilde{\mu}$. So, let $a \in S$ and let $b \in S_1$ be such that $eab, ab \in S_e$. Since $e$ is a left unity modulo $\mu$, we have $(eab, ab) \in \mu$. Hence $(ea, a) \in \tilde{\mu}$. Thus $\tilde{\mu}$ is a modular right congruence on $S$ of degree $e$.

If $\mu$ is a maximal modular right congruence on $S_e$, then we claim that $\tilde{\mu}$ is a maximal right congruence on $S$. Since Corollary 6.4 in [5] also holds for right congruences on semigroups, there exists a maximal right congruence $\lambda$ on $S$ such that $\mu \subseteq \lambda$ and $\lambda \cap S_e \times S_e = \mu$. Since $\mu \subseteq \tilde{\mu}$ and $(a_1, a_2) \in \lambda$, then $(a_1, a_2)^{S_1} \cap S_e \times S_e \subseteq \lambda \cap S_e \times S_e = \mu$. Therefore $(a_1, a_2) \in \tilde{\mu}$. Hence $\tilde{\mu} \cap S_e \times S_e \subseteq \mu$. Therefore $\mu$ is a maximal right congruence.

On the other hand, if $\mu$ is a (maximal) modular right congruence on $S$ of degree $e$, then it is an easy exercise to prove that the relation $\mu \cap S_e \times S_e$ is a (maximal) modular right congruence on $S_e$.

Theorem 2.12. Let $S = \bigcup_{s \in S} S_s$ be a regular homogeneous semigroup, $|O(S)| \leq 1$, and $e \in \Delta'$ an idempotent element. Then $P(S_e) = P(S) \cap S_e \times S_e$.

Proof. Let $(a_1, a_2) \in P(S)$. We claim that $(a_1, a_2) \in P(S_e)$. In the proof we generalize the technique used in the proof of the corresponding inclusion regarding Jacobson radicals for the case of graded rings in [9].

Suppose that $(a_1, a_2) \notin P(S)$. Then, according to Corollary 2.10 and the proof of Theorem 2.11, there exists $x$ such that $(a_1x, a_2x) \in S_e \times S_e$, but $(a_1x, a_2x) \notin P(S_e)$, where $\xi \in \Delta'$ is an idempotent element. Since we assume that $S$ is regular, we have that $\delta = \delta(x)$, $\xi$ and $e$ are mutually distinct. If $T$ is any semigroup and $A = \{ s \in T \mid TsT \subseteq O(T) \}$, then $A$ is an ideal of $T$. Moreover, $A^2 \subseteq O(T)$. Therefore, if $\langle A \rangle$ is the Jacobson radical of $A$, we have that $\langle A \rangle = A \times A$ according to Proposition 2 in [29]. We know that the Jacobson radical of a semigroup is ideally hereditary (see Lemma 3.5 in [30]), and $A \times A \subseteq \langle A \rangle$. However, if $P(T)$ denotes the classical simple radical of $T$, then, as we know, $\langle A \rangle \subseteq P(T)$ (see for instance [29]). In our case we look at the semigroup $S_e$ and conclude that there exists $y \in S_e$ such that $a_1, a_2$ is not a left zero of $S_e$ or that there exists $z \in S_e$ such that $za_2x$ is not a left zero of $S_e$. Then we have $\xi(e \delta) = \xi^2 = \xi = e \delta \notin O(S)$. Since $S$ is by assumption regular, it follows that $\xi = e$, that is, $z = e$, a contradiction. Therefore, $S_e \times S_e \cap P(S) \supseteq P(S_e)$.
Conversely, we know that \( P(S) \) coincides with the intersection of all maximal modular right congruences \( \mu \) on \( S \). Denote the set of all maximal modular right congruences on \( S \) by \( C \). Then, for every \( \epsilon^2 = \epsilon \in \Delta' \), we have that \( S_{\epsilon} \times S_{\epsilon} \cap P(S) = S_{\epsilon} \times S_{\epsilon} \cap \bigcap_{\epsilon \in C} \mu \) is, according to Theorem 2.11, a subset of \( P(S_{\epsilon}) \). Hence, \( S_{\epsilon} \times S_{\epsilon} \cap P(S) \subseteq P(S_{\epsilon}) \). \( \square \)

In order to prove an analogous result for the Jacobson radical, we add more assumptions on a regular homogeneous semigroup which make it an irreducible act over itself.

**Theorem 2.13.** Let \( S = \bigcup_{\epsilon \in \Delta} S_{\epsilon} \) be a regular homogeneous semigroup such that \( |O(S)| = 1 \). Also, let us assume that \( S_{\xi} \cap S_{\eta} = O(S) \) for all distinct \( \xi, \eta \in \Delta \), and that for every \( a \in S \setminus O(S) \) and every \( b \in S \) there exists \( s \in S \) such that \( b = as \). If \( \epsilon \in \Delta' \) is an idempotent element, then \( J(S_{\epsilon}) = J(S) \cap S_{\epsilon} \times S_{\epsilon} \).

*Proof.* Let us denote the unique left zero of \( S \) by \( z \). In Theorem 2.11 we have established a one-to-one correspondence between modular right congruences on \( S \) of degree \( \epsilon \) and modular right congruences on \( S_{\epsilon} \). Let \( \hat{\mu} \) be a modular right congruence on \( S \) of degree \( \epsilon \). Then \( S_{\epsilon} / \hat{\mu} \) is an irreducible homogeneous \( S \)-act. Indeed, according to assumptions on \( S \), we have that for all \( [x]_\epsilon \not\in F(S/\hat{\mu}) \) and all \( [y]_\hat{\mu} \in S_{\epsilon} / \hat{\mu} \), there exists \( s \in S \) such that \( [y]_\hat{\mu} = [xz]_s = [x]_\epsilon s \). Also, \( (S/\hat{\mu}) \sigma \cap (S/\hat{\mu}) \eta = [z]_\epsilon \sigma \cap (S/\hat{\mu}) \eta \) for all distinct \( \xi, \eta \in \Delta \). We claim that \( F(S/\hat{\mu}) = [z]_\epsilon \). Indeed, if \( [x]_\hat{\mu} \in F(S/\hat{\mu}) \), then \( [xz]_\epsilon = [x]_\hat{\mu} z = [z]_\epsilon \). On the other hand, \( xz \in O(S) \), and therefore \( xz = z \). Hence \( [x]_\hat{\mu} = [z]_\epsilon \). Thus \( S_{\epsilon} / \hat{\mu} \) is an irreducible \( S \)-act. Let \( \mu = \hat{\mu} \cap S_{\epsilon} \times S_{\epsilon} \) be the corresponding modular right congruence on \( S_{\epsilon} \). We claim that \( S_{\epsilon} / \mu \) is an irreducible \( S_{\epsilon} \)-act. Let us first prove that \( F(S_{\epsilon}/\mu) = 1 \), similarly to above. We know that \( [z]_\mu \not\in F(S_{\epsilon}/\mu) \). Let \( [x]_\mu \in F(S_{\epsilon}/\mu) \). Then \( [xz]_\mu = [x]_\mu z = [z]_\mu \), but \( z = xz \in O(S) \), and hence \( [x]_\mu = [z]_\mu \). Now, let \( [x]_\mu \not\in [z]_\mu \) and \( [y]_\mu \) be arbitrary elements of \( S_{\epsilon} / \mu \). We know that there exists \( s \in S \) such that \( xs = y \). Now, if \( \neq z \), then the regularity of \( S \) implies that \( s \in S_{\epsilon} \). On the other hand, if \( y = z \), then again there exists \( s \in S_{\epsilon} \) such that \( xs = z \), namely \( s = z \). Therefore there exists \( s \in S_{\epsilon} \) such that \( [y]_\mu = [x]_\mu s \). Hence \( S_{\epsilon} / \mu \) is an irreducible \( S_{\epsilon} \)-act. We have proved that the one-to-one correspondence from Theorem 2.11 is in our case a one-to-one correspondence between modular right congruences \( \hat{\mu} \) on \( S \) of degree \( \epsilon \) such that \( S_{\epsilon} / \hat{\mu} \) is irreducible and modular right congruences \( \mu \) on \( S_{\epsilon} \) such that \( S_{\epsilon} / \mu \) is irreducible. Therefore, the claim of the theorem can now be proved analogously to Theorem 2.12. \( \square \)

**Remark 2.14.** Examples of semigroups which satisfy conditions of the previous theorem exist and are among semigroups obtained from homogeneous parts of graded rings under multiplication. For instance, if \( R \) is a division ring, then the homogeneous part of the graded ring \( \left( \begin{array}{cc} R & 0 \\ 0 & R \end{array} \right) \oplus \left( \begin{array}{cc} 0 & 0 \\ 0 & R \end{array} \right) \) is such an example.

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