Hedging Error Estimate of the American Put Option Problem in Jump-Diffusion Processes

Sultan Hussain\(^a\), Salman Zeb\(^b\), Muhammad Shoaib Saleem\(^c\), Nasir Rehman\(^d\)

\(^a\)Department of Mathematics, COMSATS University Islamabad Abbottabad Campus, Abbottabad, Pakistan
\(^b\)Department of Mathematics, University of Malakand, Chakdara, Pakistan
\(^c\)Department of Mathematics, University of Okara, Okara, Pakistan
\(^d\)Department of Mathematics, Allama Iqbal Open University, Islamabad, Pakistan

Abstract. We consider discrete time hedging error of the American put option in case of brusque fluctuations in the price of assets. Since continuous time hedging is not possible in practice so we consider discrete time hedging process. We show that if the proportions of jump sizes in the asset price are identically distributed independent random variables having finite moments then the value process of the discrete time hedging uniformly approximates the value process of the corresponding continuous-time hedging in the sense of \(L_1\) and \(L_2\)-norms under the real world probability measure.

1. Introduction

Merton [20] considered incomplete markets in order to introduce jump-diffusion process in the theory of option valuation and formulated pricing formulas for European kind options assuming that the jump risk is un-priced (in that setup the sample path is not continuous). Merton work [20] was later generalized by several authors, for example Naik and Lee [21] analyzed the option price where the underlying asset is the market portfolio with discontinuous returns, Huyôn Pham [15] studied the pricing of the American put option by applying the free-boundary approach while Van Moerbeke [27] converted the American option pricing and analysed into free boundary problem in the case of diffusion model. Bayraktar and Xing [2] studied and approximated the price of the American put for jump diffusions by a sequence of functions, which converges exponentially to the price function uniformly. Hussain et al. [12] investigated regularity properties of value process of the American option problem in the jump-diffusion case.

It is shown in Shreve [25] that construction of perfect hedging of the American style options strongly depends on the first order derivative of the corresponding value function. As there does not exist perfect valuation of the American style options, so there is no perfect hedging strategy. Moreover, perfect hedging requires continuous time trading on financial markets which is not possible in practice. One can trade discrete times only. In this regards, Hussain and Shashiashvili [13] constructed discrete time hedging strategy of the American put option problem. In this setting the underlying stock neither pays dividends.

2010 Mathematics Subject Classification. 91B28, 60J75, 60G60

Keywords. American Option, Jump Diffusion Process, Discrete Time Hedging.

Received: 19 June 2017; Accepted: 03 December 2017

Communicated by Miljana Jovanović
Higher Education Commission of Pakistan

Email addresses: tauief775650@yahoo.co.in (Sultan Hussain), zebsalman@gmail.com (Salman Zeb), shaby455@yahoo.com (Muhammad Shoaib Saleem), nasirzainy@hotmail.com (Nasir Rehman)
nor having brusque variations in its price. Hussain and Rehman [14] extended the latter work to American style option to the dividends paying stock.

Financial markets are full of uncertainties which make the stock price discontinuous like release of unexpected economic figure, political changes or natural disasters etc may lead to brusque changes in asset prices. In this case the stock market is incomplete and therefore, there is no perfect hedging strategy to hedge the underlying option. However, many authors studied some admissible delta hedging strategy (see for example [19]). To hedge the jump risk, He et al. [11] explored two different hedging strategies: a semi-static approach which uses a portfolio of the underlying and traded short maturity options to hedge a long maturity option, and a dynamic technique which involves frequent trading of options and the underlying. Kennedy et al. [18] study that, in the case of incomplete market, hedging a contingent claim written on the asset is not a trivial matter, and other instruments besides the underlying must be used to hedge in order to provide adequate protection against jump risk. They devised a dynamic hedging strategy that uses a hedge portfolio consisting of the underlying asset and liquidly traded options, where transaction costs are assumed present due to a relative bid-ask spread. In our knowledge, probably no body has investigated hedging error of the American options in discrete time setting under this phenomenon.

In this article, we consider American put option on a stock the price of which has brusque variations i.e., American put option in jump-diffusion process and investigate the corresponding discrete time hedging error. Our main result shows that the value process in discrete time setting converges uniformly to the continuous-time setting in the sense of $L_1$ and $L_2$-norms under the real world probability measure.

For further details, we refer the readers to Salman [23].

2. Formulation of Main Problem and some Preliminary Results

We consider the American put option in jump diffusion processes in a financial market which is incomplete so we have brusque variations in the assets price and focus on the delta hedging which in this framework is not perfect however used by practitioners. We divide the time interval $[0, T]$ where $T$ is the time to maturity of the corresponding option, into $n$ equal parts and denote by $\delta = \frac{T}{n}$. We consider the value process of the discrete time delta hedging strategy minimising the risk at maturity in jump-diffusion process and, using purely probabilistic approach, we estimate the discrete-time hedging error and find that it is proportional to the square root of $\delta \log \frac{L}{T}$.

In an incomplete financial market, we consider the probability space $(\Omega, \mathcal{F}, P)$ and define the Wiener Process $B = (B_t)$, Poisson Process $N = (N_t)$ with parameter $\lambda$ and $(U_t)_{t \geq 1}$ a sequence of iid random variables taking values in the open interval $(−1, \infty)$ with common density $f(z)$, average expected value $\beta$ with the assumption that the mathematical expectations $E(U_1^n) < \infty$, for all $i, m \in \mathbb{N}$. We take time $T$ to be finite and $\sigma$–fields of $(B_t), (N_t), (U_t)$ to be independent.

We make the $P$–completion of our natural filtration of $(B_t), (N_t)$ and $(U_t)_{t \leq N_t}, i \geq 1$ by $(\mathcal{F}_t)$. On the so formed filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, let us consider a financial market having assets: $(M_t, Q_t, 0 \leq t \leq T)$ a bank account and the value of a stock respectively with jumps in the proportions $U_1, U_2, ..., \text{at the random times}$. We also suppose that $\tau_t, s$ represent the jump timings of $(N_t)$.

The assets $M_t$ and $Q_t$ evolve according to following ordinary and stochastic differential equations

$$dM_t = rM_t dt, \quad M_0 = 1, \quad 0 \leq t \leq T,$$  

$$dQ_t = Q_{t^-} \left( b dt + \sigma dB_t + d \left( \sum_{i=1}^{N_t} U_i \right) \right), \quad Q_0 > 0, \quad 0 \leq t \leq T,$$  

where the drift term $b$, the interest rate $r$ and the stock volatility $\sigma$ are positive constants and $Q_{t^-}$ is the left-hand limit of $Q_t$ at time $t$.

From (2), unique solution of the stock price $Q_t$ is given by

$$Q_t = Q_0 \left( \prod_{i=1}^{N_t} (1 + U_i) \right) e^{b \left( t - \tau_s \right) \frac{\sigma^2}{2} + \sigma B_t}.$$
Under the assumption that $E[|U_i|] < \infty$, where $E$ stands for the mathematical expectation, the discounted stock price $\tilde{Q}_t = e^{-rt}Q_t$ is a martingale (see, Lamberton and Lapeyre [19]) if

$$b = r - \lambda E(U_1).$$

(4)

In the corresponding framework, let $\nu(t, x), 0 \leq t \leq T, x \geq 0$ denotes arbitrary price function (as the market is incomplete, so there is no unique price of the put American option under the physical probability measure) of the American put option then, from Lamberton and Lapeyre [19], an admissible strategy minimising the risk at maturity the writer of the option should hold

$$\Delta(t) = \frac{1}{\sigma^2 + \lambda E(U_1^2)} \left[ \frac{\sigma^2}{\lambda} \frac{\partial}{\partial x} \nu(t, x) + \lambda \int Z(t, x(1 + z)) - \nu(t, x) \, df(z) \right]$$

(5)

number of stocks at time $t$, where $f(z)$ is the law of random variables $U_i$.

The above admissible delta hedging strategy is replicable (that is the payoff at maturity time will be equal to the final value of the delta hedging with $P$ a.s. (see Lamberton and Lapeyre [19] Chapter 7)) however, it requires continuous time trading on the financial market. No one can trade continuously in time.

Practically, the discrete time strategy defined through $\Delta(t)$ is the following:

$$\Delta_{\delta}(t) = \Delta(t_{i-1}), t_{i-1} \leq t < t_i, k = 1, 2, ..., n,$$

(6)

where $\delta = \frac{t_i - t_{i-1}}{n}$, the subscript $\delta$ indicates the error of approximation.

The aim here is to derive an estimate for the error due to the fact that the portfolio is re-adjusted at discrete non-random dates (say $k\delta$, $k = 0, 1, ..., n$ with $\delta = \frac{1}{n}$). Let us denote by $\Pi_0(t), 0 \leq t \leq T$, the value of the portfolio corresponds to the discrete time strategy $\Delta_{\delta}(t)$ at time $t$ defined in (6). Assume the hedger of the option starts with initial amount $\nu(0, Q_0) = \nu(0)$ and adjusts his portfolio at each time $t_0, k = 0, 1, ..., n - 1$. He holds $\Delta_{\delta}(t_k)$ number of shares from stock in time interval $[t_k, t_{k+1})$ and invest the remainder $\nu_\delta(t_k) - \Delta_{\delta}(t_k)Q_{0t}$ in money market account at time $t_k$. The value process $\Pi_0(t)$ can be expressed as

$$\nu_\delta(t) = e^{rt} \left[ \nu(0) + \int_0^t \Delta_{\delta}(u) d\tilde{Q}_u \right], 0 \leq t \leq T.$$  

(7)

Similar representation is valid for self-financing continuous time portfolio value process $\nu(t)$

$$\nu(t) = e^{rt} \left[ \nu(0) + \int_0^t \Delta(u) d\tilde{Q}_u \right].$$

(8)

Discrete time error of American option for admissible strategy (5) is then given as

$$E \sup_{0 \leq t \leq T} \left| \nu(t) - \nu_\delta(t) \right| = E \sup_{0 \leq t \leq T} e^{rt} \left| \int_0^t (\Delta(u) - \Delta_{\delta}(u)) d\tilde{Q}_u \right|,$$

(9)

where $E$ represents the mathematical expectation with respect to the real world probability measure $P$.

To estimate the above discrete-time error, we have to pass on to the new probability measure $\hat{P}$. Let $f(y)$ be a density function such that $f(y) = 0$ whenever $f(y) = 0$ and define the process

$$Z_t = e^{-\theta y - \frac{1}{2} \sigma^2 y^2} e^{(\theta - \frac{1}{2} \sigma^2)yt} \prod_{i=1}^{N_t} \frac{\lambda f(U_i)}{\hat{f}(U_i)}, 0 \leq t \leq T,$$

(10)

where $\theta = \frac{b + r + \lambda \beta}{\sigma}$ is the market price of risk, $\lambda$ is the intensity of $N_t$, $\beta$ is the average value and $\hat{f}(y)$ is the common density of $U_1, U_2, ...$ with $P$, where

$$\hat{P}(C) = \int_C Z_T dP, \text{ for all } C \in \mathcal{F}_T.$$
Since \( J \) is increments of \( \tilde{N} \) then from (13), we can write
\[
\Delta \tilde{N} \equiv \begin{cases} J \Delta & \text{if there is no jump at time } t \\
0 & \text{otherwise}
\end{cases}
\]

and in the case there is a jump at \( t \) it is clear (see, for example, Shreve [25] Theorem 11.3.1) that the compensated Poisson process
\[
\Delta \tilde{N} \equiv \begin{cases} J \Delta & \text{if there is no jump at time } t \\
0 & \text{otherwise}
\end{cases}
\]
\[
\Delta \tilde{J} \equiv \begin{cases} J \Delta & \text{if there is no jump at time } t \\
0 & \text{otherwise}
\end{cases}
\]

and in particular, \( \bar{E} Y_t = 1 \), for all \( t \geq 0 \).

**Proof.** In the proof, we follow the book by Shreve [25]. Let us define the pure jump process \( J_t \), the compensated Poisson processes \( H_t \) and \( R_t \), as
\[
J_t = \prod_{i=1}^{N_t} \left( \frac{\lambda f(U_i)}{\lambda f(U_i)} \right)^2 , \quad H_t = \sum_{i=1}^{N_t} \left( \frac{\lambda f(U_i)}{\lambda f(U_i)} \right)^2 , \quad R_t = \sum_{i=1}^{N_t} U_i , \quad 0 \leq t \leq T.
\]

As \( \Delta R_t = R_t - R_{t-} = U_{N_t}, 0 \leq t \leq T \), at the jump times of \( R \), we have
\[
\Delta J_t = \left[ \left( \frac{\lambda f(\Delta R_t)}{\lambda f(\Delta R_t)} \right)^2 - 1 \right] J_{t-} , \quad \Delta H_t = \left( \frac{\lambda f(\Delta R_t)}{\lambda f(\Delta R_t)} \right)^2 , \quad 0 \leq t \leq T.
\]

Let us calculate
\[
\bar{E} \left( \frac{\lambda f(U_i)}{\lambda f(U_i)} \right)^2 = \left( \frac{\lambda}{\lambda} \right)^2 \int_{-\infty}^{\infty} \frac{f^2(u)}{\lambda f(u)} f(u) du = \left( \frac{\lambda}{\lambda} \right)^2 \mathcal{L}.
\]

It is clear (see, for example, Shreve [25] Theorem 11.3.1) that the compensated Poisson process
\[
H_t = \frac{\lambda^2}{\lambda^2} \mathcal{L}
\]

is a martingale with \( \bar{P} \).

Note that if there is no jump at time \( t \), then \( N_t = N_{t-} \) which further gives

\[
\Delta N_t = 0 , \quad \Delta J_t = 0 \quad \text{and} \quad \Delta H_t = 0,
\]

and in the case there is a jump at \( t \) then \( \Delta N_t = 1 \) and therefore, we can write

\[
\Delta J_t = J_t - J_{t-} \Delta N_t.
\]

Thus from (13), we can write

\[
\Delta J_t = J_{t-} \Delta H_t - J_{t-} \Delta N_t , \quad 0 \leq t \leq T.
\]

Since \( J_t, H_t \) and \( N_t \) are all pure jump process, therefore the latter expression can be written as

\[
dJ_t = J_{t-} dH_t - J_{t-} dN_t , \quad 0 \leq t \leq T.
\]
Using Itô’s product rule for the jump process $\prod_{i=1}^{N_t} \left( \frac{1/f(U_i)}{1/f(U_i)} \right)^2$ and continuous process $e^{2(\lambda - \dot{\lambda}) t}$, we express

$$Y_t = Y_0 + 2(\lambda - \dot{\lambda}) \int_0^t J_s e^{2(\lambda - \dot{\lambda}) s} ds + \int_0^t e^{2(\lambda - \dot{\lambda}) s} dJ_s$$

$$= 1 - \int_0^t e^{2(\lambda - \dot{\lambda}) s} J_s dN_s + \int_0^t e^{2(\lambda - \dot{\lambda}) s} J_s d(H_s - 2\lambda s + 3\dot{\lambda} s),$$

(17)

where we have used equation (16).

From expressions (15) and (17), it is clear that $Y_t$ is a martingale if

$$H_t - \frac{\lambda^2}{\ddot{\lambda}^2} L_t = H_t - 2\lambda t + 3\dot{\lambda} t, \quad 0 \leq t \leq T,$$

from where we get the required value of $L$.

Moreover, because $Z(0) = 1$ and $Z(t)$ is a martingale, so we have $\tilde{E}Z(t) = 1$ for all $t \geq 0$ under the condition (12).

Let us define a new Brownian motion $W_t = B_t + \Theta_t, 0 \leq t \leq T$. By Girsanov theorem $W_t$ is a Wiener Process with $\tilde{P}$ with the same $\mathcal{F}_t$. Equation (2) is transformed as

$$dQ_t = Q_t \left[ (rdt + \sigma dW_t) + d \left( R_t - \dot{\lambda} \tilde{\beta} t \right) \right],$$

(18)

and the discounted stock price $\tilde{Q}_t = e^{-rt} Q_t$ satisfies

$$d\tilde{Q}_t = \tilde{Q}_t \left[ (\sigma dW_t + d \left( R_t - \dot{\lambda} \tilde{\beta} t \right)) \right], \quad \tilde{Q}_0 = Q_0.$$

(19)

Moreover, both the expectations are related as

$$EC = \tilde{E} \left( Z_T^{-3} C \right),$$

for any arbitrary square integrable $\mathcal{F}_T$-measurable random variable $C$.

Making use of (10) and (11), the latter relation gives

$$EC \leq \left( \tilde{E}Y_T \right)^{\frac{1}{3}} \left( \tilde{E}C^2 \right)^{\frac{1}{2}}.$$

(20)

To estimate explicitly $\tilde{E}Y_T$, we use the fact that the expressions (15) and $N_t - \dot{\lambda} t$ are martingales, from (17) we can write

$$\tilde{E}Y_T = 1 + \left[ \frac{\lambda^2}{\ddot{\lambda}} - 2\lambda + 3\dot{\lambda} \right] \tilde{E} \int_0^T e^{2(\lambda - \dot{\lambda}) s} J_s ds.$$

From Fubini’s theorem, Gronwall’s inequality and the process $J$ having finitely many jumps, we obtain

$$\tilde{E}Y_T = 1 + \left[ \frac{\lambda^2}{\ddot{\lambda}} - 2\lambda + 3\dot{\lambda} \right] \int_0^T \tilde{E}e^{2(\lambda - \dot{\lambda}) s} J_s ds \leq \exp \left\{ \left[ \frac{\lambda^2}{\ddot{\lambda}} - 2\lambda + 3\dot{\lambda} \right] T \right\}.$$
By using the latter estimate and the bound (20), we get the following estimate for (9)

\[ E \sup_{0 \leq t \leq T} |\pi(t) - \pi_0(t)| \leq e^{\left(\frac{\gamma}{2}\right)^2 \frac{\left(\lambda - 1\right)^2}{\sigma^2}} \left( E \int_0^T (\Lambda(u) - \Lambda_0(u))^2 \, \sigma^2 \, du \right)^{\frac{1}{2}} \]

\[ \leq 2e^{\left(\frac{\gamma}{2}\right)^2 \frac{\left(\lambda - 1\right)^2}{\sigma^2}} \left( E \sup_{0 \leq t \leq T} \left\{ \left( \int_0^T (\Lambda(u) - \Lambda_0(u)) \, \sigma^2 \, du \right)^{\frac{1}{2}} \right\} \right)^2 \]

\[ + \left( \int_0^T (\Lambda(u) - \Lambda_0(u)) \, \sigma^2 \, du \right)^{\frac{1}{2}} \left( E \sup_{0 \leq t \leq T} \left\{ \left( \int_0^T (\Lambda(u) - \Lambda_0(u)) \, \sigma^2 \, du \right)^{\frac{1}{2}} \right\} \right)^{\frac{1}{2}}. \]

Using Doob’s maximal inequality, we can write

\[ E \sup_{0 \leq t \leq T} |\pi(t) - \pi_0(t)| \leq 8e^{\left(\frac{\gamma}{2}\right)^2 \frac{\left(\lambda - 1\right)^2}{\sigma^2}} \left( E \int_0^T (\Lambda(u) - \Lambda_0(u))^2 \, \sigma^2 \, du \right)^{\frac{1}{2}} \]

\[ + \left( E \int_0^T (\Lambda(u) - \Lambda_0(u))^2 \, \sigma^2 \, du \right)^{\frac{1}{2}} \left( E \int_0^T (\Lambda(u) - \Lambda_0(u))^2 \, \sigma^2 \, du \right)^{\frac{1}{2}} \]

\[ = 8e^{\left(\frac{\gamma}{2}\right)^2 \frac{\left(\lambda - 1\right)^2}{\sigma^2}} \sqrt{\sigma^2 + \lambda \Lambda_0(T)^2} \left( E \int_0^T (\Lambda(u) - \Lambda_0(u))^2 \, \sigma^2 \, du \right)^{\frac{1}{2}} \]

where we have used the assumption that the number of jumps is finite.

Assume \( v(t, x) \) be value function of American put option at time \( t \) and \( T_{t,T} \) denotes all \( (F_u)_{0 \leq u \leq T} \)-measurable stopping times \( \tau \) from \( t \) to \( T \). Then in particular, we have

\[ v(t, x) = \sup_{\tau \in T_{t,T}} \mathbb{E} \left[ e^{-r(t-\tau)} g(\pi_0(T_{t,T})(x)) \right], 0 \leq t \leq T, x \geq 0, \]

(22)

where \( g(x), x \geq 0, \) is non-increasing bounded below convex payoff function with assumption \( g(0) = g(0+) \) and the stock price \( Q_0(t, x) \), evolves as

\[ dQ_0(t, x) = Q_0(t, x) \left( rdu + \sigma dW_u + d \left( R_u - \lambda \beta u \right) \right), t \leq u \leq T, \]

with \( Q_0(t, x) = x \).

Assume the mathematical expectation of \( U_1, U_1^2, |\ln(1 + U_1)| \) and \( \ln^2(1 + U_1) \) is finite, we come to the following result:

**Proposition 2.2.** Assume \( \Delta(t), 0 \leq t \leq T, \) represents the continuous time delta strategy minimizing the risk at maturity, defined as (5), of the American option and \( \Delta_0(t) \) the corresponding discrete time trading strategy given in (6). Then the following estimate is valid

\[ \mathbb{E} \int_0^T (\Lambda(u) - \Lambda_0(u))^2 \, \sigma^2 \, du \leq a \ln \frac{T}{\delta} \cdot \delta, \]

(23)

where \( a \) depends on \( Q_0, g(0), r, \sigma, \lambda, \beta, T, \mathbb{E}(U_1), \mathbb{E}(U_1^2), \mathbb{E}(|\ln(1 + U_1)|), \mathbb{E}(\ln^2(1 + U_1)), \) and \( N(2\sigma^2 T) \), where \( N(\cdot) \) stands for standard normal distribution.
Proof. Using expressions (5) and (6), we can express
\[
\bar{E} \int_0^T (\Delta(t) - \Delta_0(t))^2 \, Q^2 \, dt = \frac{1}{(\sigma^2 + \Lambda E(t_1))^2} \bar{E} \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \left[ \sigma Q_i \left( \frac{\partial v(t, Q_i)}{\partial x} - \frac{\partial v(t_{k-1}, Q_{h_{k-1}})}{\partial x} \right) \right] \, dt - \lambda \int z \left( v(t, Q_i) - v(t, Q_i(1 + z)) - \frac{Q_i}{Q_{h_{k-1}}} \left( v(t_{k-1}, Q_{h_{k-1}}) - v(t_{k-1}, Q_{h_{k-1}}(1 + z)) \right) \right) \, df(z) \, dt \leq \frac{2}{(\sigma^2 + \Lambda E(t_1))^2} \bar{E} \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \left[ \sigma^2 Q_i^2 \left( \frac{\partial v(t, Q_i)}{\partial x} - \frac{\partial v(t_{k-1}, Q_{h_{k-1}})}{\partial x} \right) \right] \, dt + 2\lambda^2 \int z^2 \left( \left( v(t, Q_i) - \frac{Q_i}{Q_{h_{k-1}}} v(t_{k-1}, Q_{h_{k-1}}) \right)^2 + \left( v(t, Q_i(1 + z)) - \frac{Q_i}{Q_{h_{k-1}}} v(t_{k-1}, Q_{h_{k-1}}(1 + z)) \right) \right) \, df(z) \, dt.
\]
(24)

We denote by \( \Gamma(t, y) = y \frac{\partial v(t, y)}{\partial y}, 0 \leq t < T, y > 0 \) and express
\[
\bar{E} \sum_{k=1}^n \int_{t_{k-1}}^{t_k} Q_i^2 \left( \frac{\partial v(t, Q_i)}{\partial x} - \frac{\partial v(t_{k-1}, Q_{h_{k-1}})}{\partial x} \right) \, dt = \bar{E} \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \left( \Gamma(t, Q_i) - \Gamma(t_{k-1}, Q_{h_{k-1}}) \frac{Q_i}{Q_{h_{k-1}}} \right) \, dt.
\]
(25)

Let us write
\[
\Gamma(t, Q_i) - \Gamma(t_{k-1}, Q_{h_{k-1}}) \frac{Q_i}{Q_{h_{k-1}}} = \Gamma(t, Q_i) - \Gamma(t_{k-1}, Q_i) + \Gamma(t_{k-1}, Q_i) - \Gamma(t_{k-1}, Q_{h_{k-1}}) + \Gamma(t_{k-1}, Q_{h_{k-1}}) \frac{Q_i}{Q_{h_{k-1}}},
\]
(26)

therefore, we can write
\[
\bar{E} \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \left( \Gamma(t, Q_i) - \Gamma(t_{k-1}, Q_{h_{k-1}}) \frac{Q_i}{Q_{h_{k-1}}} \right)^2 \, dt \leq 4\bar{E} \sum_{k=1}^n \left[ \int_{t_{k-1}}^{t_k} \left( \Gamma(t, Q_i) - \Gamma(t_{k-1}, Q_i) \right)^2 \, dt + \int_{t_{k-1}}^{t_k} \left( \Gamma(t_{k-1}, Q_i) - \Gamma(t_{k-1}, Q_{h_{k-1}}) \right)^2 \, dt \right] + \int_{t_{k-1}}^{t_k} \Gamma^2(t_{k-1}, Q_{h_{k-1}}) \frac{Q_{h_{k-1}} - Q_i}{Q_{h_{k-1}}} \, dt.
\]
(27)

We estimate all the integrals on the right side of previous inequality by using Propositions 3.1 and 3.4 from Hussain et al. [12] which state that the mapping \( \zeta(t, y) = y \frac{v(t, y)}{\partial y} \) is Lipschitz continuous in \( y \), that is,
\[
|\zeta(t, x) - \zeta(t, y)| \leq 2g(0)|x - y|, 0 \leq t \leq T, 0 < x \leq y < \infty,
\]
and locally Lipschitz continuous in \( t \), that is,
\[
|\zeta(t, y) - \zeta(s, y)| \leq \frac{C y}{\sqrt{T - t}} |t - s|, 0 \leq s \leq t \leq T, y > 0,
\]
and \( \Gamma(t, y) = y \frac{\partial v(t, y)}{\partial y} \) satisfies
\[
|\Gamma(t, y) - \Gamma(s, y)| \leq \frac{G + yH}{\sqrt{T - t}} |t - s|^2, 0 \leq t < T, y > 0,
\]
(28)
(29)
(30)
where the positive constants $C$, $G$ and $H$ are functions of $r$, $\sigma$, $g(0)$, $\lambda$, $E[U_{l}]$, $E[\ln(1 + U_{l})]$ and $T$ and the fact that $v(t, x) \leq g(0)$, $0 \leq t \leq T, x > 0$, we write

$$
\bar{E} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_{k}} (\Gamma(t, Q_{t}) - \Gamma(t_{k-1}, Q_{t}))^2 dt \leq \bar{E} \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_{k}} \frac{(G + Q_{t}H)^2}{T - t} (t - t_{k-1}) dt + \bar{E} \int_{t_{k-1}}^{T} (\Gamma(t, Q_{t}) - \Gamma(t_{k-1}, Q_{t}))^2 dt
$$

$$
\leq 2\delta \bar{E} \int_{0}^{T} \frac{G^2 + Q_{t}^2H^2}{T - t} dt + 36g^2(0) \cdot \delta.
$$

where $\delta = \frac{T}{n}$, $n = 1, 2,...$

Using the unique solution $Q_{t}$ of (18), we find

$$
\bar{E}Q_{t}^2 = Q_{0}^{2}e^{2r(\beta + \sigma^{2} + \lambda(\bar{E}(1 + U_{l})^{2} + 1)T)}.
$$

Therefore

$$
\bar{E} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_{k}} (\Gamma(t, Q_{t}) - \Gamma(t_{k-1}, Q_{t}))^2 dt \leq 2 \left[ (G + H^{2}Q_{0}^{2}) \exp \left[ (2r + 2\lambda\beta + \sigma^{2} + \lambda(\bar{E}(1 + U_{l})^{2} + 1)T) \right] \right] \cdot \delta \ln \frac{T}{R} + 36g^2(0) \cdot \delta. \quad (31)
$$

To estimate the second expectation on the right hand side of (27), we introduce the change of variable $z = \ln x$ and define $u(t, z)$ such that

$$
u(t, z) = v(t, e^{z}), \quad 0 \leq t \leq T, \quad -\infty < z < \infty. \quad (32)
$$

Replace the new function $u(t, z)$ in (bounds (22) in [12]) the system of inequalities

$$
\begin{cases}
-r\nu(t, x) + \frac{\partial\nu(t, x)}{\partial t} - \lambda xEU_{l}e^{-rt} \frac{\partial\nu(t, x)}{\partial x} + \frac{\sigma^{2}e^{-2rt}}{2} \frac{\partial^{2}\nu(t, x)}{\partial x^{2}} \\
-\lambda \int (v(t, x(1 + z)) - v(t, x)) df(z) \leq 0, \\
\frac{\partial^{2}\nu(t, x)}{\partial x^{2}} \geq 0,
\end{cases}
$$

a.e. in $[0, T) \times \mathbb{R}^{+}$; $x > 0$,

where $f$ is the common law of the random variables $U_{l}$’s, we obtain

$$
\begin{cases}
-r\nu(t, z) + \frac{\partial\nu(t, z)}{\partial t} - (\lambda EU_{l} + \frac{\sigma^{2}}{2} e^{-rt}) \frac{\partial\nu(t, z)}{\partial z} \\
+ \frac{\sigma^{2}}{2} e^{-2rt} \frac{\partial^{2}\nu(t, z)}{\partial z^{2}} - \lambda \int (u(t, z + \ln(1 + z)) - u(t, z)) df(z) \leq 0, \\
\frac{\partial^{2}\nu(t, z)}{\partial z^{2}} \geq \frac{\partial u(t, z)}{\partial z},
\end{cases}
$$

a.e. in $[0, T) \times \mathbb{R}$.

This system gives

$$
\left| \frac{\partial^{2}u(t, z)}{\partial z^{2}} \right| \leq \left| \frac{\partial u(t, z)}{\partial z} \right| + 2 \frac{\sigma^{2}}{2} e^{\sigma^{2}T} \left[ ru(t, z) + \left| \frac{\partial u(t, z)}{\partial t} \right| + \left( \lambda EU_{l} + \frac{\sigma^{2}}{2} \right) \left| \frac{\partial u(t, z)}{\partial z} \right| \right]
\right.
$$

$$
+ \lambda \int \left[ u(t, z + \ln(1 + z)) - u(t, z) \right] df(z).
$$

Using Proposition 2.3 from [12], which states that the mapping $u(t, z)$ satisfies

$$
\begin{align*}
|u(t, y) - u(t, z)| &\leq g(0)|y - z|, 0 \leq t \leq T, \quad y, z \in \mathbb{R}, \\
|u(t, z) - u(s, z)| &\leq \frac{A}{\sqrt{T - t}}|t - s|, 0 \leq s \leq t \leq T, \quad z \in \mathbb{R},
\end{align*}
$$

(33)
where $A$ depends on $Q_0$, $r$, $\sigma$, $g(0)$, $\lambda$, $E(U_t)$, and $T$, we obtain

$$\left| \frac{\partial^2 u(t,z)}{\partial z^2} \right| \leq \frac{F}{\sqrt{T-t}},$$

(34)

where $F$ depends on $r$, $\sigma$, $g(0)$, $\lambda$, $E(U_1)$, $E[\ln(1+U_1)]$ and $T$.

By using the continuity of $\frac{\partial u(t,z)}{\partial z}$ (see, Pham [15] and Zhang [29]) and the bound (34), we get

$$\left| \Gamma(t,x) - \Gamma(t,y) \right| \leq \frac{F}{\sqrt{T-t}} |\ln x - \ln y|, \quad 0 \leq t < T, \ x, y \in (0, \infty).$$

(35)

Thus we can write

$$\bar{E} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (\Gamma(t_{k-1},Q_i) - \Gamma(t_{k-1},Q_{b_{k-1}}))^2 \, dt \leq \bar{E} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{F^2}{T-t_{k-1}} (\ln Q_i - \ln Q_{b_{k-1}})^2 \, dt.$$

(36)

Solution of (18) further gives

$$\bar{E} (\ln Q_i - \ln Q_{b_{k-1}})^2 \leq 4 \bar{E} \left( \sum_{i=1}^{N_t} \ln(1+U_i) \right)^2 + 4 \left( r - \lambda \beta - \frac{\sigma^2}{2} \right)^2 (t - t_{k-1})^2 + 2 \bar{E} (\sigma(W_t - W_{b_{k-1}}))^2$$

$$\leq 2 \left( 4 \lambda \bar{E} \ln^2 (1+U_1) + 2 \left( r - \lambda \beta - \frac{\sigma^2}{2} \right) T + \sigma^2 \right) (t - t_{k-1}),$$

thus (36) becomes

$$\bar{E} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (\Gamma(t_{k-1},Q_i) - \Gamma(t_{k-1},Q_{b_{k-1}}))^2 \, dt$$

$$\leq 2F^2 \left( 4 \lambda \bar{E} \ln^2 (1+U_1) + 2 \left( r - \lambda \beta - \frac{\sigma^2}{2} \right)^2 T + \sigma^2 \right) \left[ \delta \int_{0}^{t_{k-1}} \frac{dt}{T-t} + \int_{t_{k-1}}^{T} \frac{t}{T-t} \, dt \right]$$

$$= 2F^2 \left( 4 \lambda \bar{E} \ln^2 (1+U_1) + 2 \left( r - \lambda \beta - \frac{\sigma^2}{2} \right)^2 T + \sigma^2 \right) (\ln \frac{T}{\delta} + 1).$$

(37)

To estimate the third integral in (27), we use (28) and the bound $v(t,x) \leq g(0), 0 \leq t \leq T, x > 0$, and write

$$\bar{E} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Gamma^2(t_{k-1},Q_i) \left( \frac{Q_{b_{k-1}} - Q_i}{Q_{b_{k-1}}} \right)^2 \, dt \leq g^2(0) \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \bar{E} \left( \frac{1 - \frac{Q_i}{Q_{b_{k-1}}} \, dt. \right.$$

Since, for $t_{k-1} \leq t \leq t_k$, we have

$$\bar{E} \left( \frac{1 - \frac{Q_i}{Q_{b_{k-1}}} \right)^2 = \bar{E} \left[ 1 - 2 \exp \left( \sum_{i=1}^{N_t} \ln(1+U_i) + \left( r - \lambda \beta - \frac{\sigma^2}{2} \right) (t - t_{k-1}) + \sigma(W_t - W_{b_{k-1}}) \right)$$

$$+ \exp \left( \sum_{i=1}^{N_t} \ln(1+U_i) + 2 \left( r - \lambda \beta - \frac{\sigma^2}{2} \right) (t - t_{k-1}) + 2\sigma(W_t - W_{b_{k-1}}) \right) \right]$$

$$= 2 \left[ 1 - \exp \left( \left( r - \lambda \beta + \lambda \bar{E} U_1 \right) (t - t_{k-1}) \right) \right]$$

$$+ \exp \left( 2r - 2\lambda \beta + \sigma^2 + \lambda \left( \bar{E}(1+U_1)^2 - 1 \right) (t - t_{k-1}) \right) - 1 \right].$$
Mean value theorem then gives
\[
E\left(1 - \frac{Q_t}{Q_{t-1}}\right)^2 \leq \left((2r + 2\lambda \beta + \sigma^2 + \lambda \overline{E}(1 + U_1)^2)e^{(2r + 2\lambda \beta + \sigma^2 + \lambda \overline{E}(1 + U_1)^2)T} \right.
\]
\[+ 2(r + \lambda \beta + \lambda \overline{E}|U_1|)e^{(r + \lambda \beta + \lambda \overline{E}|U_1|)T}\] \[\] \[(t - t_{k-1}),
\]
for \(t_{k-1} \leq t \leq t_k\).

This bound leads us
\[
E\sum_{k=1}^n \int_{t_{k-1}}^{t_k} \Gamma(t_{k-1}, Q_{t-1}) \left(\frac{Q_{t-1} - Q_t}{Q_{t-1}}\right)^2 dt \leq \alpha^2(0)\frac{2r + 2\lambda \beta + \sigma^2 + \lambda \overline{E}(1 + U_1)^2)e^{(2r + 2\lambda \beta + \sigma^2 + \lambda \overline{E}(1 + U_1)^2)T} \right.
\]
\[+ 2(r + \lambda \beta + \lambda \overline{E}|U_1|)e^{(r + \lambda \beta + \lambda \overline{E}|U_1|)T}\] \[\] \[(t - t_{k-1}),
\]
Combining (31), (37) and (38) in (27), we obtain
\[
E\sum_{k=1}^n \int_{t_{k-1}}^{t_k} \left(\Gamma(t, Q_t) - \Gamma(t_{k-1}, Q_{t-1})\right) \left(\frac{Q_{t-1} - Q_t}{Q_{t-1}}\right)^2 dt \leq \left(A_1 \ln \frac{T}{\delta} + B_1\right)\delta
\]
where the positive constants \(A_1\) and \(B_1\) are functions of the parameters \(Q_0, r, \beta, \sigma, g(0), \lambda, T, \overline{E}(U_1), \overline{E}(U_1^2), \overline{E}(\ln(1 + U_1))\) and \(\overline{E}(\ln^2(1 + U_1))\).

Using the similar expression as (26), we write
\[
E\sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_{z_{t_{k-1}}}^{z_k} \left(\frac{Q_{t-1} - Q_t}{Q_{t-1}}\right)^2 d\nu(z) dt
\]
\[\leq 4E\sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_{z_{t_{k-1}}}^{z_k} \left(\frac{Q_{t-1} - Q_t}{Q_{t-1}}\right)^2 d\nu(z) dt
\]
Combining the estimates (39), (41) and (42) in (27), we complete the proof. □
3. Main Result

In this section, we state and prove our main result for the jump-diffusion processes.

**Proposition 3.1.** If $\Pi(t)$ denotes the continuous-time portfolio for the delta hedging strategy (5) and $\Pi_\delta(t), 0 \leq t \leq T$, the portfolio value process of the corresponding discrete time hedging process defined in (6), then we have the following hedging error

$$E \sup_{0 \leq t \leq T} |\pi(t) - \pi_\delta(t)|^2 \leq d \cdot \delta \cdot \ln \frac{T}{\delta},$$

where the non negative constant $d$ depends on $L, r, \sigma, g(0), T, \tilde{E}(U_1), \tilde{E}(U_1^2), Q_0, \tilde{E}(\ln(1 + U_1)), \tilde{E}(\ln^2(1 + U_1))$ and $\mathcal{N}(2\sigma^2 T)$, where $\mathcal{N}(\cdot)$ stands for standard normal distribution.

**Proof.** Proof follows from the application of Proposition 2.2 in the estimate (21).

**Conclusion**

We consider American style option in jump-diffusion model with finite time expiry $T$, and divide the time interval $[0, T]$ into $n$ equal parts and denote by $\delta = \frac{T}{n}$. We consider the value process of the discrete time delta hedging strategy minimising the risk at maturity. Using purely probabilistic approach, we estimate the discrete time hedging error and find that it is proportional to the square root of $\delta \log \frac{T}{\delta}$.

**Financial Competing Interests**

1. We have received neither reimbursement, fees, funding, salary nor we have stocks and shares anywhere that may have gain or loss financially from the publication of this article, either now or in future.
2. We have not any patents and not applying for patents for this article.

**Non-Financial Competing Interests**

We have no non-financial competing interests (political, personal, religious, ideological, logical, intellectual, commercial etc) which are related to this manuscript.

**Contribution**

All the authors have equally contributed. All the authors read and approved the final form of the manuscript.

**References**


