Spectral Properties of the iterated Laplacian with a potential in a Punctured Domain

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Abstract. In the work we derive regularized trace formulas which were established in papers of Kanguzhin and Tokmagambetov for the Laplace and $m$-Laplace operators in a punctured domain with the fixed iterating order $m \in \mathbb{N}$. By using techniques of Sadovnichii and Lyubishkin, the authors in that papers described regularized trace formulae in the spatial dimension $d = 2$. In this note one claims that the formulas are also true for more general operators in the higher spatial dimensions, namely, $2 \leq d \leq 2m$. Also, we give the further discussions on a development of the analysis associated with the operators in punctured domains. This can be done by using so called ‘nonharmonic’ analysis.

1. Introduction

In the remark we investigate a class of elliptic differential equations in a punctured domain. For general motivation, we refer to the papers [1, 3, 4, 9, 11, 12, 18, 19] and references therein, where different differential operators with $\delta$–like potentials are studied, and spectral properties, that is, formulas for the regularized traces and resolvents are given.

In this paper we observe that the results of the work [7] are valid, even when there is a potential and, the spatial dimension is greater than two.

Let $D \subset \mathbb{R}^d$ be a simply connected domain with the smooth boundary $\partial D$. Denote by $s = (s_1, \ldots, s_d)$ a fixed point of the domain $D$. Then we define a punctured domain $D_0 := D \setminus \{s\}$. During this manuscript, we study the differential expression

$$(-\Delta)^m u + qu$$

with real valued potential $q$ in a punctured domain $D_0$. Here

$$(-\Delta)^m u := \left(-\sum_{j=1}^d \frac{\partial^2 u}{\partial x_j^2}\right)^m.$$
We assume that the operator corresponding to the equation (1) with the Dirichlet boundary condition on the “whole” domain $D$ has only discrete spectrum.

Since $D_0$ is not simply connected, we need a special functional space for (1) to define an operator correctly. For this, we introduce the functional class $F_m$ that can be represented in the following form

$$w(x) = w_0(x) + kG_m(x,s), \quad (2)$$

where $k$ is some constant. The function $w_0$ is from the functional space $F_m$ which is consisted of the functions $v \in H^{2m}(D)$ such that

$$\left(\frac{\partial}{\partial n}\right)^j v|_{\partial D} = 0, \quad (3)$$

for all $j = 0, \ldots, m - 1$, where $\frac{\partial}{\partial n}$ is the outer normal derivative. Here $H^{q}$ stands for the usual Sobolev space with the parameters $(2,q)$, and $G_m(x,s)$ is the Green function of the Dirichlet problem for the equation (1) in the whole domain $D$ with the boundary conditions (3).

Now, we define a functional for our further investigations. To this, we consider the paralleled $\Pi_{s,\delta} = \{x : -\delta \leq |x - s| \leq \delta\}$.

Then for the function $h$ from the space $F_m$ defined as (2) we introduce the following functional

$$\alpha_m(h) = \lim_{\delta \to 0^+} \int_{\partial \Pi_{s,\delta}} \left[ \frac{\partial (-\Delta)^{m-1} h(\xi)}{\partial n_\xi} \right] ds_\xi. \quad (4)$$

**Remark 1.1.** We note that the functional (4) is defined for all $d \in \mathbb{N}$. Moreover, the value of $\alpha_m$ from the function $G(x,s)$ exists.

For our convenience, we denote

$$\gamma := \alpha_m(G(\cdot,s)), \quad \alpha(\cdot) := \frac{1}{\gamma} \alpha_m(\cdot),$$

and

$$\xi^- (w) := \alpha(w), \quad \xi^+ (w) := w_0(s).$$

2. **Main Results**

In this section we repeat the results of the paper [7]. However, here we formulate them also for the case $d \leq 2m$.

Now, we are in a way in the Hilbert space $H^2(D)$ to introduce an operator associated with the differential equation (1), that is, $(-\Delta)^m u + qu$. We denote by $K_M$ the operator defined as

$$K_M u = (-\Delta)^m u + qu,$$

in the punctured domain $D_0$ for all functions $u \in F_m$. Assign $K_m$ as the restriction of the operator $K_M$ to $D(K_m) = \{u|u \in F_m, \xi^- (u) = 0, \xi^+ (u) = 0\}$.

Discussing as in the works [5, 7, 8], we get the following statements:

**Proposition 2.1.** Let $d \leq 2m$. Assume that $u, v \in F_m$. Then, we have

$$< K_M u, v > = < u, K_M v > + \xi^- (u) \xi^+ (v) - \xi^- (v) \xi^+ (u).$$
Moreover, the operator \( \mathcal{K}_0 \) defined on \( \mathcal{F}_m \) by the expression

\[
(-\Delta)^m u + q u = f,
\]

in the punctured domain \( D_0 \) with the condition

\[
\theta_1 \xi^-(u) = \theta_2 \xi^+(u)
\]

is a self-adjoint extension of \( \mathcal{K}_m \) in the functional space \( \mathcal{F}_m \). Here \( \theta = (\theta_1, \theta_2) \), \( \theta_1, \theta_2 \in \mathbb{R} \) with the property \( \theta_1^2 + \theta_2^2 \neq 0 \).

In the Hilbert space \( H^2(D) \) consider the operator

\[
\mathcal{K}_Q u(x) := [(-\Delta)^m + q] u(x), \quad x \in D_0
\]

on \( u \in \mathcal{F}_m \) with

\[
a(u) + \int_D Q(x)[(-\Delta)^m + q] u_0(x) dx = 0,
\]

where \( Q \in H^2(D) \). Here we can write

\[
\int_D Q(x)[(-\Delta)^m + q] u_0(x) dx = : \langle Q, [(-\Delta)^m + q] u_0 \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) denotes inner product of \( H^2(D) \).

Now, we consider the operator \( \mathcal{K}_Q \) as a perturbation of \( \mathcal{K}_0 \). Here \( \mathcal{K}_0 \) stands for the Dirichlet problem for \( m \)-Laplace operator in the whole domain \( D \). Then, we assume that \( \{\mu_n\}_{n=1}^\infty \) are the eigenvalues of \( \mathcal{K}_Q \) ordered in the increasing order of their absolute values taking into account the multiplicities, and suppose that \( \{\lambda_n\}_{n=1}^\infty \) are the eigenvalues of \( \mathcal{K}_0 \) ordered in the increasing order by taking into account their multiplicities.

**Theorem 2.2.** Let the spatial dimension \( d \leq 2m \). Suppose that \( p, q > 0 \) are fixed numbers. Assume that \( Q \in D(\mathcal{K}^m_0) \), \( \mathcal{K}_0^{m-1} Q \in H^p(\Pi_{x_0}) \), and \( Q(s) \neq -1 \). Then, we have the following regularized trace formula

\[
\sum_{n=1}^\infty (\mu_n - \lambda_n) = \frac{\bar{Q}(s)}{1 + Q(s)}.
\]

Here \( \bar{Q}(s) = -\lim_{x \to x_0} \mathcal{K}_0^{m-1} Q(x) \).

The proof of Theorem 2.2 follows directly from the proofs of the main theorems of the papers \([7, 17]\).

2.1. Further development

Finally, we note that Proposition 2.1 implies the following corollary, which gives a way to find out self-adjoint operators from the class of operators \( \{\mathcal{K}_Q : Q \in H^2(D)\} \), namely:

**Corollary 2.3.** Suppose that \( \theta_1 \neq 0 \) and \( Q(x) = -\mu G_m(x, s) \) with \( \mu = \theta_2/\theta_1 \). Then the operator \( \mathcal{K}_Q \) is self-adjoint with the parameter \( (\theta_1, \theta_2) \) in the space \( \mathcal{F}_m \):

\[
\mathcal{K}_{-\mu G_m} \sim \mathcal{K}_{(1, \mu)} = \mathcal{K}_{(\theta_1, \theta_2)}.
\]

Thus, we observe that the class of operators given by the equation (6) and condition (7) has a huge number of self-adjoint operators in a punctured domain. One can be started a 'nonharmonic' analysis connected with the singular, in the above sense, operators. Note, that the nonharmonic analysis is developed in the works \([2, 10, 13, 15]\) with applications given in \([14, 16]\). For more general setting of the nonharmonic analysis, see for instance \([6]\).
References