On the Existence and Stability of Solution of Boundary Value Problem for Fractional Integro-Differential Equations with Complex Order

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Abstract. In this paper, we establish sufficient conditions for the existence and stability of solutions for fractional integro-differential equations with boundary conditions involving complex order. The proofs are based upon the Banach contraction principle. An example is included to show the applicability of our results.

1. Introduction

The study of fractional differential equations (FDEs) ranges from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions. FDEs appear naturally in a number of fields such as physics, polymer rheology, regular variational in thermodynamics, biophysics, electrical circuits, electron-analytical chemistry, biology, control theory, etc. An excellent description in the study of FDEs can be found in [13, 23, 24, 28]. It is considerable that there are many works about fractional integro-differential equations (FIDEs) (see, for example, [4, 25, 30, 38]).

Ulam’s stability problem [14] has been attracted by many famous researchers, for example, see Andras, Jung and Rus [2, 19, 31]. For more recent contribution on such interesting topic, see [2, 15, 22, 29, 36, 37] and references therein. Rabha W. Ibrahim studied Ulam stability for FDEs in Complex Domain in [16]. The author also considered a generalization of the admissible functions in complex Banach spaces; one can refer to [17, 18].

The topics of FDEs, which attracted a growing interest for some time, in particular, in relation to the complex order in fractional calculus, have been rapidly developed recent years. E. R. Love [21] started the research of fractional derivatives of imaginary order. The concept is usual definitions of fractional integrals and derivatives by defining derivatives of purely imaginary orders. The notion of fractional operator of complex order, introduced by Samko et al.[32]. In this direction, several notions of fractional derivative of complex order were discussed [1, 33]. For instance, Carla M.A.Pinto [10] introduced the two approximations of the complex order van der Pol oscillator. In the paper [27], the authors investigated the existence of solutions of boundary value problems (BVPs) with complex order. Most recently, Vivek...
et al. studied the existence and stability results for pantograph equations[35] and integro-differential equations[34] with nonlocal conditions involving complex order.

Motivated by the works mentioned in [4, 21, 27, 33, 34], in this paper, we establish four types of Ulam stability, namely Ulam-Hyers(U-H) stability, generalized U-H stability, U-H-Rassias and generalized U-H-Rassias stability for the following BVPs for FIDEs with complex order

\[ D^\theta_0 x(t) = f(t, x(t), \int_0^t h(t, s, x(s))ds), \quad t \in J := [0, T], \quad \theta = m + i\alpha, \]  
\[ ax(0) + bx(T) = c, \]  
where \( D^\theta_0 \) is Caputo fractional derivative of order \( \theta \in \mathbb{C} \). Let \( \alpha \in \mathbb{R}^+, \ m \in (0, 1] \) and \( f : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \), \( h : \Delta \times \mathbb{R} \rightarrow \mathbb{R} \) are continuous. Here, \( \Delta = \{(t, s) : 0 \leq s \leq t \leq T\} \). For brevity let us take

\[ Hx(t) = \int_0^t h(t, s, x(s))ds. \]

There have been many papers (see, for example, in [3, 6–8, 11, 26, 39]) dealing with BVPs of FDEs. The paper is organized as follows. In Section 2, we recall some basic definitions from fractional calculus and establish auxiliary lemmas which play a pivotal role in the sequel. Section 3 contains existence and Ulam stability results for the problem (1)-(2).

2. Prerequisites

In this section, we recall some definitions and lemmas used further.

By \( C(J, \mathbb{R}) \) we denote the Banach space of all continuous function \( f \) into \( \mathbb{R} \) with the norm

\[ ||f||_\infty = \sup \{ |f(t)| : t \in J \}. \]

By \( L^1(J) \) we denote the space of Lebesgue-integrable function \( x : J \rightarrow \mathbb{R} \) with the norm \( ||x||_{L^1} = \int_0^T |x(t)|dt \).

**Definition 2.1.** ([28]) The Riemann-Liouville fractional integral of order \( \theta \in \mathbb{C} \), \( \text{Re}(\theta) > 0 \) of a function \( f \) is

\[ I^\theta_0 f(t) = \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s)ds. \]

**Definition 2.2.** ([28]) For a function \( f \) given by on the interval \( J \), the Caputo fractional-order \( \theta \in \mathbb{C} \), \( \text{Re}(\theta) > 0 \) of \( f \), is defined by

\[ (D^\theta_0 f)(t) = \frac{1}{\Gamma(n-\theta)} \int_0^t (t-s)^{n-\theta-1} f^{(n)}(s)ds, \]

where \( n = [\text{Re}(\theta)] + 1 \) and \([\text{Re}(\theta)]\) denotes the integral part of the real number \( \theta \).

**Definition 2.3.** ([20]) The Stirling asymptotic formula of the Gamma function for \( z \in \mathbb{C} \) is following

\[ \Gamma(z) = (2\pi)^{\frac{1}{2}} z^{\frac{1}{2}} e^{-z} \left[ 1 + O\left(\frac{1}{z}\right) \right], \quad (|\arg(z)| < \pi; |z| \rightarrow \infty), \]  
\[ \text{(3)} \]

and its results for \( |\Gamma(u + iv)| \), \( u, v \in \mathbb{R} \) is

\[ |\Gamma(u + iv)| = (2\pi)^{\frac{1}{2}} \left| v \right|^{\frac{1}{2}} e^{-\pi v/2} \left[ 1 + O\left(\frac{1}{|v|}\right) \right], \quad (v \rightarrow \infty). \]  
\[ \text{(4)} \]
Lemma 2.4. (see Lemma 7.1.1 ([14])) Let \( z,w:[0,T) \to [0,\infty) \) be continuous functions where \( T \leq \infty \). If \( w \) is nondecreasing and there are constants \( k \geq 0 \) and \( 0 < \nu < 1 \) such that

\[
z(t) \leq w(t) + k \int_0^t (t-s)^{-\nu} z(s) \, ds, \quad t \in [0,T),
\]

then

\[
z(t) \leq w(t) + \int_0^t \left( \sum_{n=1}^\infty \frac{(k\Gamma(\nu))^n}{\Gamma(n\nu)} (t-s)^{n\nu-1} w(s) \right) \, ds, \quad t \in [0,T).
\]

Remark 2.5. Under the hypothesis of Lemma 2.4, let \( w(t) \) be a nondecreasing function on \( [0,T) \). Then we have \( z(t) \leq w(t)E_{\nu,1}(k\Gamma(\nu)t^\nu) \).


Now we consider the problem (1) and the following inequalities

\[
|D_{0+}^\nu z(t) - f(t,z(t),Hz(t))| \leq \epsilon, \quad t \in J,
\]

(5)

\[
|D_{0+}^\nu z(t) - f(t,z(t),Hz(t))| \leq \epsilon \varphi(t), \quad t \in J,
\]

(6)

\[
|D_{0+}^\nu z(t) - f(t,z(t),Hz(t))| \leq \varphi(t), \quad t \in J,
\]

(7)

Definition 2.6. The equation (1) is U-H stable if there exists a real number \( C_f > 0 \) such that for each \( \epsilon > 0 \) and for each solution \( z \in C(J,\mathbb{R}) \) of the inequality (7) there exists a solution \( x \in C(J,\mathbb{R}) \) of equation (1) with

\[
|z(t) - x(t)| \leq C_f \epsilon, \quad t \in J.
\]

Definition 2.7. The equation (1) is generalized U-H stable if there exists \( \psi_f \in C([0,\infty),[0,\infty)) \), \( \psi_f(0) = 0 \) such that for each solution \( z \in C(J,\mathbb{R}) \) of the inequality (7) there exists a solution \( x \in C(J,\mathbb{R}) \) of equation (1) with

\[
|z(t) - x(t)| \leq \psi_f \epsilon, \quad t \in J.
\]

Definition 2.8. The equation (1) is U-H-Rassias stable with respect to \( \varphi \in C(J,\mathbb{R}) \) if there exists a real number \( C_f > 0 \) such that for each \( \epsilon > 0 \) and for each solution \( z \in C(J,\mathbb{R}) \) of the inequality (7) there exists a solution \( x \in C(J,\mathbb{R}) \) of equation (1) with

\[
|z(t) - x(t)| \leq C_f \epsilon \varphi(t), \quad t \in J.
\]

Definition 2.9. The equation (1) is generalized U-H-Rassias stable with respect to \( \varphi \in C(J,\mathbb{R}) \) if there exists a real number \( C_{f,\varphi} > 0 \) such that for each \( \epsilon > 0 \) and for each solution \( z \in C(J,\mathbb{R}) \) of the inequality (5) there exists a solution \( x \in C(J,\mathbb{R}) \) of equation (1) with

\[
|z(t) - x(t)| \leq C_{f,\varphi} \varphi(t), \quad t \in J.
\]

Remark 2.10. A function \( z \in C(J,\mathbb{R}) \) is a solution of the inequality (7) if and only if there exists a function \( g \in C(J,\mathbb{R}) \) (which depend on \( z \)) such that

1. \( |g(t)| \leq \epsilon, \forall \ t \in J \);
2. \( D_{0+}^\nu z(t) = f(t,z(t),Hz(t)) + g(t), \ t \in J \).

One can have similar remarks for the inequality (6) and (5).
Remark 2.11. Let \( \theta = m + ia, \ m \in (0,1] \) and \( a \in \mathbb{R}^{+} \) if \( z \in C([\theta, \mathbb{R}) \) is a solution of the inequality (\( ? \)), then \( z \) is a solution of the following integral inequality

\[
\left| z(t) - A_z - \frac{1}{\Gamma(\theta)} \int_{0}^{t} \Gamma(t-s)^{\theta-1} f(s, z(s), Hz(s)) ds \right| \leq \frac{\epsilon}{|\theta|} \frac{T^{m}}{m} \left( 1 + \frac{|b|}{|a| + |b|} \right).
\]

Indeed, by Remark 2.10, we have that

\[
D^{\theta}_{0^{+}} z(t) = f(t, z(t), Hz(t)) + g(t), \quad t \in \mathbb{J}.
\]

Then

\[
z(t) = A_z + \frac{1}{\Gamma(\theta)} \int_{0}^{t} \Gamma(t-s)^{\theta-1} f(s, z(s), Hz(s)) ds + \frac{1}{\Gamma(\theta)} \int_{0}^{t} (t-s)^{\theta-1} g(s) ds
\]

with

\[
A_z = \frac{1}{a + b} \left[ c - \frac{b}{\Gamma(\theta)} \int_{0}^{T} (T-s)^{\theta-1} f(s, z(s), Hz(s)) ds \right].
\]

From it follows that

\[
\left| z(t) - A_z - \frac{1}{\Gamma(\theta)} \int_{0}^{t} \Gamma(t-s)^{\theta-1} f(s, z(s), Hz(s)) ds \right|
\]

\[
= \left| \frac{1}{\Gamma(\theta)} \int_{0}^{t} \Gamma(t-s)^{\theta-1} f(s, z(s), Hz(s)) ds - \frac{b}{a + b} \frac{1}{\Gamma(\theta)} \int_{0}^{T} (T-s)^{\theta-1} g(s) ds \right|
\]

\[
\leq \frac{1}{|\theta|} \left| \int_{0}^{t} \Gamma(t-s)^{\theta-1} f(s, z(s), Hz(s)) ds \right| + \frac{|b|}{|a + b|} \frac{1}{|\theta|} \left| \int_{0}^{T} (T-s)^{\theta-1} g(s) ds \right|
\]

\[
\leq \frac{\epsilon}{|\theta|} \frac{T^{m}}{m} \left( 1 + \frac{|b|}{|a| + |b|} \right).
\]

Remark 2.12. Clearly,

1. Definition 2.6 \( \Rightarrow \) Definition 2.7.
2. Definition 2.8 \( \Rightarrow \) Definition 2.9.

Remark 2.13. A solution of the FIDEs with complex order inequality (\( ? \)) is called an fractional e-solution of the problem (1)-(2).

3. Existence and U-H stability results

Lemma 3.1. Let \( \theta = m + ia, \ 0 < m \leq 1, \ a \in \mathbb{R}^{+} \) and \( f : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \), \( h : \Delta \times \mathbb{R} \rightarrow \mathbb{R} \) be continuous functions. Then the FIDEs with complex order

\[
D^{\theta}_{0^{+}} x(t) = f(t, x(t), Hx(t)), \quad t \in \mathbb{J},
\]

\[
ax(0) + bx(T) = c
\]

has a unique solution which is given by

\[
x(t) = \frac{1}{\Gamma(\theta)} \int_{0}^{t} \Gamma(t-s)^{\theta-1} f(s, x(s), Hx(s)) ds
\]

\[
= \frac{1}{a + b} \left[ \frac{b}{\Gamma(\theta)} \int_{0}^{T} (T-s)^{\theta-1} f(s, x(s), Hx(s)) ds - c \right].
\]
Proof. By integration of eqn. (??), we obtain
\[ x(t) = x_0 + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), Hx(s)) ds. \]  
(11)

We use condition (8) to compute the constant \( x_0 \), so we have
\[ ax(0) = ax_0 \quad \text{and} \quad bx(T) = bx_0 + \frac{b}{\Gamma(\theta)} \int_0^T (T-s)^{\alpha-1} f(s, x(s), Hx(s)) ds, \]
then
\[ ax(0) + bx(T) = c, \]
since
\[ x_0 = -\frac{1}{(a+b)} \left[ \frac{b}{\Gamma(\theta)} \int_0^T (T-s)^{\alpha-1} f(s, x(s), Hx(s)) ds - c \right]. \]

Substituting in Eqn.(11) leads to formula (10).

First, we give the following result based on Banach contraction principle.

Lemma 3.2. Assume hypotheses

\( H1 \) The function \( f : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous and there exists a constant \( L_1 > 0 \) such that
\[ \| f(t, x_1, y_1) - f(t, x_2, y_2) \| \leq L_1 \left[ |x_1 - x_2| + |y_1 - y_2| \right] , \forall x_1, x_2, y_1, y_2 \in \mathbb{R}. \]

\( H2 \) The function \( h : \Delta \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous and there exists a constant \( H_1 > 0 \) such that
\[ |h(t, s, x_1) - h(t, s, x_2)| \leq H_1 |x_1 - x_2| , \quad \forall x_1, x_2 \in \mathbb{R}. \]

If
\[ \left( \frac{L_1(1 + H_1)}{m \Gamma(\theta)} \right)^n \left[ 1 + \frac{|b|}{|a + b|} \right] < 1, \]  
(12)
the problem (1)-(2) has a unique solution.

Proof. Transform the problem (1)-(2) into a fixed point problem. Consider the operator \( P : C(J, \mathbb{R}) \rightarrow C(J, \mathbb{R}) \) defined by
\[ (Px)(t) = \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), Hx(s)) ds 
- \frac{1}{a+b} \left[ \frac{b}{\Gamma(\theta)} \int_0^T (T-s)^{\alpha-1} f(s, x(s), Hx(s)) ds - c \right]. \]  
(13)

Clearly, the fixed points of the operator \( P \) are solution of the problem (1)-(2). We shall use the Banach contraction principle to prove that \( P \) defined by (13) has a fixed point. We shall show that \( P \) is a contraction.
Let \( x, y \in C(J, \mathbb{R}) \). Then, for each \( t \in J \) we have
\[
\begin{align*}
\| (Px)(t) - (Py)(t) \| &\leq \frac{1}{|\Gamma(\theta)|} \int_0^T \| (t-s)^{\theta-1} \| f(s, x(s), Hx(s)) - f(s, y(s), Hy(s)) \| ds \\
&\quad + \frac{|b|}{|\Gamma(\theta)|} \int_0^T \| (T-s)^{\theta-1} \| f(s, x(s), Hx(s)) - f(s, y(s), Hy(s)) \| ds \\
&\leq \frac{L_1(1+H_1)}{|\Gamma(\theta)|} \int_0^x \| (t-s)^{\theta-1} \| \| x(s) - y(s) \| ds + \frac{L_1(1+H_1)|b|}{|\Gamma(\theta)|} \int_0^T \| (T-s)^{\theta-1} \| \| x(s) - y(s) \| ds \\
&\leq \frac{L_1(1+H_1)}{|m|\Gamma(\theta)|} |\Gamma(\theta)| \int_0^T \| (t-s)^{\theta-1} \| ds + \frac{L_1(1+H_1)|b|}{|\Gamma(\theta)|} \int_0^T \| (T-s)^{\theta-1} \| ds \\
&\leq \left( \frac{L_1(1+H_1)}{|m|\Gamma(\theta)|} \int_0^T \| (t-s)^{\theta-1} \| ds \right) \left( 1 + \frac{|b|}{|a+b|} \right) \| x - y \|_{\infty}.
\end{align*}
\]
Thus
\[
\| P_x - P_y \|_{\infty} \leq \left( \frac{L_1(1+H_1)}{|m|\Gamma(\theta)|} \int_0^T \| (t-s)^{\theta-1} \| ds \right) \left( 1 + \frac{|b|}{|a+b|} \right) \| x - y \|_{\infty}.
\]
From (12), it follows that \( P \) has a unique fixed point which is solution of the problem (1)-(2). \( \square \)

**Theorem 3.3.** *In the conditions (H1), (H2) and (12), the problem (1)-(2) is U-H stable.*

**Proof.** Let \( z \in C(J, \mathbb{R}) \) be a solution of the inequality (3). Denote by \( x \in C(J, \mathbb{R}) \) the unique solution of the following problem
\[
\begin{align*}
D^\alpha_{\theta} x(t) &= f(t, x(t), Hx(t)); \quad t \in J, \quad \theta = m + i\alpha, \\
x(0) &= z(0), \quad x(T) = z(T),
\end{align*}
\]
where \( m \in (0, 1), \alpha \in \mathbb{R}^+ \).
Using Lemma 3.1, we have that
\[
x(t) = A_x + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s, x(s), Hx(s)) ds,
\]
with
\[
A_x = \frac{1}{a+b} \left[ c - \frac{b}{\Gamma(\theta)} \int_0^T (T-s)^{\theta-1} f(s, x(s), Hx(s)) ds \right].
\]
On the other hand, if \( x(T) = z(T) \) and \( x(0) = z(0) \), then \( A_x = A_z \).
Indeed,
\[
\begin{align*}
|A_x - A_z| &\leq \frac{|b|}{|a+b| |\Gamma(\theta)|} \int_0^T (T-s)^{\theta-1} \| f(s, x(s), Hx(s)) - f(s, z(s), Hz(s)) \| ds \\
&\leq \frac{L_1(1+H_1)|b|}{|a+b|} \| x(T) - z(T) \|_{C},
\end{align*}
\]
Thus
\[
A_x = A_z.
\]
Then, we have
\[ x(t) = A_z + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s, x(s), Hx(s))ds. \]

By integration of the inequality (10) and using Remark 2.11, we have
\[
|z(t) - A_z - \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s, z(s), Hz(s))ds| \leq \frac{e^{T^m}}{m \Gamma(\theta)} \left( 1 + \frac{|b|}{|a + b|} \right) T^m.
\]

for all \( t \in J \). From above it follows:
\[
|z(t) - x(t)| = |z(t) - A_z - \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s, z(s), Hz(s))ds| \leq \frac{e^{T^m}}{m \Gamma(\theta)} \left( 1 + \frac{|b|}{|a + b|} \right) T^m.
\]

By Lemma 2.4(Gronwall inequality) and Remark 2.5, for all \( t \in J \), we have that
\[
|z(t) - x(t)| \leq \frac{e^{T^m}}{m \Gamma(\theta)} \left( 1 + \frac{|b|}{|a + b|} \right) T^m.
\]

Thus, the problem (1)-(2) is U-H stable.

We have the following generalized U-H -Rassias stability results.

**Theorem 3.4.** In the conditions (H1), (H2), (12) and

(H3) There exists an increasing function \( \varphi \in C(J, \mathbb{R}) \) and there exists \( \lambda_\varphi > 0 \) such that for any \( t \in J \)
\[
I_{\theta,0}^\varphi x(t) \leq \lambda_\varphi \varphi(t),
\]

the problem (1)-(2) is generalized U-H-Rassias stable.

**Proof.** Let \( z \in C(J, \mathbb{R}) \) be solution of the following inequality
\[
|D_0^\theta z(t) - f(t, z(t), Hx(t))| \leq \epsilon \varphi(t), \quad t \in J, \quad \epsilon > 0,
\]

and let \( x \in C(J, \mathbb{R}) \) be the unique solution of the following problem
\[
\begin{cases}
D_0^\theta x(t) = f(t, x(t), Hx(t)); & t \in J, \quad \theta = m + ia, \\
x(0) = z(0), & x(T) = z(T),
\end{cases}
\]

where \( m \in (0, 1], a \in \mathbb{R}^+ \).

By Lemma 3.1,
\[
x(t) = A_z + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s, x(s), Hx(s))ds,
\]

with
\[
A_z = \frac{1}{a + b} \left[ c - \frac{b}{\Gamma(\theta)} \int_0^T (T-s)^{\theta-1} f(s, x(s), Hx(s))ds \right].
\]
By integration of the inequality (14), we obtain
\[
|z(t) - x(t)| \leq |z(t) - A z - \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s, z(s), Hz(s)) ds| \\
\leq e^{\lambda \varphi(t)} \left(1 + \frac{|b|}{|a + b|}\right) \left|\frac{L_1(1 + H_1)}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} |s - x(s)| ds\right|.
\]

We have for any \( t \in J \)
\[
|z(t) - x(t)| \leq e^{\lambda \varphi(t)} \left(1 + \frac{|b|}{|a + b|}\right) \left|\frac{L_1(1 + H_1)}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} |s - x(s)| ds\right|.
\]

Using Lemma 2.4 (Gronwall inequality) and Remark 2.5, we obtain
\[
|z(t) - x(t)| \leq e^{\lambda \varphi(t)} \left(1 + \frac{|b|}{|a + b|}\right) \left|\frac{L_1(1 + H_1)}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} |s - x(s)| ds\right|, \quad t \in J.
\]

Thus, the problem (1)-(2) is generalized U-H-Rassias stable. \( \Box \)

**Remark 3.5.** ([9]) The boundary value problem (1)-(2) are appropriate for the following problems:

1. **Initial value problem:** \( a = 1, b = 0, c = 0 \).
2. **Terminal value problem:** \( a = 0, b = 1, c \) arbitrary.
3. **Anti-periodic problem:** \( a = 1, b = -1, c = 0 \).

However, they are not for the periodic problem, i.e., for \( a = 1, b = -1, c = 0 \).

**4. An example**

Consider the following fractional integro-differential equation with complex order
\[
D^\theta x_p(t) = \frac{t |x_p|}{10 (1 + |x_p|)} + \frac{1}{2 p} \int_0^t e^{-(t-s)} x_p(s) ds, \quad t \in J := [0, 1],
\]
\[
x_p(0) = 0, \quad x_p(1) = 0,
\]
where \( \theta = \alpha + im, m = \frac{1}{2} \) and \( \alpha = 1 \).

Set
\[
f_p = \frac{t |x_p|}{10 (1 + |x_p|)} + \frac{1}{2 p} \int_0^t e^{-(t-s)} x_p(s) ds,
\]
\[
H_p(x) = \frac{1}{2 p} \int_0^t e^{-(t-s)} x_p(s) ds,
\]
\[
k(t, s) = e^{-(t-s)}.
\]

Let \( x_p, y_p \in \mathbb{R} \) and \( t \in J \). Then, we have
\[
|f(t, x_p(t), (Hx_p)(t)) - f(t, y_p(t), (Hy_p)(t))| \leq L_1 \left(|x_p - y_p| + |Hx_p - Hy_p|\right),
\]
\[
\leq L_1 (1 + H_1) |x_p - y_p|,
\]
where \( L_1 = \frac{1}{m} \) and \( H_1 = \frac{1}{2} \).

Hence the conditions (H1), (H2) hold with \( L_1 = \frac{1}{m} \) and \( H_1 = \frac{1}{2} \). We shall check that condition (12) is satisfied for suitable values of \( \alpha = 1, m = \frac{1}{2} \) with \( a = b = T = 1 \). Indeed,

\[
\left( \frac{L_1(1 + H_1)}{m [L((1T)^{m})]} \right)^{T} \left[ 1 + \frac{|b|}{|a + b|} \right] < 1.
\]

It follows from Lemma 3.2 that the problem (15)-(16) has a unique solution on \( J \). In addition, Theorem (3.3) implies that the problem (15)-(16) is Ulam-Hyers stable.

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All of the authors equally contributed to the conception and development of this manuscript.

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