Notes on the Results of Lower Bounds for a Class of Harmonic Functions in the Half Space

David Simms


The origin of our work lies in Zhang, Deng and Kou [5]. In [5] Lemmas 1 and 2 and therefore also Theorem 1 are erroneous. We give now the correction of these statements. The present notation and terminology in the same as used in [5].

To this end, we start with an auxiliary proposition. Actually, this proposition is a direct corollary of [2, p. 3296], in which harmonic majorization Theorems with respect to a half-space and their applications were introduced. But it plays an important role in our discussions.

Proposition 1. Let $H$ be an admissible domain with boundary $\partial H$ in $\mathbb{R}^n$. If $u$ and $v$ are two harmonic functions in $H$, then we have
\[
\int_{\partial H} \left( u(x) \frac{\partial v(x)}{\partial n} - v(x) \frac{\partial u(x)}{\partial n} \right) d\sigma(x) = 0,
\]
where $d\sigma(x)$ is the surface element of sphere in $H$ and $\partial/\partial n$ denotes differentiation along the inward normal into $H$.

We now return to [5, Lemma 1] and give a corrected proof of it. This result does not seem easy to be proved, hence we refer to utilize a slightly different approach. For more details about this procedure we refer to [1], where a different problem is studied by a similar argument.

Lemma 1. Let $u(x)$ be a harmonic function in the upper half space $\mathbb{R}^n_+$ and continuous on $\partial \mathbb{R}^n_+$. Then
\[
\int_{\{x \in \mathbb{R}^n_+ : |x| = r\}} \frac{n x_n}{R^{n+1}} d\sigma(x) + \int_{\{x \in \mathbb{R}^n_+ : r < |x| < R\}} u(x') \left( \frac{1}{|x'|} - \frac{1}{R^n} \right) d\sigma' = c_1(r) + \frac{c_2(r)}{R^n}
\]
for $0 < r < R$, where
\[
c_1(r) = \int_{\{x \in \mathbb{R}^n_+ : |x| = r\}} \left( \frac{n - 1}{r^{n+1}} u(x) + \frac{x_n}{r^n} \frac{\partial u(x)}{\partial n} \right) d\sigma(x)
\]
and
\[
c_2(r) = \int_{\{x \in \mathbb{R}^n_+ : |x| = r\}} \left( \frac{x_n}{r} u(x) - x_n \frac{\partial u(x)}{\partial n} \right) d\sigma(x).
\]

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Email address: david.simms821@gmail.com (David Simms)
Remark 1. In [5, Lemma 1] the definition of $\partial u/\partial n$ is inaccurate, the expressions of $c_1(r)$ and $c_2(r)$ are incorrect.

Proof. Put

\[ v(x) = \frac{x_n}{|x|^n} - \frac{x_n}{R^n} \]

in

\[ B^+ (r, R) = \{ x \in \mathbb{R}^n : r < |x| < R \}. \]

It is easy to see that $v(x)$ is a harmonic function in $B^+(r, R)$. It follows that

\[ v(x) = 0, \quad \frac{\partial v(x)}{\partial n} = \frac{nx_n}{R^{n+1}} \tag{2} \]

on the half sphere $\{ x \in \mathbb{R}^n : |x| = R \}$,

\[ \frac{\partial v(x)}{\partial n} = -\frac{x_n}{r} \left( \frac{n-1}{r^n} + \frac{1}{R^n} \right) \tag{3} \]

on the half sphere $\{ x \in \mathbb{R}^n : |x| = r \}$ and

\[ v(x) = 0, \quad \frac{\partial v(x)}{\partial n} = \frac{1}{|x|^n} - \frac{1}{R^n} \tag{4} \]

on the set $\{ x \in \mathbb{R}^n : r < |x'| < R \}$.

By applying Proposition 1 to two harmonic functions $u(x)$ and $v(x)$ in $B^+(r, R)$, we obtain that

\[ U_1 + U_2 + U_3 = 0, \tag{5} \]

where

\begin{align*}
U_1 &= \int_{\{x \in \mathbb{R}^n : |x| = R\}} \left( u(x) \frac{\partial v(x)}{\partial n} - v(x) \frac{\partial u(x)}{\partial n} \right) d\sigma(x), \\
U_2 &= \int_{\{x \in \mathbb{R}^n : |x| = r\}} \left( u(x) \frac{\partial v(x)}{\partial n} - v(x) \frac{\partial u(x)}{\partial n} \right) d\sigma(x) \\
\end{align*}

and

\begin{align*}
U_3 &= \int_{\{x \in \mathbb{R}^n : r < |x| < R\}} \left( u(x) \frac{\partial v(x)}{\partial n} - v(x) \frac{\partial u(x)}{\partial n} \right) d\sigma(x).
\end{align*}

It follows that

\begin{align*}
U_1 &= \int_{\{x \in \mathbb{R}^n : |x| = R\}} u(x) \frac{nx_n}{R^{n+1}} d\sigma(x), \quad U_2 = -c_1(r) - \frac{c_2(r)}{R^n} \\
\end{align*}

and

\begin{align*}
U_3 &= \int_{\{x \in \mathbb{R}^n : r < |x| < R\}} u(x') \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx',
\end{align*}

from (2), (3) and (4), respectively, which together with (5) give that (1) holds.

This lemma is proved. \Box

The proof of [5, Lemma 2] fails at Line 3, p. 1491. The formula

\[ G^+_n(x, y) = G^+_n(x, y) - G^+_n(x', y) \]

should read

\[ G^+_n(x, y) = G^+_n(x', y) - G^+_n(x, y'). \]

More importantly, the definition of the set $B^+_R$ is incorrect. Moreover, the hypothesis $n > 2$ should be added in Lemma 2.

A correction of Lemma 2 reads as follows, which improve the corresponding one established by Kuran in [2].
Lemma 2. Let \( n > 2 \) and \( u(x) \) be defined as in Lemma 1. Then

\[
    u(x) = \int_{\{ y \in \mathbb{R}^n \mid |y| = R \}} \left( \frac{R^2 - |x|^2}{a_n R} - \frac{1}{|y - x'|^n} \right) u(y) d\sigma(y) + \frac{2x_n}{a_n} \int_{\{ y' \in \mathbb{R}^n \mid |y'| < R \}} \left( \frac{1}{|y' - x'|^n} - \frac{R^n}{|x|^n} \right) u(y') dy'
\]

for any

\[
    x \in \{ x \in \mathbb{R}^n : |x| \leq R \},
\]

where \( \bar{x} = R^2 x/|x|^2 \) and \( x' = (x', -x_n) \).

Finally, what we get instead of [5, Theorem 1] is the following. The proof of it is carried out in the same way as for Theorem 1 in [5], except that instead of the erroneous Lemmas 1 and 2 their corrected versions above are used.

Theorem 1. Let \( u(x) \) be a harmonic function in \( \mathbb{R}^n_+ \) and continuous on \( \partial \mathbb{R}^n_+ \). Suppose that

\[
    u(x) \leq Kr^\rho, \quad x \in \mathbb{R}^n_+, \quad r = |x| > 1, \quad \rho > 1
\]

and

\[
    u(x) \geq -K, \quad |x| \leq 1, \quad x_n \geq 0.
\]

Then the result in [5, Theorem 1] holds.

Remark 2. Conditions (6) and (7) are weaker than conditions (1) and (2) in [5, Theorem 1]. For the conical version of Theorem 1, we refer the reader to the paper by Armitage [1] and Li & Vetro [3].

References