Bilateral Set-Valued Stochastic Integral Equations

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Abstract. We investigate bilateral set-valued stochastic integral equations and these equations combine widening and narrowing set-valued stochastic integral equations studied in literature. An existence and uniqueness theorem is established using approximate solutions. In addition stability of the solution with respect to small changes of the initial state and coefficients is established, also we provide a result on boundedness of the solution, and an estimate on a distance between the exact solution and the approximate solution is given. Finally some implications for deterministic set-valued integral equations are presented.

1. Introduction

Set-valued analysis arises naturally in physics, economics, optimization; see e.g. [5, 10] and the references therein. It is fundamental in the theory of set-valued differential equations [24], which are mathematical models of dynamic systems with incomplete information and this theory was used to study properties of solutions for differential inclusions [33]. Set-valued differential equation were considered in [7–9, 13]. Existence of solutions was discussed in [1] while in [2, 3, 6, 14, 16, 18, 32, 35] stability results were provided. Also in the literature there are results using the monotone iterative technique [12], the variation of constants formula [17], monotone flows [25], quasilinearization [34], and periodic solutions were studied in [20].

Set-valued differential equations including equations with causal operators [11, 14, 22], equations with second type Hukuhara derivative [27, 28], equations on time scales [19, 26, 35] were also considered.

In this paper we consider bilateral set-valued integral equations in a stochastic context. More precisely we consider an equation of the form

\[ X(t) + (S) \int_0^t \tilde{F}(s, X(s))ds + (S) \int_0^t \tilde{G}(s, X(s))d\tilde{B}(s) = X_0 + (S) \int_0^t \tilde{F}(s, X(s))ds + (S) \int_0^t \tilde{G}(s, X(s))d\tilde{B}(s), \]

where \( t \) runs over an interval, \( X_0 \) is a set, \( \tilde{F}, \tilde{F}, \tilde{G}, \tilde{G} \) are some set-valued stochastic processes, \( B, \tilde{B} \) are Brownian motions and all the integrals are set-valued. Unfortunately this equation cannot easily be reduced to the equation of the above type with only one side. The difficulty lies in the issue of the difference of sets i.e. this difference may not exist. Also each side of the equation has a different effect on the properties of the solution (i.e. a different effect on the behavior of a function whose values are the diameter of the solution at time \( t \)); the right-hand side drives an increase in diameter while the integrals on the left forces the diameter...
to decrease. Such an observation with equations only with the left-hand side was explicitly emphasized in [31]. The equation above combines two different types of equations previously investigated in the literature, i.e., widening equations [29] and [30] and narrowing equations [31], so in this case the solutions can change the type of monotonicity of diameter over time.

In this paper we consider the existence of a unique solution under a Lipschitz type condition by first discussing and defining a sequence of approximate set-valued solutions. We also give some estimates on the distance between the approximate and the exact solution. An estimate on the magnitude of the solution is presented which allows us to discuss the solution’s boundedness. A justification of low sensitivity of the solution to slight changes in the initial value and coefficients of the equation is also presented. In the last part of the paper, some implications specific to deterministic set-valued equations are discussed which are of interest in their own right [24] and in this setting we show that a certain restrictive condition used for stochastic equations can be replaced by a more convenient one.

2. Preliminaries

In this part of the paper we collect some notions and properties concerning set-valued mappings and integrals; see [29, 31]. It is done for the convenience of the reader.

Let \( (X, \| \cdot \|_X) \) be a separable Banach space. By the symbol \( \mathcal{K}^c(X) \) we denote the family of all nonempty closed bounded and convex subsets of \( X \). In \( \mathcal{K}^c(X) \), the Hausdorff metric \( H_X \) is considered

\[
H_X(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \| a - b \|_X, \sup_{b \in B} \inf_{a \in A} \| a - b \|_X \right\}.
\]

Then \( (\mathcal{K}^c(X), H_X) \) is a complete metric space (see [21]). Moreover the family of nonempty, closed and convex subsets of a separable and reflexive Banach space \( X \) supplied with the Mosco topology \( \tau_{M_X} \) is a Polish topological space. The Mosco topology is metrizable and weaker than the topology \( \tau_{H_X} \) generated by the Hausdorff metric \( H_X \).

In the set \( \mathcal{K}^c(X) \) one defines addition and scalar multiplication as follows: for \( A, B \in \mathcal{K}^c(X) \) and \( r \in \mathbb{R} \) we have \( A + B = \{ a + b : a \in A, b \in B \} \), \( rA = \{ ra : a \in A \} \). The notion of difference of sets of \( A, B \in \mathcal{K}^c(X) \) used in this paper is considered as the Hukuhara difference, i.e. \( A \ominus B \in \mathcal{K}^c(X) \) is a set such that \( (A \ominus B) + B = A \). If \( A \ominus B \) exists, it is unique.

For the metric \( H_X \) and \( A, B, C, D \in \mathcal{K}^c(X) \) and \( \mu \in \mathbb{R} \) the following properties hold

(P1) \( H_X(A + B, C + D) \leq H_X(A, C) + H_X(B, D) \),

(P2) \( H_X(A + C, B + C) = H_X(A, B) \),

(P3) if \( A \ominus B \) and \( C \ominus D \) exist then \( H_X(A \ominus B, C \ominus D) \leq H_X(A, C) + H_X(B, D) \).

Let \( (Z, \mathcal{Z}, \mu) \) be a measure space. A set-valued mapping \( F : Z \to \mathcal{K}^c(X) \) is called \( \mathcal{Z} \)-measurable (or set-valued random variable) if it satisfies:

\[
\{ z \in Z : F(z) \cap O \neq \emptyset \} \in \mathcal{Z} \quad \text{for every open set } O \subset X.
\]

A set-valued random variable \( F \) is \( L^p \)-integrally bounded (\( p \geq 1 \)), if \( z \mapsto H_X(F(z), \{0\}) \) belongs to \( L^p(Z, \mathcal{Z}, \mu; \mathbb{R}) \).

Define \( I = [0, T] \), where \( T < \infty \), and by \( \beta_t \) we denote the Borel \( \sigma \)-algebra of subsets of \( I \). By \( (\Omega, \mathcal{A}, \{ \mathcal{A}_t \}_{t \in \mathbb{R}}, P) \) we denote a complete filtered probability space satisfying the usual hypotheses, i.e. \( \{ \mathcal{A}_t \}_{t \in \mathbb{R}} \) is an increasing and right continuous family of sub-\( \sigma \)-algebras of \( \mathcal{A} \) and \( \mathcal{A}_0 \) contains all \( P \)-null sets. Let \( \mathcal{N} \) denote the \( \sigma \)-algebra of the nonanticipating elements in \( I \times \Omega \), i.e.

\[
\mathcal{N} = \{ A \in \beta_I \otimes \mathcal{A} : A^t \in \mathcal{A}_t \text{ for every } t \in I \},
\]

where \( A^t = \{ \omega : (t, \omega) \in A \} \).
Let \( \{B(t)\}_{t \in \mathbb{I}} \) be an \( \{\mathcal{A}_t\}\)-Brownian motion. By \( \lambda \) we denote the Lebesgue measure on \((I, \beta_I)\). Consider the space

\[
L^2_N(\lambda \times P) := L^2(I \times \Omega, \mathcal{N}, \lambda \times P; \mathbb{R}^d).
\]

Then for every \( f \in L^2_N(\lambda \times P) \) and \( \tau, t \in I, \tau < t \) the Itô stochastic integral \( \int_{\tau}^{t} f(s)dB(s) \) exists and one has \( \int_{\tau}^{t} f(s)dB(s) \in L^2(\Omega, \mathcal{F}_t; \mathbb{R}^d) \subset L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d) \).

Let \( F: I \times \Omega \to \mathcal{K}^d(\mathbb{R}^d) \) be a set-valued stochastic process, i.e. a family of \( \mathcal{A}\)-measurable set-valued mappings \( F(t, \cdot): \Omega \to \mathcal{K}^d(\mathbb{R}^d), t \in I \). We call \( F \) nonanticipating if \( F(\cdot, \cdot) \) is an \( \mathcal{N}\)-measurable set-valued mapping. Let us define the set

\[
S^2_N(F, \lambda \times P) := \{ f \in L^2_N(\lambda \times P) : f \in F, \lambda \times P\text{-a.e.}\}.
\]

If \( F: I \times \Omega \to \mathcal{K}^d(\mathbb{R}^d) \) is nonanticipating and \( L^2_N(\lambda \times P)\)-integrally bounded, then by the Kuratowski and Ryll-Nardzewski Selection Theorem (see [23]) it follows that \( S^2_N(F, \lambda \times P) \neq \emptyset \). For such a set-valued stochastic process \( F \) we can define the set-valued stochastic Itô trajectory integral. Namely, for \( \tau, t \in I, \tau < t \), by this integral we mean the set

\[
(S) \int_{\tau}^{t} F(s)dB(s) := \left\{ \int_{\tau}^{t} f(s)dB(s) : f \in S^2_N(F, \lambda \times P) \right\}.
\]

From this definition we have \( \int_{\tau}^{t} F(s)dB(s) \subset L^2(\Omega, \mathcal{A}, P; \mathbb{R}^d) \subset L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d) \). For the set-valued stochastic process \( F \) we can also define the following set denoted by \( (S) \int_{\tau}^{t} F(s)ds \) and called the set-valued stochastic Aumann integral

\[
(S) \int_{\tau}^{t} F(s)ds := \left\{ \int_{\tau}^{t} f(s)ds : f \in S^2_N(F, \lambda \times P) \right\}.
\]

Obviously \( (S) \int_{\tau}^{t} F(s)ds \subset L^2(\Omega, \mathcal{A}, P; \mathbb{R}^d) \subset L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d) \).

The following properties of stochastic trajectory integrals (see e.g. [29, 30]) are useful in studying set-valued stochastic integral equations.

**Lemma 2.1.** Let \( F, G: I \times \Omega \to \mathcal{K}^d(\mathbb{R}^d) \) be nonanticipating and \( L^2_N(\lambda \times P)\)-integrally bounded set-valued stochastic processes. Let \( \tau, t \in I, \tau < t \). Then

\[
H^2_{L^2}(S) \int_{\tau}^{t} F(s)dB(s), (S) \int_{\tau}^{t} G(s)dB(s) \leq \int_{[\tau, t] \times \Omega} H^2_{\mathbb{R}^d}(F, G)ds \times dP,
\]

and

\[
H^2_{L^2}(S) \int_{\tau}^{t} F(s)ds, (S) \int_{\tau}^{t} G(s)ds \leq (t - \tau) \int_{[\tau, t] \times \Omega} H^2_{\mathbb{R}^d}(F, G)ds \times dP.
\]

**Lemma 2.2.** Under assumptions of Lemma 2.1, the mappings

\[
[\tau, T] \ni t \mapsto (S) \int_{\tau}^{t} F(s)dB(s) \in \mathcal{K}^d(L^2), \quad [\tau, T] \ni t \mapsto (S) \int_{\tau}^{t} F(s)ds \in \mathcal{K}^d(L^2)
\]

are \( H^2_{L^2}\)-continuous.
3. Main Results

For abbreviation, we write $L^2$ instead of $L^2(\Omega, \mathcal{A}, P; \mathbb{R}^d)$ and $L^2_t$ instead of $L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d)$ where $t \in I$. Let $\Theta, \theta$ denote the zero elements in $L^2$ and $\mathbb{R}^d$, respectively. In this part of the paper we assume that the $\sigma$-algebras $\mathcal{A}, \mathcal{A}_t$ are separable with respect to the probability measure $P$. This way the spaces $L^2$ and $L^2_t$ are separable.

Let $F, \tilde{F}, G, \tilde{G}: I \times \Omega \times \mathcal{K}_c^2(\mathbb{R}) \to \mathcal{K}_c^2(\mathbb{R})$ and $X_0 \in \mathcal{K}_c^2(L^2_0)$ be given. By a bilateral set-valued stochastic integral equation we mean the following relation in the metric space $(\mathcal{K}_c^2(L^2), H_2)$:

$$X(t) + (S) \int_0^t F(s, X(s))ds + (S) \int_0^t G(s, X(s))dB(s) = X_0 + (S) \int_0^t \tilde{F}(s, X(s))ds + (S) \int_0^t \tilde{G}(s, X(s))dB(s) \text{ for } t \in I.$$  

(3.1)

Notice that if $F \equiv [0]$ and $G \equiv [0]$ then the equation written above takes the form

$$X(t) = X_0 + (S) \int_0^t \tilde{F}(s, X(s))ds + (S) \int_0^t \tilde{G}(s, X(s))dB(s) \text{ for } t \in I,$$

(3.2)

which is a natural extension of classical single-valued stochastic integral equations [4, 15] to a set-valued framework. Such equations were studied in [29], for example. Also, if $\tilde{F} \equiv [0]$ and $\tilde{G} \equiv [0]$ then (3.1) reduces to

$$X(t) + (S) \int_0^t F(s, X(s))ds + (S) \int_0^t G(s, X(s))dB(s) = X_0 \text{ for } t \in I.$$  

(3.3)

This equation, investigated in [31], is quite different from (3.2). It also includes classical single-valued stochastic integral equations but in a rather different way than in (3.2). The main difference between (3.2) and (3.3) can be seen from the geometrical properties of their solutions. Namely, solutions $X$ to equation (3.2) possess the property that the function $t \mapsto \text{diam}(X(t))$ is nonincreasing, while solutions $X$ to equation (3.3) satisfy the property that $t \mapsto \text{diam}(X(t))$ is nonincreasing. Hence we call (3.2) and (3.3) the widening equation and the narrowing equation, respectively. These equations can be useful in the mathematical description of stochastic dynamics of real life phenomena when some additional nonstochastic uncertainties of initial values or imprecise parameter values are taken into account.

For example, consider a situation when a microbiologist grows a population of microorganisms in a limited area. Suppose that the number of individuals can depend on random factors and stochastic noises and the microbiologist has the ability to control the population growth by changing the doses of food. In such a setting, the number of individuals at the instant $t \in I$, denoted by $x(t)$, is random and can be described by a controlled stochastic integral equation

$$x(t) = x_0 + \int_0^t f(x(s), u(s))ds + \int_0^t g(x(s), u(s))dB(s), \text{ for } t \in I, \text{ P-a.e.,}$$

(3.4)

where $x_0: \Omega \to \mathbb{R}$ is the initial number of individuals, $f: \mathbb{R}^2 \to \mathbb{R}$ denotes the drift coefficient, $g: \mathbb{R}^2 \to \mathbb{R}$ is a diffusion coefficient, $u$ is a feeding strategy, $u \in U$, and $U$ is a set of controls. Assuming that $x(t) \in L^2(\Omega, \mathcal{A}, P; \mathbb{R})$ for $t \in I$, equation (3.4) can be transformed to an equation in the space $L^2(\Omega, \mathcal{A}, P; \mathbb{R})$, i.e. to the equation

$$x(t) = x_0 + \int_0^t f(s, x(s), u)ds + \int_0^t g(s, x(s), u)dB(s), \text{ for } t \in I,$$

where the coefficients $f, g: I \times \Omega \times L^2 \times U \to \mathbb{R}$ are defined as

$$f(s, \omega, \xi, u) := f(\xi(\omega), u(s, \omega)) \text{ and } g(s, \omega, \xi, u) := g(\xi(\omega), u(s, \omega)).$$
In most cases, a microbiologist cannot accurately determine \( x_0 \). Suppose the microbiologist only knows that \( x_0 \) is an \( \mathcal{A}_0 \)-measurable random variable whose values are bounded by a fixed number \( a > 0 \). In this way, in the presence of additional nonstochastic uncertainty, the initial number of individuals can be viewed as the following set

\[
X_0 := \{ x_0 \in L^2_0 : 0 \leq x_0 \leq a \} \in \mathcal{K}^c(L^2_0).
\]

The dynamics of the uncertain number of individuals \( X(t) \) can be described by the set-valued stochastic integral equation

\[
X(t) = X_0 + (S) \int_0^t \tilde{f}(s, X(s))ds + (S) \int_0^t \tilde{G}(s, X(s))dB(s), \quad t \in I,
\]

where \( \tilde{f}, \tilde{G} : I \times \Omega \times \mathcal{K}^c(L^2) \to \mathcal{K}^c(\mathbb{R}) \) are defined as

\[
\tilde{f}(s, \omega, A) := \overline{\bigcup_{\xi \in \mathbb{C}} \bigcup_{u \in U} f(s, \omega, \xi, u)} \quad \text{and} \quad \tilde{G}(s, \omega, A) := \overline{\bigcup_{\xi \in \mathbb{C}} \bigcup_{u \in U} g(s, \omega, \xi, u)};
\]

here \( \overline{B} \) denotes the closed convex hull of the set \( B \). Now we observe that equation (3.5) is a set-valued stochastic integral equation of the type (3.2). Hence for its solution \( X \), which denotes an uncertain number of individuals, one has that \( \text{diam}(X(t)) \) starts from \( \text{diam}(X_0) \) and cannot decrease. The number \( \text{diam}(X(t)) \) can be interpreted by the microbiologist as a level of uncertainty of the number of individuals. It seems natural that in the case of a large number of microorganisms, uncertainty should be greater than in the case of a small number. However, if one wants to achieve the dynamics of the number of individuals with decreasing uncertainty, one should replace equation (3.2) with equation (3.3) and with

\[
F(s, \omega, A) := \overline{\bigcup_{\xi \in \mathbb{C}} \bigcup_{u \in U} f(s, \omega, \xi, u)} \quad \text{and} \quad G(s, \omega, A) := \overline{\bigcup_{\xi \in \mathbb{C}} \bigcup_{u \in U} g(s, \omega, \xi, u)}.
\]

Then the function \( t \mapsto \text{diam}(X(t)) \) is nonincreasing. However, if one wants to keep the uncertainty at a certain fixed level through an appropriate control strategy or if one wants this uncertainty in the time set to be the same as at the initial moment, neither the widening equation (3.2) nor the narrowing equation (3.3) would be adequate. Considering bilateral set-valued stochastic integral equations (3.1) gives a possibility to handle such situations. Returning to the example of microorganisms growth, one might study the separate influences of births and deaths of individuals on the number of individuals. Suppose that a drift of births is described by \( \tilde{f} \) and diffusion by \( \tilde{G} \) and the volatility is driven by the Brownian motion \( \tilde{B} \), simultaneously assume that a drift of deaths is described by \( F \) and with a diffusion part \( G \) is driven by the Brownian motion \( B \). Notice that in general case the Brownian motions \( \tilde{B} \) and \( B \) can be as the model needs. They can be equal or correlated either independent (and in (3.1) we consider \( \tilde{B} = B \) as freely chosen). Here, in this example, it is reasonable to assume that Brownian motions are independent. Obviously, births force the number of individuals to increase and as before to increase a level of uncertainty \( \text{diam}(X(t)) \), while deaths decrease the number of individuals and force \( \text{diam}(X(t)) \) to be decreasing. Hence we arrive exactly to the bilateral equation (3.1). The set-valued solution \( t \mapsto X(t) \) can give information on approximate dynamics of the population growth \( t \mapsto x(t) \).

This paper includes both the widening and narrowing properties and allows us to have solutions with varying diameter of their values. It also motivates a future new research direction in the field of set-valued integral equations in the stochastic and deterministic context. For example with this formulation it may be possible in the future to consider periodic solutions to set-valued equations.

We begin our formal study. First we say what is meant by a solution to (3.1).

**Definition 3.1.** By a global solution to (3.1) we mean a \( H_{\mathcal{C}} \)-continuous set-valued mapping \( X : I \to \mathcal{K}^c(L^2) \) that satisfies (3.1) for every \( t \in I \). A global solution \( X : I \to \mathcal{K}^c(L^2) \) to (3.1) is unique if \( X(t) = Y(t) \) for every \( t \in I \) where \( Y : I \to \mathcal{K}^c(L^2) \) is any solution of (3.1).
Let $J = [0, \hat{T}] \subset I = [0, T]$, where $\hat{T} < T$.

**Definition 3.2.** A set-valued mapping $X : I \to \mathcal{K}^c_c(L^2)$ is said to be a local solution to (3.1) if it is $H^2_{L^2}$-continuous and satisfies (3.1) for $t \in I$.

The uniqueness of a local solution is defined in an obvious way. We begin the analysis of the bilateral set-valued stochastic integral equations with the existence and uniqueness of a solution to equation (3.1).

Suppose that the coefficients of equation (3.1), i.e. $F, \hat{F}, G, \hat{G} : I \times \Omega \times \mathcal{K}^c_c(L^2) \to \mathcal{K}^c_c(R^d)$, satisfy

1. **(H1)** the set-valued mappings $F(\cdot, \cdot, \cdot), \hat{F}(\cdot, \cdot, \cdot), G(\cdot, \cdot, \cdot), \hat{G}(\cdot, \cdot, \cdot) : I \times \Omega \times \mathcal{K}^c_c(L^2) \to \mathcal{K}^c_c(R^d)$ are $N \times \beta(\tau_{M, 2})$-measurable, where $\beta(\tau_{M, 2})$ is the Borel $\sigma$-algebra induced by the Mosco topology $\tau_{M, 2}$.

2. **(H2)** there exist $K_F, K_{\hat{F}}, K_G, K_{\hat{G}} \in L^2(I \times \Omega, \beta_I \otimes \mathcal{A}, \lambda \times P; R)$ such that $\lambda \times P$-a.e. for every $A, B \in \mathcal{K}^c_c(L^2)$
   \[
   H^2_{L^2}(F(t, \omega, A), F(t, \omega, B)) \leq K_F(t, \omega)H^2_{L^2}(A, B),
   \]
   \[
   H^2_{L^2}(\hat{F}(t, \omega, A), \hat{F}(t, \omega, B)) \leq K_{\hat{F}}(t, \omega)H^2_{L^2}(A, B),
   \]
   \[
   H^2_{L^2}(G(t, \omega, A), G(t, \omega, B)) \leq K_G(t, \omega)H^2_{L^2}(A, B),
   \]
   \[
   H^2_{L^2}(\hat{G}(t, \omega, A), \hat{G}(t, \omega, B)) \leq K_{\hat{G}}(t, \omega)H^2_{L^2}(A, B),
   \]

3. **(H3)** there exist $C_F, C_{\hat{F}}, C_G, C_{\hat{G}} \in L^1(I \times \Omega, \beta_I \otimes \mathcal{A}, \lambda \times P; R)$ such that $\lambda \times P$-a.e.
   \[
   H^2_{L^2}(F(t, \omega, \{\theta\}), \{\theta\}) \leq C_F(t, \omega),
   \]
   \[
   H^2_{L^2}(\hat{F}(t, \omega, \{\theta\}), \{\theta\}) \leq C_{\hat{F}}(t, \omega),
   \]
   \[
   H^2_{L^2}(G(t, \omega, \{\theta\}), \{\theta\}) \leq C_G(t, \omega),
   \]
   \[
   H^2_{L^2}(\hat{G}(t, \omega, \{\theta\}), \{\theta\}) \leq C_{\hat{G}}(t, \omega),
   \]

4. **(H4)** there exists $\hat{T} \in (0, T]$ such that the sequence $\{X_n\}_{n=0}^\infty$ described by
   \[
   X_0(t) = X_0, \quad t \in J := [0, \hat{T}],
   \]
   and for $n = 1, 2, \ldots$,
   \[
   X_n(t) = \left[ X_0 + (S) \int_0^t F(s, X_{n-1}(s))ds + (S) \int_0^t G(s, X_{n-1}(s))dB(s) \right] 
   \]
   \[
   \oplus \left[ (S) \int_0^t \hat{F}(s, X_{n-1}(s))ds + (S) \int_0^t \hat{G}(s, X_{n-1}(s))dB(s) \right],
   \quad t \in J
   \]
   can be defined, i.e. the Hukuhara differences exist.

Assumptions (H2) and (H3), i.e. the Lipschitz condition and the boundedness condition, are formulated with integrable stochastic processes $K_F, K_{\hat{F}}, K_G, K_{\hat{G}}, C_F, C_{\hat{F}}, C_G, C_{\hat{G}}$ and this is more general than considering these processes to be constant. Condition (H4) may look restrictive. However, if one takes a closer look at equation (3.1) one sees that it is extremely important and natural. Note (3.1) can be rewritten as

\[
X(t) = \left[ X_0 + (S) \int_0^t F(s, X(s))ds + (S) \int_0^t G(s, X(s))dB(s) \right] 
\]
\[
\oplus \left[ (S) \int_0^t \hat{F}(s, X(s))ds + (S) \int_0^t \hat{G}(s, X(s))dB(s) \right] \quad \text{for } t \in I.
\] (3.6)
Lemma 3.3. Assume that $F, \tilde{F}, G, \tilde{G} : I \times \Omega \times \mathcal{K}_c^c(L^2) \to \mathcal{K}_c^c(\mathbb{R}^d)$ satisfy hypotheses (H1)-(H4). Then each $X_n : J \to \mathcal{K}_c^c(L^2)$ is a well-defined $H_{1,2}^\infty$-continuous set-valued mapping.

Proof. First, notice that $X_0(\cdot)$ is a well-defined mapping, since it is constantly equal to $X_0, X_0 \in \mathcal{K}_c^c(L_2^2)$. Next, using the measurability condition (H1), we can infer that the set-valued mappings $\tilde{F}(\cdot, X_0), G(\cdot, X_0), \tilde{G}(\cdot, X_0) : I \times \Omega \to \mathcal{K}_c^c(\mathbb{R}^d)$ are nonanticipating. It can be checked, using (H2) and (H3), that the following inequalities hold $\lambda \times P$-a.e.

\[
\begin{align*}
H^2_{\mathcal{K}}(F(t, \omega, X_0), \Theta) \leq 2K_F(t, \omega)H^2_{\mathcal{K}}(X_0, \Theta) + 2C_F(t, \omega), \\
H^2_{\mathcal{K}}(\tilde{F}(t, \omega, \cdot), \Theta) \leq 2K_{\tilde{F}}(t, \omega)H^2_{\mathcal{K}}(X_0, \Theta) + 2C_{\tilde{F}}(t, \omega), \\
H^2_{\mathcal{K}}(G(t, \omega, \cdot), \Theta) \leq 2K_G(t, \omega)H^2_{\mathcal{K}}(X_0, \Theta) + 2C_G(t, \omega), \\
H^2_{\mathcal{K}}(\tilde{G}(t, \omega, \cdot), \Theta) \leq 2K_{\tilde{G}}(t, \omega)H^2_{\mathcal{K}}(X_0, \Theta) + 2C_{\tilde{G}}(t, \omega).
\end{align*}
\]

Thus $F(\cdot, X_0), \tilde{F}(\cdot, X_0), G(\cdot, X_0)$ and $\tilde{G}(\cdot, X_0)$ are $L^2_{\mathcal{K}}(\lambda \times P)$-integrally bounded. Next we claim that the set-valued stochastic trajectory integrals in the formulation of $X_1(t)$ are well-defined and are elements of the set $\mathcal{K}_c^c(L^2_t)$. Since $X_0 \in \mathcal{K}_c^c(L_2^2) \subset \mathcal{K}_c^c(L^2_t)$ and it is assumed that the Hukuhara differences in (H4) exist, we obtain that $X_1(t) \in \mathcal{K}_c^c(L^2_t)$ for every $t \in J$. Moreover, the mapping $t \mapsto X_1(t)$ is $H_{1,2}^\infty$-continuous from Lemma 2.2. Since the Mosco topology $\tau_{M,2}$ is is weaker than the topology generated by the Hausdorff metric $H_{1,2}^\infty$, the mapping $t \mapsto X_1(t)$ is continuous with respect to topology $\tau_{M,2}$ as well. Hence the set-valued mappings $(t, \omega) \mapsto F(t, \omega, X_1(t)), (t, \omega) \mapsto \tilde{F}(t, \omega, X_1(t)), (t, \omega) \mapsto G(t, \omega, X_1(t))$ and $(t, \omega) \mapsto \tilde{G}(t, \omega, X_1(t))$ are nonanticipating. Observing that

\[
\begin{align*}
H^2_{\mathcal{K}}(F(t, \omega, X_1(t)), \Theta) &\leq 2K_F(t, \omega)\sup_{t \in J}H^2_{\mathcal{K}}(X_1(t), \Theta) + 2C_F(t, \omega), \\
H^2_{\mathcal{K}}(\tilde{F}(t, \omega, X_1(t)), \Theta) &\leq 2K_{\tilde{F}}(t, \omega)\sup_{t \in J}H^2_{\mathcal{K}}(X_1(t), \Theta) + 2C_{\tilde{F}}(t, \omega), \\
H^2_{\mathcal{K}}(G(t, \omega, X_1(t)), \Theta) &\leq 2K_G(t, \omega)\sup_{t \in J}H^2_{\mathcal{K}}(X_1(t), \Theta) + 2C_G(t, \omega), \\
H^2_{\mathcal{K}}(\tilde{G}(t, \omega, X_1(t)), \Theta) &\leq 2K_{\tilde{G}}(t, \omega)\sup_{t \in J}H^2_{\mathcal{K}}(X_1(t), \Theta) + 2C_{\tilde{G}}(t, \omega),
\end{align*}
\]

and $\sup_{t \in J}H^2_{\mathcal{K}}(X_1(t), \Theta) < \infty$, we get that $(t, \omega) \mapsto F(t, \omega, X_1(t)), (t, \omega) \mapsto \tilde{F}(t, \omega, X_1(t)), (t, \omega) \mapsto G(t, \omega, X_1(t))$ and $(t, \omega) \mapsto \tilde{G}(t, \omega, X_1(t))$ are $L^2_{\lambda \times P}(\lambda \times P)$-integrally bounded. This allows us to infer that $X_2$ is well-defined and $H_{1,2}^\infty$-continuous. Proceeding recursively we see that every mapping $X_n$ is well-defined and $H_{1,2}^\infty$-continuous. □

Theorem 3.4. Let the assumptions of Lemma 3.3 be satisfied. Then equation (3.1) has a unique (possibly local) solution.
Proof. From Lemma 3.3 we see that each $X_n$ is continuous with respect to the metric $H_{L^2}$. Consider the space $C([K^0_2(L^2)])$ endowed with the supremum metric. We now show that $\{X_n\}_{n=0}^\infty$ is a Cauchy sequence in this metric space.

Applying properties (P3), (P2) and (P1) we have for $t \in J$ that

$$H^2_{L^2}(X_1(t), X_0(t)) = H^2_{L^2}\left[\int_0^t \hat{F}(s, X_0)ds + (S) \int_0^t \hat{G}(s, X_0)d\hat{B}(s)\right]$$

$$\Theta \left[ \int_0^t F(s, X_0)ds + (S) \int_0^t G(s, X_0)d\hat{B}(s), X_0 \right]$$

$$\lesssim 2H^2_{L^2}(\int_0^t \hat{F}(s, X_0)ds + (S) \int_0^t \hat{G}(s, X_0)d\hat{B}(s), X_0)$$

$$+2H^2_{L^2}(\Theta \left[ \int_0^t F(s, X_0)ds + (S) \int_0^t G(s, X_0)d\hat{B}(s), \Theta \right])$$

$$\lesssim 4H^2_{L^2}(\int_0^t \hat{F}(s, X_0)ds, \Theta) + 4H^2_{L^2}(\Theta \left[ \int_0^t \hat{G}(s, X_0)d\hat{B}(s), \Theta \right])$$

$$+4H^2_{L^2}(\Theta \left[ \int_0^t F(s, X_0)ds, \Theta \right]) + 4H^2_{L^2}(\Theta \left[ \int_0^t G(s, X_0)d\hat{B}(s), \Theta \right]).$$

Using Lemma 2.1 we get

$$H^2_{L^2}(X_1(t), X_0(t)) \lesssim 4t \int_{[0,1] \times \Omega} H^2_{L^2}(\hat{F}(s, X_0), \Theta)ds \times d\Omega + 4t \int_{[0,1] \times \Omega} H^2_{L^2}(\hat{G}(s, X_0), \Theta)ds \times d\Omega$$

$$+8t \int_{[0,1] \times \Omega} H^2_{L^2}(\hat{F}(s, X_0), \hat{F}(s, \Theta))ds \times d\Omega + 8t \int_{[0,1] \times \Omega} H^2_{L^2}(\hat{G}(s, X_0), \hat{G}(s, \Theta))ds \times d\Omega$$

$$+8t \int_{[0,1] \times \Omega} H^2_{L^2}(\hat{F}(s, X_0), \hat{G}(s, \Theta))ds \times d\Omega + 8t \int_{[0,1] \times \Omega} H^2_{L^2}(\hat{G}(s, X_0), \hat{G}(s, \Theta))ds \times d\Omega$$

and by hypotheses (H2) and (H3) we have

$$H^2_{L^2}(X_1(t), X_0(t)) \lesssim 16(t + 1)H^2_{L^2}(X_0, \Theta) \int_{[0,1] \times \Omega} (K_F(s) + K_G(s) + K_F(s) + K_G(s))ds \times d\Omega$$

$$+16(t + 1) \int_{[0,1] \times \Omega} (C_F(s) + C_G(s) + C_F(s) + C_G(s))ds \times d\Omega$$

$$\lesssim M_1,$$

where

$$M_1 = 16(T + 1)[H^2_{L^2}(X_0, \Theta) \int_{[0,1] \times \Omega} (K_F(s) + K_G(s) + K_F(s) + K_G(s))ds \times d\Omega$$

$$+ \int_{[0,1] \times \Omega} (C_F(s) + C_G(s) + C_F(s) + C_G(s))ds \times d\Omega] < \infty.$$
Considering $n \geq 2$ we get

\[
H_{L^2}^4(X_n(t), X_{n-1}(t)) \leq 4t \int_{[0,1] \times \Omega} K_f(s) H_{L^2}^2(X_{n-1}(s), X_{n-2}(s)) ds \times dP
\]

\[
+ 4 \int_{[0,1] \times \Omega} K_G(s) H_{L^2}^2(X_{n-1}(s), X_{n-2}(s)) ds \times dP
\]

\[
+ 4t \int_{[0,1] \times \Omega} K_f(s) H_{L^2}^2(X_{n-1}(s), X_{n-2}(s)) ds \times dP
\]

\[
+ 4 \int_{[0,1] \times \Omega} K_G(s) H_{L^2}^2(X_{n-1}(s), X_{n-2}(s)) ds \times dP
\]

\[
\leq \left[ 4t \left( \int_{[0,1] \times \Omega} K_f^2(s) ds \times dP \right) \right]^{1/2}
\]

\[
+ 4 \left( \int_{[0,1] \times \Omega} K_G^2(s) ds \times dP \right)^{1/2}
\]

\[
+ 4 \left( \int_{[0,1] \times \Omega} K_f^2(s) ds \times dP \right)^{1/2}
\]

\[
\times \left( \int_{[0,1] \times \Omega} H_{L^2}^4(X_{n-1}(s), X_{n-2}(s)) ds \times dP \right)^{1/2}
\]

Thus

\[
H_{L^2}^4(X_n(t), X_{n-1}(t)) \leq M_2 \int_0^t H_{L^2}^4(X_{n-1}(s), X_{n-2}(s)) ds,
\]

where

\[
M_2 = \left[ 4t \left( \int_{[0,1] \times \Omega} K_f^2(s) ds \times dP \right) \right]^{1/2}
\]

\[
+ 4 \left( \int_{[0,1] \times \Omega} K_G^2(s) ds \times dP \right)^{1/2}
\]

\[
+ 4 \left( \int_{[0,1] \times \Omega} K_f^2(s) ds \times dP \right)^{1/2}
\]

\[
\times \left( \int_{[0,1] \times \Omega} H_{L^2}^4(X_{n-1}(s), X_{n-2}(s)) ds \times dP \right)^{1/2}.
\]  

This leads us to the conclusion that

\[
H_{L^2}(X_n(t), X_{n-1}(t)) \leq \left( M_2^2 (M_2 T)^{n-1} \right)^{1/4}
\]

and

\[
\sup_{t \in J} H_{L^2}(X_n(t), X_{n-1}(t)) \leq \left( M_2^2 (M_2 T)^{n-1} \right)^{1/4}.
\]

Hence for $m < n$ we have

\[
\sup_{t \in J} H_{L^2}(X_n(t), X_m(t)) \leq \sum_{k=0}^{n-1} \left( M_1^2 (M_2 T)^k \right)^{1/4}
\]

and this allows us to infer that $\{X_n\}$ is a Cauchy sequence in the set $C(J, \mathcal{K}_b^2(L_2^1))$ supplied with the supremum metric. Thus there exists $X \in C(J, \mathcal{K}_b^2(L_2^1))$ such that

\[
\sup_{t \in J} H_{L^2}(X_n(t), X(t)) \to 0 \quad \text{as} \quad n \to \infty.
\]

Since $(\mathcal{K}_b^2(L_2^1), H_{L^2})$ is a complete metric space for each $t \in J$ and $X_n(t), X_m(t) \in \mathcal{K}_b^2(L_2^1)$ and

\[
H_{L^2}(X_n(t), X_m(t)) = H_{L^2}(X_n(t), X_m(t)) \to 0 \quad \text{as} \quad n, m \to \infty \quad \text{for every} \quad t \in J,
\]

we obtain that $X(t) \in \mathcal{K}_b^2(L_2^1)$ for every $t \in J$. 

Furthermore we have $X$ will be a local solution. Notice that for every $t \in J$, $X$ will be a global solution, otherwise if $J \not\subseteq I$, $X$ will be a local solution. Notice that for every $t \in J$ we get

$$H^2_{L_2}\left[ X_0 + \left( S \int_0^t \hat{f}(s, X(s))ds + (S) \int_0^t \hat{G}(s, X(s))dB(s) \right) \right] \supseteq \left[ (S) \int_0^t F(s, X(s))ds + (S) \int_0^t G(s, X(s))dB(s) \right], X(t)$$

where

$$R_n(t) = H^2_{L_2}\left[ X_0 + \left( S \int_0^t \hat{f}(s, X(s))ds + (S) \int_0^t \hat{G}(s, X(s))dB(s) \right) \right] \supseteq \left[ (S) \int_0^t F(s, X(s))ds + (S) \int_0^t G(s, X(s))dB(s) \right], X_0 + \left( S \int_0^t \hat{f}(s, X_{n-1}(s))ds + (S) \int_0^t \hat{G}(s, X_{n-1}(s))dB(s) \right) \supseteq \left[ (S) \int_0^t F(s, X_{n-1}(s))ds + (S) \int_0^t G(s, X_{n-1}(s))dB(s) \right].$$

Observe that

$$R_n(t) \leq 2H^2_{L_2}\left( S \int_0^t \hat{f}(s, X_{n-1}(s))ds + (S) \int_0^t \hat{G}(s, X_{n-1}(s))dB(s), (S) \int_0^t F(s, X(s))ds \right) + 2H^2_{L_2}\left( (S) \int_0^t F(s, X_{n-1}(s))ds + (S) \int_0^t G(s, X_{n-1}(s))dB(s), (S) \int_0^t G(s, X(s))dB(s) \right)$$

Furthermore we have

$$R_n(t) \leq [4t \left( \int_{[0,t] \times \Omega} K^2_f(s)ds \times dP \right)^{1/2} + 4 \left( \int_{[0,t] \times \Omega} K^2_G(s)ds \times dP \right)^{1/2} + 4 \left( \int_{[0,t] \times \Omega} K^2_G(s)ds \times dP \right)^{1/2}] \times \left( \int_{[0,t] \times \Omega} H^2_{L_2}(X_{n-1}(s), X(s))ds \times dP \right)^{1/2}.$$
As a consequence we obtain
\[ R_n^2(t) \leq M_2 \int_0^t H^4_{L^2}(X_{n-1}(s), X(s))ds \leq M_2 T \sup_{t \in J} H^4_{L^2}(X_{n-1}(t), X(t)), \]
where \( M_2 \) is like in (3.8). Since \( \sup_{t \in J} H^4_{L^2}(X_{n-1}(t), X(t)) \) \( \xrightarrow{n \to \infty} 0 \), we have \( R_n(t) \xrightarrow{n \to \infty} 0 \) for every \( t \in J \). This allows us to infer that
\[
H^2_{L^2}(X_0 + (S) \int_0^t \tilde{F}(s, X(s))ds + (S) \int_0^t \tilde{G}(s, X(s))dB(s) + (S) \int_0^t F(s, X(s))ds + (S) \int_0^t G(s, X(s))dB(s), X(t)) = 0
\]
for every \( t \in J \), which means that \( X \) is a solution (possibly local) to (3.1).

We now show that the solution \( X \) is unique. In order to see that, suppose that \( X : J \to K^b_c(L^2) \) and \( Y : J \to K^b_c(L^2) \) are two solutions to (3.1). Then we can check for \( t \in J \) that
\[
H^4_{L^2}(X(t), Y(t)) \leq M_2 \int_0^t H^4_{L^2}(X(s), Y(s))ds,
\]
where \( M_2 \) is like in (3.8). Thus, by applying the Gronwall inequality, we get
\[
H^4_{L^2}(X(t), Y(t)) = 0 \text{ for every } t \in J.
\]
It follows that \( X(t) = Y(t) \) for every \( t \in J \), which proves uniqueness of the solution \( X \). \( \square \)

The mapping \( X_n \) approximates the exact solution \( X \) of (3.1). We now give an estimate on the error between \( X_n \) and \( X \). From the result below we see that the convergence of \( \{X_n\} \) to the solution \( X \) is exponential.

**Proposition 3.5.** Let the assumptions of Lemma 3.3 be satisfied. Then
\[
\sup_{t \in J} H^4_{L^2}(X_n(t), X(t)) \leq 2^{3/4} \left( M_1 \left( \frac{M_2 \|T\|}{n!} \right)^{1/2} \right) \exp[2M_2 \|T\|] \text{ for every } n \in \mathbb{N},
\]
where the constants \( M_1 \) and \( M_2 \) are defined like in (3.7) and (3.8), respectively.

**Proof.** Similar calculations to those in the proof of Theorem 3.4 lead us to the inequality
\[
H^2_{L^2}(X_n(t), X(t)) \leq 4t \left( \int_{[0,t] \times \Omega} K^2_f(s)ds \times dP \right)^{1/2} + 4 \left( \int_{[0,t] \times \Omega} K^2_g(s)ds \times dP \right)^{1/2}
\]
\[
+ 4t \left( \int_{[0,t] \times \Omega} K^2_f(s)ds \times dP \right)^{1/2} + 4 \left( \int_{[0,t] \times \Omega} K^2_g(s)ds \times dP \right)^{1/2}
\]
\[
\times \left( \int_{[0,t] \times \Omega} H^4_{L^2}(X_{n-1}(s), X(s))ds \times dP \right)^{1/2}
\]
for \( t \in J \). Thus
\[
H^4_{L^2}(X_n(t), X(t)) \leq M_2 \int_0^t H^4_{L^2}(X_{n-1}(s), X(s))ds
\]
\[
\leq 8M_2 \int_0^t H^4_{L^2}(X_{n-1}(s), X(s))ds + 8M_2 \int_0^t H^4_{L^2}(X_n(s), X(s))ds
\]
and applying (3.9) we obtain
\[
H^4_{L^2}(X_n(t), X(t)) \leq 8M_2 \left( \frac{M_2 \|T\|}{n!} \right)^{1/2} + 8M_2 \int_0^t H^4_{L^2}(X_n(s), X(s))ds,
\]
Proof. Using (P3) and (P1) we have for
\[H_{L^2}(X(t), X(t)) \leq 8M_1^4 \frac{(M_2 T)^n}{n!} \exp[8M_2 t] \text{ for every } t \in I.\]

Now the claim follows immediately. \(\Box\)

The next result gives an estimate confirming the boundedness of the solution to equation (3.1).

**Proposition 3.6.** Under the assumptions of Lemma 3.3 for the solution \(X\) to (3.1) we have the estimate
\[
\sup_{t \in I} H_{L^2}(X(t), [\Theta]) \leq M_3 \exp[M_4 T],
\]
where
\[
M_3 = (18H_{L^2}^4(X_0, [\Theta]) + 3(12T) \int_{\gamma \times \Omega} C_f(s) ds \times dP + 12 \int_{\gamma \times \Omega} C_G(s) ds \times dP + 8T \int_{\gamma \times \Omega} C_f(s) ds \times dP + 8 \int_{\gamma \times \Omega} C_G(s) ds \times dP)^{1/4}
\]
and
\[
M_4 = \left( \frac{3}{4}(12T) \left( \int_{\gamma \times \Omega} C_f(s) ds \times dP \right)^{1/2} + 12 \left( \int_{\gamma \times \Omega} C_G(s) ds \times dP \right)^{1/2} + 8T \left( \int_{\gamma \times \Omega} C_f(s) ds \times dP \right)^{1/2} + 8 \left( \int_{\gamma \times \Omega} C_G(s) ds \times dP \right)^{1/2} \right)^{2}.
\]

Proof. Using (P3) and (P1) we have for \(t \in I\) that
\[
H_{L^2}(X(t), [\Theta]) \leq 6H_{L^2}^2(X_0, [\Theta]) + 6H_{L^2}^2 \left( S \int_0^t F(s, X(s)) ds, [\Theta] \right) + 4H_{L^2}^2 \left( S \int_0^t G(s, X(s)) ds, [\Theta] \right).
\]

Applying Lemma 2.1 we get
\[
H_{L^2}^2(X(t), [\Theta]) \leq 6H_{L^2}^2(X_0, [\Theta]) + 12T \left[ \int_{[0,t] \times \Omega} H_{R^2}^2(\hat{F}(s, X(s)), \hat{F}(s, [\Theta])) ds \times dP \right] + 8T \left[ \int_{[0,t] \times \Omega} H_{R^2}^2(\hat{G}(s, X(s)), \hat{G}(s, [\Theta])) ds \times dP \right] + 8T \left[ \int_{[0,t] \times \Omega} H_{R^2}^2(\tilde{F}(s, X(s)), \tilde{F}(s, [\Theta])) ds \times dP \right] + 8T \left[ \int_{[0,t] \times \Omega} H_{R^2}^2(\tilde{G}(s, X(s)), \tilde{G}(s, [\Theta])) ds \times dP \right].
\]
From assumptions (H2) and (H3) we have

\[ H_{L^2}^2(X(t), \{\Theta\}) \leq 6H_{L^2}^2(X_0, \{\Theta\}) \]

\[ + 12l \int \left[ \int_{[0,l] \times \Omega} K_F(s)H_{L^2}^2(X(s), \{\Theta\})ds \right] dP + 12l \int \left[ \int_{[0,l] \times \Omega} C_F(s)ds \right] dP \]

\[ + 12 \int \left[ \int_{[0,l] \times \Omega} K_G(s)H_{L^2}^2(X(s), \{\Theta\})ds \right] dP + 12 \int \left[ \int_{[0,l] \times \Omega} C_G(s)ds \right] dP \]

\[ + 8l \int \left[ \int_{[0,l] \times \Omega} K_F(s)H_{L^2}^2(X(s), \{\Theta\})ds \right] dP + 8l \int \left[ \int_{[0,l] \times \Omega} C_F(s)ds \right] dP \]

\[ + 8 \int \left[ \int_{[0,l] \times \Omega} K_G(s)H_{L^2}^2(X(s), \{\Theta\})ds \right] dP + 8 \int \left[ \int_{[0,l] \times \Omega} C_G(s)ds \right] dP \]

\[ \leq 6H_{L^2}^2(X_0, \{\Theta\}) \]

\[ + 12l \int \left[ \int_{[0,l] \times \Omega} C_F(s)ds \right] dP + 12 \int \left[ \int_{[0,l] \times \Omega} C_G(s)ds \right] dP \]

\[ + 8l \int \left[ \int_{[0,l] \times \Omega} C_F(s)ds \right] dP + 8 \int \left[ \int_{[0,l] \times \Omega} C_G(s)ds \right] dP \]

\[ + \left( 12l \int \left[ \int_{[0,l] \times \Omega} K_F^2(s)ds \right] dP \right)^{1/2} + 12 \left( \int \left[ \int_{[0,l] \times \Omega} K_G^2(s)ds \right] dP \right)^{1/2} \]

\[ + 8l \left( \int \left[ \int_{[0,l] \times \Omega} K_F^2(s)ds \right] dP \right)^{1/2} + 8 \left( \int \left[ \int_{[0,l] \times \Omega} K_G^2(s)ds \right] dP \right)^{1/2} \]

\[ \times \left( \int \left[ \int_{[0,l] \times \Omega} H_{L^2}^4(X(s), \{\Theta\})ds \right] dP \right)^{1/2} . \]

Thus

\[ H_{L^2}^4(X(t), \{\Theta\}) \leq 18H_{L^2}^4(X_0, \{\Theta\}) \]

\[ + 3\left( 12l \int \left[ \int_{[0,l] \times \Omega} C_F(s)ds \right] dP + 12 \int \left[ \int_{[0,l] \times \Omega} C_G(s)ds \right] dP \right) \]

\[ + 8l \left( \int \left[ \int_{[0,l] \times \Omega} C_F(s)ds \right] dP \right)^2 + 8 \left( \int \left[ \int_{[0,l] \times \Omega} C_G(s)ds \right] dP \right)^2 \]

\[ + 3 \left( 12 \left( \int \left[ \int_{[0,l] \times \Omega} K_F^2(s)ds \right] dP \right)^{1/2} + 12 \left( \int \left[ \int_{[0,l] \times \Omega} K_G^2(s)ds \right] dP \right)^{1/2} \right) \]

\[ + 8 \left( \int \left[ \int_{[0,l] \times \Omega} K_F^2(s)ds \right] dP \right)^{1/2} + 8 \left( \int \left[ \int_{[0,l] \times \Omega} K_G^2(s)ds \right] dP \right)^{1/2} \]

\[ \times \left( \int \left[ \int_{[0,l] \times \Omega} H_{L^2}^4(X(s), \{\Theta\})ds \right] dP \right)^{1/2} . \]

\[ \leq M_1^4 + 4M_1 \int_0^d H_{L^2}^4(X(s), \{\Theta\})ds . \]

From Gronwall’s inequality we get

\[ H_{L^2}^4(X(t), \{\Theta\}) \leq M_2^4 \exp \left[ 4M_2(t) \right] \text{ for every } t \in J, \]

and this inequality leads us to the assertion easily. \( \Box \)

Now we focus on the property of continuous dependence of the solution with respect to the initial value and the coefficients of the equation. This property is needed to ensure that the theory of bilateral set-valued stochastic integral equations is well-posed.
Let us denote by $J$ and (3.10), respectively, $J$.

Using Lemma 2.1 and assumptions (H2) and (H3) we get

$$X(t) + (S) \int_0^t F(s, X(s))ds + (S) \int_0^t G(s, X(s))dB(s)$$

$$= \tilde{X}_0 + (S) \int_0^t \tilde{F}(s, X(s))ds + (S) \int_0^t \tilde{G}(s, X(s))d\tilde{B}(s) \text{ for } t \in I. \quad (3.10)$$

Let us denote by $X: I_1 \to \mathcal{K}_c^2(L^2)$ and $Y: I_2 \to \mathcal{K}_c^2(L^2)$ the unique solutions (if they exist) to equations (3.1) and (3.10), respectively, $I_1 = [0, \bar{T}_1], \ I_2 = [0, \bar{T}_2]$ for some $\bar{T}_1, \bar{T}_2 \in (0, T)$. Let $I = I_1 \cap I_2$.

**Theorem 3.7.** Let $F, \tilde{F}, G, \tilde{G}$ satisfy the conditions (H1)-(H3). Assume also that $F, \tilde{F}, G, \tilde{G}$ satisfy (H4) with $X_0$, and that $F, \tilde{F}, G, \tilde{G}$ satisfy (H4) with $\tilde{X}_0$, too. Then

$$\sup_{t \in I} H_{L^2}(X(t), Y(t)) \leq 12^{1/4} H_{L^2}(X_0, \tilde{X}_0) \exp[\tilde{M} \min\{T_1, T_2\}],$$

where

$$\tilde{M} = \frac{1}{2} \left(6 \min\{T_1, T_2\} \left( \int_{[0,t]} K^2_{f}(s)ds \times dP \right)^{1/2} \right) + 6 \left( \int_{[0,t]} K^2_{f}(s)ds \times dP \right)^{1/2}$$

$$+ 4 \min\{T_1, T_2\} \left( \int_{[0,t]} K^2_{\tilde{f}}(s)ds \times dP \right)^{1/2} + 4 \left( \int_{[0,t]} K^2_{\tilde{f}}(s)ds \times dP \right)^{1/2}.$$

**Proof.** Notice that

$$H_{L^2}^2(X(t), Y(t)) \leq 6H_{L^2}^2(X_0, \tilde{X}_0)$$

$$+ 6H_{L^2}^2 \left( \int_0^t \tilde{F}(s, X(s))ds, \int_0^t \tilde{F}(s, Y(s))ds \right)$$

$$+ 6H_{L^2}^2 \left( \int_0^t \tilde{G}(s, X(s))d\tilde{B}(s), \int_0^t \tilde{G}(s, Y(s))d\tilde{B}(s) \right)$$

$$+ 4H_{L^2}^2 \left( \int_0^t F(s, X(s))ds, \int_0^t F(s, Y(s))ds \right)$$

$$+ 4H_{L^2}^2 \left( \int_0^t G(s, X(s))d\tilde{B}(s), \int_0^t G(s, Y(s))d\tilde{B}(s) \right).$$

Using Lemma 2.1 and assumptions (H2) and (H3) we get

$$H_{L^2}^2(X(t), Y(t)) \leq 6H_{L^2}^2(X_0, \tilde{X}_0)$$

$$+ 4t \int_{[0,t]} K_{f}(s)H_{L^2}^2(X(s), Y(s))ds \times dP$$

$$+ 4t \int_{[0,t]} K_{\tilde{f}}(s)H_{L^2}^2(X(s), Y(s))ds \times dP$$

$$+ 4t \int_{[0,t]} K_{f}(s)H_{L^2}^2(X(s), Y(s))ds \times dP$$

$$+ 4t \int_{[0,t]} K_{\tilde{f}}(s)H_{L^2}^2(X(s), Y(s))ds \times dP.$$
Hence
\[ H_{L}^{1}(X(t), Y(t)) \leq 12H_{L}^{1}(X_0, \bar{X}_0) + 2\left(6\left(\int_{\Gamma} K_s^2(s)ds \times dP\right)^{1/2} + 6\left(\int_{\Omega} K_s^2(s)ds \times dP\right)^{1/2}
\right.
+ 4\left(\int_{\Gamma} K_s^2(s)ds \times dP\right)^{1/2} + 4\left(\int_{\Omega} K_s^2(s)ds \times dP\right)^{1/2}\right)^2
\int_{0}^{t} H_{L}^{1}(X(s), Y(s))ds.

Gronwall’s inequality allows us to infer that
\[ H_{L}^{1}(X(t), Y(t)) \leq \tilde{M}_5 \exp[\tilde{M}_5t] \text{ for every } t \in I,
\]
where \( \tilde{M}_5 = 12H_{L}^{1}(X_0, \bar{X}_0) \) and
\[ \tilde{M}_6 = 2\left(6\min(T_1, T_2)\left(\int_{\Gamma} K_s^2(s)ds \times dP\right)^{1/2} + 6\left(\int_{\Omega} K_s^2(s)ds \times dP\right)^{1/2}
\right.
+ 4\min(T_1, T_2)\left(\int_{\Gamma} K_s^2(s)ds \times dP\right)^{1/2} + 4\left(\int_{\Omega} K_s^2(s)ds \times dP\right)^{1/2}\right)^2.
\]
Therefore we get
\[ \sup_{t \in I} H_{L}^{1}(X(t), Y(t)) \leq (\tilde{M}_5)^{1/4} \exp[\tilde{M}_6 \min(T_1, T_2)/4],
\]
which ends the proof. \( \square \)

Now consider equation (3.1) and equations (for \( n \in \mathbb{N} \))
\[ X(t) + (S) \int_{0}^{t} F_n(s, X(s))ds + (S) \int_{0}^{t} G_n(s, X(s))dB(s)
= X_0 + (S) \int_{0}^{t} \tilde{F}_n(s, X(s))ds + (S) \int_{0}^{t} \tilde{G}_n(s, X(s))d\tilde{B}(s) \text{ for } t \in I \tag{3.11}
\]
with other coefficients \( F_n, \tilde{F}_n, G_n, \tilde{G}_n \). Let us denote by \( X \) and \( Y \) unique solutions (if they exist) to equations (3.1) and (3.11), respectively. Suppose that they all are defined on a common interval \( I = [0, T] \) with \( \tilde{T} \in (0, T] \).

**Theorem 3.8.** Let \( X_0, F, \tilde{F}, G, \tilde{G} \) satisfy the conditions (H1)-(H4). Assume also that \( X_0, F_n, \tilde{F}_n, G_n, \tilde{G}_n \) satisfy (H1)-(H4), in particular the conditions (H2) and (H3) are satisfied with the processes \( K_F, K_{\tilde{F}}, K_G, K_{\tilde{G}}, C_F, C_{\tilde{F}}, C_G, C_{\tilde{G}} \), respectively. Assume that there exist constants \( S_F, S_{\tilde{F}}, S_G, S_{\tilde{G}} > 0 \) such that for every \( n \in \mathbb{N} \)
\[ \int_{\Gamma} K_{F_n}^2(s)ds \times dP \leq S_F, \int_{\Omega} K_{G_n}^2(s)ds \times dP \leq S_G,
\]
\[ \int_{\Gamma} K_{\tilde{F}_n}^2(s)ds \times dP \leq S_{\tilde{F}} \text{ and } \int_{\Omega} K_{\tilde{G}_n}^2(s)ds \times dP \leq S_{\tilde{G}}.
\]
Suppose that for every \( A \in \mathcal{K}^2_{\mathcal{L}}(L^2) \)
\[ \int_{\Gamma} H^{2}_{\mathcal{L}}(F_n(s, A), F(s, A))ds \times dP \to 0 \text{ as } n \to \infty,
\]
\[ \int_{\Omega} H^{2}_{\mathcal{L}}(G_n(s, A), G(s, A))ds \times dP \to 0 \text{ as } n \to \infty,
\]
\[ \int_{\Gamma} H^{2}_{\mathcal{L}}(\tilde{F}_n(s, A), \tilde{F}(s, A))ds \times dP \to 0 \text{ as } n \to \infty \text{ and }
\]
\[ \int_{\Omega} H^{2}_{\mathcal{L}}(\tilde{G}_n(s, A), \tilde{G}(s, A))ds \times dP \to 0 \text{ as } n \to \infty.
\]
Then for the local solution \(X\) to (3.1) and solutions \(X_n\) to (3.11) we have

\[
\sup_{t \in J} H_{L^2}(Y_n(t), X(t)) \to 0 \text{ as } n \to \infty.
\]

Proof. Using the form (3.6) of the solutions \(X\) and \(Y_n\) and properties (P3), (P2) and (P1) we have for \(t \in J\) that

\[
H_{L^2}^2(Y_n(t), X(t)) \leq 4H_{L^2}^2 \left( \int_0^t \tilde{F}_n(s, Y_n(s))ds, \int_0^t \tilde{F}(s, X(s))ds \right)
+ 4H_{L^2}^2 \left( \int_0^t \tilde{G}_n(s, Y_n(s))d\tilde{B}(s), \int_0^t \tilde{G}(s, X(s))d\tilde{B}(s) \right)
+ 4H_{L^2}^2 \left( \int_0^t F_n(s, Y_n(s))ds, \int_0^t F(s, X(s))ds \right)
+ 4H_{L^2}^2 \left( \int_0^t G_n(s, Y_n(s))dB(s), \int_0^t G(s, X(s))dB(s) \right)
\leq 8H_{L^2}^2 \left( \int_0^t \tilde{F}_n(s, Y_n(s))ds, \int_0^t \tilde{F}_n(s, X(s))ds \right)
+ 8H_{L^2}^2 \left( \int_0^t F_n(s, Y_n(s))ds, \int_0^t F(s, X(s))ds \right)
+ 8H_{L^2}^2 \left( \int_0^t \tilde{G}_n(s, Y_n(s))d\tilde{B}(s), \int_0^t \tilde{G}_n(s, X(s))d\tilde{B}(s) \right)
+ 8H_{L^2}^2 \left( \int_0^t F_n(s, Y_n(s))ds, \int_0^t F_n(s, X(s))ds \right)
+ 8H_{L^2}^2 \left( \int_0^t G_n(s, Y_n(s))dB(s), \int_0^t G_n(s, X(s))dB(s) \right)
+ 8H_{L^2}^2 \left( \int_0^t G_n(s, X(s))dB(s), \int_0^t G(s, X(s))dB(s) \right).
\]

Applying Lemma 2.1 we obtain the estimate

\[
H_{L^2}^2(Y_n(t), X(t)) \leq 8t \int_{[0,t] \times \Omega} H_{W^2}^2(F_n(s, Y_n(s)), F_n(s, X(s)))ds \times dP
+ 8t \int_{[0,t] \times \Omega} H_{W^2}^2(F_n(s, X(s)), F(s, X(s)))ds \times dP
+ 8 \int_{[0,t] \times \Omega} H_{W^2}^2(G_n(s, X_n(s)), \tilde{G}_n(s, X(s)))ds \times dP
+ 8 \int_{[0,t] \times \Omega} H_{W^2}^2(\tilde{G}_n(s, X(s)), \tilde{G}(s, X(s)))ds \times dP
+ 8t \int_{[0,t] \times \Omega} H_{W^2}^2(F_n(s, Y_n(s)), F_n(s, X(s)))ds \times dP
\]
where.

Therefore

\[ H_{1,2}^2(Y_n(t), X(t)) \leq M_7(n) + M_8(n) \left( \int_0^T H_{1,2}^4(Y_n(s), X(s))ds \right)^{1/2}, \]

where

\[ M_7(n) = 8T \int_{\mathcal{J} \times \Omega} H^2_{\Re}(F_n(s, X(s)), \tilde{F}(s, X(s)))ds \times dP \]

\[ + 8 \int_{\mathcal{J} \times \Omega} H^2_{\Re}(G_n(s, X(s)), \tilde{G}(s, X(s)))ds \times dP \]

\[ + 8 T \int_{\mathcal{J} \times \Omega} H^2_{\Re}(F_n(s, X(s)), \tilde{F}(s, X(s)))ds \times dP \]

\[ + 8 \int_{\mathcal{J} \times \Omega} H^2_{\Re}(G_n(s, X(s)), \tilde{G}(s, X(s)))ds \times dP \]

and

\[ M_8(n) = 8T\left( \int_{\mathcal{J} \times \Omega} K^2_{F_n}(s)ds \times dP \right)^{1/2} + 8\left( \int_{\mathcal{J} \times \Omega} K^2_{G_n}(s)ds \times dP \right)^{1/2} \]

\[ + 8T\left( \int_{\mathcal{J} \times \Omega} K^2_{F_n}(s) \times dP \right)^{1/2} + 8\left( \int_{\mathcal{J} \times \Omega} K^2_{G_n}(s) \times dP \right)^{1/2}. \]
Thus
\[
H_{12}^4(Y_n(t), X(t)) \leq 2M_2^2(n) + 2M_8^2(n) \int_0^t H_{12}^4(Y_n(s), X(s))ds
\]

Applying the Gronwall inequality we get for every \( t \in J \) that
\[
H_{12}^4(Y_n(t), X(t)) \leq 2M_2^2(n) \exp[2M_8^2(n)t].
\]

Hence
\[
\sup_{t \in J} H_{12}^4(Y_n(t), X(t)) \leq 2M_2^2(n) \exp[2M_8^2(n)T].
\]

From our assumptions we obtain that \( M_2(n) \to 0 \) as \( n \to \infty \) and \( M_8(n) \leq 8T \sqrt{S_F} + 8 \sqrt{S_G} + 8T \sqrt{S_F} + 8 \sqrt{S_G} \).

Thus
\[
\sup_{t \in J} H_{12}^4(Y_n(t), X(t)) \to 0
\]
which completes the proof. \( \square \)

Although condition (H4), which requires the existence of some Hukuhara differences, seems to be restrictive, as we mentioned before it is important and indispensable in the study of bilateral set-valued integral equations. It is needed, since in the representation (3.6) of equation (3.1) some Hukuhara differences are involved. In general, having in the background subsets of infinitely dimensional space \( L^2 \), some assumption concerning the existence of Hukuhara’s differences must appear. An assumption of this kind, in the authors’ opinion, is also needed in considering subsets of at least a two-dimensional space. Our reasoning is from the fact that in such spaces there is no convenient way to check the existence of Hukuhara’s differences. Some slightly different conditions that lead to (H4) are presented below.

**Remark 3.9.** In each of the conditions mentioned below one can replace (H4) in the study of the bilateral set-valued stochastic integral equation (3.1).

(H41) There exists \( T \in (0, T] \) such that for every \( t \in [0, T] \) and for every \( H_{12} \)-continuous mapping \( X: [0, T] \to \mathcal{K}_b^c(L^2) \) satisfying \( X(t) \in \mathcal{K}_b^c(L^2), \ t \in [0, T] \), there exist Hukuhara’s differences
\[
\left[ X_0 + (S) \int_0^t \tilde{F}(s, X(s))ds + (S) \int_0^t \tilde{G}(s, X(s))dB(s) \right] \in \left[ (S) \int_0^t F(s, X(s))ds + (S) \int_0^t G(s, X(s))dB(s) \right].
\]

(H42) There exists \( \tilde{T} \in (0, T] \) such that for every \( t \in [0, \tilde{T}] \) and for every \( A \in \mathcal{K}_b^c(L^2) \), there exist Hukuhara’s differences
\[
\left[ X_0 + (S) \int_0^t \tilde{F}(s, A)ds + (S) \int_0^t \tilde{G}(s, A)dB(s) \right] \in \left[ (S) \int_0^t F(s, A)ds + (S) \int_0^t G(s, A)dB(s) \right].
\]

Unfortunately, checking the existence of differences in the case of subsets of the abstract space \( L^2 \) is still a very difficult task.

In the next section we show that (H4) can be replaced with another condition that guarantees the existence of the desired Hukuhara differences. However, we limit ourselves to considering subsets of a one-dimensional space. For subsets of such a space, there is a convenient criterion for checking the existence of the Hukuhara difference.
4. Implications for Deterministic Bilateral Set-Valued Integral Equations

Since $\mathbb{R}$ can be embedded into $L^2(\Omega, \mathcal{A}, P; \mathbb{R})$, the following deterministic, bilateral set-valued integral equation

$$X(t) + (S) \int_0^t \Psi(s, X(s))ds = X_0 + (S) \int_0^t \Phi(s, X(s))ds \quad \text{for} \ t \in I, \hspace{1cm} (4.1)$$

where $\Psi, \Phi: I \times \mathcal{K}_i^c(\mathbb{R}) \to \mathcal{K}_i^c(\mathbb{R})$, $X_0 \in \mathcal{K}_i^c(\mathbb{R})$ and the integral is the set-valued Aumann integral, is a particular case of equation (3.1). If $\Psi \equiv 0$ then (4.1) constitutes an integral form of set-valued differential equations studied in [24]. We show that assuming a kind of boundedness of the coefficient $\Psi$ we get the existence of the Hukuhara differences written in (H4).

We assume that $\Psi, \Phi: I \times \mathcal{K}_i^c(\mathbb{R}) \to \mathcal{K}_i^c(\mathbb{R})$ satisfy

(S1) for every $A \in \mathcal{K}(\mathbb{R})$ the set-valued mappings $\Psi(\cdot, A), \Phi(\cdot, A): I \to \mathcal{K}_i^c(\mathbb{R})$ are $\beta_i$-measurable,

(S2) there exist $K_\Psi, K_\Phi \in L^2(I, \beta_i, \lambda; \mathbb{R})$ such that $\lambda$-a.e. for every $A, B \in \mathcal{K}_i^c(\mathbb{R})$

$$H_{R} \left( \Psi(t, A), \Psi(t, B) \right) \leq K_\Psi(t)H_{R}(A, B),$$

$$H_{R} \left( \Phi(t, A), \Phi(t, B) \right) \leq K_\Phi(t)H_{R}(A, B),$$

(S3) there exist $C_\Psi, C_\Phi \in L^1(I, \beta_i, \lambda; \mathbb{R})$ such that $\lambda$-a.e.

$$H_{R} \left( \Psi(t, [0]), [0] \right) \leq C_\Psi(t),$$

$$H_{R} \left( \Phi(t, [0]), [0] \right) \leq C_\Phi(t).$$

Additionally suppose that $\Psi$ satisfies the following condition

(S4) for every $N \in \mathbb{N}$ there exists a positive constant $M_N$ such that for every $t \in I$ and for every $A \in \mathcal{K}_i^c(\mathbb{R})$ we have

$$H_{R}(A, [0]) \leq N \implies H_{R}(\Psi(t, A), [0]) \leq M_N. \hspace{1cm} (4.2)$$

We claim in this case that the sequence $\{X_n\}$ described in (H4) by $X_0(t) = X_0$ and

$$X_n(t) = \left( X_0 + (S) \int_0^t \Phi(s, X_{n-1}(s))ds \right) \ominus (S) \int_0^t \Psi(s, X_{n-1}(s))ds$$

(for $n \in \mathbb{N}$) is well defined on an interval $J \subset I$. Indeed, there exists $N_0 \in \mathbb{N}$ such that $H_{R}(X_0, [0]) \leq N_0$ and this means, in view of (4.2), that there exists $M_{N_0}$ such that $H_{R}(\Psi(t, X_0), [0]) \leq M_{N_0}$ for every $t \in I$. Hence for $t \in [0, \text{diam}X_0/(2M_{N_0})]$ we have

$$\text{diam} \left( S \int_0^t \Psi(s, X_0)ds \right) \leq 2 \int_0^t H_{R}(\Psi(s, X_0), [0])ds \leq 2M_{N_0}t \leq \text{diam}X_0 \leq \text{diam}X_0 + \text{diam} \left( S \int_0^t \Phi(s, X_0)ds \right).$$

In the hyperspace $\mathcal{K}_i^c(\mathbb{R})$ we have: if $\text{diam}A \geq \text{diam}B$ then $A \ominus B$ exists, $A, B \in \mathcal{K}_i^c(\mathbb{R})$. Hence the Hukuhara difference $\left( X_0 + (S) \int_0^t \Phi(s, X_0)ds \right) \ominus (S) \int_0^t \Psi(s, X_0)ds$ exists for each $t \in [0, \text{diam}X_0/(2M_{N_0})]$ which means that
Let the assumptions of Proposition 4.1 be satisfied. Then for every $n \in \mathbb{N}$, Proposition 4.2.

The boundedness of the solution can also be obtained.

Proposition 4.3. Under the assumptions of Proposition 4.1 for the local solution $X$ to (4.1) we have

$$\sup_{t \in J} H_{R}(X(t), \{0\}) \leq k_{4},$$

where $k_{4}$ is a positive constant.
To consider the sensitivity of solutions to small changes of initial value, we study equation (4.1) and the same equation with another initial value $\tilde{X}_0$

$$X(t) + (S) \int_0^t \Psi(s, X(s))ds = \tilde{X}_0 + (S) \int_0^t \Phi(s, X(s))ds \quad \text{for} \quad t \in I. \quad (4.3)$$

Let $X: J_1 \to \mathcal{K}_c^b(\mathbb{R})$ and $Y: J_2 \to \mathcal{K}_c^b(\mathbb{R})$ denote the unique solutions (if they exist) to these equations, respectively, $J_1 = [0, \tilde{T}_1], J_2 = [0, \tilde{T}_2]$ for some $\tilde{T}_1, \tilde{T}_2 \in (0, T]$. Let $J = J_1 \cap J_2$.

**Proposition 4.4.** Let $X_0, \tilde{X}_0 \in \mathcal{K}_c^b(\mathbb{R})$. Assume that $\Psi, \Phi$ satisfy the conditions (S1)-(S4). Then

$$\sup_{t \in J} H_R(X(t), Y(t)) \leq k_5 H_R(X_0, \tilde{X}_0) \exp[k_6 \min\{T_1, T_2\}],$$

where $k_5, k_6$ are some positive constants.

To consider stability of solutions to (4.1) with respect to small changes of coefficients $\Psi$ and $\Phi$, we consider equation (4.1) and equations (for $n \in \mathbb{N}$)

$$X(t) + (S) \int_0^t \Psi_n(s, X(s))ds = X_0 + (S) \int_0^t \Phi_n(s, X(s))ds \quad \text{for} \quad t \in I \quad (4.4)$$

with other coefficients $\Psi_n$ and $\Phi_n$ than $\Psi$ and $\Phi$ in (4.1). Let $X, Y_n$ denote the unique solutions (if they exist) to these equations, respectively. Assume that they all are defined on a common interval $J = [0, \tilde{T}]$ with $\tilde{T} \in (0, T]$.

**Proposition 4.5.** Let $X_0 \in \mathcal{K}_c^b(\mathbb{R})$. Suppose that $\Psi, \Phi$ satisfy the conditions (S1)-(S4). Assume also that $\Psi_n, \Phi_n$ satisfy (S1)-(S4), in particular the conditions (S2) and (S3) are satisfied with the functions $K_{\Psi_n}, K_{\Phi_n}, C_{\Psi_n}, C_{\Phi_n}$, respectively. Assume that there exist constants $S_\Psi, S_\Phi > 0$ such that for every $n \in \mathbb{N}$

$$\int_I K_{\Psi_n}^2(s)ds \leq S_\Psi, \quad \int_I K_{\Phi_n}^2(s)ds \leq S_\Phi.$$

Suppose that for every $A \in \mathcal{K}_c^b(\mathbb{R})$

$$\int_I H_R(\Psi_n(s, A), \Psi(s, A))ds \to 0 \quad \text{as} \quad n \to \infty,$$

$$\int_I H_R(\Phi_n(s, A), \Phi(s, A))ds \to 0 \quad \text{as} \quad n \to \infty.$$

Then for the solution $X$ to (4.1) and the solutions $X_n$ to (4.4) we have

$$\sup_{t \in J} H_R(Y_n(t), X(t)) \to 0 \quad \text{as} \quad n \to \infty.$$

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