Linear Combinations of a Class of Harmonic Univalent Mappings

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Abstract. A planar harmonic mapping is a complex-valued function $f : \mathbb{U} \to \mathbb{C}$ of the form $f(x + iy) = u(x, y) + iv(x, y)$, where $u$ and $v$ are both real harmonic. Such a function can be written as $f = h + \overline{g}$, where $h$ and $g$ are both analytic; the function $\omega = g'/h'$ is called the dilatation of $f$. We consider the linear combinations of planar harmonic mappings that are the vertical shears of the asymmetrical vertical strip mappings $\varphi_j(z) = \frac{1}{2\pi i} \log \left( \frac{1 + ze^{i\alpha_j}}{1 + ze^{-i\alpha_j}} \right)$ with various dilatations, where $\alpha_j \in \left[ \frac{\pi}{2}, \pi \right)$, $j = 1, 2$. We prove sufficient conditions for the linear combination of this class of harmonic univalent mappings to be univalent and convex in the direction of the imaginary axis.

1. Introduction

A continuous complex-valued function $f = \mu + iv$ defined in a simply connected domain $\Omega \subset \mathbb{C}$ is said to be harmonic in $\Omega$ if both $\mu$ and $v$ are real harmonic in $\Omega$. In any simply connected domain $\Omega$, we can write

\[ f = h + \overline{g}, \tag{1.1} \]

where $h$ and $g$ are analytic in $\Omega$. A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $\Omega$ is that $|h'| > |g'|$ in $\Omega$.

Denote by $S_H$ the class of functions $f$ of the form (1.1) that are harmonic univalent and sense-preserving in the unit disc $\mathbb{U} = \{ z : |z| < 1 \}$ and normalized by $f(0) = f_z(0) = 0$. It is obvious that the normalization condition is equivalent to

\[ h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \]

Furthermore, let $S_{H1}^0$ be the subclass of $S_H$ consisting of $f$ with $b_1 = 0$. The classical family $S$ of analytic univalent, normalized functions on $\mathbb{U}$ is the subclass of $S_H$ in which $b_n = 0$ for all $n$. In 1984, Clunie and Sheil-Small [5] investigated the class $S_H$ for the first time. Since then, harmonic mappings have been an
area of active research. Many remarkable results for harmonic mappings can be found in the literature [1-3, 5-11, 13-18, 20, 21, 24, 25].

A domain \( \Omega \subset \mathbb{C} \) is said to be convex in the direction \( \gamma \), if for all \( z_0 \in \mathbb{C} \), the set \( \Omega \cap \{ z_0 + te^{i\gamma} : t \in \mathbb{R} \} \) is either connected or empty. Particularly, a domain is convex in the direction of the real (resp. imaginary) axis if its intersection with each horizontal (resp. vertical) line is connected. A function is convex in the direction of real (resp. imaginary) axis if it maps \( U \) onto a domain convex in the direction of real (resp. imaginary) axis. The following result due to Hengarther and Schober [12] is very useful in checking if an analytic function is convex in the direction of the imaginary axis.

**Lemma 1.1.** Suppose \( f \) is analytic and non constant in \( U \). Then

\[
\Re[(1 - z^2)f'(z)] \geq 0, \quad z \in U
\]

if and only if

(i) it is univalent in \( U \);

(ii) it is convex in the direction of the imaginary axis;

(iii) there exist sequences \( \{z_n^1\} \) and \( \{z_n^2\} \) converging to \( z = 1 \) and \( z = -1 \), respectively, such that

\[
\lim_{n \to \infty} \Re(f(z_n^1)) = \sup_{|z|<1} \Re(f(z)),
\]

\[
\lim_{n \to \infty} \Re(f(z_n^2)) = \inf_{|z|<1} \Re(f(z)).
\]

Clunie and Sheil-Small [5] introduced the shear construction method that produces a harmonic univalent mapping with a given dilatation onto domains convex in one direction. This result is given in Lemma 1.2.

**Lemma 1.2.** A locally univalent harmonic function \( f = h + \overline{g} \) in \( U \) is a univalent harmonic mapping of \( U \) onto a domain convex in the direction of the imaginary (resp. real) axis if and only if \( h + g \) (resp. \( h - g \)) is an analytic univalent mapping of \( U \) onto a domain convex in the direction of the imaginary (resp. real) axis.

A linear combination is an important method to construct a new function. MacGregor [19] showed that the linear combination \( tf + (1 - t)g \) for \( 0 \leq t \leq 1 \) of analytic functions need not to be univalent even if \( f \) and \( g \) are convex functions. Some results on the linear combinations of analytic functions are obtained in [4, 19, 23]. Recently, the linear combinations of harmonic mappings have started to be studied [6, 16, 24, 25].

It is interesting and meaningful to investigate the classes of harmonic functions that map \( U \) onto specific domains. One can refer to [1, 7, 8, 10, 11, 17, 18]. Specifically, the collection of functions \( f = h + \overline{g} \in S^o_H \) that map \( U \) onto the right half-plane, \( R = \{w : \Re w > -1/2\} \), have the form

\[
h(z) + g(z) = \frac{z}{1 - z}
\]

and those that map \( U \) onto the vertical strip, \( \Omega_\alpha = \{w : \frac{\alpha - \pi}{2 \sin \alpha} < \Re w < \frac{\alpha - \pi}{2 \sin \alpha}\} \), where \( \frac{\pi}{2} \leq \alpha < \pi \), have the form

\[
h(z) + g(z) = \frac{1}{2i \sin \alpha} \log \left( \frac{1 + ze^{i\alpha}}{1 + ze^{-i\alpha}} \right).
\]

In this paper, we derive several sufficient conditions for the combination \( f = tf_1 + (1 - t)f_2 \) to be univalent and convex in the imaginary direction, where \( f_1 \) and \( f_2 \) are univalent harmonic mappings obtained by shearing of \( h + g \) as given in (1.3).
2. Main Results

Theorem 2.1. Let \( f_i = h_i + \overline{g_i} \in S_H \), where \( h_j(z) + g_j(z) = \frac{1}{2\sin \alpha_j} \log \left( \frac{1 + \zeta e^{\alpha_j}}{1 + \zeta e^{-\alpha_j}} \right) \), \( \alpha_j \in \left[ \frac{\pi}{2}, \pi \right] \) for \( j = 1, 2 \). Then \( f = tf_1 + (1-t)f_2 \in S_H \) is convex in the direction of the imaginary axis for \( 0 \leq t \leq 1 \), if \( f \) is locally univalent and sense-preserving.

Proof. Define \( F = tF_1 + (1-t)F_2 \), where \( F_j = h_j + g_j = \frac{1}{2\sin \alpha_j} \log \left( \frac{1 + \zeta e^{\alpha_j}}{1 + \zeta e^{-\alpha_j}} \right) \) for \( j = 1, 2 \). Let

\[
q_j(z) := (1 - z^2)F_j'(z) = \frac{1 - z^2}{(1 + \zeta e^{\alpha_j})(1 + \zeta e^{-\alpha_j})}.
\]

It is easy to verify that \( q_j(0) = 0 \) and \( \Re[q_j(z)] = 0 \) for \( |z| = 1 \). By the minimum principle for harmonic functions with \( z \in \mathbb{U} \), we have

\[
\Re[q_j(z)] = \Re[(1 - z^2)F_j'(z)] > 0.
\]

Therefore,

\[
\Re[(1 - z^2)F'(z)] = t\Re[q_1(z)] + (1-t)\Re[q_2(z)] > 0,
\]

for all \( z \in \mathbb{U} \). It follows from Lemma 1.1 that \( F = tF_1 + (1-t)F_2 = th_1 + (1-t)h_2 + tg_1 + (1-t)g_2 \) is analytic and convex in the direction of the imaginary axis. Therefore if

\[
f = tf_1 + (1-t)f_2 = th_1 + (1-t)h_2 + tg_1 + (1-t)g_2
\]

is locally univalent and sense-preserving, then by Lemma 1.2 we have \( f \in S_H \) is convex in the direction of the imaginary axis. \( \square \)

In view of Theorem 2.1, the main object we need to focus on is the condition \( f \) is locally univalent and sense-preserving. Actually, we just need \( \omega \), the dilation of \( f \), to satisfy \( |\omega| < 1 \). We begin by finding an expression for \( \omega \).

Lemma 2.2. Let \( f_i = h_i + \overline{g_i} \in S_H \), where \( h_j(z) + g_j(z) = \frac{1}{2\sin \alpha_j} \log \left( \frac{1 + \zeta e^{\alpha_j}}{1 + \zeta e^{-\alpha_j}} \right) \), \( \alpha_j \in \left[ \frac{\pi}{2}, \pi \right] \) for \( j = 1, 2 \). If \( \omega_j = \frac{g_j'}{h_j'} \) are dilatation functions of \( f_j \), \( j = 1, 2 \), respectively, then the dilatation function \( \omega \) of \( f = tf_1 + (1-t)f_2 \), \( 0 \leq t \leq 1 \), is given by

\[
\omega = \frac{t(1 + \omega_2)(1 + 2z\cos \alpha_2 + z^2)\omega_1 + (1-t)(1 + \omega_1)(1 + 2z\cos \alpha_1 + z^2)\omega_2}{t(1 + \omega_2)(1 + 2z\cos \alpha_2 + z^2) - (1-t)(1 + \omega_1)(1 + 2z\cos \alpha_1 + z^2)}.
\]

(2.1)

Proof. Since \( f = tf_1 + (1-t)f_2 = th_1 + (1-t)h_2 + tg_1 + (1-t)g_2 \),

\[
\omega = \frac{tg'_1 + (1-t)g'_2}{th'_1 + (1-t)h'_2} = \frac{t\omega_1 h'_1 + (1-t)\omega_2 h'_2}{th'_1 + (1-t)h'_2}.
\]

(2.2)

From \( h_j(z) + g_j(z) = \frac{1}{2\sin \alpha_j} \log \left( \frac{1 + \zeta e^{\alpha_j}}{1 + \zeta e^{-\alpha_j}} \right) \) and \( \omega_j = \frac{g_j'}{h_j'} \), \( j = 1, 2 \), we have

\[
h'_j(z) = \frac{1}{(1 + \omega_j(z))(1 + 2z\cos \alpha_j + z^2)}.
\]

Substituting these into (2.2), we get (2.1). \( \square \)

If \( \omega_1 = \omega_2 \), then (2.1) reduces to \( \omega = \omega_1 = \omega_2 \). Considering Theorem 2.1, we get the following result.
Theorem 2.3. Let \( f_j = h_j + \overline{g}_j \in S_{Hj} \), where \( h_j(z) + g_j(z) = \frac{1}{2 \sin \alpha_j} \log \left( \frac{1 + z^{\alpha_j}}{1 + z^{-\alpha_j}} \right) \), \( \alpha_j \in [\frac{\pi}{2}, \pi) \) for \( j = 1, 2 \). If \( \alpha_1 = \alpha_2 \), then \( f = tf_1 + (1-t)f_2 \in S_{Hj} \) is monotonic in \( t \) for \( 0 \leq t \leq 1 \).

If \( \alpha_1 = \alpha_2 \), we have the following theorem.

Theorem 2.4. Let \( f_j = h_j + \overline{g}_j \in S_{Hj} \), where \( h_j(z) + g_j(z) = \frac{1}{2 \sin \alpha_j} \log \left( \frac{1 + z^{\alpha_j}}{1 + z^{-\alpha_j}} \right) \), \( \alpha_j \in [\frac{\pi}{2}, \pi) \) for \( j = 1, 2 \). If \( \alpha_1 = \alpha_2 \), then \( f = tf_1 + (1-t)f_2 \in S_{Hj} \) is convex in the direction of the imaginary axis for \( 0 \leq t \leq 1 \).

Proof. In view of Theorem 2.1, it suffices to show that \( f \) is locally univalent and sense-preserving. Substituting \( \alpha_1 = \alpha_2 \) into (2.1), we have

\[
\omega = \frac{t(1 + \omega_1)\omega_2 + (1-t)(1 + \omega_1)\omega_2}{t(1 + \omega_2) + (1-t)(1 + \omega_1)} = \frac{t\omega_1 + (1-t)\omega_2 + \omega_1\omega_2}{1 + t\omega_2 + (1-t)\omega_1}.
\]

(2.3)

Let

\[
I := |1 + t\omega_2 + (1-t)\omega_1|^2 - |t\omega_1 + (1-t)\omega_2 + \omega_1\omega_2|^2
\]

\[
= 1 - 2t(1 - |\omega_1|^2)\Re \omega_2 + 2(1-t)(1 - |\omega_2|^2)\Re \omega_1
\]

\[
+ (1 - 2t)(|\omega_1|^2 - |\omega_2|^2) - |\omega_1|^2|\omega_2|^2
\]

and \( \omega_j = \rho_j e^{i\alpha_j} \), \( j = 1, 2 \). Then

\[
I = 1 - 2t(1 - \rho_1^2)\rho_2 \cos \theta_2 + 2(1-t)(1 - \rho_2^2)\rho_1 \cos \theta_1 + (1 - 2t)( \rho_1^2 - \rho_2^2 - \rho_1^2 \rho_2^2
\]

\[
\geq 1 - 2t(1 - \rho_1^2)\rho_2 - 2(1-t)(1 - \rho_2^2)\rho_1 + (1 - 2t)( \rho_1^2 - \rho_2^2 - \rho_1^2 \rho_2^2
\]

\[
= (1 - \rho_1^2)(1 - \rho_2^2) + 2t(\rho_1 - \rho_2)[1 + \rho_1 \rho_2 - (\rho_1 + \rho_2)] \geq 0.
\]

It is easy to verify that \( 1 + \rho_1 \rho_2 - (\rho_1 + \rho_2) > 0 \) for \( \rho_j \in [0, 1), j = 1, 2 \). Actually, \( 0 < 1 + \rho_1 \rho_2 - (\rho_1 + \rho_2) \leq 1 \) for all \( \rho_j \in [0, 1), j = 1, 2 \). If \( \rho_1 - \rho_2 \geq 0 \), then

\[
I \geq II \geq (1 - \rho_2^2)(1 - \rho_1^2) > 0.
\]

If \( \rho_1 - \rho_2 < 0 \), then

\[
I \geq II \geq (1 - \rho_2^2)(1 - \rho_1^2) + 2(\rho_1 - \rho_2)[1 + \rho_1 \rho_2 - (\rho_1 + \rho_2)]
\]

\[
= (1 - \rho_1^2)(1 - \rho_2^2) > 0.
\]

Therefore, \( I > 0 \) which implies \( |\omega| < 1 \). Hence \( f \) is locally univalent and sense-preserving. \( \Box \)

Remark 2.5. In an earlier paper [24, Thm. 3], it was claimed that for \( \omega \) given in (2.3), \( \omega < 1 \). The authors assumed that the function \( I \) is monotonic in \( t \) for \( t \in [0, 1] \) and this is not the case. Our proof establishes the validity of the result.

The following lemma is popularly known as Cohn’s Rule.

Lemma 2.6 (Cohn’s Rule, see [22, p. 375]). Given a polynomial \( p(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n \) of degree \( n \), let

\[
p'(z) = z^n p\left( \frac{1}{z} \right) = \overline{a}_n + \overline{a}_{n-1} z + \overline{a}_{n-2} z^2 + ... + \overline{a}_0 z^n.
\]

Denote by \( r \) and \( s \) the number of zeros of \( p \) inside and on the circle \( |z| = 1 \), respectively. If \( |a_0| < |a_n| \), then

\[
p_1 = \frac{\overline{a}_n p(z) - a_0 p'(z)}{z}
\]

is of degree \( n - 1 \) and has \( r_1 = r - 1 \) and \( s_1 = s \) number of zeros inside and on the unit circle \( |z| = 1 \), respectively.
In order to get our next result, we need Lemma 6.

**Lemma 2.7.** Let \( a \in (-1, 0) \cup (0, 1) \), \( t \in (0, 1) \) and \( \alpha_1, \alpha_2 \in [\frac{\pi}{2}, \pi) \). If \( a(\alpha_1 - \alpha_2) > 0 \), then

\[
\begin{align*}
(1) \quad & a(2t - 1) + 1 > a(2t - 1) + 2at(1 - t)(\cos \alpha_1 - \cos \alpha_2); \\
(2) \quad & a(2t - 1) + 1 + at(1 - t)(\cos \alpha_1 - \cos \alpha_2) > [at + 1]t \cos \alpha_2 + (1 - t)(at + 1 - a) \cos \alpha_1. 
\end{align*}
\]

**Proof.** (1) It is obvious that \( a(2t - 1) + 1 > 0 \) holds for \( a \in (-1, 0) \cup (0, 1) \) and \( t \in (0, 1) \). Thus we just need to prove the following double inequality

\[
a(2t - 1) + 1 > a(2t - 1) + 1 + 2at(1 - t)(\cos \alpha_1 - \cos \alpha_2) > -[a(2t - 1) + 1].
\]

That is,

\[
0 > at(1 - t)(\cos \alpha_1 - \cos \alpha_2) > -[a(2t - 1) + 1].
\]

First, because \( \alpha_1, \alpha_2 \in [\frac{\pi}{2}, \pi) \), then \( a(\alpha_1 - \alpha_2) > 0 \) is equivalent to \( a(\cos \alpha_1 - \cos \alpha_2) < 0 \). Therefore, for \( t \in (0, 1) \) we have

\[
0 > at(1 - t)(\cos \alpha_1 - \cos \alpha_2).
\]

Next, to prove the second inequality of (2.6), we consider two subcases.

**Subcase 1:** if \( a \in (0, 1) \) and \( \alpha_1, \alpha_2 \in [\frac{\pi}{2}, \pi) \), then \( a(\alpha_1 - \alpha_2) > 0 \) implies \( -1 < \cos \alpha_1 - \cos \alpha_2 < 0 \). Thus,

\[
at(1 - t)(\cos \alpha_1 - \cos \alpha_2) > -at(1 - t) > -[a(2t - 1) + 1]
\]

holds for \( t \in (0, 1) \). The last inequality holds because of \( a(t^2 + t - 1) > -1 \) for \( t \in (0, 1) \) and \( a \in (0, 1) \).

**Subcase 2:** if \( a \in (-1, 0) \) and \( \alpha_1, \alpha_2 \in [\frac{\pi}{2}, \pi) \), then \( a(\alpha_1 - \alpha_2) > 0 \) implies \( 0 < \cos \alpha_1 - \cos \alpha_2 < 1 \). Thus,

\[
at(1 - t)(\cos \alpha_1 - \cos \alpha_2) > at(1 - t) > -[a(2t - 1) + 1]
\]

holds for \( t \in (0, 1) \). The last inequality holds because of \( a(-t^2 + 3t - 1) > -1 \) for \( t \in (0, 1) \) and \( a \in (-1, 0) \).

Therefore, the second inequality of the double inequality (2.6) follows from inequalities (2.8) and (2.9).

(2) If \( a(\alpha_1 - \alpha_2) > 0 \), then in view of inequality (2.6) we know that \( a(2t - 1) + 1 + at(1 - t)(\cos \alpha_1 - \cos \alpha_2) > 0 \) for \( a \in (-1, 0) \cup (0, 1) \), \( t \in (0, 1) \), and \( \alpha_1, \alpha_2 \in [\frac{\pi}{2}, \pi) \). So inequality (2.5) is equivalent to the double inequality

\[
a(2t - 1) + 1 + at(1 - t)(\cos \alpha_1 - \cos \alpha_2) > [at + 1]t \cos \alpha_2 + (1 - t)(at + 1 - a) \cos \alpha_1
\]

or

\[
=a(2t - 1) + 1 + at(1 - t)(\cos \alpha_1 - \cos \alpha_2)
\]

\[
> -[a(2t - 1) + 1 + at(1 - t)(\cos \alpha_1 - \cos \alpha_2)].
\]

Now, let

\[
f(a, t) := a(2t - 1) + 1 + at(1 - t)(\cos \alpha_1 - \cos \alpha_2)
\]

\[
= [at + 1]t \cos \alpha_2 + (1 - t)(at + 1 - a) \cos \alpha_1
\]

\[
= (1 - a)(1 - \cos \alpha_1)
\]

\[
+ t[(1 + a)(\cos \alpha_1 - \cos \alpha_2) + 2a(1 - \cos \alpha_1)].
\]

It follows that

\[
\frac{\partial f(a, t)}{\partial a} = -(1 - \cos \alpha_1) + t[\cos \alpha_1 - \cos \alpha_2 + 2(1 - \cos \alpha_1)];
\]

\[
\frac{\partial f(a, t)}{\partial t} = (1 + a)(\cos \alpha_1 - \cos \alpha_2) + 2a(1 - \cos \alpha_1).
\]
Let
\[ \frac{\partial f(a, t)}{\partial a} = 0 \quad \text{and} \quad \frac{\partial f(a, t)}{\partial t} = 0. \]
Then we have
\[ a = a_0 = \frac{\cos \alpha_2 - \cos \alpha_1}{2 - \cos \alpha_1 + \cos \alpha_2} \quad \text{and} \quad t = t_0 = \frac{1 - \cos \alpha_1}{2 - \cos \alpha_1 + \cos \alpha_2}. \]
Therefore, it is obvious that
\[ f(a, t) \geq f(a_0, t_0) = \frac{2(1 - \cos \alpha_1)(1 - \cos \alpha_2)}{2 - \cos \alpha_1 + \cos \alpha_2} > 0. \tag{2.11} \]
Inequality (2.11) implies that the first inequality of the double inequality (2.10) holds. Next, let
\[ I := (at + 1)t \cos \alpha_2 + (1 - t)(at + 1 - a) \cos \alpha_1 \\ + at(2t - 1) + 1 + at(1 - t)(\cos \alpha_1 - \cos \alpha_2) \\ = [at(2t - 1) + 1][1 + \cos \alpha_1 - t(\cos \alpha_1 - \cos \alpha_2)]. \tag{2.12} \]
If \( \cos \alpha_1 - \cos \alpha_2 > 0 \), then
\[ I > [at(2t - 1) + 1][1 + \cos \alpha_1 - \cos \alpha_2] \\ = [at(2t - 1) + 1](1 + \cos \alpha_2) > 0. \tag{2.13} \]
If \( \cos \alpha_1 - \cos \alpha_2 < 0 \), then
\[ I > [at(2t - 1) + 1](1 + \cos \alpha_1) > 0. \tag{2.14} \]
Therefore, \( I > 0 \) follows from inequalities (2.13) and (2.14) and the second inequality of the double inequality (2.10) is proved. \( \square \)

**Theorem 2.8.** Let \( f_j = h_j + \log g_j \in S_H \), where \( h_j(z) + g_j(z) = \frac{1}{\sin \alpha_j} \log \left( \frac{1 + \alpha_j z}{1 + \alpha_j z^*} \right) \), \( \alpha_j \in [\frac{\pi}{2}, \pi) \) for \( j = 1, 2 \). If \( \omega_1(z) = z \), \( \omega_2(z) = \frac{z + a}{1 + z} \), \( a \in (-1, 1) \), then \( f = f_1 + (1 - t)f_2 \in S_H \) \((0 < t < 1)\) is convex in the direction of the imaginary axis provided \( a(\alpha_1 - \alpha_2) \geq 0 \).

**Proof.** By Theorem 2.1, we just need to show that \( |a| < 1 \) in \( \mathbb{U} \). If \( a = 0 \), then \( \omega_2(z) = \omega_1(z) = z \) and this case can follow from Theorem 2.3. If \( \alpha_1 = \alpha_2 \), then the case follows from Theorem 2.4. Therefore, we shall only consider the case when \( \alpha(\alpha_1 - \alpha_2) > 0 \).

Setting \( \omega_1(z) = z \) and \( \omega_2(z) = \frac{z + a}{1 + z} \) in (2.1), we get
\[ \omega(z) = \frac{(1 + a^2)(1 + 2z \cos \alpha_2 + z^2)z + (1 - t)(1 + z)(1 + 2z \cos \alpha_1 + z^2)}{(1 + a^2)(1 + 2z \cos \alpha_2 + z^2) + (1 - t)(1 + z)(1 + 2z \cos \alpha_1 + z^2)} \\ = \frac{t(1 + a)(1 + 2z \cos \alpha_2 + z^2)z + (1 - t)(1 + 2z \cos \alpha_1 + z^2)(z + a)}{(1 + a)(1 + 2z \cos \alpha_2 + z^2) + (1 - t)(1 + 2z \cos \alpha_1 + z^2)(1 + z^2)} \\ = \frac{p(z)}{p^*(z)} \]
where
\[ p(z) = (at + 1)z^3 + [2t(1 + a) \cos \alpha_2 + 2(1 - t) \cos \alpha_1 + a(1 - t)]z^2 \\ + [1 + at + 2a(1 - t) \cos \alpha_1]z + a(1 - t) \\ := a_0 z^3 + a_1 z^2 + a_1 z + a_0 \]
and

\[ p'(z) = a(1-t)z^2 + [1 + at + 2a(1-t) \cos \alpha_1]z^2 \\
+ [2t(1+a) \cos \alpha_2 + 2(1-t) \cos \alpha_1 + a(1-t)]z + (at + 1) \]

\[ = z^2 p(\frac{1}{z}). \]

Thus if \( z_0 \) is a zero of \( p \) and \( z_0 \neq 0 \), then \( 1/z_0 \) is a zero of \( p' \), and we can rewrite

\[ \omega(z) = \frac{(z + A)(z + B)(z + C)}{(1 + Az)(1 + Bz)(1 + Cz)}. \]

For \( |\beta| \leq 1 \), the function \( \phi(z) = \frac{z + \beta}{1 + z \beta} \) maps \( \overline{\mathbb{U}} = \{ z : |z| \leq 1 \} \) onto \( \overline{\mathbb{U}} \). Hence, to prove that \( |\omega| < 1 \) in \( \mathbb{U} \), it suffices to show that \( |A| \leq 1, |B| \leq 1, |C| \leq 1 \) with at least one of these having modulus strictly less than one. As \( |\alpha_0| = at + 1 > |\alpha_0| = |\alpha(1-t)| \) holds for all \( a \in (-1, 0) \cup (0, 1) \) and \( t \in (0, 1) \), we can apply Cohn's Rule to \( p \), and thus it is sufficient to show that all the zeros of \( p_1 \) lie inside or on \( |z| = 1 \), where

\[ p_1(z) = a_0 p(z) - a_0 p'(z) \]

and

\[ \overline{p}_1(z) = \frac{a(2t - 1) + 1}{z^2} + \frac{2(1-t)(at+1-a) \cos \alpha_1 + 2t(at+1) \cos \alpha_2}{z} \]

\[ = b_2 z^2 + b_1 z + b_0. \]

By Lemma 2.7(1), we have \( |b_2| > |b_0| \). So we can use Cohn's Rule again on \( \overline{p}_1 \). Now, we need to show that all the zeros of \( p_2 \) lie inside or on \( |z| = 1 \), where

\[ p_2(z) = \frac{b_2 \overline{p}_1(z) - b_0 \overline{p}_1(z)}{z} = -4at(1-t)(\cos \alpha_1 - \cos \alpha_2) \overline{p}_2(z) \]

and

\[ \overline{p}_2(z) = \frac{a(2t - 1) + 1 + at(1-t)(\cos \alpha_1 - \cos \alpha_2)}{z} \]

\[ + (at + 1) \cos \alpha_2 + (1-t)(at + 1-a) \cos \alpha_1 \]

\[ := c_1 z + c_0. \]

By Lemma 2.7(2), we have \( |c_1| > |c_0| \). Hence, the zeros of \( \overline{p}_2, p_2, \overline{p}_1 \), and \( p_1 \) lie in \( |z| < 1 \). Therefore, \( |\omega| < 1 \). □

**Theorem 2.9.** Let \( f_j = h_j + \overline{g}_j \in S_H \) with \( \omega_j = e^{\theta_j} z \), where \( h_j(z) + g_j(z) = \frac{1}{2 i \sin \alpha_j} \log \left( \frac{1 + e^{i\gamma_j}}{1 + e^{-i\gamma_j}} \right), \alpha_j \in \left[ \frac{\pi}{2}, \pi \right) \) and \( \theta_j \in [0, 2\pi) \) for \( j = 1, 2 \). For each case, if the stated conditions are satisfied, then \( f = tf_1 + (1-t)f_2 \in S_H \) is convex in the direction of the imaginary axis for \( 0 < t < 1 \).

**Case (1):** \( \theta_1 = \theta_2 \) or \( \alpha_1 = \alpha_2 \);

**Case (2):**

(a) \( \cos \alpha_1 - \cos \alpha_2)(\cos \alpha_1 - \cos \alpha_2 + \cos \theta_1 - \cos \theta_2) < 0 \), and

(b) \( |\cos \alpha_1 - \cos \alpha_2 + \cos \theta_1 - \cos \theta_2| > |\cos \alpha_1 e^{i\theta_1} - \cos \alpha_2 e^{i\theta_2} + i \sin(\theta_1 - \theta_2)|. \)

**Proof.** By Theorem 2.1, we just need to show that dilatation \( \omega \) of \( f \) satisfies \( |\omega| < 1 \) in \( \mathbb{U} \). By Theorem 2.3 and 2.4, respectively, Case (1) is true.
If \( \theta_1 \neq \theta_2 \) and \( \alpha_1 \neq \alpha_2 \), by substituting \( \omega_j = e^{i\theta_j}z, j = 1, 2 \) into the equation (2.1), we derive

\[
\omega(z) = ze^{i(\theta_1 + \theta_2)} p(z)
p'(z)
\]

where

\[
p(z) = z^3 + [(1 - t)e^{-i\theta_1} + te^{-i\theta_2} + 2(t \cos \alpha_2 + (1 - t) \cos \alpha_1)]z^2
+ [2t \cos \alpha_2 e^{-i\theta_2} + 2(1 - t) \cos \alpha_1 e^{-i\theta_1} + 1]z + (1 - t)e^{-i\theta_1} + te^{-i\theta_2} := a_3z^3 + a_2z^2 + a_1z + a_0
\]

and

\[
p'(z) = z^3 p'(\frac{1}{z}) = \overline{a}_3z^3 + \overline{a}_2z^2 + \overline{a}_1z + \overline{a}_0.
\]

Because of \( t \in (0, 1) \) and \( \theta_1 \neq \theta_2 \), we have \( |(1 - t)e^{-i\theta_1} + te^{-i\theta_2}| < 1 \). Hence, \( |a_0| < |a_3| \). By Cohn’s Rule, we have

\[
p_1(z) = \frac{\overline{p}_3z - \overline{a}_0p'(z)}{z} = 2(1 - t)(e^{-i\theta_1} - e^{-i\theta_2})\overline{p}_1(z),
\]

where

\[
\overline{p}_1(z) = \frac{1}{2}(e^{i\theta_1} - e^{i\theta_2})z^3 + (\cos \alpha_1 e^{i\theta_1} - \cos \alpha_2 e^{i\theta_2})z
+ \cos \alpha_1 - \cos \alpha_2 + \frac{1}{2}(e^{i\theta_1} - e^{i\theta_2})
:= b_2z^2 + b_1z + b_0.
\]

In view of condition (a) in Case (2), we have

\[
|b_0|^2 - |b_2|^2 = |\cos \alpha_1 - \cos \alpha_2 + \frac{1}{2}(e^{i\theta_1} - e^{i\theta_2})|^2 - \frac{1}{4}|e^{i\theta_1} - e^{i\theta_2}|^2 = (\cos \alpha_1 - \cos \alpha_2)(\cos \alpha_1 - \cos \alpha_2 + \cos \theta_1 - \cos \theta_2) < 0.
\]

It follows \( |b_0| < |b_2| \). Therefore, we can make use of Cohn’s Rule again. Let

\[
\overline{p}_1(z) = z^3 \overline{p}_1(\frac{1}{z}) = \overline{b}_0z^2 + \overline{b}_1z + \overline{b}_2.
\]

By direct computation, we have

\[
p_2(z) := \frac{\overline{b}_2\overline{p}_1(z) - \overline{b}_0\overline{p}_1(\frac{1}{z})}{z}
= -(\cos \alpha_1 - \cos \alpha_2)\overline{p}_2(z) = -(\cos \alpha_1 - \cos \alpha_2)(c_1z + c_0),
\]

where

\[
c_1 = \cos \alpha_1 - \cos \alpha_2 + \cos \theta_1 - \cos \theta_2, \quad c_0 = \cos \alpha_1 e^{-i\theta_1} - \cos \alpha_2 e^{-i\theta_2} + i \sin(\theta_1 - \theta_2).
\]

Because of condition (a) in Case (2), we have \( |c_1| > |c_0| \). So the zero(s) of \( \overline{p}_2 \) and \( \overline{p}_1 \) both lie inside the unit circle \( |z| = 1 \). Therefore, it follows \( |c| < 1 \). □

If we take specific values for \( \theta_1 \) and \( \theta_2 \), we have the following corollary.
Corollary 2.10. Let \( f_j = h_j + \overline{g_j} \in S_H \), where \( h_j(z) + g_j(z) = \frac{1}{\pi \sin \pi_j} \log \left( \frac{1 + z e^{Property\}}{1 + z e^{-Property\}} \right), \alpha_j \in [\frac{\pi}{2}, \pi) \) for \( j = 1, 2 \). For each case, if the stated conditions are satisfied, then \( f = tf_1 + (1-t)f_2 \in S_H \) is convex in the direction of the imaginary axis for \( 0 < t < 1 \).

Case (1): For \( \omega_1(z) = z e^{\theta + \frac{\pi}{2}}, \omega_2(z) = z e^{\theta} \).

\( \begin{align*} 
& (a) \quad (\cos \alpha_1 - \cos \alpha_2)(\cos \alpha_1 - \cos \alpha_2 - \sin \theta - \cos \theta) < 0, \text{ and} \\
& (b) \quad (\cos \alpha_1 + \sin \theta)(\cos \theta - \cos \alpha_2) > (\sin \theta + \cos \theta)(\cos \alpha_1 - \cos \alpha_2).
\end{align*} \)

Case (2): For \( \omega_1(z) = z e^{\theta + \frac{\pi}{2}}, \omega_2(z) = z e^{\theta} \).

\( \begin{align*} 
& (a) \quad (\cos \alpha_1 - \cos \alpha_2)(\cos \alpha_1 - \cos \alpha_2 - 2 \cos \theta) < 0, \text{ and} \\
& (b) \quad (\cos \alpha_1 - \cos \theta)(\cos \alpha_2 + \cos \theta) < 0.
\end{align*} \)

Case (3): For \( \omega_1(z) = z, \omega_2(z) = -z \).

\( \begin{align*} 
& (a) \quad \alpha_1 > \alpha_2.
\end{align*} \)

Theorem 2.11. Let \( f_j = h_j + \overline{g_j} \in S_H \), where \( h_j(z) + g_j(z) = \frac{1}{\pi \sin \pi_j} \log \left( \frac{1 + z e^{Property\}}{1 + z e^{-Property\}} \right), \alpha_j \in [\frac{\pi}{2}, \pi) \) for \( j = 1, 2 \). For each case, if the stated conditions are satisfied, then \( f = tf_1 + (1-t)f_2 \in S_H \) is convex in the direction of the imaginary axis for \( 0 < t < 1 \).

Case (1): For \( \omega_1(z) = z, \omega_2(z) = z^2, \text{ and } \alpha_1 < \alpha_2 \).

Case (2): For \( \omega_1(z) = z, \omega_2(z) = -z^2, \text{ and } \alpha_1 > \alpha_2 \).

Proof. As the proof is similar to the proof of Theorem 2.8 and Theorem 2.9, it is omitted.

3. Example

In this section, we give two examples to illustrate our results.

Example 3.1. If \( \alpha_1 = \alpha_2 = \frac{\pi}{2} \), then \( h_j(z) + g_j(z) = \frac{1}{\pi} \log \left( \frac{1 + z e^{Property\}}{1 + z e^{-Property\}} \right) \) for \( j = 1, 2 \). Taking \( \omega_1(z) = z \), we get

\( h_1(z) = \frac{1}{4} \log \left( \frac{1 + z e^{Property\}}{1 + z e^{-Property\}} \right) + \frac{1}{2} \arctan z, \)

\( g_1(z) = -\frac{1}{4} \log \left( \frac{1 + z e^{Property\}}{1 + z e^{-Property\}} \right) + \frac{1}{2} \arctan z. \)

Taking \( \omega_2(z) = -z^2 \), we obtain

\( h_2(z) = \frac{1}{4} \log \left( \frac{1 + z e^{Property\}}{1 + z e^{-Property\}} \right) + \frac{1}{2} \arctan z, \)

\( g_2(z) = -\frac{1}{4} \log \left( \frac{1 + z e^{Property\}}{1 + z e^{-Property\}} \right) + \frac{1}{2} \arctan z. \)

Let \( f_j = h_j + \overline{g_j} \) for \( j = 1, 2 \) and \( f = tf_1 + (1-t)f_2 \). Then by Theorem 2.4, we know that \( f \) is in \( S_H \) and is convex in the direction of the imaginary axis. The images of \( \cup \) under \( f \) with \( t = 0, \frac{1}{2} \) and 1, respectively, are shown in Figure 1.
Example 3.2. Letting $\alpha_1 = \frac{3\pi}{4}$, we have

$$h_1(z) + g_1(z) = \frac{1}{\sqrt{2}i} \log \left( \frac{\sqrt{2} + (-1 + i)z}{\sqrt{2} - (1 + i)z} \right).$$

With $\omega_1(z) = z$, we have

$$h_1(z) = \frac{2-\sqrt{2}}{4} \log \frac{(1+z)^2}{1-\sqrt{2}z^2} + \frac{(1+\sqrt{2})\arctan(\sqrt{2}z-1)}{2+\sqrt{2}} + \frac{(1+\sqrt{2})n}{4(2+\sqrt{2})},$$
$$g_1(z) = -\frac{2-\sqrt{2}}{4} \log \frac{(1+z)^2}{1-\sqrt{2}z^2} + \frac{(1+\sqrt{2})\arctan(\sqrt{2}z-1)}{2+\sqrt{2}} + \frac{(1+\sqrt{2})n}{4(2+\sqrt{2})}.$$

With $\alpha_2 = \frac{\pi}{2}$, then $h_2(z) + g_2(z) = \frac{1}{2} \log \left( \frac{1+z}{1-z} \right)$. Taking $\omega_2(z) = -z$, we get

$$h_2(z) = \frac{1}{4} \log \left( \frac{1+z}{1-z} \right) + \frac{1}{2} \arctan z,$$
$$g_2(z) = -\frac{1}{4} \log \left( \frac{1+z}{1-z} \right) + \frac{1}{2} \arctan z.$$

Let $f_j = h_j + \overline{g_j}$ for $j = 1, 2$ and $f = tf_1 + (1-t)f_2$. Then Corollary 2.10 gives us that $f$ is in $S_H$ and is convex in the direction of the imaginary axis. The images of $U$ under $f$ with $t = 0$, $\frac{1}{2}$, and 1, respectively, are shown in Figure 2.
References