On Filter Convergence of Nets in Uniform Spaces

Ekrem Savas\textsuperscript{a}, Ulaş Yamancı\textsuperscript{b}

\textsuperscript{a}Uşak University, Department of Mathematics, Uşak, Turkey

\textsuperscript{b}Süleyman Demirel University, Faculty of Arts and Sciences, Department of Statistics, Isparta, Turkey

Abstract. In this paper, we introduce $F$-convergent and $F_{st}$-fundamental nets in uniform spaces and study some their properties.

1. Introduction and Notations

The concept of statistical convergence was introduced by Fast [7] and Schonberg [21], and its topological properties were discussed by Fridy [8], Salat [18] and Maddox [15]. Fridy [8] also introduced the concept of statistically fundamental sequence and showed its equivalence to statistical convergence with respect to numerical sequences. This problem on the uniform space was raised in [16]. The authors [16] showed that if the sequence $\{x_n\}_{n \in \mathbb{N}}$ is statistically convergent in a uniform space, then it is statistically fundamental.

Recently, Bilalov and Nazarova [3] gave the concept of $F_{st}$-fundamental sequences in uniform spaces and obtain some results related with this concept.

Kostyrko et al. [12] introduced the notion of $I$-convergence of sequences in a metric space and discussed some properties of such convergence. Recall that $I$-convergence is a generalization of statistical convergence. Some problems about the ideals or filters can be found in [4, 5, 13, 14].

We now recall some concepts of ideal and filter [3, 12, 17].

A family of sets $I \subset 2^\mathbb{N}$ is said to be an ideal if (i) \( \emptyset \in I \); (ii) \( A, B \in I \) imply \( A \cup B \in I \); (iii) \( A \in I \), \( B \subset A \) imply \( B \in I \).

A family of sets $F \subset 2^\mathbb{N}$ is said to be a filter if (i) \( \emptyset \not\in F \); (ii) \( A, B \in F \) imply \( A \cap B \in F \); (iii) \( A \in F \), \( A \subset B \) imply \( B \in F \).

If filter $F$ satisfy the following axioms:

(iv) if \( A_1 \supset A_2 \supset ... \) and \( A_n \in F \) for all \( n \in \mathbb{N} \), then there exists \( \{n_m\}_{m \in \mathbb{N}} \subset \mathbb{N} \); \( n_1 < n_2 < ... \) such that \( \bigcup_{m=1}^{\infty} (\alpha_m, \alpha_{m+1}] \cap A(n) \in F \),

(v) \( F(N,F) \in F \) for any finite subset \( F \subset \mathbb{N} \),

then filter $F$ is said to be a monotone closed filter and a right filter, respectively [2, 3].

An ideal $I$ is said to be non-trivial if \( I \neq \emptyset \) and \( I \neq \mathbb{N} \). \( I \subset 2^\mathbb{N} \) is a non-trivial ideal if and only if \( F = F(I) = \{N \setminus A : A \in I\} \) is a filter. A non-trivial ideal $I$ is said to be admissible if $I \supset \{n : n \in \mathbb{N}\}$. Filter convergence was introduced in [1] and described in details in the paper [9]. Convergence with respect to

2010 Mathematics Subject Classification. Primary 40A35; Secondary 54A20, 54E15

Keywords. Ideal, filter, nets, $F$-convergence, $F_{st}$-convergence

Received: 02 June 2017; Revised: 11 October 2017; Accepted: 17 October 2017
Communicated by Ljubiša D.R. Kočinac

Email addresses: ekremsavas@yahoo.com (Ekrem Savas), ulasyamanci@sdu.edu.tr (Ulaş Yamancı)
set of filters was studied in the paper [11]. More information about filters and convergence with respect to filters can be found in [1, 12, 17, 19, 20].

Now we recall the definition of uniformity on a set X [6, 10].

\[ \Lambda = \{(x, x) : x \in X\} \] is said to be a diagonal or the identity relation. If \( U \subset X \times X \) is a relation, then the inverse of this relation \( U^{-1} \) is defined as the set of all pairs \((x, y)\) such that \((y, x) \in U\), that is, \(U^{-1} = \{(x, y) : (y, x) \in U\}\). Let \( U, V \subset X \times X \) be some relations. The composition \( U \circ V \) of the relations \( U \) and \( V \) is defined as the set of all pairs \((x, z)\), we get \((x, y) \in V\) and \((y, z) \in U\) for some \( y \in X\), that is, \( U \circ V = \{(x, z) : \exists y \in X, (x, y) \in V \land (y, z) \in U\}\). Let \( K \subset X \) be some set and \( U \subset X \times X \) be a relation. Assume \( U[K] = \{y \in X : \exists x \in K \implies (x, y) \in U\}\). For \( K = \{x\}\) suppose \( U[K] = U[x] \). Uniformity on the set \( X \) is a non-empty family \( \Omega \subset 2^{X \times X} \) which satisfies the following axioms:

(a) \( \Lambda \subset U, \forall U \in \Omega; \)
(b) \( U \in \Omega \) imply \( U^{-1} \in \Omega; \)
(c) \( U \in \Omega \) imply \( \exists V \in \Omega \) such that \( V \circ V \subset U; \)
(d) \( U, V \in \Omega \) imply \( U \cap V \in \Omega; \)
(e) \( U \in \Omega \) and \( U \subset V \subset X \times X \) imply \( V \in \Omega. \)

\((X, \Omega)\) is said to be a uniform space. Subfamily \( \Delta \subset \Omega \) of the uniformity \( \Omega \) is said to be its base if any element of the family \( \Omega \) contains an element of the family \( \Delta \).

Let \((X, \Omega)\) be a uniform space. The topology \( \tau \), associated with a uniformity \( \Omega \), is the family of all sets \( K \subset X \) such that for each \( x \in K \) there exists a \( U \in \Omega \) such that \( U[x] \subset K \).

The uniform space \((X, \Omega)\) is called Hausdorff if \( \cap_{U \in \Omega} U = \Lambda \). Let \((X, \Omega)\) be a uniform space and \( \{x_n\}_{n \in \mathbb{N}} \) be some sequence. \( \{x_n\}_{n \in \mathbb{N}} \) is called fundamental if \( \forall U \in \Omega \), there exists a \( n_0 \in \mathbb{N} \) such that \( (x_n, x_n) \in U \) for all \( n, m \geq n_0 \).

Throughout the paper \((D, \geq)\) will denote a directed set and \( I \) a non-trivial proper ideal of \( D \). A net is a mapping from \( D \) to \( X \) and will be denoted by \( \{s_\alpha : \alpha \in D\} \). Let \( D_\alpha = \{\beta \in D : \beta \geq \alpha\} \) for \( \alpha \in D \). Then the collection \( \mathcal{F}_0 = \{A \subset D : A \supset D_\alpha \text{ for some } \alpha \in D\} \) forms a filter in \( D \). Let \( \mathcal{I}_0 = \{A \subset D : A^c \in \mathcal{F}_0\} \). Then \( \mathcal{I}_0 \) is a non-trivial ideal of \( D \). A nontrivial ideal \( I \) of \( D \) will be said to be \( D\)-admissible if \( D_\alpha \in \mathcal{F} \) for all \( \alpha \in D \). A net \( \{s_\alpha : \alpha \in D\} \) in a topological space \((X, \tau)\) is called \( \mathcal{F}\)-convergent to \( s \in X \) if \( \{\alpha \in D : s_\alpha \in U\} \subset \mathcal{F} \) for any open set \( U \) containing \( s \).

2. Main Results

In this section, we introduce \( \mathcal{F}\)-convergent and \( \mathcal{F}_{st}\)-fundamental nets in uniform spaces and study some of their properties.

Now we introduce our main definitions.

**Definition 2.1.** Let \((X, \Omega)\) be a uniform space and \( \{s_\alpha : \alpha \in D\} \) a net in \( X \). The net \( \{s_\alpha : \alpha \in D\} \) is said to be \( \mathcal{F}\)-convergent to \( s \) (in short, \( \mathcal{F}\)-lim \( s_\alpha = s \)) if for every \( U \in \Omega \), \( \{\alpha \in D : (s_\alpha, s) \in U\} \in \mathcal{F} \). In other words, for \( \forall U \in \Omega \), \( \{\alpha \in D : s_\alpha \in U[s] \} \subset \mathcal{F} \).

**Definition 2.2.** Let \((X, \Omega)\) be a uniform space and \( \{s_\alpha : \alpha \in D\} \) a net in \( X \). The net \( \{s_\alpha : \alpha \in D\} \) is said to be \( \mathcal{F}_{st}\)-fundamental in \( X \) if for every \( U \in \Omega \), there exist a \( \alpha_0 \in D \) such that \( \{\alpha \in D : s_\alpha \in U[s_{\alpha_0}] \} \subset \mathcal{F} \).

**Lemma 2.3.** Let \((X, \Omega)\) be a Hausdorff uniform space and \( \{s_\alpha : \alpha \in D\} \) a net in \( X \). If there exists \( \mathcal{F}\)-lim \( s_\alpha \), then it is unique.

**Proof.** Let \((X, \Omega)\) be a Hausdorff uniform space. Accordingly, \( s = \cap_{U \in \Omega} U[s] \). Let \( \{s_\alpha : \alpha \in D\} \) be a net in \( X \). We prove that if there exists \( \mathcal{F}\)-lim \( s_\alpha \), then it is unique. Suppose to contrary, that is, \( \mathcal{F}\)-lim \( s_\alpha \) has two values \( t_1 \neq t_2 \). Then it is obvious that there exists a \( U_k \in \Omega \) such that \( t_1 \notin U_k[t_2] \) and \( t_2 \notin U_k[t_1] \). If \( U = U_1 \cap U_2 \), then \( U \in \Omega \). Furthermore, \( t_1 \notin U[t_2] \) and \( t_2 \notin U[t_1] \). Since \( U \in \Omega \), there exists a \( V \in \Omega \) such that \( V \subset U \) and \( V = V^{-1} \). It is clear that \( t_1 \notin V[t_2] \) and \( t_2 \notin V[t_1] \). Suppose that \( A_1 = \{\alpha \in D : s_\alpha \in V[t_1]\} \).
and
\[ A_2 = \{ \alpha \in D : s_\alpha \in V[t_2] \} . \]

If \( A_1, A_2 \in \mathcal{F} \), then \( A_1 \cap A_2 \in \mathcal{F} \). On the other hand, \( A_1 \cap A_2 = \emptyset \in \mathcal{F} \). If \( A_1 \cap A_2 \neq \emptyset \), then there exists a \( \alpha_0 \in D \) such that \( s_{\alpha_0} \in A_1 \cap A_2 \). Moreover, \( (s_{\alpha_0}, t_1) \in V \) and \( (s_{\alpha_0}, t_2) \in V \). From the symmetry of \( V \), we have \((t_1, t_2), (t_2, t_1) \in V \). Consequently, \((t_1, t_2) \in V \circ V \subset U \). This is a contradiction, that is, \( \mathcal{F} - \text{lim} s_\alpha \) is unique. \( \square \)

**Theorem 2.4.** Let \((X, \Omega)\) be a Hausdorff uniform space and \( \{ s_\alpha : \alpha \in D \} \) be a net in \( X \) which is \( \mathcal{F} \)-convergent. Then \( \{ s_\alpha : \alpha \in D \} \) is \( \mathcal{F}_0 \)-fundamental.

**Proof.** Let \((X, \Omega)\) be a uniform space, \( \{ s_\alpha : \alpha \in D \} \) be a net in \( X \) and \( \mathcal{F}_0 \)-lim \( s_\alpha = s \). Now we prove that \( \{ s_\alpha : \alpha \in D \} \) is \( \mathcal{F}_0 \)-fundamental. Let \( U \in \Omega \). Then there exists a \( V \in \Omega \) such that \( V \circ V \subset U \) and \( V = V^{-1} \). Take \( \alpha_0 \in \{ \alpha \in D : s_\alpha \in V[s] \} \). It is obvious that
\[ \{ \alpha \in D : s_\alpha \in V[s] \} \in \mathcal{F}. \]

If \( s_\alpha \in V[s] \), then \((s_\alpha, s_\alpha) \in V \circ V \subset U \). As a result,
\[ \{ \alpha \in D : s_\alpha \in V[s] \} \subset \{ \alpha \in D : s_\alpha \in U[s_\alpha] \} \]

and so
\[ \{ \alpha \in D : s_\alpha \in U[s_\alpha] \} \in \mathcal{F}. \]

Hence, the theorem is proved. \( \square \)

**Theorem 2.5.** Let \((X, \Omega)\) be a Hausdorff complete uniform space with a countable base and \( \{ s_\alpha : \alpha \in D \} \) be a net in \( X \). If the net \( \{ s_\alpha : \alpha \in D \} \) is \( \mathcal{F}_0 \)-fundamental, then there exists \( s \in X \) such that \( \mathcal{F} \)-lim \( s_\alpha = s \).

**Proof.** Let \((X, \Omega)\) be a complete uniform space. We suppose that \((X, \Omega)\) has a countable base and it is Hausdorff. Then, there exists \( U_\alpha \in \Omega \) such that \( \cap_{\alpha \in D} U_\alpha = \Lambda \) and \( U_\alpha \subset U \) for all \( \alpha \in D \). Without loss of generality, we suppose that \( U^{(a_1)} \circ U^{(a_1)} \subset U^{(a)} \) and \( U^{(a)} = \left( U^{(a)} \right)^{-1} \). Let \( \{ s_\alpha : \alpha \in D \} \) be \( \mathcal{F}_0 \)-fundamental in \( X \). Hence, by definition there exists \( A_1 \in D \) such that \( A_1 \in \mathcal{F} \), where \( A_i = \{ \alpha \in D : s_\alpha \in U^{(a)[s_\alpha]} \} \) for \( i = 1, 2 \). It is obvious that \( A_{(1)} = A_1 \cap A_2 \in \mathcal{F} \). Let \( B_1 = U^{(1)}[s_\alpha] \cap U^{(2)}[s_\alpha] \). Clearly, \( s_\alpha \in B_1 \) for all \( \alpha \in A_{(1)} \). Likewise, there exists \( A_2 \in D \) such that \( A_2 = \{ \alpha \in D : s_\alpha \in U^{(a_2)[s_\alpha]} \} \in \mathcal{F} \). Suppose that \( A_{(2)} = A_{(1)} \cap A_3 \). It is obvious that \( A_{(2)} \in \mathcal{F} \). Put \( B_2 = B_1 \cap U^{(a_2)[s_\alpha]} \). As a result, \( B_2 \neq \emptyset \) and so \( s_\alpha \in B_2 \) for all \( \alpha \in A_{(2)} \). Continuing in the same way, we get the net of open non-empty sets \( \{ B_\alpha \}_{\alpha \in D} \subset X \) such that
\[ B_1 \supset B_2 \supset \ldots, B_\alpha \subset U^{(a_\alpha)}[s_{k_{a_\alpha}}] \]

for all \( \alpha \in D \), such that \( A_{(1)} \in \mathcal{F} \) such that \( A_{(1)} = \{ k \in D : s_k \in B_k \} \) for all \( k \in D \). Take \( \tilde{s}_\alpha \in B_\alpha \) for all \( \alpha \in D \). Now we prove that \( \{ s_\alpha : \alpha \in D \} \) is a fundamental element. Then, it is clear that there exists \( a_0 \in D \) such that \( U^{(a_0)} \subset U \) for \( a \geq a_0 \). Let \( a \geq a_0 \) be arbitrary. We obtain \( s_{a \uparrow p} \in B_{a \uparrow p} \subset B_a \) for all \( p \in D \). Since, we have \( B_a \subset U^{(a_0)}[s_{k_{a_0}}] \), it is obvious that \( \tilde{s}_{a \uparrow p} \in U^{(a_0)}[s_{k_{a_0}}] \) and \( \tilde{s}_{a \uparrow p} \in U^{(a_0)}[s_{k_{a_0}}] \). Moreover, \( \tilde{s}_{a \uparrow p} \in U^{(a_0)} \circ U^{(a_0)} \subset U^{(a_0)} \) for all \( p \in D \). As a result, \( \tilde{s}_{a \uparrow p} \in U^{(a_0)} \) for all \( a \geq a_0 \) and \( p \in D \). Since \( U \) is arbitrary, the net \( \{ s_\alpha : \alpha \in D \} \) is fundamental in \((X, \Omega)\) and let \( \text{lim} \, s_\alpha = s \). Now prove that \( \mathcal{F} \)-lim \( s_\alpha = s \). Take \( U \in \Omega \). Then, there exists \( a_0 \in D \) such that \( U^{(a_0)} \subset U \) for all \( a \geq a_0 \). Since \( B_a \subset U^{(a_0)}[s_{k_{a_0}}] \), we have
\[ A_{(a)} = \{ \alpha \in D : s_\alpha \in U^{(a_0)}[s_{k_{a_0}}] \} \in \mathcal{F} \]

for all \( \alpha \in D \). Let \( A_1 \in D \) such that \( \tilde{s}_k \in U^{(a_0)}[s] \) for all \( k \geq a_1 \). Without loss of generality, we suppose that \( a_1 \geq a_0 + 1 \). As a result, \( s_{a_1} \in B_{a_1} \subset U^{(a_0)}[s_{k_{a_0}}] \). We put \( \{ s_{a_1}, s_{k_{a_0}} \} \in U^{(a_1)} \). Then \( (s_{a_1}, s_{k_{a_0}}) \in U^{(a_0)} \circ U^{(a_0)} \subset U^{(a_1)} \). Since, \( (s_{k_{a_0}}, s_{k_{a_0}}) \in U^{(a_0)} \circ U^{(a_0)} \subset U^{(a_1)} \), then it is obvious that
\[ \{ s_{a_1}, s_{k_{a_0}} \} \in U^{(a_1)} \circ U^{(a_0)} \subset U^{(a_0)} \subset U. \]
This implies that
\[ \{ \alpha \in D : s_\alpha \in B_{t_0} \} \subset \{ \alpha \in D : s_\alpha \in U[s] \}. \]
Therefore,
\[ A_{(a)} = \{ \alpha \in D : s_\alpha \in B_{t_0} \} \in \mathcal{F}. \]
From the previous inclusion it follows that
\[ \{ \alpha \in D : s_\alpha \in U[s] \} \in \mathcal{F}. \]
Since \( U \) was arbitrary, we have \( \mathcal{F} \)-\( \text{lim} \) \( s_\alpha = s \).

**Theorem 2.6.** Let \((X, \Omega)\) be a uniform space with a countable base and let \( \{s_\alpha : \alpha \in D\} \) be an \( \mathcal{F}_{st} \)-fundamental net in \( X \). Then:

i) if \( \mathcal{F} \) is monotone closed filter and \( \mathcal{F} \)-\( \text{lim} \) \( s_\alpha = s \), then there exists \( \{t_\alpha\}_{\alpha \in D} \subset X \) such that \( \lim t_\alpha = s \) and \( \{ \alpha \in D : s_\alpha = t_\alpha \} \in \mathcal{F} \);

ii) if \( \mathcal{F} \) is a right filter and \( \lim t_\alpha = s \) and \( \{ \alpha \in D : s_\alpha = t_\alpha \} \in \mathcal{F} \), then \( \mathcal{F} \)-\( \text{lim} \) \( s_\alpha = s \).

**Proof.**

i) Suppose that the net \( \{s_\alpha : \alpha \in D\} \) is \( \mathcal{F}_{st} \)-fundamental, \( \mathcal{F} \) is monotone closed filter and the space \((X, \Omega)\) has a countable base. Consider the net \( \{A_{(a)}\}_{a \in D} \), constructed in the proof of Theorem 2.5. We get
\[ A_{(1)} \supset A_{(2)} \supset \ldots \text{ and } A_{(a)} \in \mathcal{F} \text{ for } a \in D. \]

Then by condition (ii) of filter we get \( \{a_m : a_1 < a_2 < \ldots\} \) such that
\[ \bigcup_{m=1}^{\infty} \left( (a_m, a_{m+1}] \cap A_{(m)} \right) \in \mathcal{F}. \]

Suppose that
\[ D_0 = \left\{ k \in D : k \in (a_m, a_{m+1}] \cap A_{(m)}^{c}, m \in D \right\} \cup \left\{ 1, a_1 \right\}. \]

Define
\[ t_k = \begin{cases} s, & k \in D_0 \\ s_k, & k \notin D_0 \end{cases}, \]
where \( s = \mathcal{F} \)-\( \text{lim} \) \( s_\alpha \). Now we prove that \( \lim t_k = s \). Let \( U \in \Omega \) be an arbitrary element. If \( k \in D_0 \), then it is obvious that \( t_k \in U[s] \). If \( k \notin D_0 \) then there exists a \( m \in D \) such that \( a_m < k \leq a_{m+1} \) and \( k \notin A_{(m)}^{c} \). Moreover, if \( k \in A_{(m)} \), then \( s_k \in B_{t_0} \). Let \( a_0 \in D \) be a number such that \( U_{(a_0-1)} \subset U \). Let \( k \) be sufficiently large \( m \geq a_0 \). We get \( s_k \in U_{(a_0)}[s] \) and so \( s_k \in U_{(a_0+1)}[s_k_{a_0+1}] \) and \( s_k_{a_0+1} \in U_{(a_0+1)}[s] \). Hence, \( (t_k, s) \in U_{(a_0)} \subset U \), since, in this case \( s_k = t_k \). Since \( U \) is arbitrary, \( \lim t_k = s \). Now we prove that \( \tilde{A} = \{ k \in D : s_k = t_k \} \in \mathcal{F} \). It is clear that
\[ \bigcup_{m=1}^{\infty} ( (a_m, a_{m+1}] \cap A_{(m)} ) \subset \tilde{A}. \]

Hence, \( \bigcup_{m=1}^{\infty} ( (a_m, a_{m+1}] \cap A_{(m)} ) \in \mathcal{F} \) and we obtain \( \tilde{A} \in \mathcal{F} \) from the condition (iii) of filter. Therefore, if \( \mathcal{F} \)-\( \text{lim} \) \( s_\alpha = s \), then there exists an \( \tilde{A} \in \mathcal{F} \) such that \( \lim t_\alpha = s \) and \( s_\alpha = t_\alpha \) for all \( \alpha \in \tilde{A} \).

ii) Suppose that \( \lim t_\alpha = s, \tilde{A} = \{ \alpha \in D : s_\alpha = t_\alpha \} \in \mathcal{F} \) and \( \mathcal{F} \) is a right filter. Let \( U \in \Omega \) be arbitrary. Then there exists \( a_0 \in D \) such that \( t_\alpha \in U[s] \) for all \( \alpha \geq a_0 \). We get
\[ \{ \alpha \in D : \alpha \geq a_0 \} \cap \tilde{A} \subset \{ \alpha \in D : s_\alpha \in U[s] \}. \]

It is obvious that
\[ \{ \alpha \in D : \alpha \geq a_0 \} \cap \tilde{A} \in \mathcal{F}. \]
Then we have \( \{ \alpha \in D : s_\alpha \in U[s] \} \in \mathcal{F} \) from the condition (iii) of filter. \( \square \)
The following results are immediate consequences of Theorems 2.5 and 2.6.

**Corollary 2.7.** Let \((X, \Omega)\) be a uniform space with a countable base, \(\{s_\alpha : \alpha \in D\}\) be a net in \(X\) and \(F\) be a monotone closed and a right filter. Then the followings are equivalent:

i) \(F\)-\(\lim s_\alpha = s\),

ii) \(\{s_\alpha : \alpha \in D\}\) is \(F\)-st-fundamental,

iii) \(\lim t_\alpha = s\) and \(\{\alpha \in D : s_\alpha = t_\alpha\} \in F\).

**Corollary 2.8.** Let \((X, \Omega)\) be a uniform space with a countable base, \(\{s_\alpha : \alpha \in D\}\) be an \(F\)-st-fundamental net in \(X\), and \(F\) be a right filter. If \(F\)-\(\lim s_\alpha = s\), then there exists a \(\{\alpha_k : \alpha_1 < \alpha_2 < \ldots\}\) in \(F\) such that \(\lim s_{\alpha_k} = s\).

**Acknowledgement**

The authors are most grateful to anonymous referees for careful reading of the manuscript and valuable suggestions.

**References**


