Reliable Numerical Algorithm for Handling Fuzzy Integral Equations of Second Kind in Hilbert Spaces

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Abstract. Integral equations under uncertainty are utilized to describe different formulations of physical phenomena in nature. This paper aims to obtain analytical and approximate solutions for a class of integral equations under uncertainty. The scheme presented here is based upon the reproducing kernel theory and the fuzzy real-valued mappings. The solution methodology transforms the linear fuzzy integral equation to crisp linear system of integral equations. Several reproducing kernel spaces are defined to investigate the approximate solutions, convergence and the error estimate in terms of uniform continuity. An iterative procedure has been given based on generating the orthonormal bases that rely on Gram-Schmidt process. Effectiveness of the proposed method is demonstrated using numerical experiments. The gained results reveal that the reproducing kernel is a systematic technique in obtaining a feasible solution for many fuzzy problems.

1. Introduction

Fuzzy integral equations are an influential part of the theory of uncertainty analysis that have become indispensable mathematical tools to describe and analyze real-world problems [14, 22, 25, 29, 40, 41]. They have a wide proportion of applications in different fields of science and engineering, which have been utilized to introduce uncertain parameter values in modelling many physical processes in nature [4, 26, 35, 42]. In many applications, some of parameters are typically given in uncertainty, so it is necessary to develop a suitable model and a viable algorithm to address these issues. Uncertainty during the modeling stages comes from many sources such as the errors and approximations in estimation values, data collection, initial data measurement, model parameters, model structure, and so on. However, it is too complicated to insert an explicit formula for these equations, thereby it is need reliable approximations to treat the complexities of uncertain data. Usually, by using parametrization of the fuzzy numbers, the fuzzy integral equation may be converted to a crisp system of integral equations that can be solved approximately.

Numerical simulations for fuzzy integral equations have continued to grow and attracted the attention of mathematicians and physicists. Over the past decades, various techniques have been proposed to deal with the uncertain integral equations. Based on this trend, the iterative procedure based on quadrature formula is described in [27, 37], while the successive approximations procedure is presented in [19, 20]. The numerical-analytical procedures to solve fuzzy integral equations including fuzzy Laplace transform method [36],
homotopy analysis method [39], fuzzy differential transform method [32], Adomian decomposition method [38], finite differences method [15] and variational iteration method [34] are also applied. Other numerical techniques developed for fuzzy integral equations based on fuzzy Haar wavelets, Legendre wavelets, Legendre and Chebyshev interpolations, Galerkin, triangular and block-pulse functions can be found in [16, 18, 21, 28, 31].

This paper aims to extend the application of the reproducing kernel method (RKM) in finding approximate solutions for a class of linear fuzzy Fredholm-Volterra integral equations in the following form

\[ F(\tau) = g(\tau) + \lambda_1 \int_a^\tau h(\tau, \rho)F(\rho)d\rho + \lambda_2 \int_a^\tau k(\tau, \rho)F(\rho)d\rho, \quad a \leq \rho < \tau \leq b, \]  

where \( \lambda_1, \lambda_2 \) are positive parameters, \( g : [a, b] \rightarrow \mathbb{R}_F \) is continuous fuzzy-valued function, \( h(\tau, \rho), k(\tau, \rho) \) are positive crisp kernel functions over the square \( a \leq \rho < \tau \leq b \), and \( a, b \) are real finite constants in which \( \mathbb{R}_F \) refers to the set of fuzzy numbers on \( \mathbb{R} \).

The theory of the RKM has a wide range of applications in applied mathematics, numerical analysis, financial mathematics, machine learning, statistical learning, control theory, global optimization and so on. Recently, the reproducing kernel theory has been used to treat many nonlinear problems such that nonlinear boundary value problems, nonlinear partial differential equations, nonlinear oscillator models, nonlinear differential-difference equations, nonlinear integral and integro-differential equations, nonlinear fractional differential equations, and nonlinear fuzzy models [1, 5, 6, 12, 17, 30]. The RKM possess several advantages: it is accurate and need less effort to achieve the results, it has excellent properties in error estimation reflects high accuracy and reliability, provides uniform convergence rates to the exact solution, it is possible to pick any point in the interval of integration and as well the approximate solution will be applicable, and offers a straightforward assumption to deal with various boundary conditions [2, 3, 7, 9–11, 13, 23].

The organization of this work is as follows. In the next section, preliminaries about fuzzy calculus theory are given. Section 3 contains the formulation of model problem and some necessary concepts of the reproducing kernel theory. Section 4 is devoted to computational algorithm and to error estimates. Simulation results are presented in Section 5 to guarantee the RK procedure and to confirm its performance. Finally, the last section includes concluding observations.

2. Preliminaries

In this section, we briefly recall some definitions and results related to fuzzy analysis needed throughout the paper. We start with the basic definition of fuzzy numbers.

Definition 2.1. ([25]) A fuzzy number is a mapping \( \nu : \mathbb{R} \rightarrow [0, 1] \) that satisfies normal, fuzzy convex, upper semicontinuous and compactly supported.

An equivalent parametric definition is given by Kaleva [29] and Friedman et al. [22] as follows:

Definition 2.2. A fuzzy number \( \nu \) in parametric form is a pair \((\nu^-, \nu^+)\) of functions \( \nu^-, \nu^+ \), \( 0 \leq r \leq 1 \), which satisfy the following properties:

- \( \nu^- \) is bounded left continuous non-decreasing function over \([0, 1]\),
- \( \nu^+ \) is bounded left continuous non-increasing function over \([0, 1]\),
- \( \nu^- \leq \nu^+ \leq 1 \).

For \( 0 < r \leq 1 \), the \( r \)-level set of a fuzzy number \( \nu \) is defined as \([\nu]^r = \{ \alpha \in \mathbb{R} : \nu(\alpha) \geq r \}\), which is a closed and bounded interval \([\nu^-_r, \nu^+_r]\), where \( \nu^-_r \) and \( \nu^+_r \) denote the left-hand and the right-hand endpoint of the \( r \)-level set, respectively. Also, \([\nu]^0 = \{ \alpha \in \mathbb{R} : \nu(\alpha) > 0 \}\) is a closure of the support of \( \nu \), which is a compact set. The set of all normal, convex, and upper semi-continuous fuzzy numbers with bounded \( r \)
level intervals is denoted by $\mathbb{R}_F$. For arbitrary any real number $\mu \in \mathbb{R}$ can be interpreted as a fuzzy number $\tilde{\mu} = \chi_{[\mu]}$, where $\chi_{[\mu]}$ is the characteristic function at $\mu$, therefore $\mathbb{R} \subset \mathbb{R}_F$.

For arbitrary fuzzy numbers $v = (v_{1r}, v_{2r})$, $w = (w_{1r}, w_{2r})$ and real number $\mu$, the addition $v \oplus w$ and scalar multiplication $\mu \odot v$ are defined as follows:

- $v = w$ if and only if $v_{1r} = w_{1r}$ and $v_{2r} = w_{2r}$.
- $v \oplus w = (v_{1r} + w_{1r}, v_{2r} + w_{2r})$.
- $\mu \odot v = \begin{cases} (\mu v_{1r}, \mu v_{2r}), & \mu \geq 0 \\ (\mu v_{2r}, \mu v_{1r}), & \mu < 0 \end{cases}$

Supremum metric structure on $\mathbb{R}_F$ is the most commonly metric used and is given by the following definition:

**Definition 2.3.** ([41]) For arbitrary fuzzy numbers $v = (v_{1r}, v_{2r})$, $w = (w_{1r}, w_{2r})$, the quantity $d(v, w) = \sup_{r \in [0,1]} \max \{|v_{1r} - w_{1r}|, |v_{2r} - w_{2r}|\}$ is the distance between $v$ and $w$. It is demonstrated that $(d, \mathbb{R}_F)$ is a complete metric space [29] and the following properties are realized:

- $d(v \oplus u, w \oplus u) = d(v, w)$, $\forall v, w, u \in \mathbb{R}_F$.
- $d(v \oplus u, w \oplus u) \leq d(v, w) + d(u, z)$, $\forall v, w, u, z \in \mathbb{R}_F$.
- $d(\mu \odot v, \mu \odot w) = |\mu|d(v, w)$, $\forall \mu \in \mathbb{R}$, $\forall v, w \in \mathbb{R}_F$.
- $d(\mu_1 \odot v, \mu_2 \odot w) = |\mu_1 - \mu_2|d(v, 0)$, $\forall \mu_1, \mu_2 \in \mathbb{R}$, $\mu_1 \mu_2 \geq 0$, $\forall v \in \mathbb{R}_F$.

**Definition 2.4.** ([14]) A fuzzy-valued function $g : [a, b] \to \mathbb{R}_F$ is said to be continuous at $\tau_0 \in [a, b]$ if for each fixed $\epsilon > 0$, there is $\delta > 0$ such that $d(g(\tau), g(\tau_0)) < \epsilon$ whenever $|\tau - \tau_0| < \delta$. If $g$ is continuous for each $\tau_0 \in [a, b]$, then $g$ is said to be continuous on $[a, b]$.

A fuzzy number $v$ is upper bound for $g : [a, b] \to \mathbb{R}_F$ if $g_{1r}(\tau) \leq v_{1r}$ and $g_{2r}(\tau) \leq v_{2r}$ for all $\tau \in [a, b]$, $r \in [0,1]$, and $v$ is lower bound for $g : [a, b] \to \mathbb{R}_F$ if $v_{1r} \leq g_{1r}(\tau)$ and $v_{2r} \leq g_{2r}(\tau)$ for all $\tau \in [a, b]$, $r \in [0,1]$. Further, a fuzzy-valued function $g : [a, b] \to \mathbb{R}_F$ is said to be bounded if it has a lower and an upper bound. The following definition is presented for fuzzy integral based on Riemann integral concept. It is worth noting that all the different concepts of fuzzy integral yield the same results in the sense of continuity of the fuzzy-valued function. The concept defined in this paper is more appropriate for numerical computations.

**Definition 2.5.** ([29]) Let $\mathcal{F} : \Omega \to \mathbb{R}_F$. For a partition $\mathcal{P}_r^\infty = \{\tau_0, \tau_1, \ldots, \tau_n\}$ of $\Omega$ and for arbitrary points $\xi_i \in [\tau_{i-1}, \tau_i]$, $i = 1, 2, \ldots, n$. Let $\theta_r = \sum_{i=1}^n (\mathcal{F}(\xi_i)(\tau_i - \tau_{i-1}))$ and $\Delta_r = \max_{1 \leq i \leq n} |\tau_i - \tau_{i-1}|$, then the definite integral of $\mathcal{F}(\tau)$ over $\Omega$ is defined by $\int_\Omega \mathcal{F}(\tau) d\tau = \lim_{\Delta_r \to 0} \theta_r$ provided the limit exists in the metric space $(\mathbb{R}_F, d)$.

**Theorem 2.1.** ([29]) Let $\mathcal{F} : \Omega \to \mathbb{R}_F$ be continuous fuzzy-valued function and put $[\mathcal{F}(\tau)]' = [\mathcal{F}_{1r}(\tau), \mathcal{F}_{2r}(\tau)]$ for each $r \in [0,1]$. Then $\int_\Omega \mathcal{F} d\tau \in \mathbb{R}_F$ exist, $\mathcal{F}_{1r}(\tau), \mathcal{F}_{2r}(\tau)$ are integrable functions on $\Omega$, and $\int_\Omega [\mathcal{F}(\tau)]' d\tau = \int_\Omega [\mathcal{F}_{1r}(\tau) d\tau, \int_\Omega \mathcal{F}_{2r}(\tau) d\tau]$. 

M. Al-Smadi / Filomat 33:2 (2019), 583–597

585
3. Fuzzy Integral Equation in Reproducing Kernel Spaces

In this section, the original fuzzy Fredholm-Volterra integral equation is converted equivalently into crisp Fredholm-Volterra integral equations system based on the use of Riemann integrability. However, to do this, consider the standard form of Fredholm-Volterra integral equation as follows:

\[ \mathcal{F}(\tau) = g(\tau) + \lambda_1 \int_a^b h(\tau, \rho) F(\rho) \, d\rho + \lambda_2 \int_a^b k(\tau, \rho) F(\rho) \, d\rho, \quad a \leq \rho < \tau \leq b, \]  
(2)

where \( g : [a, b] \to \mathbb{R} \) is continuous real-valued function and \( h(\tau, \rho), k(\tau, \rho) \) are arbitrary crisp kernel functions over the square \( a \leq \rho < \tau \leq b \), \( (a, b \in \mathbb{R}) \), such that \( h(\tau, \rho) \) is non-negative for \( a \leq \rho \leq c_1 \) and is non-positive for \( c_1 \leq \rho \leq b \); and \( k(\tau, \rho) \) is non-negative for \( a \leq \rho \leq c_2 \) and is non-positive for \( c_2 \leq \rho \leq \tau \). Consequently, if \( g(\tau) \) is a crisp function, then the solution \( F(\tau) \) of Eq. (2) is a crisp. Otherwise if \( g(\tau) \) is a fuzzy function, then Eq. (2) may possess only fuzzy solution \( F(\tau) \). Now, we assume that \( g(\tau) \) is a fuzzy function, \( H(\tau, \rho, F(\rho)) = h(\tau, \rho) F(\rho) \) and \( K(\tau, \rho, F(\rho)) = k(\tau, \rho) F(\rho) \). Let \([g_1(\tau), g_2(\tau)]\) and \([F_1(\tau), F_2(\tau)]\) be the parametric forms of \( g(\tau) \) and \( F(\tau) \), respectively; then the parametric form of Eq. (2) is given by

\[ [\mathcal{F}(\tau)]' = [g(\tau)]' + \lambda_1 \int_a^b [H(\tau, \rho, F(\rho))]' \, d\rho + \lambda_2 \int_a^b [K(\tau, \rho, F(\rho))]' \, d\rho, \]  
(3)

where

\[ [H(\tau, \rho, F(\rho))]' = \begin{bmatrix} H_{11}(\tau, \rho, F_{11}(\rho), F_{21}(\rho)) \\ H_{21}(\tau, \rho, F_{11}(\rho), F_{21}(\rho)) \end{bmatrix}, \]

\[ [K(\tau, \rho, F(\rho))]' = \begin{bmatrix} K_{11}(\tau, \rho, F_{11}(\rho), F_{21}(\rho)) \\ K_{21}(\tau, \rho, F_{11}(\rho), F_{21}(\rho)) \end{bmatrix}. \]

such that

\[ H_{11}(\tau, \rho, F_{11}(\rho), F_{21}(\rho)) = \min \{ h(\tau, \rho) F_{11}(\rho), h(\tau, \rho) F_{21}(\rho) \}, \]

\[ H_{21}(\tau, \rho, F_{11}(\rho), F_{21}(\rho)) = \max \{ h(\tau, \rho) F_{11}(\rho), h(\tau, \rho) F_{21}(\rho) \}, \]

\[ K_{11}(\tau, \rho, F_{11}(\rho), F_{21}(\rho)) = \min \{ k(\tau, \rho) F_{11}(\rho), k(\tau, \rho) F_{21}(\rho) \}, \]

\[ K_{21}(\tau, \rho, F_{11}(\rho), F_{21}(\rho)) = \max \{ k(\tau, \rho) F_{11}(\rho), k(\tau, \rho) F_{21}(\rho) \}. \]

Thereby the parametric forms of the fuzzy Fredholm-Volterra integral equations system can be converted as follows:

\[ F_{11}(\tau) = g(\tau)_{11} + \lambda_1 \left( \int_a^{c_1} h(\tau, \rho) F_{11}(\rho) \, d\rho + \int_{c_1}^b h(\tau, \rho) F_{21}(\rho) \, d\rho \right) + \lambda_2 \left( \int_a^{c_2} k(\tau, \rho) F_{11}(\rho) \, d\rho + \int_{c_2}^b k(\tau, \rho) F_{21}(\rho) \, d\rho \right), \]  
(4)

\[ F_{21}(\tau) = g(\tau)_{21} + \lambda_1 \left( \int_a^{c_1} h(\tau, \rho) F_{11}(\rho) \, d\rho + \int_{c_1}^b h(\tau, \rho) F_{11}(\rho) \, d\rho \right) + \lambda_2 \left( \int_a^{c_2} k(\tau, \rho) F_{21}(\rho) \, d\rho + \int_{c_2}^b k(\tau, \rho) F_{21}(\rho) \, d\rho \right). \]  
(5)

**Definition 3.1.** ([5]) Let \( S \) be a nonempty abstract set. A function \( \varphi : S \times S \to \mathcal{C} \) is a reproducing kernel of the Hilbert space \( \mathcal{H} \) if (i) \( \forall \tau \in S, \varphi(\cdot, \tau) \in \mathcal{H} \) and (ii) \( \forall \psi \in \mathcal{H}, \forall \psi' \in \mathcal{H}, \langle \psi(\cdot), \varphi(\cdot, \tau) \rangle = \psi(\tau) \). The condition (ii) is “the reproducing property”. If there exists a reproducing kernel \( \varphi \) of \( \mathcal{H} \), then \( \mathcal{H} \) is called a reproducing kernel Hilbert space (RKHS).
Next, a complete reproducing kernel space \( \mathcal{W}_2^{m}[a, b] = \mathcal{H}_2^m[a, b] \oplus \mathcal{H}_2^m[a, b] \) is constructed to represent the solution of Eq. (2).

**Definition 3.2.** The inner product space \( \mathcal{W}_2^{m}[a, b] \) is defined as \( \mathcal{W}_2^{m}[a, b] = \{ f | f, f', ..., f^{(m-1)} \text{ are absolutely continuous functions, } f^{(m)} \in L^2[a, b] \} \) where \( L^2[a, b] = \{ f | \int_a^b f^2(\rho)d\rho < \infty \} \). The inner product and norm are given, respectively, by \( \langle f(t), g(t) \rangle_{\mathcal{W}_2^{m}[a, b]} = \sum_{i=0}^{m-1} f^{(i)}(a)g^{(i)}(a) + \int_a^b f^{(m)}(\rho)g^{(m)}(\rho)d\rho \) and \( ||f||_{\mathcal{W}_2^{m}} = \sqrt{\langle f, f \rangle_{\mathcal{W}_2^{m}}} \), \( f, g \in \mathcal{W}_2^{m}[a, b] \).

**Theorem 3.1.** The space \( \mathcal{W}_2^{m}[a, b] \) is a RKHS. The reproducing kernel function \( Q^m_\tau(s) \) can be written as follows

\[
Q^m_\tau(s) = \begin{cases} \frac{1}{(m-1)!}\sum_{i=0}^{m-1} f^{(i)}(a)(s-a)^i + \frac{1}{(m-1)!}\sum_{i=0}^{m-1} f^{(i)}(a)(s-a)^i, s \leq \tau \\
\frac{1}{(m-1)!}\sum_{i=0}^{m-1} f^{(i)}(a)(s-a)^i + \frac{1}{(m-1)!}\sum_{i=0}^{m-1} f^{(i)}(a)(s-a)^i, s > \tau \end{cases}
\]

Proof. Let \( \{ \xi_n \}_{n=1}^{\infty} \) be a Cauchy sequence in the space \( \mathcal{W}_2^{m}[a, b] \), then \( ||\xi_{n+1} - \xi_n||_{\mathcal{W}_2^{m}} = \sum_{i=1}^{m-1} \int_a^b (\xi_n^{(i)}(a) - \xi_n^{(i)}(a))^2 + \int_a^b (\xi_n^{(m)}(\rho) - \xi_n^{(m)}(\rho))^2 d\rho \to 0 \) as soon as \( n \to \infty \). Anyhow, \( \{ \xi_n(a) \}_{n=1}^{\infty}, \{ \xi_n'(a) \}_{n=1}^{\infty}, ..., \{ \xi_n^{(m-1)}(a) \}_{n=1}^{\infty} \) be the all Cauchy sequence of real number and \( \{ \xi_n^{(m)}(\tau) \}_{n=1}^{\infty} \) be a Cauchy sequence in \( L^2[a, b] \). Thus, there exist a real number \( \eta_i, i = 0, 1, ..., m-1 \) and a real-valued function \( \xi \in L^2[a, b] \) such that \( \xi_n(a) = \eta_0, \xi_n'(a) = \eta_1, ..., \xi_n^{(m-1)}(a) = \eta_{m-1} \), and \( \int_a^b (\xi_n^{(m)}(\rho) - \xi(\rho))^2 d\rho \to 0 \) as \( n \to \infty \). Indeed, let \( v^{(m)}(\tau) = \xi(\tau) \in L^2[a, b] \), then

\[
v^{(m)}(\tau) = \eta_0 + \eta_1(\tau - a) + ... + \eta_{m-1}(\tau - a)^{m-1} (m-1)! + \int_a^\tau (\tau - \rho)^{m-1}(m-1)! \xi(\rho)d\rho.
\]

Clearly, \( v^{(i)}(\tau), i = 0, 1, ..., m-1 \) are absolutely continuous functions such that \( v(\tau) \in \mathcal{W}_2^{m}[a, b], v^{(i)}(\tau) = \eta_i, i = 0, 1, ..., m-1 \), and \( ||v_n - v||_{L^2[a, b]} \to 0 \) as \( n \to \infty \). So, \( \mathcal{W}_2^{m}[a, b] \) is complete Hilbert space.

For the conduct of proceedings in the proof, for any \( f \in \mathcal{W}_2^{m}[a, b] \), it yields that

\[
f(\tau) = \sum_{i=0}^{m-1} \frac{f^{(i)}(a)(\tau - a)^{(m-1)}}{i!} + \frac{1}{(m-1)!} \int_a^\tau (\tau - \rho)^{m-1} f^{(m)}(\rho)d\rho.
\]

Consequently, from Cauchy-Schwarz inequality, one can obtain that

\[
|f(\tau)| \leq \left| \sum_{i=0}^{m-1} \frac{f^{(i)}(a)(\tau - a)^{(m-1)}}{i!} \right| + \frac{1}{(m-1)!} \int_a^\tau (\tau - \rho)^{m-1} f^{(m)}(\rho)d\rho
\]

\[
\leq \left( \sum_{i=0}^{m-1} \frac{(\tau - a)^{2i}}{(i!)^2} \left( \sum_{i=0}^{m-1} \frac{(f^{(i)}(a))^2}{i!} \right)^{\frac{i}{2}} + \frac{1}{(m-1)!} \left( \int_a^b (b - \rho)^{2(m-1)}d\rho \right)^{\frac{1}{2}} \left( \int_a^b (f^{(m)}(\rho))^2 d\rho \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}
\]

\[
\leq \left( \sum_{i=0}^{m-1} \frac{(b-a)^{2i}}{(i!)^2} + \frac{(b-a)^{m-\frac{i}{2}}}{(m-1)!} \right) ||f||_{\mathcal{W}_2^{m}}.
\]

Hence, for any \( f \in \mathcal{W}_2^{m}[a, b] \), one can write
The image of the set

\[ \{ f(s), Q^n(s) \} = \sum_{i=0}^{m-1} f(i) Q^n(i) + \int_a^\tau f^{(m)}(s) Q^{(m)}(s) ds \]

\[ = \sum_{i=0}^{m-1} f(i)(\tau - a)^i/i! (s-a)^i + \frac{1}{(m-1)!} \int_a^\tau (\tau - s)^{m-1} f^{(m)}(s) ds \]

(9)

So, the proof of the theorem is completely constructed. \( \square \)

**Corollary 3.1.** The reproducing kernel functions \( Q^2(s) \) and \( Q^1(s) \) on the interval \([a, b]\) can be written, respectively, by

\[
Q^2(s) = \begin{cases} 
1 + (\tau - a)(s - a) - \frac{1}{6}(\tau - a)^2(2a + \tau - 3s), & s \leq \tau \\
1 + (\tau - a)(s - a) - \frac{1}{3}(s - a)^2(2a + s - 3\tau), & s > \tau 
\end{cases}
\]  

(10)

and

\[
Q^1(s) = \begin{cases} 
1 - a + \tau, & s \leq \tau \\
1 - a + s, & s > \tau 
\end{cases}
\]  

(11)

The lower and upper bounds of the kernel functions \( Q^1 \) and \( Q^2 \) on \([a, b]\) can be given as in the following theorem:

**Theorem 3.2.** The image of the set \([a, b]\) under the mapping \( Q^2 : [a, b] \times [a, b] \to \mathbb{R} \) and \( Q^1 : [a, b] \times [a, b] \to \mathbb{R} \) are obtained as \( Q^2_{[a,b]}([a, b]) = 1 + (b - a)^2 \left[ \frac{1}{2} (b - a), 1 + \frac{1}{2} (b - a) \right] \) and \( Q^1_{[a,b]}([a, b]) = 1 + [0, b - a] \), respectively.

**Proof.** From the interval analysis theory; the replacement of the two variables \( s \) and \( \tau \) by the corresponding interval variables will gives the following natural inclusion function at \([a, b]\):

\[
Q^2_{[a,b]}([a, b]) = 1 + [(a, b) - a][(a, b) - a] - \frac{1}{6}((a, b) - a)^2(2a + [a, b] - 3[a, b])
\]

\[
= 1 + [0, b - a][0, b - a] - \frac{1}{6}[0, b - a]^2[3(a - b), b - a]
\]

\[
= 1 + [0, (b - a)^2] - \frac{1}{6}[0, (b - a)^2][3(a - b), b - a]
\]

\[
= 1 + [0, (b - a)^2] - \frac{1}{6}[(b - a)^2(3(a - b), b - a)^2]
\]

\[
= 1 + [0, (b - a)^2] - \frac{1}{6}[-3(b - a)^3, (b - a)^3]
\]

\[
= 1 + [0, (b - a)^2] + \frac{1}{6}[-(b - a)^3, 3(b - a)^3]
\]

\[
= 1 + [0 - \frac{1}{6}(b - a)^3, (b - a)^2 + 3/6(b - a)^3]
\]

\[
= 1 + \left[ -\frac{1}{6}(b - a)^3, (b - a)^2 + \frac{1}{2}(b - a)^3 \right]
\]

\[
= 1 + (b - a)^2 \left[ -\frac{1}{6}(b - a), 1 + \frac{1}{2}(b - a) \right].
\]  

(12)

In similar fashion, it follows that \( Q^1_{[a,b]}([a, b]) = 1 + [0, b - a]. \) \( \square \)
Definition 3.3. The inner product space \( \mathcal{W}_s^m[a,b] \) is defined as \( \mathcal{W}_s^m[a,b] = \{ f = (f_1, f_2) \mid f_1, f_2 \in \mathcal{W}_s^m[a,b] \} \) and \( \| f \|_{\mathcal{W}_s^m} = \sqrt{\sum_{i=1}^{s} \| f_i \|_{\mathcal{W}_s^m}^2}, f, g \in \mathcal{W}_s^m[a,b] \).

4. Analysis of Reproducing Kernel Method

In this section, the solution formulation of model problem (2) is acquired in the \( \mathcal{W}_2^2[a,b] \). First of all, the linear operator \( \mathcal{L} : \mathcal{W}_2^2[a,b] \rightarrow \mathcal{W}_1^1[a,b] \) is described by

\[
\mathcal{L} \mathcal{F}_r(\tau) = \mathcal{F}_r(\tau) - \lambda_1 \int_a^b h(\tau, \rho) \mathcal{F}_r(\rho) d\rho - \lambda_2 \int_a^b k(\tau, \rho) \mathcal{F}_r(\rho) d\rho = G_r(\tau)
\]

where

\[
\mathcal{F}_r = \begin{bmatrix} f_{1r} & f_{2r} \\ g_{1r} & g_{2r} \end{bmatrix}, \quad G_r = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix}, \quad l_{ij} : \mathcal{W}_2^2[a,b] \rightarrow \mathcal{W}_2^2[a,b], \quad i, j = 1, 2, \text{ so that (13) }
\]

\[
l_{ij}f(\tau) = \begin{cases} f(\tau) - \int_a^b h(\tau, \rho)f(\rho) d\rho - \int_a^b k(\tau, \rho)f(\rho) d\rho, & i = j \\ -\int_a^b h(\tau, \rho)f(\rho) d\rho - \int_a^b k(\tau, \rho)f(\rho) d\rho, & i \neq j \end{cases}
\]

From definition of the inner product and norm of the \( \mathcal{W}_2^2[a,b] \) and \( \mathcal{W}_1^1[a,b] \) as well as by using Schwarz inequality, it is easy to show that \( l_{ij} : \mathcal{W}_2^2[a,b] \rightarrow \mathcal{W}_2^2[a,b], \quad i, j = 1, 2, \) is a bounded linear operator.

**Lemma 4.1.** The \( \mathcal{L} : \mathcal{W}_2^2[a,b] \rightarrow \mathcal{W}_1^1[a,b] \) is a bounded linear operator.

**Proof.** It is clear that \( \mathcal{L} \) is a linear operator since \( l_{ij} \) is linear for \( i, j = 1, 2 \). For boundedness of \( \mathcal{L} \), it follows that

\[
\| \mathcal{L} \mathcal{F}_r \|_{\mathcal{W}_1} = \sqrt{\sum_{i=1}^{2} \| \sum_{j=1}^{2} l_{ij} f_{jr} \|_{\mathcal{W}_2^2}^2} \leq \sqrt{\sum_{i=1}^{2} \left( \sum_{j=1}^{2} |l_{ij}| \| f_{jr} \|_{\mathcal{W}_2^2} \right)^2} \leq \sqrt{\sum_{i=1}^{2} \left( \sum_{j=1}^{2} |l_{ij}|^2 \| f_{jr} \|_{\mathcal{W}_2^2}^2 \right)}
\]

From definition of the inner product and norm of the \( \mathcal{W}_2^2[a,b] \) and \( \mathcal{W}_1^1[a,b] \) as well as by using Schwarz inequality, it is easy to show that \( l_{ij} : \mathcal{W}_2^2[a,b] \rightarrow \mathcal{W}_2^2[a,b], \quad i, j = 1, 2, \) is a bounded linear operator.

**Lemma 4.2.** Let \( \{ \tau_k \}_{k=1}^{\infty} \) be a dense subset of \( [a,b] \), then \( [l_{ij}^r Q_1^s(\eta)] = [l_{ij}^s Q_1^r(\tau)] \) for \( i, j = 1, 2 \).

**Proof.** From the reproducing property of \( \mathcal{W}_2^2[a,b] \), we have \( [l_{ij}^r Q_1^s(\eta)] = \langle [l_{ij}^r Q_1^s(\eta)], Q_1^r(\tau) \rangle_{\mathcal{W}_2^2} = \langle Q_1^r(\tau), [l_{ij}^s Q_1^r(\eta)] \rangle_{\mathcal{W}_2^2} = [l_{ij}^r Q_1^s(\tau)] \) for \( i, j = 1, 2 \) where \( l_{ij}^r \) is an adjoint operator of \( l_{ij} \).

Let \( \{ \tau_k \}_{k=1}^{\infty} \) be a countable dense set on \( [a,b] \), put \( \psi_{ij}(\tau) = \begin{bmatrix} l_{ij}^1 Q_1(s) \end{bmatrix}_{s=1} \), where \( \psi_{ij} \) is an adjoint operator of \( l_{ij} \).

**Theorem 4.1.** Let \( \{ \tau_k \}_{k=1}^{\infty} \) be a dense subset of \( [a,b] \), then \( \{ \psi_{ij}(\tau) \}_{k=1}^{\infty} \) is linearly independent in the \( \mathcal{W}_2^2[a,b] \).

**Proof.** Assume that \( \psi_{i_1 j_1}(\tau), \psi_{i_2 j_2}(\tau), \psi_{i_3 j_3}(\tau) \) are linearly dependent, then there exists \( c_{ij} \) (where \( i, j = 1, 2, \ldots, a \)) of \( a \) not all zero such that \( \sum_{k=1}^{a} c_{ij} \psi_{ij} = 0 \). Thus, for any \( \mathcal{F}_r \in \mathcal{W}_2^2[a,b] \), we have

\[
\langle \mathcal{L} \mathcal{F}_r(s), \begin{bmatrix} \sum_{i=1}^{a} c_1 a_i(\sigma) \\ \sum_{i=1}^{a} c_2 a_i(\sigma) \end{bmatrix} \rangle_{\mathcal{W}_1} = \langle \begin{bmatrix} f_{1r}(s) \\ f_{2r}(s) \end{bmatrix}, \begin{bmatrix} \sum_{i=1}^{a} c_1 a_i(\sigma) \\ \sum_{i=1}^{a} c_2 a_i(\sigma) \end{bmatrix} \rangle_{\mathcal{W}_1} = \langle \begin{bmatrix} \sum_{i=1}^{a} l_{11} f_{jr}(s) \\ \sum_{i=1}^{a} l_{12} f_{jr}(s) \end{bmatrix}, \begin{bmatrix} \sum_{i=1}^{a} c_1 a_i(\sigma) \\ \sum_{i=1}^{a} c_2 a_i(\sigma) \end{bmatrix} \rangle_{\mathcal{W}_1} = 0
\]
Theorem 4.2. The set 

\[ \{ \psi_{ij}(t) \}_{i,j=1}^{\infty} \]

is a complete function system in the \( W_{2}^{2}[a,b] \).

Proof. For each \( F_{r} \in W_{2}^{2}[a,b] \), we have

\[ \langle F_{r}(t), \psi_{ij}(t) \rangle_{W_{2}^{2}} = \left[ \begin{array}{c} f_{1r}(t) \\ f_{2r}(t) \end{array} \right] \left[ \begin{array}{cc} 1_{ij}Q_{1}^{2}(s)_{i=1}^{n} \\ 1_{ij}Q_{2}^{2}(s)_{i=1}^{n} \end{array} \right]_{W_{2}^{2}} = \langle f_{1r}(t), l_{1j}Q_{1}^{2}(s)_{i=1}^{n} \rangle_{W_{2}^{2}} + \langle f_{2r}(t), l_{2j}Q_{2}^{2}(s)_{i=1}^{n} \rangle_{W_{2}^{2}} \]

(17)

Since \( |r_{1}| = |r_{2}| = 1 \), then \( l_{1j}f_{1r}(t) + l_{2j}f_{2r}(t) = 0 \) and \( l_{1j}f_{2r}(t) + l_{2j}f_{2r}(t) = 0 \). Hence, \( F_{r}(t) = \left[ \begin{array}{c} f_{1r}(t) \\ f_{2r}(t) \end{array} \right] = 0 \).

Let \( \xi_{2i-1} \) be \( \psi_{ij}(t) \) for \( i = 1, 2, ..., j = 1, 2, \) then the sequence \( \{ \psi_{ij}(t) \}_{i,j=1}^{\infty} \) can be arranged as

\[ \{ \psi_{11}(t), \psi_{12}(t), \psi_{21}(t), \psi_{22}(t), ... \} = \{ \xi_{1}(t), \xi_{2}(t), \xi_{3}(t), ... \} \]

Anyhow, the orthonormal system \( \{ \xi_{i}(t) \}_{i=1}^{\infty} \) can be derived from Gram-Schmidt process of \( \{ \xi_{i}(t) \}_{i=1}^{\infty} \) by

\[ \xi_{i}(t) = \sum_{k=1}^{i} \beta_{ik} \xi_{k}(t), \quad i = 1, 2, ... \]

(18)

where \( \beta_{ik} \) are orthogonalization coefficients given by \( \beta_{11} = \| \xi_{1} \|_{W_{2}^{2}}^{2} \), \( \beta_{ij} = (\Delta_{i})^{-\frac{1}{2}} \sum_{k=1}^{i} (-b_{ik} \beta_{ik}) \) for \( i > j \),

\[ \Delta_{i} = (\| \xi_{i} \|_{W_{2}^{2}}^{2} - \sum_{k=1}^{i-1} b_{ik}^{2}) \] and \( b_{ik} = \langle \xi_{i}, \xi_{k} \rangle_{W_{2}^{2}} \).

Theorem 4.3. If \( F_{r}(t) \) is the solution of Eq. (2) is unique, then

\[ F_{r}(t) = \sum_{i=1}^{\infty} \left( \sum_{k=1}^{i} \beta_{ik} \alpha_{k} \right) \xi_{i}(t) \]

(19)

where \( \{ \xi_{i} \}_{i=1}^{\infty} \) is dense in \( [a,b] \) and \( \alpha_{k} = \langle F_{r}(t), \xi_{k}(t) \rangle_{W_{2}^{2}} \)
Proof. From Theorem 4.2, we acquire that $\{\xi_i\}_{i=1}^\infty$ is the complete orthonormal basis of $W^2[a,b]$. Let $F_r(\tau)$ be the exact solution of Eq. (2), then $F_r(\tau)$ can be expanded as a Fourier series in terms of the complete orthonormal basis $\{\xi_i\}_{i=1}^\infty$ of the space $W^2[a,b]$ as,

$$F_r(\tau) = \sum_{i=1}^\infty \langle F_r(\tau), \xi_i(\tau) \rangle_{W^2} \xi_i(\tau)$$

(20)

Based on Eq. (18), $\langle F_r(\tau), \xi_i(\tau) \rangle_{W^2}$ can be written by

$$\langle F_r(\tau), \xi_i(\tau) \rangle_{W^2} = \langle F_r(\tau), \sum_{k=1}^j \beta_k \xi_k(\tau) \rangle_{W^2} = \sum_{k=1}^j \beta_k \langle F_r(\tau), \xi_k(\tau) \rangle_{W^2} = \sum_{k=1}^j \beta_k \alpha_k$$

$$= \sum_{k \text{ is odd}} \beta_k f_1(\gamma_k) + \sum_{k \text{ is even}} \beta_k f_1(\gamma_k).$$

\[\blacksquare\]

Remark 4.1. The approximate solution $F^n_r(\tau)$ of Eq. (2) can be obtained directly by truncating the infinite series representation form of $F_r(\tau)$ as follows

$$F^n_r(\tau) = \sum_{i=1}^\infty \left( \sum_{k=1}^j \beta_k \alpha_k \right) \xi_i(\tau).$$

(21)

Theorem 4.4. The approximate solution $F^n_r(\tau)$ of Eq. (2) is converges uniformly to the exact solution $F_r(\tau)$ as $n \to \infty$.

Proof. By the expression form of the reproducing kernel function $Q^2_r(s)$, one can write

$$|f^n_r(\tau) - f_r(\tau)| = |\langle f^n_r(s), Q^2_r(s) \rangle - \langle f_r(s), Q^2_r(s) \rangle| \leq \|f^n_r - f_r\|_{W^2} \|Q^2_r\|_{W^2} \leq \sqrt{Q^2_r(\tau)} \|f^n_r - f_r\|_{W^2} \leq M_0 \|f^n_r - f_r\|_{W^2},$$

(22)

where $M_0 \in R$.

Thereby $f^n_r(\tau)$ converges uniformly to $f_r(\tau), i = 1, 2$ on $[a,b]$ as $n \to \infty$. Hence, $F^n_r(\tau)$ converges uniformly to $F_r(\tau)$ as $n \to \infty$.

\[\blacksquare\]

Corollary 4.1. Let $e_n = \|F_r - F^n_r\|_{W^2[a,b]}$. Then, the sequence $\{e_n\}$ is decreasing in the sense of $\|\|_{W^2[a,b]}$ such that $e_n \to 0$ as soon as $n \to \infty$.

Proof. From Eq. (19) and Eq. (21), we have

$$\epsilon_n^2 = \sum_{i=n+1}^\infty \langle F_r(\tau), \xi_i(\tau) \rangle_{W^2} \xi_i(\tau) \|_{W^2[a,b]}^2 = \sum_{i=n+1}^\infty \langle F_r(\tau), \xi_i(\tau) \rangle_{W^2[a,b]}^2$$

$$\epsilon_{n-1}^2 = \sum_{i=n}^\infty \langle F_r(\tau), \xi_i(\tau) \rangle_{W^2} \xi_i(\tau) \|_{W^2[a,b]}^2 = \sum_{i=n}^\infty \langle F_r(\tau), \xi_i(\tau) \rangle_{W^2[a,b]}^2$$

Consequently, it obvious that $\epsilon_{n-1} \geq \epsilon_n$ and the sequence $\{e_n\}$ is decreasing in the sense of $\|\|_{W^2[a,b]}$. On the other hand, since $\|\sum_{i=1}^\infty \langle F_r(\tau), \xi_i(\tau) \rangle_{W^2} \xi_i(\tau)\|_{W^2[a,b]}$ is convergent, then $\epsilon_n^2 = \sum_{i=n+1}^\infty \langle F_r(\tau), \xi_i(\tau) \rangle_{W^2[a,b]}^2 \to 0$ as soon as $n \to \infty$.
5. Numerical Results

In this section, the RKHS method is applied for solving the linear fuzzy integral equation. For the purpose of comparison, the method is compared systematically with the methods used in [8, 24, 33]. The results are very encouraging and are found outperforms in terms of accuracy and efficiency. In the process of computation, all the symbolic and numerical computations are performed by using MAPLE 13 software package.

Algorithm 5.1 To approximate the solution \( F^n(\tau) \) of Eq. (2), we do the following main steps:

**Input:** The dependent interval \([a, b]\), the unit truth interval \([0, 1]\), the integers \(n, m\), the kernel functions \( Q_1^1(\tau), Q_2^1(\tau) \), the linear operator \( L \), the crisp kernel functions \( h(\tau, \rho), k(\tau, \rho) \), and the nonhomogeneous term \( g(\tau) \).

**Output:** The RKHS solution \( F^n(\tau) \) of \( F(\tau) \) for Eq. (2).

**Step 1:** Set \( F(\tau) = \begin{bmatrix} f_1(\tau) \\ f_2(\tau) \end{bmatrix} \) and \( F^n(\tau) = \begin{bmatrix} f_1^n(\tau) \\ f_2^n(\tau) \end{bmatrix} \).

**Step 2:** Fixed \( \tau \ in [a, b] \) and set \( s \in [a, b] \);
- If \( s \leq \tau \), then let \( Q_2^1(\tau) = 1 + (\tau - a)(s - a) - \frac{1}{6}(\tau - a)^2(2a + \tau - 3s) \);
- Else let \( Q_2^1(\tau) = 1 + (\tau - a)(s - a) - \frac{1}{6}(s - a)^2(2a + s - 3\tau) \);
- For \( i = 1, 2, ..., n \) and \( j = 1, 2 \), do the following:

  - Set \( \tau_i = \frac{i - 1}{n - 1} \);
  - Set \( \xi_i = \frac{h - 1}{m - 1} \);
  - Set \( \psi_{ij}(\tau_i) = \begin{bmatrix} \int Q_1^1(s)Q_2^1(s)|_{s=\tau_i} \\ \int Q_2^1(s)Q_2^1(s)|_{s=\tau_i} \end{bmatrix} \);

**Step 3:** For \( l = 2, 3, ..., n \) and \( k = 1, 2, ..., l \), do the following:

  - Set \( \eta_l = \frac{\|\xi_l\|^2_{W^2} - \sum_{p=1}^{l-1} c_p^2}{\sum_{p=1}^{l-1} c_p^2} \);
  - Set \( c_{\xi_l} = \langle \xi_l, \xi_l \rangle_{W^2} \);
  - If \( k \neq l \), then set \( \beta_{lk} = -\frac{1}{\eta_l} \sum_{p=k}^{l-1} c_p \beta_{pk} \);
  - Else set \( \beta_{ll} = \eta_l^{-1} \);
  - Else set \( \beta_{11} = \||\xi_1||_{W^2}^{-1} \).

**Step 4:** For \( l = 2, 3, ..., n \) and \( k = 1, 2, ..., l \), do the following:

  - Set \( \eta_l = \frac{\|\xi_l\|^2_{W^2} - \sum_{p=1}^{l-1} c_p^2}{\sum_{p=1}^{l-1} c_p^2} \);
  - Set \( c_{\xi_l} = \langle \xi_l, \xi_l \rangle_{W^2} \);
  - If \( k \neq l \), then set \( \beta_{lk} = -\frac{1}{\eta_l} \sum_{p=k}^{l-1} c_p \beta_{pk} \);
  - Else set \( \beta_{ll} = \eta_l^{-1} \);
  - Else set \( \beta_{11} = \||\xi_1||_{W^2}^{-1} \).

**Output:** The complete orthogonal system \( \psi_{ij}(\tau_i) \).

Output: The orthonormal function system \( \xi_l(\tau_i) \).
Step 5: Set $F_n^0(\tau_1) = 0$; For $i = 1, 2, ..., n$, do the following:

Set $a_i = \left\{ \begin{array}{ll}
f_1(x_i), & \gamma_0 = \frac{k+1}{2}, k \text{ is odd} \\
f_2(x_i), & \gamma_0 = \frac{k-1}{2}, k \text{ is even} \\
\end{array} \right.$

Set $F_n^a(\tau_i) = \sum_{i=1}^{n} \beta_k a_k \xi_i(\tau_i)$;

Output: the RKHS solution $F_n^a(\tau)$ of $F(\tau)$ for Eq. (2).

Example 5.1. ([8]) Consider the following linear fuzzy integral equation of Volterra type:

$$F(\tau) = (\gamma + 1)\tau + \tau^2 \varphi(\tau) + \int_{0}^{\tau} \tau^2 (1 - 2\rho) F(\rho) d\rho, 0 \leq \rho < \tau \leq 1,$$

where $\gamma \in \mathbb{R}$, $\varphi(\tau)$ is continuous fuzzy-valued function such that

$$[\varphi(\tau)]_r = \left[ -\frac{2}{5} r^3 + \frac{4}{3} \tau^3 + \frac{1}{2} 2r^2 - \tau^2 + \frac{1}{12} \right] \cdot r \in (0, 1)$$

The corresponding exact solution is $F(\tau) = (\gamma + 1)\tau$, where $\gamma(s) = \max(0, 1 - |s|)$.

If we set $r = 1 - |s|$, that is, $s = r - 1$ or $s = 1 - r$, then $[\gamma]_r = [r - 1, 1 - r]$. The crisp kernel function $k(\tau, \rho) = \tau^2 (1 - 2\rho)$ is nonnegative on $0 \leq \rho \leq \frac{1}{2}$, and nonpositive on $\frac{1}{2} \leq \rho \leq 1$. Anyhow, at the full truth degree, $r = 1$, the model (24) will be transformed into crisp integral equation

$$F(\tau) = \frac{2}{3} \tau^5 - \frac{1}{2} \tau^4 + \tau + \int_{0}^{\tau} \tau^2 (1 - 2\rho) F(\rho) d\rho$$

which has the exact solution $F(\tau) = \tau$. Consequently, the corresponding parametric form of Eq. (24) can be written by the following system of crisp integral equations

$$F(\tau) = rt + \tau^2 \left( -\frac{2}{3} r^3 + \frac{4}{3} \tau^3 + \frac{1}{2} 2r^2 - \tau^2 + \frac{1}{12} \right) \cdot r \in (0, 1)$$

Using RKHS method, taking $\tau_i = \frac{i}{n-1}, i = 1, 2, ..., n$ and $\tau_h = \frac{h}{m-1}, h = 1, 2, ..., m$.

Our next goal is to illustrate numerical results of the RKHS algorithm. Table 1 present the approximate solutions of problem (24) for $\tau = 1$ and different values of $r$. From this table, one can see that the results obtained at $\tau = 1$ satisfy the convex symmetric triangular fuzzy number due to fuzzy coefficients appeared in problem (24), as well as the branch solutions $f_1, f_2$ satisfy the inequality $f_1(1) < f_2(1)$ for $r \in (0, 1)$ and equality hold for $r = 1$. Anyhow, a homotopy perturbation HP [13] and analysis HA [7] methods are applied to problem (24). Our results are better than the results obtained by these methods. Table 2 shows a numerical comparison between the errors for approximating $[\varphi(\tau)]_r = [f_1(\tau), f_2(\tau)]$ with the HP method [40] and the HA method [41] in the sense of the metrix $d_{\infty}(F(\tau), F^*(\tau))$. For the HA and HP solutions, one can conclude that, the accuracy of certain node is inversely proportional to its distance from the endpoint of $[0, 1]$, while on the other hand, a steady state accuracy is notable in the RKHS solution at all nodes of $[0, 1]$. 

approximating RKHS solution will be applicable in the whole solution domain. Next, the absolute errors of numerically if we set $r$

Example 5.2. ([33]) Consider the following linear fuzzy integral equation of Fredholm-Volterra type:

$$\mathcal{F}(\tau) = -\frac{1}{3} \tau^3 + \frac{2}{3} \tau^2 - \frac{1}{3} \tau - \frac{1}{4} + \gamma - 4\gamma \tau + \int_0^1 (\rho + \tau)\mathcal{F}(\rho)d\rho + \int_0^\tau \mathcal{F}(\rho)d\rho, 0 \leq \rho < \tau \leq 1, \quad (27)$$

The corresponding exact solution is $\mathcal{F}(\tau) = \tau^2 + 2\gamma$, where $\gamma(s) = \begin{cases} 10s - 9, & 0.9 \leq s \leq 1, \\ 5 - 4s, & 1 \leq s \leq 1.25, \\ 0, & \text{otherwise}. \end{cases}$ Here, if we set $r = 10s - 9$, that is, $s = 0.9 + 0.1r$ and if we set $r = 5 - 4s$, that is, $s = 1.25 - 0.25r$, then $[\gamma]' = [0.9 + 0.1r, 1.25 - 0.25r]$. For the conduct of proceedings in the RKHS solution, we have the following system of crisp integral equations

$$f_1(t) = -\frac{1}{3} \tau^3 + \frac{2}{3} \tau^2 - \frac{1}{3} \tau - \frac{1}{4} + (0.9 + 0.1r)\tau - 4(1.25 - 0.25r) + \int_0^1 (\rho + \tau)f_1(\rho)d\rho + \int_0^\tau f_1(\rho)d\rho,$$

$$f_2(t) = -\frac{1}{3} \tau^3 + \frac{2}{3} \tau^2 - \frac{1}{3} \tau - \frac{1}{4} + (1.25 - 0.25r)\tau - 4(0.9 + 0.1r) + \int_0^1 (\rho + \tau)f_2(\rho)d\rho + \int_0^\tau f_2(\rho)d\rho. \quad (28)$$

As we mentioned earlier, it is possible to pick any point in the independent interval of $t$ and as well the RKHS solution will be applicable in the whole solution domain. Next, the absolute errors of numerically approximating $[\mathcal{F}(\tau)]'$ for FIE (27) have been calculated for various $\tau$ and $r$ as shown in Tables 3 and 4. In fact, results obtained by considering the corresponding crisp system of Eq. (28).
Finally, the approximate solution set \([\mathcal{F}(\tau)]\) of Eq. (27) has been plotted in Figure 1.a at various values of \(\tau\) and \(r\). As the plots show, while the value of \(r\) increases on \([0,1]\), the set \([\mathcal{F}(\tau)]\) decreasing for each \(\tau\); that is, the \(r\)-cut representation form of the RKHS solution is valid level sets on the specific domain. Whilst on the other aspect as well, the geometric behavior of the support and the core of the RKHS solution is studied in Figure 1.b. Whereas the set of elements in \(\mathbb{R}\) that have one membership in \(\mathcal{F}(\tau)\) is called the core, denoted by \((\mathcal{F}(\tau)) = \{s \in \mathbb{R} : \mathcal{F}(s) = 1\}\), whilst the set of elements in \(\mathbb{R}\) that have membership in between \([0,1]\) is called the support, denoted by \(\text{support}(\mathcal{F}(\tau)) = \{s \in \mathbb{R} : \mathcal{F}(s) > 0\}\). Indeed, these sets associated with 1-cut and 0-cut representation form of \([\mathcal{F}(\tau)]\), respectively, that is, \(\text{core}(\mathcal{F}(\tau)) = [\mathcal{F}(\tau)]^1\) and \(\text{support}(\mathcal{F}(\tau)) = \{\min[\mathcal{F}(\tau)], \max[\mathcal{F}(\tau)]\}\). The scenario of Figure 1.b is to plot the core and the support of the RKHS solution for various \(\tau\).
6. Conclusion

In this paper, we studied a numerical algorithm to acquire solution for fractional fuzzy integral equation of Fredholm-Volterra type based on the reproducing kernel method. The analytic and numeric solutions are represented in series form in term of their parametric form in the space $W^2_2[\alpha, b]$. The convergence analysis and error estimation is presented to capture the behavior of a solution for the fuzzy-valued mappings. Numerical results, compared with other methods, indicate that the proposed method is of higher precision and is easy to implement to get these results.

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