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Reeb Lie derivatives on real hypersurfaces in complex hyperbolic two-plane Grassmannians

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Abstract. In complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2}) = SU_{2+m}/S(U_2{\cdot}U_m)$, it is known that a real hypersurface satisfying the condition $(\hat{\mathcal{L}}^{(k)}_{\xi} R_{\xi})Y = (\mathcal{L}_{\xi} R_{\xi})Y$ is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. In this paper, as an abient space, we consider a complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$ and give a complete classification of Hopf real hypersurfaces in $SU_{2,m}/S(U_2{\cdot}U_m)$ with the above condition.

1. Introduction

For a real hypersurface with parallel symmetric tensor, many differential geometers studied in complex projective spaces or in quaternionic projective spaces([6, 11, 12]), which are Hermitian symmetric spaces of rank 1. By means of Hopf hypersurfaces, Kimura([7]) asserted that there do not exist any real hypersurfaces with parallel Ricci tensor, that is $\nabla S = 0$, where S denotes the Ricci tensor of a Hopf hypersurface M in complex projective spaces.

From a different point of view, it is interesting to consider a Hermitian symmetric space of rank 2 with certain conditions. For instance, there are some results of parallel structure Jacobi operator(for more detail, see [4, 5]). A complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ is a typical example of a symmetric space of compact type.

When we think about the Reeb vector field ξ in the expression of the curvature tensor R for a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, the structure Jacobi operator R_{ξ} can be defined as

$$R_{\xi}(X) = R(X, \xi)\xi,$$

for any tangent vector field X on M.

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Instead of the Levi-Civita connection for real hypersurfaces in Kähler manifolds, let us consider another new connection named *generalized Tanaka-Webster connection* (in short, say the *GTW connection*) $\hat{\nabla}^{(k)}$ for a non-zero real number k ([8]). This connection $\hat{\nabla}^{(k)}$ can be regarded as a natural extension of Tanno's generalized Tanaka-Webster connection $\hat{\nabla}$ for contact metric manifolds. In fact, Tanno([17]) introduced the generalized Tanaka-Webster connection $\hat{\nabla}$ for contact Riemannian manifolds using the canonical connection on a nondegenerate, integrable CR manifold.

On the other hand, the original *Tanaka-Webster connection*([16, 18]) is given as a unique affine connection on a non-degenerate, pseudo-Hermitian *CR* manifold associated with the almost contact structure. In particular, if a real hypersurface in a Kähler manifold satisfies $\phi A + A\phi = 2k\phi$ $(k \neq 0)$, then the GTW connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection.

Related to GTW connection, according to Jeong, Pak and Suh([2, 3]), the GTW Lie derivative was defined by

$$\hat{\mathcal{L}}^{(k)}_X Y = \hat{\nabla}^{(k)}_X Y - \hat{\nabla}^{(k)}_Y X,$$

where $\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$, $k \in \mathbb{R} \setminus \{0\}$.

As a previous result, using the GTW Lie derivative and the Lie derivative, we want to consider a condition that the *GTW Reeb Lie derivative of the structure Jacobi operator coincides with the Reeb Lie derivative*, that is,

$$(\hat{\mathcal{L}}^{(k)}_\xi R_\xi)Y = (\mathcal{L}_\xi R_\xi)Y,$$

for any tangent vector field Y on a real hypersurface of $G_2(\mathbb{C}^{m+2})$. Using the above condition, Pak, Kim and Suh([10]) proved following :

Theorem 1.1. *Let M be a connected orientable Hopf hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the GTW Reeb Lie derivative of the structure Jacobi operator coincides with the Reeb Lie derivative and the Reeb curvature is non-vanishing constant along the Reeb vector field, then M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

As an ambient space, the complex hyperbolic two-plane Grassmannian consists of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}_1$. This is a Riemannian symmetric space of noncompact. Especially, it is a irreducible Riemannian manifold which has both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$. Then, naturally we could consider two geometric conditions for a hypersurface M in a complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$, namely, that the 1-dimensional distribution $[\xi] = \mathrm{Span}\{\xi\}$ and the 3-dimensional distribution $\mathcal{Q}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ are both invariant under the shape operator A of M([1]).

Using above geometric conditions, Berndt and Suh([1]) gave a classification theorem as follows :

Theorem 1.2. *Let M be a connected hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 2$. Then the maximal complex subbundle $\mathcal{C}$ of TM and the maximal quaternionic subbundle $\mathcal{Q}$ of TM are both invariant under the shape operator of M if and only if M is congruent to an open part of one of the following hypersurfaces :*

(A) *a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$;*

(B) *a tube around a totally geodesic $\mathbb{H}H^n$ in $SU_{2,2n}/S(U_2U_{2n})$, $m = 2n$;*

(C) *a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular ;*

or the following exceptional case holds :

(D) *The normal bundle νM of M consists of singular tangent vectors of type $JX \perp \mathfrak{J}X$. Moreover, M has at least four distinct principal curvatures, three of which are given by*

$$\alpha = \sqrt{2}\,,\ \gamma = 0\,,\ \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus (\mathcal{C} \cap \mathcal{Q})\ ,\ T_\gamma = J(TM \ominus \mathcal{Q})\ ,\ T_\lambda \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

If μ is another (possibly nonconstant) principal curvature function, then we have $T_\mu \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}$, $JT_\mu \subset T_\lambda$ and $\mathfrak{J}T_\mu \subset T_\lambda$.

Now in this paper let us consider a real hypersurface M in $SU_{2,m}/S(U_2{\cdot}U_m)$ with $(\hat{\mathcal{L}}^{(k)}_\xi R_\xi)Y = (\mathcal{L}_\xi R_\xi)Y$ for any vector field Y on M, that is, the *GTW* Reeb Lie derivative of the structure Jacobi operator coincides with the Reeb Lie derivative. Then by virtue of Theorem 1.2, we can assert the following :

Theorem 1.3. *Let M be a Hopf hypersurface in a complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$. If the GTW Reeb Lie derivative of the structure Jacobi operator coincides with the Reeb Lie derivative and the Reeb curvature is non-vanishing constant along the Reeb vector field, then M is locally congruent to an open part of a tube around some totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ or a horosphere whose center at infinity with $JX{\in}\mathfrak{J}X$ is singular.*

In section 2, we will give some basic formulas which will be widely used in $SU_{2,m}/S(U_2\ {\cdot}U_m)$ given by [15]. Related to the structure Jacobi operator in section 3, we will introduce basic equations and prove the two key lemmas which will be used in the proof of our Theorem 1.3.

By virtue of these two lemmas in section 3, we can consider two cases, that is, the Reeb vector field ξ either belongs to $\mathcal{Q}$ or $\mathcal{Q}^\perp$. We will treat each case in sections 4 and 5 respectively, which give a complete proof of our Theorem 1.3.

2. Basic Equations

Let M be a real hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$, that is, a submanifold in $SU_{2,m}/S(U_2\ {\cdot}U_m)$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Levi Civita covariant derivative of (M, g). We denote by $\mathcal{C}$ and $\mathcal{Q}$ the maximal complex and quaternionic subbundle of the tangent bundle TM of M, respectively. Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N$$

for any tangent vector field X of a real hypersurface M in $SU_{2,m}/S(U_2{\cdot}U_m)$, where ϕX denotes the tangential component of JX and N a unit normal vector field of M in $SU_{2,m}/S(U_2{\cdot}U_m)$.

From the Kähler structure J of $SU_{2,m}/S(U_2{\cdot}U_m)$ there exists an almost contact metric structure (ϕ, ξ, η, g) induced on M in such a way that

$$\phi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \phi\xi = 0, \quad \text{and} \quad \eta(X) = g(X, \xi)$$

for any vector field X on M and $\xi = -JN$.

If M is orientable, then the vector field ξ is globally defined and said to be the induced *Reeb vector field* on M. Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces a local almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$, on M. Locally, $\mathcal{C}$ is the orthogonal complement in TM of the real span of ξ, and $\mathcal{Q}$ the orthogonal complement in TM of the real span of $\{\xi_1, \xi_2, \xi_3\}$.

Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then the quaternionic Kähler structure J_ν of $SU_{2,m}/S(U_2{\cdot}U_m)$, together with the condition

$$J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$$

in section 1, induces an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M as follows :

$$\begin{aligned}
&\phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu,\ \phi_\nu\xi_\nu = 0,\ \eta_\nu(\xi_\nu) = 1\\
&\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2},\\
&\phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu,\\
&\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}
\end{aligned}$$

for any vector field X tangent to M. The tangential and normal component of the commuting identity $JJ_\nu X = J_\nu JX$ give

$$\phi\phi_\nu X - \phi_\nu\phi X = \eta_\nu(X)\xi - \eta(X)\xi_\nu \quad \text{and} \quad \eta_\nu(\phi X) = \eta(\phi_\nu X). \tag{1}$$

The last equation implies $\phi_\nu\xi = \phi\xi_\nu$. The tangential and normal component of $J_\nu J_{\nu+1}X = J_{\nu+2}X = -J_{\nu+1}J_\nu X$ give

$$\phi_\nu\phi_{\nu+1}X - \eta_{\nu+1}(X)\xi_\nu = \phi_{\nu+2}X = -\phi_{\nu+1}\phi_\nu X + \eta_\nu(X)\xi_{\nu+1}$$

and

$$\eta_\nu(\phi_{\nu+1}X) = \eta_{\nu+2}(X) = -\eta_{\nu+1}(\phi_\nu X).$$

Putting $X = \xi_\nu$ and $X = \xi_{\nu+1}$ into the first one of these two equations yields $\phi_{\nu+2}\xi_\nu = \xi_{\nu+1}$ and $\phi_{\nu+2}\xi_{\nu+1} = -\xi_\nu$, respectively. Using the Gauss and Weingarten formulas, the tangential and normal component of the Kähler condition $(\bar\nabla_X J)Y = 0$ give $(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi$ and $(\nabla_X\eta)Y = g(\phi AX, Y)$. The last equation implies $\nabla_X\xi = \phi AX$. By the expression of the curvature tensor (see [1]), we have the equation of Gauss([15]) as follows:

$$\begin{aligned}
R(X, Y)Z = -\frac{1}{2}\Big[& g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\
& + \sum_{\nu=1}^{3}\Big\{g(\phi_\nu Y, Z)\phi_\nu X - g(\phi_\nu X, Z)\phi_\nu Y - 2g(\phi_\nu X, Y)\phi_\nu Z\Big\} + \sum_{\nu=1}^{3}\Big\{g(\phi_\nu\phi Y, Z)\phi_\nu\phi X - g(\phi_\nu\phi X, Z)\phi_\nu\phi Y\Big\} \\
& - \sum_{\nu=1}^{3}\Big\{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X - \eta(X)\eta_\nu(Z)\phi_\nu\phi Y\Big\} - \sum_{\nu=1}^{3}\Big\{\eta(X)g(\phi_\nu\phi Y, Z) - \eta(Y)g(\phi_\nu\phi X, Z)\Big\}\xi_\nu\Big] \\
& + g(AY, Z)AX - g(AX, Z)AY
\end{aligned}$$

for any vector fields X, Y, and Z on M. From now on, unless otherwise stated, we will use these basic equations as stated above frequently without mention it.

3. Key Lemmas

In this section, together with some conditions, we give some important lemmas which will be used in the proof of our Theorem 1.3.

First, using the structure Jacobi operator given in the introduction, structure Jacobi operator is given by

$$\begin{aligned}
R_\xi X = -\frac{1}{2}\Big[& X - \eta(X)\xi - \sum_{\nu=1}^{3}\Big\{\eta_\nu(X)\xi_\nu - \eta(X)\eta_\nu(\xi)\xi_\nu + 3g(\phi_\nu X, \xi)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi X\Big\}\Big] \\
& + \alpha AX - \alpha^2\eta(X)\xi,
\end{aligned} \tag{2}$$

for any tangent field X on M.

In [2], they defined the GTW Lie derivative as follows:

$$\hat{\mathcal{L}}^{(k)}_X Y = \hat\nabla^{(k)}_X Y - \hat\nabla^{(k)}_Y X,$$

where $\hat\nabla^{(k)}_X Y = \nabla_X Y + F^{(k)}_X Y$, $F^{(k)}_X Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$. The operator $F^{(k)}_X Y$ said to be the *generalized Tanaka-Webster operator* (in short, GTW operator). Putting $X = \xi$ and $Y = \xi$, the GTW operator is written as

$$F^{(k)}_\xi Y = -k\phi Y \ \text{ and } \ F^{(k)}_X \xi = -\phi AX, \ \text{respectively.} \tag{3}$$

For an (1-1) type tensor R_ξ, this condition $(\hat{\mathcal{L}}^{(k)}_X R_\xi)Y = (\mathcal{L}_X R_\xi)Y$ is equivalent to

$$F^{(k)}_X(R_\xi Y) - F^{(k)}_{R_\xi Y}X - R_\xi F^{(k)}_X Y + R_\xi F^{(k)}_Y X = 0. \tag{4}$$

Putting $X = \xi$ in (4), we get

$$-k\phi R_\xi Y + \phi A R_\xi Y + kR_\xi \phi Y - R_\xi \phi A Y = 0. \tag{5}$$

Since R_ξ is a symmetric tensor field, taking the transpose part of (5), we have

$$kR_\xi \phi Y - R_\xi A\phi Y - k\phi R_\xi Y + A\phi R_\xi Y = 0. \tag{6}$$

Subtracting (6) from (5), we obtain

$$(\phi A - A\phi)R_\xi Y = R_\xi(\phi A - A\phi)Y. \tag{7}$$

Therefore, this condition that the GTW Reeb Lie derivative of the structure Jacobi operator coincides with the Reeb Lie derivative has a geometric meaning such that the operator $(\phi A - A\phi)$ commutes with the structure Jacobi operator R_ξ.

Putting $Y = \xi$ in (4) and using (3), it is replaced by

$$R_\xi(\phi AX) - kR_\xi(\phi X) = 0. \tag{8}$$

By taking the symmetric part on (8), we get

$$-A\phi R_\xi X + k\phi R_\xi X = 0.$$

By using these equations, we can give two lemmas as follows:

Lemma 3.1. *Let M be a Hopf hypersurface M in $SU_{2,m}/S(U_2{\cdot}U_m)$. If the GTW Reeb Lie derivative of the structure Jacobi operator coincides with the Reeb Lie derivative of this operator and the principal curvature α is constant along the direction of the Reeb vector field ξ, then the Reeb vector field ξ belongs to the distribution $\mathcal{Q}$ or the distribution $\mathcal{Q}^\perp$*

Proof. Suppose $\xi = \eta(X_0)X_0 + \eta_1(\xi_1)\xi_1$, for some unit vector fields $X_0 \in \mathcal{Q}$ and $\xi_1 \in \mathcal{Q}^\perp$.
If $\alpha = 0$, then $\xi \in \mathcal{Q}$ or $\xi \in \mathcal{Q}^\perp$, which is proved by the equation([13]) $\operatorname{grad}\alpha = (\xi\alpha)\xi - 2\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi\xi_\nu$.

Now let us consider the other case $\alpha \neq 0$.
Putting $X = \xi_1$ into (2) and using $A\xi_1 = \alpha\xi_1$, we have

$$R_\xi(\xi_1) = \alpha^2\xi_1 - \alpha^2\eta(\xi_1)\xi. \tag{9}$$

Replacing $X = \phi\xi_1$ into (2), (2) becomes

$$R_\xi(\phi\xi_1) = (\alpha^2 - 4\eta^2(X_0))\phi_1\xi. \tag{10}$$

Putting $X = \xi$ into (4) and using (3), (2.6) is written as

$$-k\phi R_\xi Y + \phi A R_\xi Y + kR_\xi(\phi Y) - R_\xi(\phi A Y) = 0.$$

Substituting of $Y = \xi_1$ in the above equation and using (9), (10), it becomes

$$4(\alpha - k)\eta^2(X_0)\phi_1\xi = 0. \tag{11}$$

Taking the inner product of (11) with $\phi_1\xi$, we get

$$4(\alpha - k)\eta^4(X_0) = 0. \tag{12}$$

This equation induces that $k = \alpha$ or $\eta^4(X_0) = 0$. Therefore, it completes the proof of our Lemma. □

In next section, we will give a complete proof of our Theorem 1.3. In order to do this, first we consider the case that $\xi \in \mathcal{Q}^{\perp}$. Without loss of generosity, we may put $\xi = \xi_1$.

Lemma 3.2. *Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$ with non-vanishing Reeb curvature. If the GTW Reeb Lie derivative of the structure Jacobi operator coincides with the Reeb Lie derivative of this operator and the Reeb vector field ξ is belong to the distribution $\mathcal{Q}^{\perp}$, then the shape operator A commutes with the structure tensor ϕ.*

Proof. Putting $\xi = \xi_1$ in (2), we get

$$R_\xi X = -\frac{1}{2}X + \frac{1}{2}\eta(X)\xi + \frac{1}{2}\phi_1\phi X + \alpha AX - \alpha^2\eta(X)\xi - \eta_2(X)\xi_2 - \eta_3(X)\xi_3. \tag{13}$$

Let us replace X with AX in (13), it is written as

$$R_\xi AX = -\frac{1}{2}AX + \frac{1}{2}\alpha\eta(X)\xi + \frac{1}{2}\phi_1\phi AX + \alpha A^2X - \alpha^3\eta(X)\xi - \eta_2(AX)\xi_2 - \eta_3(AX)\xi_3. \tag{14}$$

And applying the shape operator A on (13), it follows

$$AR_\xi X = -\frac{1}{2}AX + \frac{1}{2}\alpha\eta(X)\xi + \frac{1}{2}A\phi_1\phi X + \alpha A^2X - \alpha^3\eta(X)\xi - \eta_2(X)A\xi_2 - \eta_3(X)A\xi_3. \tag{15}$$

On the other hand, applying the structure tensor field ϕ to the equation (1.8) in [9], we get

$$AX = \alpha\eta(X)\xi + 2\eta_2(AX)\xi_2 + 2\eta_3(AX)\xi_3 - \phi\phi_1 AX. \tag{16}$$

Taking the symmetric part of (16), we obtain

$$AX = \alpha\eta(X)\xi + 2\eta_2(X)A\xi_2 + 2\eta_3(X)A\xi_3 - A\phi_1\phi X. \tag{17}$$

Putting $\nu = 1$ in the equation (1), it becomes

$$\phi\phi_1 X = \phi_1\phi X. \tag{18}$$

Subtracting (15) from (14), together with (16) and (17), we have

$$R_\xi AX = AR_\xi X. \tag{19}$$

From (19) and putting $Y = X$, (2.6) is written as

$$A(R_\xi\phi - \phi R_\xi)X = (R_\xi\phi - \phi R_\xi)AX. \tag{20}$$

By inserting $X = \phi X$ in (13), we have

$$R_\xi\phi X = -\frac{1}{2}\phi X + \frac{1}{2}\phi_1\phi^2 X + \alpha A\phi X - \eta_2(\phi X)\xi_2 - \eta_3(\phi X)\xi_3. \tag{21}$$

When we apply the structure tensor field ϕ to (13), we get

$$\phi R_\xi X = -\frac{1}{2}\phi X + \frac{1}{2}\phi\phi_1\phi X + \alpha\phi AX + \eta_2(X)\xi_3 - \eta_3(X)\xi_2. \tag{22}$$

Subtracting (22) from (21), we obtain

$$(R_\xi\phi - \phi R_\xi)X = \alpha(A\phi - \phi A)X. \tag{23}$$

From this, using the equation (23), the equivalent condition of (20) is this one as

$$\alpha A(A\phi - \phi A)X = \alpha(A\phi - \phi A)AX. \tag{24}$$

By our assumption $\alpha \neq 0$, the above equation can be replaced by

$$A(A\phi - \phi A)X = (A\phi - \phi A)AX. \tag{25}$$

Note that two tensors $A\phi - \phi A$ and A are symmetric each other, where the structure tensor ϕ is skew-symmetric. Because of (25), using the method of simultaneously diagonalization, there is a common basis $\{e_i \mid i = 1, ..., 4m-1\}$ for these tensors such that

$$Ae_i = \lambda_i e_i \tag{26}$$

and

$$(A\phi - \phi A)e_i = \gamma_i e_i. \tag{27}$$

Consequently, using (26), (27) becomes

$$\gamma_i e_i = A\phi e_i - \phi A e_i = A\phi e_i - \lambda_i \phi e_i. \tag{28}$$

Taking the inner product with e_i, we get $\gamma_i = 0$.
Since the eigenvalue γ_i vanishes for all i, from (27) we conclude that

$$A\phi - \phi A = 0. \tag{29}$$

Therefore, we proved this lemma. □

4. The case: $\xi \in \mathcal{Q}^\perp$

Let us consider a Hopf hypersurface M in $SU_{2,m}/S(U_2{\cdot}U_m)$ with $(\hat{\mathcal{L}}^{(k)}_\xi R_\xi)Y = (\mathcal{L}_\xi R_\xi)Y$.
By Lemma 3.1 in the section 3, we can conclude that the Reeb vector field ξ in M belongs either to the distribution $\mathcal{Q}$ or $\mathcal{Q}^\perp$.
Now, we check the first case $\xi \in \mathcal{Q}^\perp$ in our consideration.

Theorem 1.2 and Lemma 3.2 assert that if $\xi \in \mathcal{Q}^\perp$, then M is locally congruent to the model space of $\mathcal{T}_A$ or $\mathcal{H}_A$. We have to check if the model spaces of $\mathcal{T}_A$ and $\mathcal{H}_A$ satisfy the condition $(\hat{\mathcal{L}}^{(k)}_\xi R_\xi)Y = (\mathcal{L}_\xi R_\xi)Y$ or not respectively.

Proposition 4.1. *Let M be a connected real hypersurface in a complex hyperbolic two-plane Grassamnnian $SU_{2,m}/S(U_2U_m)$, $m \geq 3$. Assume that the maximal complex subbundle $\mathcal{C}$ of TM and the maximal quaternionic subbundle $\mathcal{Q}$ of TM are both invariant under the shape operator of M. If $JN \in \mathfrak{J}N$, then one of the following statements holds :*

($\mathcal{T}_A$) *M has exactly four distinct constant principal curvatures*

$$\alpha = 2\coth(2r),\ \beta = \coth(r),\ \lambda_1 = \tanh(r),\ \lambda_2 = 0,$$

and the corresponding principal curvature spaces are

$$T_\alpha = TM \ominus \mathcal{C},\ T_\beta = \mathcal{C} \ominus \mathcal{Q},\ T_{\lambda_1} = E_{-1},\ T_{\lambda_2} = E_{+1}.$$

The principal curvature spaces T_{λ_1} and T_{λ_2} are complex (with respect to J) and totally complex (with respect to $\mathfrak{J}$).

($\mathcal{H}_A$) *M has exactly three distinct constant principal curvatures*

$$\alpha = 2,\ \beta = 1,\ \lambda = 0$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus \mathcal{C},\ T_\beta = (\mathcal{C} \ominus \mathcal{Q}) \oplus E_{-1},\ T_\lambda = E_{+1}.$$

Here, E_{+1} and E_{-1} are the eigenbundles of $\phi\phi_1|_{\mathcal{Q}}$ with respect to the eigenvaleus $+1$ and -1, respectively.

First, we check the $\mathcal{T}_A$-case by using information of Proposition 4.1. Putting $X = \xi$ in (4), we get the equivalent condition of $(\hat{\mathcal{L}}^{(k)}_\xi R_\xi)Y = (\mathcal{L}_\xi R_\xi)Y$ as follows:

$$-k\phi R_\xi Y + \phi AR_\xi Y + kR_\xi\phi Y - R_\xi\phi AY = 0. \tag{30}$$

On the other hand, since ξ in $\mathcal{Q}^\perp$, putting $\xi = \xi_1$ into (2), we get

$$R_\xi X = -\frac{1}{2}X + \frac{1}{2}\eta(X)\xi + \frac{1}{2}\phi_1\phi X - \eta_2(X)\xi_2 - \eta_3(X)\xi_3 + \alpha AX - \alpha^2\eta(X)\xi. \tag{31}$$

Putting $X = \phi X$ into (31), we get

$$R_\xi\phi X = -\frac{1}{2}\phi X - \eta_2(\phi X)\xi_2 - \eta_3(\phi X)\xi_3 - \frac{1}{2}\phi_1 X + \frac{1}{2}\eta(X)\phi_1\xi + \alpha A\phi X. \tag{32}$$

So we calculate the structure Jacobi operator for all eigenspaces in $\mathcal{T}_A$:

$$R_\xi X = \begin{cases} 0, & \text{if } X \in T_\alpha \\ (\alpha\beta - 1)\xi_\nu, & \text{if } X = \xi_\nu \in T_\beta \\ \alpha\lambda_1 X - X, & \text{if } X \in T_{\lambda_1} \\ \alpha\lambda_2 X, & \text{if } X \in T_{\lambda_2}. \end{cases} \tag{33}$$

and

$$R_\xi\phi X = \begin{cases} 0, & \text{if } X \in T_\alpha \\ -(\alpha\beta - 1)\xi_3, & \text{if } X = \xi_2 \in T_\beta \\ (\alpha\beta - 1)\xi_2, & \text{if } X = \xi_3 \in T_\beta \\ -\phi X + \alpha\lambda_1\phi X, & \text{if } X \in T_{\lambda_1} \\ \alpha\lambda_2^2\phi X, & \text{if } X \in T_{\lambda_2}. \end{cases} \tag{34}$$

Using (30) and (31), we get the following result:

$$-k\phi(R_\xi Y) + \phi A(R_\xi Y) + R_\xi k\phi Y - R_\xi\phi AY = \begin{cases} 0, & \text{if } X \in T_\alpha \\ 0, & \text{if } X \in T_\beta \\ 0, & \text{if } X \in T_\lambda \\ 0, & \text{if } X \in T_\mu. \end{cases} \tag{35}$$

Similarly, when we consider the $\mathcal{H}_A$-case, the model space of $\mathcal{H}_A$ also satisfies the condition $(\hat{\mathcal{L}}^{(k)}_\xi R_\xi)Y = (\mathcal{L}_\xi R_\xi)Y$. Therefore, we can assert that if ξ in $\mathcal{Q}^\perp$, then M is locally congruent to an open part of a tube around some totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ or a horosphere whose center at infinity with $JX\in\mathfrak{J}X$ is singular. □

5. The case: $\xi \in \mathcal{Q}$

If the Reeb vector field $\xi \in \mathcal{Q}$, due to [14], we can assert that M is locally congruent to the model space of $\mathcal{T}_B$ or $\mathcal{H}_B$ or $\mathcal{E}$. It remains whether $\mathcal{T}_B$ or $\mathcal{H}_B$ or $\mathcal{E}$ satisfies this condition $(\hat{\mathcal{L}}^{(k)}_X R_\xi)Y = (\mathcal{L}_X R_\xi)Y$. Also, by using information of these spaces in Proposition 5.1, we can check this problem.

Proposition 5.1. *Let M be a connected hypersurface in $SU_{2,m}/S(U_2U_m)$, $m \geq 3$. Assume that the maximal complex subbundle $\mathcal{C}$ of TM and the maximal quaternionic subbundle $\mathcal{Q}$ of TM are both invariant under the shape operator of M. If $JN \perp \mathfrak{J}N$, then one of the following statements holds:*

($\mathcal{T}_B$) *M has five (four for* $r = \sqrt{2}\tanh^{-1}(1/\sqrt{3})$ *in which case* $\alpha = \lambda_2$*) distinct constant principal curvatures*

$$\alpha = \sqrt{2}\tanh(\sqrt{2}r),\ \beta = \sqrt{2}\coth(\sqrt{2}r),\ \gamma = 0,$$
$$\lambda_1 = \frac{1}{\sqrt{2}}\tanh(\frac{1}{\sqrt{2}}r),\ \lambda_2 = \frac{1}{\sqrt{2}}\coth(\frac{1}{\sqrt{2}}r),$$

and the corresponding principal curvature spaces are

$$T_\alpha = TM \ominus \mathcal{C},\ T_\beta = TM \ominus \mathcal{Q},\ T_\gamma = J(TM \ominus \mathcal{Q}) = JT_\beta.$$

The principal curvature spaces T_{λ_1} *and* T_{λ_2} *are invariant under* $\mathfrak{J}$ *and are mapped onto each other by J. In particular, the quaternionic dimension of* $SU_{2,m}/S(U_2U_m)$ *must be even.*

($\mathcal{H}_B$) *M has exactly three distinct constant principal curvatures*

$$\alpha = \beta = \sqrt{2},\ \gamma = 0,\ \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus (\mathcal{C} \cap \mathcal{Q}),\ T_\gamma = J(TM \ominus \mathcal{Q}),\ T_\lambda = \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

($\mathcal{E}$) *M has at least four distinct principal curvatures, three of which are given by*

$$\alpha = \beta = \sqrt{2},\ \gamma = 0,\ \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus (\mathcal{C} \cap \mathcal{Q}),\ T_\gamma = J(TM \ominus \mathcal{Q}),\ T_\lambda \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

If μ *is another (possibly nonconstant) principal curvature function, then* $JT_\mu \subset T_\lambda$ *and* $\mathfrak{J}T_\mu \subset T_\lambda$*. Thus, the corresponding multiplicities are*

$$m(\alpha) = 4, \quad m(\gamma) = 3, \quad m(\lambda), \quad m(\mu).$$

We suppose that these spaces $\mathcal{T}_B$, $\mathcal{H}_B$ and $\mathcal{E}$ satisfy $(\hat{\mathcal{L}}_\xi^{(k)} R_\xi)Y = (\mathcal{L}_\xi R_\xi)Y$. Then, as an equivalent condition, these spaces must satisfy

$$-k\phi(R_\xi Y) + \phi A(R_\xi Y) + R_\xi k\phi Y - R_\xi \phi AY = 0. \tag{36}$$

Since ξ is belong to $\mathcal{Q}$, the structure Jacobi operator in $SU_{2,m}/S(U_2{\cdot}U_m)$ can be replaced as follows :

$$R_\xi X = -\frac{1}{2}\Big[X - \eta(X)\xi - \sum_{\nu=1}^{3}\Big\{3g(\phi_\nu X, \xi)\phi_\nu\xi\Big\}\Big] + \alpha AX - \alpha^2\eta(X)\xi. \tag{37}$$

Applying $Y = \phi_1\xi$ into (36) and using (37), we get

$$-k(\frac{3}{2} - \alpha\beta)\xi_1 = 0. \tag{38}$$

But eigenvalues of each case in Proposition 5.1 do not satisfy above equation, which gives a contradiction. So we can assert that a real hypersurface cannot be a space of $\mathcal{T}_B$, $\mathcal{H}_B$ and $\mathcal{E}$.

Summing up these assertions, we have given a complete proof of our Theorem 1.3 in the introduction. □

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