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Existence, uniqueness and stability results for neutral stochastic differential equations with random impulses

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Abstract. This manuscript is devoted to the study of existence, uniqueness and stability results of neutral stochastic differential equations with random impulses. The existence results are obtained by considering sufficient conditions and applying fixed point theory to the aforementioned system. The stability results are proved via continuous dependence on initial value. Finally an example is provided to show the effectiveness of the obtained result.

1. Introduction

Mathematical modelling of real-life problems in engineering and scientific disciplines usually results in functional equations like differential, integral, integro-differential and stochastic equations. The deterministic system often experiences fluctuations due to environmental noise. Thus the Stochastic differential equations(SDE's) indeed gained its importance. SDE contains the stochastic term composed of noise, which is being an advantageous factor in describing uncertain factors of environmental noise in the real world. This principal factor makes its use in various fields of science and engineering. Due to its widespread applications it is more appropriate to move from deterministic models to stochastic ones. Stochastic differential Equations captures disturbances from random factors. A real world system can be made comprehensible by integrating its stochastic process into mathematical models. For the fundamental study of the theory of SDE's we infer to [8, 10, 20, 30]. To get a clear view of SDE's readers may refer to the monographs and the references therein [5, 6, 9, 11, 21, 22].

Recently, differential equation with fixed moments of impulses has become a natural framework in modelling several processes in the fields of economics, physics and population dynamics. Differential equations with instantaneous impulses are investigated by authors see [25, 26, 31]. The impulses in usual exists at deterministic or random points. The properties of fixed type impulses are established in many articles [1, 13, 17, 24]. Wu and Meng [29] were the first to consider a random impulsive ordinary differential system and established boundedness of solutions to the model by Liapunov's direct function.

Stochastic functional differential equations with impulse exist in many evolution processes. It has wide implementation in modelling systems in the fields of medicine and biology, mechanics, economics, telecommunication and electronics refer [7, 17, 19]. Impulse may appear at random points i.e.,impulse time t_k

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is a random variable for $k = 1, 2, ...$ and impulsive function $b_i(.)$ being a random variable. Stability analysis for Impulsive Stochastic Differential Systems (ISDS) and Impulsive Stochastic Functional Differential Systems (ISFDS) has attracted attention among researchers. Hu and Zhu [14] have studied the exponential stability of stochastic differential equations ith impulse effects at random effects using Lyaponov method. Also, Hu and Zhu have established stability analysis by considering impulsive stochastic differential systems using Lyapunov and Razhumikhin technique refer [15, 16]. Sakthivel and Luo [23] studied the existence and asymptotical stability for mild solutions with ISDS's. Li et al. [18] investigated existence and Hyers-Ulam stability of random impulsive stochastic functional differential equations with finite delays.

Moreover, Gowrishankar et. al [12] established the stability results of random impulsive semilinear differential systems. Also, Anguraj and Vinodkumar [4] investigated the existence and stability results of random impulsive semilinear differential systems by Contraction Principle. Vinodkumar et al. [28] studied the existence and stability results on nonlinear delay integro-differential equations with random impulses. Anguraj and Vinodkumar [3] investigated the existence and uniqueness of neutral functional differential equations with random impulses. Also, Vinodkumar et al. [27] established the existence and stability results on random impulsive neutral partial differential equations. Recently, many monographs have been focusing their attention towards the theory and applications of SDEs with random impulses see (see monographs [2**?**]). However to the best of author's knowledge there has not been any papers that study the neutral stochastic differential equations with random impulses. Thus motivated by the above facts in this paper we study the existence, uniqueness and stability of random impulsive neutral stochastic differential equations.

The considered neutral stochastic differential equations with random impulse is of the form:

$$d[x(t) + \mathsf{h}(t, x_t)] = \mathsf{f}(t, x_t)dt + \mathsf{g}(t, x_t)dw(t), \qquad t \neq \xi_k,\ t \geq 0, \tag{1}$$

$$x(\xi_k^-) = b_k(\delta_k)x(\xi_k^-), \qquad k = 1, 2, ..., \tag{2}$$

$$x(t_0) = \phi, \tag{3}$$

where δ_k is a random variable defined from Ω to $\mathfrak{D}_k =^{def} (0, d_k)$ with $0 < d_k < +\infty$ for $k = 1, 2, ...$ Suppose that δ_i and δ_j are independent of each other as $i \neq j$ for $i, j = 1, 2, ...$ Here, suppose $T \in (t_0, +\infty)$, $\mathsf{f} : [t_0, T] \times \mathfrak{C} \to \mathbb{R}^d$, $\mathsf{g} : [t_0, T] \times \mathfrak{C} \to \mathbb{R}^{d\times m}$, $\mathsf{h} : [t_0, T] \times \mathfrak{C} \to \mathbb{R}^d$ and $b_k : \mathfrak{D}_k \to \mathbb{R}^{d\times d}$, and x_t is $\mathbb{R}^d$-valued stochastic process such that $x_t \in \mathbb{R}^d$, $x_t = \{x(t + \theta) : -\delta \leq \theta \leq 0\}$. The impulsive moments ξ_k from a strictly increasing sequence, i.e., $\xi_0 < \xi_1 < \cdots < \xi_k < \cdots < \lim_{k\to\infty} \xi_k = \infty$, and $x(\xi_k^-) = \lim_{t\to\xi_k-0} x(t)$. We assume that $\xi_0 = t_0$ and $\xi_k = \xi_{k-1} + \delta_k$ for $k = 1, 2, ...$ Obviously, $\{\xi_k\}$ is a processes with independent increments. We suppose that $\{N(t), t \geq 0\}$ is the simple counting process generated by $\{\xi_k\}$, and $\{w(t) : t \geq 0\}$ is a given m-dimensional Wiener process. We denote $\mathfrak{I}_t^{(1)}$ the σ-algebra generated by $\{N(t), t \geq 0\}$, and denote $\mathfrak{I}_t^{(2)}$ the σ-algebra generated by $\{w(s), s \leq t\}$. We assume that $\mathfrak{I}_\infty^{(1)}, \mathfrak{I}_\infty^{(2)}$ and ξ are mutually independent.

The arrangement of the rest of the paper is as follows. In section 2, some preliminaries and results applied in the later part of the paper are presented. Section 3 is devoted to the study of existence and uniqueness of mild solution of the system (1)-(3). In Section 4 the stability of the mild solution of the system (1)-(3) is studied.

2. Preliminaries

Let $(\Omega, \mathfrak{I}, \mathbb{P})$ is a probability space with filtration $\{\mathfrak{I}_t\}$, $t \geq 0$ satisfying $\mathfrak{I}_t = \mathfrak{I}_t^{(1)} \vee \mathfrak{I}_t^{(2)}$. Let $\mathcal{L}^2(\Omega, \mathbb{R}^d)$ be the collection of all strongly measurable, $\mathfrak{I}_t$ measurable, $\mathbb{R}^d$-valued random variables x with norm $\|x\|_{\mathcal{L}^2} = \left(\mathbf{E}\,\|x\|^2\right)^{\frac{1}{2}}$, where the expectation $\mathbf{E}$ is defined by $\mathbf{E}x = \int_\Omega x d\mathbb{P}$. Let $\delta > 0$ denote the Banach space of all piecewise continuous $\mathbb{R}^d$-valued stochastic process $\left\{\xi(t), t \in [-\delta, 0]\right\}$ by $\mathfrak{C}([-\delta, 0], \mathcal{L}^2(\Omega, \mathbb{R}^d))$ equipped with the norm

$$\left\|\psi\right\|_{\mathfrak{C}} = \sup_{\theta\in[-\delta,0]} \left(\mathbf{E}\left\|\psi(\theta)\right\|^2\right)^{\frac{1}{2}},\ \psi(\theta) \in \mathfrak{C}.$$

Let $T \in (t_0, +\infty)$, $\mathsf{f} : [t_0, T] \times \mathfrak{C} \to \mathbb{R}^d$, $\mathsf{g} : [t_0, T] \times \mathfrak{C} \to \mathbb{R}^{d\times m}$ and $\mathsf{h} : [t_0, T] \times \mathfrak{C} \to \mathbb{R}^d$ be Borel measurable. The initial data

$$x_{t_0} = \xi = \{\xi(\theta) : -\delta \le \theta \le 0\} \tag{4}$$

is an $\mathfrak{I}_{t_0}$ measurable, $[-\delta, 0]$ to $\mathbb{R}^d$-valued random variable such that $\mathbf{E}\,\|\xi\|^2 < \infty$.

Definition 2.1. *For a given $T \in (t_0, +\infty)$, a $\mathbb{R}^d$-valued stochastic process $x(t)$ on $t_0 - \delta \le t \le T$ is called a solution to (1)-(3) with the initial data (4) if $\forall\ t_0 \le t \le T$, $x(t_0) = \phi$, $\{x_t\}_{t_0 \le t \le T}$ is $\mathfrak{I}_t$-adapted and*

$$\begin{aligned} x(t) \;=\; & \sum_{k=0}^{\infty}\Big[\prod_{i=1}^{k} b_i(\delta_i)[\phi(0) + \mathsf{h}(0,\phi)] - \prod_{i=1}^{k} b_i(\delta_i)\mathsf{h}(t, x_t) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i} \mathsf{f}(s, x_s)ds + \int_{\xi_k}^{t} \mathsf{f}(s, x_s)ds \\ & + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i} \mathsf{g}(s, x_s)dw(s) + \int_{\xi_k}^{t} \mathsf{g}(s, x_s)dw(s)\Big] I_{[\xi_k,\xi_{k+1})}(t), \ a.s. \end{aligned} \tag{5}$$

where

$$\prod_{j=i}^{k} b_j(\delta_j) = b_k(\delta_k)b_{k-1}(\delta_{k-1})\cdots b_i(\delta_i),$$

and $I_{(A)}(.)$ is the index function, i.e.,

$$I_{\mathrm{A}}(t) = \begin{cases} 1, & if\ \ t \in \mathrm{A}, \\ 0, & if\ \ t \notin \mathrm{A}. \end{cases}$$

Lemma 2.2. *For any $r \ge 1$ and for arbitrary $\mathcal{L}_2^0$- valued predictable process $\Phi(.)$*

$$\sup_{s\in[0,t]} \mathbb{E}\left\|\int_0^s \Phi(u)dw(u)\right\|_{\mathbb{X}}^{2r} = (r(2r-1))^r\left(\int_0^t (\mathbb{E}\,\|\Phi(s)\|_{\mathcal{L}_2^0}^{2r})ds\right)^r$$

3. Existence and Uniqueness

In order to derive the existence and uniqueness of the system (1)-(3), we shall impose the following assumptions:

(H1) The functions $\mathsf{f} : [t_0, T] \times \mathfrak{C} \to \mathbb{R}^d$, $\mathsf{g} : [t_0, T] \times \mathfrak{C} \to \mathbb{R}^{d\times m}$ and $\mathsf{h} : [t_0, T] \times \mathfrak{C} \to \mathbb{R}^d$ satisfies the Lipschitz condition such that there exist constants $\mathcal{L}_{\mathsf{f}} = \mathcal{L}_{\mathsf{f}}(T) > 0$, $\mathcal{L}_{\mathsf{g}} = \mathcal{L}_{\mathsf{g}}(T) > 0$ and $\mathcal{L}_{\mathsf{h}} = \mathcal{L}_{\mathsf{h}}(T) > 0 \ni$,

$$\mathbb{E}\left\|\mathsf{f}(t, x_t) - \mathsf{f}(t, y_t)\right\|^2 \le \mathcal{L}_{\mathsf{f}}\mathbb{E}\left\|x - y\right\|_t^2,$$
$$\mathbb{E}\left\|\mathsf{g}(t, x_t) - \mathsf{g}(t, y_t)\right\|^2 \le \mathcal{L}_{\mathsf{g}}\mathbb{E}\left\|x - y\right\|_t^2,$$
$$\mathbb{E}\left\|\mathsf{h}(t, x_t) - \mathsf{h}(t, y_t)\right\|^2 \le \mathcal{L}_{\mathsf{h}}\mathbb{E}\left\|x - y\right\|_t^2,$$

for $x, y \in \mathfrak{C}$, $t \in [t_0, T]$.

(H2) For all $t \in [t_0, T]$, it follows that $\mathsf{f}(t, 0), \mathsf{g}(t, 0)$ and $\mathsf{h}(t, 0) \in \mathcal{L}^1$, $\ni$,

$$\mathbb{E}\,\|\mathsf{f}(t, 0)\|^2 \le \kappa_{\mathsf{f}}, \quad \mathbb{E}\,\|\mathsf{g}(t, 0)\|^2 \le \kappa_{\mathsf{g}}, \quad \mathbb{E}\,\|\mathsf{h}(t, 0)\|^2 \le \kappa_{\mathsf{h}},$$

where $\kappa_{\mathsf{f}}, \kappa_{\mathsf{g}}$ and κ_{h} are constants.

(H3) The condition $\mathbb{E}\left\{\max_{i,k}\{\prod_{j=i}^{k}\left\|b_j(\delta_j)\right\|\}\right\}$ is uniformly bounded. That is, there exist constant $C > 0$ such that,

$$\mathbb{E}\left\{\max_{i,k}\{\prod_{j=i}^{k}\left\|b_j(\delta_j)\right\|\}\right\} \;\le\; C$$

for all $\delta_j \in \mathfrak{D}_j$, $j = 1, 2, 3...$

Theorem 3.1. *Let the hypotheses (H1)-(H3) holds, there exists a unique continuous mild solution to the system (1)-(3) for any initial value (t_0,ϕ) with $t_0 \geq 0$ and $\phi \in \mathcal{B}$.*

Proof. Let $\mathcal{B}_T$ be the space $\mathcal{B}_T = \mathfrak{C}([t_0-\delta,T],\mathcal{L}^2(\Omega,\mathbb{R}^d))$ endowed with the norm

$$\|x\|^2_{\mathcal{B}_T} = \sup_{t\in[t_0,T]} \|x_t\|^2_{\mathfrak{C}},$$

where $\|x_t\|_{\mathfrak{C}} = \sup_{t-\delta\leq s\leq t} \mathbf{E}\|x(s)\|^2$.
We define the operator $\Phi : \mathcal{B}_T \to \mathcal{B}_T$ by

$$(\Phi x)(t) = \begin{cases} \phi(t-t_0) & t\in(+\infty,t_0], \\ \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\delta_i)[\phi(0)+\mathsf{h}(0,\phi)] - \prod_{i=1}^{k} b_i(\delta_i)\mathsf{h}(t,x_t) & \\ +\left[\sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\mathsf{f}(s,x_s)ds + \int_{\xi_k}^{t}\mathsf{f}(s,x_s)ds\right] & \\ +\left[\sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\mathsf{g}(s,x_s)dw(s) + \int_{\xi_k}^{t}\mathsf{g}(s,x_s)dw(s)\right]\Big]I_{[\xi_k,\xi_{k+1})}(t), & t\in[t_0,T]. \end{cases}$$

Now we need to prove, Φ maps $\mathcal{B}_T$ into itself.

$$\begin{aligned}
\|(\Phi x)(t)\|^2 &= \Big\|\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\delta_i)[\phi(0)+\mathsf{h}(0,\phi)] - \prod_{i=1}^{k} b_i(\delta_i)\mathsf{h}(t,x_t) + \Big[\sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\mathsf{f}(s,x_s)ds \\
&+ \int_{\xi_k}^{t}\mathsf{f}(s,x_s)ds\Big] + \Big[\sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\mathsf{g}(s,x_s)dw(s) + \int_{\xi_k}^{t}\mathsf{g}(s,x_s)dw(s)\Big]\Big]I_{[\xi_k,\xi_{k+1}}(t)\Big\|^2 \\
&\leq 4\Big[\sum_{k=0}^{+\infty}\left[\prod_{i=1}^{k}\|b_i(\delta_i)\|^2\left\|\phi(0)+\mathsf{h}(0,\phi)\right\|^2 I_{[\xi_k,\xi_{k+1})}(t)\right] + 4\Big[\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\|\mathsf{h}(t,x_t)\|^2 I_{[\xi_k,\xi_{k+1})}(t)\Big]\Big] \\
&+ 4\left[\max_{i,k}\left\{1,\prod_{j=i}^{k}\left\|b_j(\delta_j)\right\|\right\}\right]^2\left(\int_{t_0}^{t}\|\mathsf{f}(s,x_s)\|\,dsI_{[\xi_k,\xi_{k+1})}(t)\right)^2 + 4\left[\max_{i,k}\left\{1,\prod_{j=i}^{k}\left\|b_j(\delta_j)\right\|\right\}\right]^2 \\
&\times \left(\int_{t_0}^{t}\|\mathsf{g}(s,x_s)\|\,dw(s)I_{[\xi_k,\xi_{k+1})}(t)\right)^2\Big] \\
&\leq 8\left[\max_{k}\left\{\prod_{i=1}^{k}\|b_i(\delta_i)\|^2\right\}\right]\left[\left\|\phi(0)\right\|^2+\left\|\mathsf{h}(0,\phi)\right\|^2\right] + 8\left[\max_{k}\left\{\prod_{i=1}^{k}\|b_i(\delta_i)\|^2\right\}\right] \\
&\times \left[\|\mathsf{h}(t,x_t)-\mathsf{h}(t,0)\|^2+\|\mathsf{h}(t,0)\|^2\right] + 8\left[\max_{i,k}\left\{1,\prod_{j=i}^{k}\left\|b_j(\delta_j)\right\|^2\right\}\right] \\
&\times (t-t_0)\int_{t_0}^{t}\left[\|\mathsf{f}(s,x_s)-\mathsf{f}(s,0)\|^2+\|\mathsf{f}(s,0)\|^2\right]ds + 8\left[\max_{i,k}\left\{1,\prod_{j=i}^{k}\left\|b_j(\delta_j)\right\|^2\right\}\right] \\
&\times (t-t_0)\int_{t_0}^{t}\left[\|\mathsf{g}(s,x_s)-\mathsf{g}(s,0)\|^2+\|\mathsf{g}(s,0)\|^2\right]ds
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbb{E}\,\|(\Phi x)(t)\|_t^2 \quad &\le \quad 8C^2\mathbb{E}\left[\left\|\phi(0)\right\|^2 + \left\|\mathsf{h}(0,\phi)\right\|^2\right] + 8C^2\left[\mathcal{L}_{\mathsf{h}}\mathbb{E}\,\|x\|_t^2 + \kappa_{\mathsf{h}}\right] \\
&+ \quad 8max\left\{1, C^2\right\}(T - t_0)\int_{t_0}^{t}\left[\mathcal{L}_{\mathsf{f}}\mathbb{E}\,\|x\|_s^2 + \kappa_{\mathsf{f}}\right]ds + 8max\left\{1, C^2\right\}(T - t_0)\mathbb{C}_2 \\
&\times \quad \int_{t_0}^{t}\left[\mathcal{L}_{\mathsf{g}}\mathbb{E}\,\|x\|_s^2 + \kappa_{\mathsf{g}}\right]ds \\
&\le \quad 8C^2\mathbb{E}\left[\left\|\phi(0)\right\|^2 + \left\|\mathsf{h}(0,\phi)\right\|^2\right] + 8C^2\kappa_{\mathsf{h}} + 8C^2\mathcal{L}_{\mathsf{h}}\mathbb{E}\,\|x\|_t^2 \\
&+ \quad 8max\left\{1, C^2\right\}(T - t_0)\int_{t_0}^{t}\mathcal{L}_{\mathsf{f}}\mathbb{E}\,\|x\|_s^2\,ds + 8max\left\{1, C^2\right\}(T - t_0)^2\kappa_{\mathsf{f}} \\
&+ \quad 8max\left\{1, C^2\right\}(T - t_0)\mathbb{C}_2\int_{t_0}^{t}\mathcal{L}_{\mathsf{g}}\mathbb{E}\,\|x\|_s^2\,ds + 8max\left\{1, C^2\right\}(T - t_0)^2\mathbb{C}_2\kappa_{\mathsf{g}}
\end{aligned}$$

Taking supremum over t, we get

$$\begin{aligned}
\sup_{t\in[t_0,T]}\mathbb{E}\,\|(\Phi x)(t)\|_t^2 \quad &\le \quad 8C^2\mathbb{E}\left[\left\|\phi(0)\right\|^2 + \left\|\mathsf{h}(0,\phi)\right\|^2\right] + 8C^2\kappa_{\mathsf{h}} + 8C^2\mathcal{L}_{\mathsf{h}}\sup_{t\in[t_0,T]}\mathbb{E}\,\|x\|_t^2 \\
&+ \quad 8max\left\{1, C^2\right\}(T - t_0)\int_{t_0}^{t}\mathcal{L}_{\mathsf{f}}\sup_{t\in[t_0,T]}\mathbb{E}\,\|x\|_s^2\,ds + 8max\left\{1, C^2\right\}(T - t_0)^2\kappa_{\mathsf{f}} \\
&+ \quad 8max\left\{1, C^2\right\}(T - t_0)\mathbb{C}_2\int_{t_0}^{t}\mathcal{L}_{\mathsf{g}}\sup_{t\in[t_0,T]}\mathbb{E}\,\|x\|_s^2\,ds + 8max\left\{1, C^2\right\}(T - t_0)^2\mathbb{C}_2\kappa_{\mathsf{g}} \\
&\le \quad 8\Big[C^2\mathbb{E}\left[\left\|\phi(0)\right\|^2 + \left\|\mathsf{h}(0,\phi)\right\|^2\right] + C^2\kappa_{\mathsf{h}} + max\left\{1, C^2\right\}(T - t_0)^2\left[\kappa_{\mathsf{f}} + \mathbb{C}_2\kappa_{\mathsf{g}}\right]\Big] \\
&+ \quad 8\Big[C^2\mathcal{L}_{\mathsf{h}} + max\left\{1, C^2\right\}(T - t_0)^2\left[\mathcal{L}_{\mathsf{f}} + \mathbb{C}_2\mathcal{L}_{\mathsf{g}}\right]\Big]\|x\|_t^2
\end{aligned}$$

Thus we obtain,

$$\|\Phi x\|_{\mathcal{B}}^2 \le m_1 + m_2\,\|x\|_{\mathcal{B}_T}^2$$

where,

$$\begin{aligned}
m_1 \quad &= \quad 8\Big[C^2\mathbb{E}\left[\left\|\phi(0)\right\|^2 + \left\|\mathsf{h}(0,\phi)\right\|^2\right] + C^2\kappa_{\mathsf{h}} + max\left\{1, C^2\right\}(T - t_0)^2\left[\kappa_{\mathsf{f}} + \mathbb{C}_2\kappa_{\mathsf{g}}\right]\Big], \\
m_2 \quad &= \quad 8\Big[C^2\mathcal{L}_{\mathsf{h}} + max\left\{1, C^2\right\}(T - t_0)^2\left[\mathcal{L}_{\mathsf{f}} + \mathbb{C}_2\mathcal{L}_{\mathsf{g}}\right]\Big]
\end{aligned}$$

where m_1 and m_2 are constants. Hence Φ is bounded.
Now we have to prove that Φ is a contraction mapping. For any $x, y \in \mathcal{B}_T$, we have

$$\begin{aligned}
\left\|(\Phi x)(t) - (\Phi y)(t)\right\|^2 \quad &\le \quad 3\left[\max_k\left\{\prod_{i=1}^{k}\|b_i(\delta_i)\|^2\right\}\left\|\mathsf{h}(t, x_t) - \mathsf{h}(t, y_t)\right\|^2 I_{[\xi_k,\xi_{k+1})}\right] \\
&+ \quad 3\left[\max_{i,k}\left\{1, \prod_{j=i}^{k}\left\|b_j(\delta_j)\right\|^2\right\}\int_{t_0}^{t}\left\|\mathsf{f}(s, x_s) - \mathsf{f}(s, y_s)\right\|ds I_{[\xi_k,\xi_{k+1})}\right]^2
\end{aligned}$$

$$+ \quad 3\left[\max_{i,k}\left\{1, \prod_{j=i}^{k}\left\|b_j(\delta_j)\right\|^2\right\}\int_{t_0}^{t}\left\|\mathsf{g}(s,x_s)-\mathsf{g}(s,y_s)\right\| dw(s) I_{[\xi_k,\xi_{k+1})}\right]^2,$$

$$\begin{aligned}
\mathbb{E}\left\|(\Phi x)(t)-(\Phi y)(t)\right\|^2 &\leq 3C^2\mathbb{E}\left\|\mathsf{h}(t,x_t)-\mathsf{h}(t,y_t)\right\|^2 + 3\max\left\{1,C^2\right\}(T-t_0)\\
&\times \int_{t_0}^{t}\mathbb{E}\left\|\mathsf{f}(s,x_s)-\mathsf{f}(s,y_s)\right\|^2 ds + 3max\left\{1,C^2\right\}(T-t_0)\\
&\times \mathbb{C}_2\int_{t_0}^{t}\mathbb{E}\left\|\mathsf{g}(s,x_s)-\mathsf{g}(s,y_s)\right\|^2 ds\\
&\leq 3C^2\mathcal{L}_{\mathsf{h}}\left\|x-y\right\|_t^2 + 3max\left\{1,C^2\right\}(T-t_0)\int_{t_0}^{t}\mathcal{L}_{\mathsf{f}}\mathbb{E}\left\|x-y\right\|_s^2 ds\\
&+ 3max\left\{1,C^2\right\}(T-t_0)\mathbb{C}^2\int_{t_0}^{t}\mathcal{L}_{\mathsf{g}}\mathbb{E}\left\|x-y\right\|_s^2 ds,
\end{aligned}$$

Moreover,

$$\begin{aligned}
\sup_{t\in[t_0,T]}\mathbb{E}\left\|(\Phi x)(t)-(\Phi y)(t)\right\|^2 &\leq 3C^2\mathcal{L}_{\mathsf{h}}\sup_{t\in[t_0,T]}\left\|x-y\right\|_t^2 + 3max\left\{1,C^2\right\}(T-t_0)^2\mathcal{L}_{\mathsf{f}}\sup_{t\in[t_0,T]}\mathbb{E}\left\|x-y\right\|_s^2\\
&+ 3max\left\{1,C^2\right\}(T-t_0)^2\mathbb{C}^2\mathcal{L}_{\mathsf{g}}\sup_{t\in[t_0,T]}\mathbb{E}\left\|x-y\right\|_s^2\\
&\leq \left\{3C^2\mathcal{L}_h + 3max\left\{1,C^2\right\}(T-t_0)^2[\mathcal{L}_{\mathsf{f}}+\mathbb{C}_2\mathcal{L}_{\mathsf{g}}]\right\}\sup_{t\in[t_0,T]}\mathbb{E}\left\|x-y\right\|_t^2
\end{aligned}$$

Thus,

$$\left\|(\Phi x)-(\Phi y)\right\|_{\mathcal{B}}^2 \leq \Upsilon(T)\left\|x-y\right\|_{\mathcal{B}_T}^2$$

with

$$\Upsilon(T) = 3C^2\mathcal{L}_h + 3max\left\{1,C^2\right\}(T-t_0)^2[\mathcal{L}_{\mathsf{f}}+\mathbb{C}_2\mathcal{L}_{\mathsf{g}}]$$

By considering suitable $0 < T-1 < T$ sufficiently small $\ni$, $\Upsilon(T_1) < 1$. Hence Φ is a contraction on $\mathcal{B}_{T_1}$ where $\mathcal{B}_{T_1}$ denotes $\mathcal{B}_T$ with T substituted by T_1. By Banach Contraction Principle, a unique fixed point $x \in \mathcal{B}_{T_1}$ is obtained for the operator Φ and therefore $\Phi x = x$ is a mild solution of the system (1)-(3). The solution can be extended to the entire interval $(-\infty, T]$ in finitely many steps which completes the proof for the existence and uniqueness of mild solutions on the entire interval $(-\infty, T]$. □

4. Stability

The stability through continuous dependence of solutions on initial condition are investigated.

Definition 4.1. *A mild solution $x(t)$ of the system (1)-(3) with initial condition ϕ satisfies (4) is said to be stable in the mean square if for all $\epsilon > 0$ there exist, $\eta > 0 \ni$,*

$$\begin{aligned}
\mathbb{E}\left\|x(t)-\widehat{x}(t)\right\|^2 &\leq \epsilon \text{ whenever,}\\
\mathbb{E}\left\|\phi-\widehat{\phi}\right\|^2 &\leq \eta \text{ for all } t\in[t_0,T].
\end{aligned}$$

where $\widehat{x}(t)$ is another mild solution of the system (5) with initial value ϕ defined in (4).

Theorem 4.2. *Let $x(t)$ and $y(t)$ be mild solution of the system (1)-(3) with initial conditions ϕ_1 and ϕ_2 respectively. If the assumptions (H1)-(H3) gets satisfied, the mean solution of the system (1)-(3) is stable in the mean square.*

Proof. By assumptions, $x(t)$ and $y(t)$ be two mild solutions of the system (1)-(3) with initial values ϕ_1 and ϕ_2 respectively.

$$\begin{aligned} x(t)-y(t) &= \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\delta_i)\Big[[\phi_1-\phi_2]+[\mathsf{h}(0,\phi_1)-\mathsf{h}(0,\phi_2)]\Big]-\prod_{i=1}^{k} b_i(\delta_i)\,[\mathsf{h}(t,x_t)-\mathsf{h}(t,y_t)] \\ &+ \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}[\mathsf{f}(s,x_s)-\mathsf{f}(s,y_s)]\,ds+\int_{\xi_k}^{t}[\mathsf{f}(s,x_s)-\mathsf{f}(s,y_s)]\,ds+\sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\delta_j) \\ &\times \int_{\xi_{i-1}}^{\xi_i}[\mathsf{g}(s,x_s)-\mathsf{g}(s,y_s)]\,dw(s)+\int_{\xi_k}^{t}[\mathsf{g}(s,x_s)-\mathsf{g}(s,y_s)]\,dw(s)\Big]I_{[\xi_k,\xi_{k+1})}(t). \end{aligned}$$

Then,

$$\begin{aligned} \mathbb{E}\left\|x(t)-y(t)\right\|^2 &\le 8C^2\mathbb{E}\left\|\phi_1-\phi_2\right\|^2+8C^2\mathbb{E}\left\|\mathsf{h}(0,\phi_1)-\mathsf{h}(0,\phi_2)\right\|^2+4C^2\mathbb{E}\left\|\mathsf{h}(t,x_t)-\mathsf{h}(t,y_t)\right\|^2 \\ &+ 4max\left\{1,C^2\right\}(t-t_0)\int_{t_0}^{t}\mathbb{E}\left\|\mathsf{f}(s,x_s)-\mathsf{f}(s,y_s)\right\|^2 ds+4max\left\{1,C^2\right\}(t-t_0)\mathbb{C}_2 \\ &\times \int_{t_0}^{t}\mathbb{E}\left\|\mathsf{g}(s,x_s)-\mathsf{g}(s,y_s)\right\|^2 ds \\ &\le 8C^2\mathbb{E}\left\|\phi_1-\phi_2\right\|^2+8C^2\mathcal{L}_{\mathsf{h}}\mathbb{E}\left\|\phi_1-\phi_2\right\|^2+4C^2\mathcal{L}_{\mathsf{h}}\mathbb{E}\left\|x-y\right\|_t^2+4max\left\{1,C^2\right\}(T-t_0) \\ &\times \mathcal{L}_{\mathsf{f}}\int_{t_0}^{t}\mathbb{E}\left\|x-y\right\|_s^2 ds+4max\left\{1,C^2\right\}(T-t_0)\mathbb{C}_2\mathcal{L}_{\mathsf{g}}\int_{t_0}^{t}\mathbb{E}\left\|x-y\right\|_s^2 ds \\ &\le 8C^2\mathbb{E}\left\|\phi_1-\phi_2\right\|^2[1+\mathcal{L}_{\mathsf{h}}]+4C^2\mathcal{L}_{\mathsf{h}}\mathbb{E}\left\|x-y\right\|_t^2+4max\left\{1,C^2\right\}(T-t_0) \\ &\times \mathcal{L}_{\mathsf{f}}\int_{t_0}^{t}\mathbb{E}\left\|x-y\right\|_s^2 ds+4max\left\{1,C^2\right\}(T-t_0)\mathbb{C}_2\mathcal{L}_{\mathsf{g}}\int_{t_0}^{t}\mathbb{E}\left\|x-y\right\|_s^2 ds \end{aligned}$$

Furthermore,

$$\begin{aligned} \sup_{t\in[t_0,T]}\mathbb{E}\left\|x(t)-y(t)\right\|^2 &\le 8C^2\mathbb{E}\left\|\phi_1-\phi_2\right\|^2[1+\mathcal{L}_{\mathsf{h}}]+4C^2\mathcal{L}_{\mathsf{h}}\sup_{t\in[t_0,T]}\mathbb{E}\left\|x-y\right\|_t^2+4max\left\{1,C^2\right\}(T-t_0)^2 \\ &\times \mathcal{L}_{\mathsf{f}}\sup_{t\in[t_0,T]}\mathbb{E}\left\|x-y\right\|_t^2+4max\left\{1,C^2\right\}(T-t_0)^2\mathbb{C}_2\mathcal{L}_{\mathsf{g}}\sup_{t\in[t_0,T]}\mathbb{E}\left\|x-y\right\|_t^2 \\ &\le \Big[8C^2[1+\mathcal{L}_{\mathsf{h}}]+4C^2\mathcal{L}_{\mathsf{h}}+4max\left\{1,C^2\right\}(T-t_0)^2\Big[\mathcal{L}_{\mathsf{f}}+\mathcal{L}_{\mathsf{g}}\mathbb{C}_2\Big]\Big]\sup_{t\in[t_0,T]}\mathbb{E}\left\|x-y\right\|_t^2. \end{aligned}$$

Thus,

$$\sup_{t\in[t_0,T]}\mathbb{E}\left\|x(t)-y(t)\right\|_t^2\le\beta\mathbb{E}\left\|\phi_1-\phi_2\right\|^2$$

where,

$$\beta=8C^2[1+\mathcal{L}_{\mathsf{h}}]+4C^2\mathcal{L}_{\mathsf{h}}+4max\left\{1,C^2\right\}(T-t_0)^2[\mathcal{L}_{\mathsf{f}}+\mathcal{L}_{\mathsf{g}}\mathbb{C}_2].$$

Given $\epsilon>0$ choose $\eta=\frac{\epsilon}{\beta}$ such that $\mathbb{E}\left\|\phi_1-\phi_2\right\|^2<\eta$. Then,

$$\left\|x-y\right\|_{\mathbb{B}_T}^2\le\epsilon.$$

This completes the proof.

5. Illustration

In this section, the results obtained are applied to a stochastic partial differential equations with random impulses. Let us consider a space $H = \mathcal{L}^2([0,\pi])$.
For $z \in D(\mathfrak{A})$,

$$\mathfrak{A}z = -\sum_{n=1}^{\infty} n^2 < z, z_n > z_n,$$

where $\{z_n : n \in \mathbb{Z}\}$ is an orthonormal basis of H, $z_n(x) := \frac{1}{\sqrt{2\pi}} e^{inx}, n \in \mathbb{Z}^+, x \in [0,\pi]$. It is known that $\mathfrak{A}$ generates strongly continuous operators $C(t)$ and $\mathcal{S}(t)$ in a Hilbert space H, such that

$$C(t)z = \sum_{n=1}^{\infty} cos(nt) < z, z_n > z_n, \qquad \mathcal{S}(t)z = \sum_{n=1}^{\infty} sin(nt)/n < z, z_n > z_n,$$

for $t \in \mathbb{R}$. And we assume that $\mathcal{S}(t)$ is not a compact semigroup and $\vartheta(\mathcal{S}(t)D) \leq \vartheta(D)$, where $D \in H$ denotes a bounded set, ϑ is the Hausdroff measure of non-compactness.
In the sequel, we may consider second-order neutral stochastic functional differential equation of the form,

$$\begin{aligned}
\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t} z(t,x) - \frac{\mathfrak{m}_1}{5}\int_{-r}^{0} \varepsilon_1(s) z(t+s,x)ds\right] &= \left[\frac{\mathfrak{m}_2}{5}\int_{-r}^{0} \varepsilon_2(s) z(t+s)ds\right]dt \\
&+ \frac{m_3}{5}\int_{-r}^{0} \varepsilon_3(s) z(t+s)d\omega(t),\ t \geq t_0,\ t \neq \xi_k,\ x \in [0,\pi], \\
z(\xi_k, x) &= \varrho(k)\delta_k z(\xi_k^-, x), k = 1,2,3..., \qquad (6)\\
\frac{\partial}{\partial t} z(\xi_k, x) &= \varrho(k)\delta_k \frac{\partial}{\partial t} z(\xi_k^-, x), \\
z(t_0, x) &= \phi(\theta, x), \theta \in [-r, 0],\ x \in [0,\pi], r > 0, \\
\frac{\partial}{\partial t} z(t_0, x) &= \varphi(x),\ x \in [0,\pi], \\
z(t,0) &= z(t,\pi) = 0.
\end{aligned}$$

Let δ_k be a random variable defined on $D_k \equiv (0, d_k)$ where, $0 < d_k < +\infty$, for $k = 1,2,\cdots$. $\xi_0 = t_0 > 0$ and $\xi_k = \xi_{k-1} + \delta_k$ for $k = 1,2,\cdots$. $\omega(t)$ denotes a standard cylindrical Weiner process in H. Furthermore, let ϱ be a function of k. $\varepsilon_i : [-r, 0] \to \mathbb{R}$ are positive functions and $\mathfrak{m}_i > 0$ for $i = 1,2,3$. $\|C(t)\|, \|\mathcal{S}(t)\|$ are bounded on $\mathbb{R}$. $\|C(t)\| \leq e^{-\pi^2 t}$ and $\|\mathcal{S}(t)\| \leq e^{-\pi^2 t} (t \geq 0)$. We may assume that

(i) The function $\varepsilon(\theta) \geq 0$ is continuous on $[-r, 0]$, $\int_{-r}^{0} \varepsilon_i^2(\theta)d\theta < \infty (i = 1,2,3.)$

(ii) $\max_{i,k} = \{\prod_{j=i}^{k} \mathbb{E}[\left\|\varrho(j)\delta_j\right\|^2]\} < \mathcal{N}$.

Using above assumptions and functions $\varepsilon_1, \varepsilon_2, \varepsilon 3, \varrho$ we can show that $\mathcal{L}_{\mathfrak{g}} = \frac{r\mathfrak{m}_1}{25}\int_{-r}^{0} \varepsilon_1^2(\theta)d\theta$,$\mathcal{L}_{\mathfrak{f}} = \frac{r\mathfrak{m}_2}{25}\int_{-r}^{0} \varepsilon_1^2(\theta)d\theta$ and $\mathcal{L}_{\mathfrak{h}} = \frac{r\mathfrak{m}_3}{25}\int_{-r}^{0} \varepsilon_1^2(\theta)d\theta$. Hence stability in mean square of mild solution (6) is obtained. □

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