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Existence and uniqueness of global solutions for non-autonomous evolution equations with state-dependent nonlocal conditions

Jianbo Zhu[a]

[a]*College of Mathematics Science, Inner Mongolia Center for Applied Mathematics, Inner Mongolia Normal University, Hohhot 010022, P. R. China*

Abstract. In this paper, we consider the existence and uniqueness of global solutions for non-autonomous evolution equations with state-dependent nonlocal conditions, in which the undelayed part admits an evolution operator. We discuss the problems by utilizing theory of evolution operators, Schauder fixed point theorem and Banach fixed point theorem. Some new results on existence and uniqueness of solutions of the considered equation are obtained on the infinite internal $[0, +\infty)$. In the end, the obtained results are applied to a class of non-autonomous heat equations with state-dependent nonlocal conditions.

1. Introduction

Partial differential equations (PDEs) without or with time delay have attracted great interest because of its wide practical applications in many areas such as chemistry, physics, social sciences and other areas of science and engineering. Until now, one of the important approaches to deal with PDEs is to rewrite them as abstract evolution equations in Banach space X. For example, in [24, 31] considered the following gas flow in a large container model

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) + c^2\dfrac{\partial p}{\partial x}(t,x) = 2t\sin^3(x^2+1), \quad t>0,\ 0<x<+\infty, \\ \dfrac{\partial p}{\partial t}(t,x) + c^2\dfrac{\partial v}{\partial x}(t,x) = -t\cos(x^2+1), \quad t>0,\ 0<x<+\infty, \\ v(0,x) = h_1(x),\ p(0,x) = h_2(x), \quad 0<x<+\infty, \\ v(t,0) = p(t,0) = 0, \quad t>0, \end{cases} \tag{1}$$

where v is the velocity of the gas and p is the variation in density. Eq. (1) can be expressed equivalently as

$$\begin{cases} \dfrac{\partial}{\partial t}\begin{bmatrix} v \\ p \end{bmatrix}(t,x) = \begin{bmatrix} 0 & -c^2\frac{\partial}{\partial x} \\ -c^2\frac{\partial}{\partial x} & 0 \end{bmatrix}\begin{bmatrix} v \\ p \end{bmatrix}(t,x) + \begin{bmatrix} 2t\sin^3(x^2+1) \\ -t\cos(x^2+1) \end{bmatrix}, \quad t>0,\ x>0, \\ \begin{bmatrix} v \\ p \end{bmatrix}(0,x) = \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix}, \quad x>0. \end{cases} \tag{2}$$

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Email address: jbzhu@imnu.edu.cn (Jianbo Zhu)

In order to rewrite Eq. (2) as abstract evolution equation, we consider the Banach space

$$\mathcal{H} = L^2(0,+\infty;\mathbb{R}) \times L^2(0,+\infty;\mathbb{R})$$

equipped with the inner product

$$\left\langle \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} v_1^* \\ v_2^* \end{bmatrix} \right\rangle_{\mathcal{H}} = \int_0^{+\infty} [v_1 v_1^* + v_2 v_2 *] dx.$$

Let the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ be given by

$$\mathcal{A}\begin{bmatrix} v \\ p \end{bmatrix} = \begin{bmatrix} -c^2 \frac{\partial p}{\partial x} \\ -c^2 \frac{\partial v}{\partial x} \end{bmatrix}$$
$$D(\mathcal{A}) = H^1(0,+\infty;\mathbb{R}) \times H^1(0,+\infty;\mathbb{R}).$$

The mapping $f : [0,+\infty) \to \mathcal{H}$ by

$$f(t) = \begin{bmatrix} 2t \sin^3(\cdot^2 + 1) \\ -t\cos(\cdot^2 + 1) \end{bmatrix}.$$

We put $u(t)(\cdot) = u(t,\cdot) = \begin{bmatrix} v(t,\cdot) \\ p(t,\cdot) \end{bmatrix} \in \mathcal{H}, \ \ t > 0$, then Eq. (2) can be rewritten as the abstract form

$$\begin{cases} u'(t) = \mathcal{A}u(t) + f(t), \ \ t > 0, \\ u(0) = u_0 = \begin{bmatrix} h_1(\cdot) \\ h_2(\cdot) \end{bmatrix}, \end{cases} \tag{3}$$

in X. For more related examples, we refer to [19, 43]. In recent years, some topics for evolution equations (3) on a finite internal $[0, T]$, such as existence and uniqueness, regularity, (almost) periodicity and controllability, have been discussed by many researchers, see [3, 14, 27, 39, 42] and the references therein.

We observe that among the above researches, most of authors focus on the case that the differential operators in the main parts are independent of time t, it means that the problems under consideration are autonomous Eq. (3). Nevertheless, when we treat some non-autonomous evolution equations, it is usually supposed that the differential operators depend on time t (i.e. $A = A(t)$), since this class of operators appear frequently in the applications [37, 41]. Thus, it is important and meaningful to research non-autonomous evolution equations. Here we just mention the works [1, 2, 4, 5, 10, 13, 15, 23, 29, 30, 32, 33, 35] on issues related to non-autonomous evolution equations.

On the other hand, it is well-known that in many cases the nonlocal initial condition can be applied in physics with much better than the classical initial condition $x(0) = x_0$. For instance, Deng [16] described the diffusion phenomenon of a small amount of gas in a transparent tube by making use of the formula

$$g(x) = \sum_{i=1}^{p} c_i x(t_i), \tag{4}$$

where c_i, $i = 1, \ldots, p$ are given constants and $0 < t_1 < \cdots < t_p < a$. In this case, condition (4) allows the additional measurement at t_i, $i = 1, \ldots, p$, which is more accurate than the measurement just at $t = 0$. The pioneering work on evolution equations with the nonlocal condition (4) is due to Byszewski [8]. Since then evolution equations with the nonlocal condition (4) have been studied by many researchers and a lot of works on some topics of nonlocal problems have also been investigated in these years, see [6, 11, 12, 18, 20, 34, 45–47] among others. Dong and Li [18] proved the existence of mild solutions for semilinear differential equations with nonlocal conditions in Banach spaces using the measure of noncompactness and fixed point theory. Zhu et al. [47] considered the following abstract semilinear evolution equations in Banach spaces of the form

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), \quad t \in [0, 1], \\ x(0) + g(x) = x_0, \end{cases} \tag{5}$$

where A is the infinitesimal generator of a C_0-semigroup in Banach space X, $f(\cdot, \cdot)$ and $g(\cdot)$ are given X-valued functions. They studied the existence of mild solutions of Eq. (5) with the help of generalized Darbo's fixed point theorem.

Especially, in paper [26], by using Banach fixed point theorem Hernández and O'Regan have studied the existence of mild and strict solutions for the following semilinear evolution equations

$$u'(t) = Au(t) + F(t, u(r(t))), \quad t \in [0, T],$$

with state-dependent nonlocal conditions

$$u(0) = H(\sigma(u), u), \tag{6}$$

where A generates an analytic semigroup on Banach space X. $F(\cdot, \cdot)$, $H(\cdot, \cdot)$ and $\sigma(\cdot)$ are suitable continuous functions. It is noticed that the state-dependent nonlocal condition (6) is clearly more general than the nonlocal condition (4) and classical initial condition $x(0) = x_0$. In addition, from the mathematical point of view, some other boundary and initial conditions, such as periodic boundary condition and integral initial condition, are also special cases of the state-dependent nonlocal condition (6). For more details about the condition (6), we refer the reader to [7, 9, 17, 25, 28].

Motivated by the above mentioned works, in this paper we study the existence and uniqueness of global solutions for semilinear non-autonomous evolution equations with state-dependent nonlocal conditions of the form

$$\begin{cases} x'(t) = A(t)x(t) + G(t, x(h(t))), \quad t \geq 0, \\ x(0) + H(\sigma(x), x) = x_0, \end{cases} \tag{7}$$

where $x(\cdot)$ is the state variable taking values in a Banach space X. $\{A(t) : 0 \leq t < +\infty\}$ is a family of closed linear operators depending on time t and having constant domain $D(A)$. The function $h(\cdot) : [0, +\infty) \to [0, +\infty)$ is continuous and satisfies $0 \leq h(t) \leq t$, which is regarded as a delay function. $G(\cdot, \cdot)$, $H(\cdot, \cdot)$ and $\sigma(\cdot)$ are given functions to be specified later.

It is worth to mention that the following two aspects should be considered: on the one hand, to the best of the author's knowledge, most existing articles, such as [18, 47], are only devoted to researching the existence of mild solutions for autonomous evolution equations with nonlocal conditions on the finite interval. However, the study of the existence of mild solutions for non-autonomous evolution equations with state-dependent nonlocal conditions on the infinite interval is an untreated topic in the literature. On the other hand, Hernández [25] discussed the existence results of mild solutions for semilinear evolution equations with state-dependent nonlocal conditions by assuming that the state-dependent nonlocal function $H(\cdot, \cdot)$ is compact, which is a very strong assumption. How to remove the restriction of compactness condition on $H(\cdot, \cdot)$ is a main motivation for writing this paper.

In this work, based on the above mentioned aspects, we firstly study the existence of global mild solutions for semilinear non-autonomous evolution equation with state-dependent nonlocal conditions (7) by using Schauder fixed point theorem and Lemma 2.3 (see Section 2). In this case, we do not require that the state-dependent nonlocal function $H(\cdot, \cdot)$ satisfies compactness condition. In order to gain the existence of global mild solutions for Eq. (7), we assume that the state-dependent nonlocal function $H(\cdot, \cdot)$ entirely determined by the values on interval $[\delta, +\infty)$ for some $\delta > 0$ (see (H_3)) and adopt an approximation technique to prove the existence result. After that, we obtain the uniqueness of global mild solutions for Eq. (7) under the situation that the nonlinear function $G(\cdot, \cdot)$ and state-dependent nonlocal function $H(\cdot, \cdot)$ satisfy Lipschitz condition. The essential tool in our discussion is the theory of evolution operators and

Banach fixed point theorem. Obviously, the obtained results extend and develop the corresponding ones existing in literature, such as [18, 25, 47].

The rest of this paper is organized as follows: In Section 2, we introduce some notations and lemmas, and necessary preliminaries on the basic theory of evolution operators. Then in Section 3, we establish the existence and uniqueness of mild solutions of Eq. (7) on $[0, +\infty)$. In Section 4, at last, we provide an example to illustrate the applications of the obtained results.

2. Preliminaries

Let X and Z be two Banach spaces with norm $\|\cdot\|$. The Banach space of bounded linear operators from X into Z is denoted by $\mathscr{L}(X, Z)$ endowed with the general operator norm, and this notation is written as $\mathscr{L}(X)$ if $X = Z$. Hereafter $C([0, +\infty); X)$ denotes the space consisting of continuous functions from $[0, +\infty)$ to X. The Banach space $C_b([0, +\infty); X)$ is composed of the functions $x \in C([0, +\infty); X)$ such that $\|x\|_\infty = \sup_{t\geq 0} \|x(t)\| < +\infty$, endowed with the norm $\|\cdot\|_\infty$.

Throughout this paper, we always impose the following restrictions on the family $\{A(t) : 0 \leq t < \infty\}$ of linear operators.

- (P_1) The domain $D(A)$ of $A(t)$ is dense in X and independent of t, $A(t)$ is closed linear operator for $0 \leq t < +\infty$.
- (P_2) For each $t \in [0, +\infty)$, the resolvent $R(\lambda, A(t))$ exists for all λ with $Re\lambda \geq 0$ and there exists $C_0 > 0$ such that $\|R(\lambda, A(t))\| \leq C_0/(|\lambda| + 1)$.
- (P_3) There exists $0 < \alpha \leq 1$ and $C_1 > 0$ such that $\|(A(t) - A(s))A^{-1}(\tau)\| \leq C_1|t - s|^\alpha$ for all $t, s, \tau \in [0, +\infty)$.
- (P_4) For each $t \in [0, +\infty)$ and some $\lambda \in \rho(A(t))$ $\big($the resolvent set of $A(t)\big)$, the resolvent $R(\lambda, A(t))$ is a compact operator.

Then the family $\{A(t) : 0 \leq t < +\infty\}$ generates a unique linear evolution operator $\{U(t, s), 0 \leq s \leq t < +\infty\}$ satisfying the following properties:

- (a) $U(t, s) \in \mathscr{L}(X)$ for $0 \leq s \leq t < +\infty$.
- (b) The mapping $(t, s) \to U(t, s)$ is strongly continuous for $0 \leq s \leq t < +\infty$.
- (c) $U(t, s)U(s, \tau) = U(t, \tau)$ for $0 \leq \tau \leq s \leq t < +\infty$.
- (d) $U(t, t) = I$ for $t \geq 0$.
- (e) $U(t, s)$ is a compact operator whenever $t > s$.
- (f) $\frac{\partial}{\partial t}U(t, s) = A(t)U(t, s)$ for $0 \leq s < t < +\infty$.
- (g) If $0 < h < 1$, $|t - s| > h$, and $0 < \gamma < 1$, then $\|U(t + h, s) - U(t, s)\| \leq \frac{C'h^\gamma}{|t-s|^\gamma}$ for some $C' > 0$.
- (h) If $f(t)$ is continuous on $[0, +\infty)$, then the function $t \to \int_0^t U(t, s)f(s)ds$ is Hölder continuous with any exponent $0 < \gamma < 1$.

Remark 2.1. *The condition (P_4) ensures that the evolution operator $U(\cdot, \cdot)$ satisfies (e) (see [21], Proposition 2.1). Moreover, the compactness of $U(t, s)$ for $t > s$ implies the continuity in uniform operator topology (see [23], Proposition 2.1).*

For more details on the above preliminaries, we refer to [22, 36, 40].

The following lemmas are needed in our main results.

Lemma 2.2. *([20, Lemma 12]) Let $\{R_m\}_{m\geq 1}$ be a sequence of bounded linear maps on X converging pointwise to $R \in \mathscr{L}(X)$. Then for any compact set K in X, R_m converges to R uniformly in K, namely,*

$$\sup_{x\in K} \|R_m x - Rx\| \to 0, \quad as \quad m \to \infty.$$

Lemma 2.3. *([44, Lemma 3.4]) Let $V \subset C_b([0, +\infty); X)$ be a set. If the following conditions are fulfilled:*

(1) *V is a locally equicontinuous family of functions, i.e., for any constant $T > 0$, the functions in V are equicontinuous on $[0, T]$.*
(2) *For any $t \in [0, +\infty)$, $V(t) = \{x(t) : x \in V\}$ is relatively compact in X.*
(3) *$\lim_{t,t' \to +\infty} \|x(t) - x(t')\| = 0$ uniformly for $x \in V$, that is, given $\varepsilon > 0$, there is a $N > 0$ such that $\|x(t) - x(t')\| < \varepsilon$ for any $t, t' \geq N$ and $x \in V$.*

Then V is relatively compact in $C_b([0, +\infty); X)$.

3. Main results

The purpose of this section is to discuss the existence and uniqueness of mild solutions for the equation (7) on $[0, +\infty)$ by Schauder fixed point theorem and Banach fixed point theorem. The mild solutions of this state-dependent nonlocal equation expressed by the evolution operator are defined as follows.

Definition 3.1. *A function $x(\cdot) \in C([0, +\infty); X)$ is said to be a mild solution of Eq. (7), if it verifies*

$$x(t) = U(t, 0)\, [x_0 - H(\sigma(x), x)] + \int_0^t U(t, s)G(s, x(h(s)))ds, \quad for\ t \geq 0.$$

To ensure the existence of global mild solutions, we impose the following restrictions on Eq. (7).

(H_1) The evolution operator $\{U(t, s) : 0 \leq s \leq t < \infty\}$ satisfies

$$\|U(t, s)\| \leq Me^{-\gamma(t-s)}, \ \ 0 \leq s \leq t,$$

for some $M \geq 1$ and $\gamma > 0$.

(H_2) The function $G(\cdot, \cdot) : [0, +\infty) \times X \to X$ satisfies the following conditions:
(i) For each $t \in [0, +\infty)$, the function $G(t, \cdot) : X \to X$ is continuous and for each $x \in X$ the function $G(\cdot, x) : [0, +\infty) \to X$ is strongly measurable;
(ii) For any $k > 0$, there is a positive continuous function $W_k(\cdot)$ such that

$$\sup_{\|x\| \leq k} \|G(t, x)\| \leq W_k(t), \ \text{for } t \in [0, +\infty),$$

and

$$\int_0^{+\infty} e^{\gamma s} W_k(s) := \rho < +\infty.$$

(H_3) The function $H(\cdot, \cdot) : [0, +\infty) \times C_b([0, +\infty); X) \to X$ and the function $\sigma(\cdot) : C_b([0, +\infty); X) \to [0, +\infty)$ are both continuous, and there exists a constant $L > 0$ such that, for any $x \in C_b([0, +\infty); X)$,

$$\|H(\sigma(x), x)\| \leq L\|x\|_\infty.$$

Moreover, there is a $\delta = \delta(k) \in (0, +\infty)$ such that $H(\sigma(u), u) = H(\sigma(v), v)$ for any $u, v \in B_k$ with $u(s) = v(s), s \in [\delta, +\infty)$, where $B_k = \{x \in C_b([0, +\infty); X) : \|x\|_\infty \leq k, \ k > 0\}$.

The first result of this section is

Theorem 3.2. *Let $x(0) \in X$ and assume that the conditions $(H_1) - (H_3)$ hold true, then the equation (7) has a mild solution provided that*

$$M^2 L < 1,$$

and

$$\limsup_{k \to +\infty} \Big(k\Big(1 - M^2 L\Big) - M\rho\Big) = +\infty. \tag{8}$$

As we pointed out in Section 1, we can't prove this theorem by directly employing Schauder fixed point theorem due to the fact that the state-dependent nonlocal function $H(\cdot,\cdot)$ does not satisfies the compactness condition. Thus, we will adopt the approximation method to prove this result. To this end, for a fixed $n \in \mathbb{N}^+$, we first consider the existence of mild solutions for the state-dependent nonlocal Cauchy problem

$$\begin{cases} x'(t) = A(t)x(t) + G(t, x(h(t))), \ \ t \geq 0, \\ x(0) + U\left(\frac{1}{n}, 0\right)H(\sigma(x), x) = x_0. \end{cases} \tag{9}$$

Then, we have that

Lemma 3.3. *Assume that the conditions of Theorem 3.2 are fulfilled, then for any $n \in \mathbb{N}^+$ the state-dependent nonlocal Cauchy problem* (9) *has at least one mild solution $x_n(\cdot) \in C_b([0, +\infty); X)$.*

Proof. We define the operator $\Gamma_n : C_b([0, +\infty); X) \to C_b([0, +\infty); X)$ by

$$(\Gamma_n x)(t) = U(t, 0)\left[x_0 - U\left(\frac{1}{n}, 0\right)H(\sigma(x), x)\right] + \int_0^t U(t, s)G(s, x(h(s)))ds. \tag{10}$$

Then it is clear that the mild solution of Eq. (9) is equivalent to the fixed point of operator Γ_n defined by (10). In what follows, we shall prove that Γ_n has a fixed point on some B_k by employing the well-known Schauder fixed point theorem, here B_k is given in (H_3) which is obviously a bounded, closed and convex set in $C_b([0, +\infty); X)$ for any $k > 0$.

Firstly, we prove that $\Gamma_n x \in C_b([0, +\infty); X)$ for all $x \in C_b([0, +\infty); X)$. For $0 \leq \tau < t < +\infty$, by the property (b) of evolution operator $U(\cdot,\cdot)$, (H_1) and (10), we obtain that

$$\begin{aligned} \|(\Gamma_n x)(t) - (\Gamma_n x)(\tau)\| \leq & \left\| U(t, 0)\left[x_0 - U\left(\frac{1}{n}, 0\right)H(\sigma(x), x)\right] - U(\tau, 0)\left[x_0 - U\left(\frac{1}{n}, 0\right)H(\sigma(x), x)\right]\right\| \\ & + \int_0^\tau \|U(t, s)G(s, x(h(s))) - U(\tau, s)G(s, x(h(s)))\| \, ds \\ & + M\int_\tau^t \|G(s, x(h(s)))\| \, ds \\ & \to 0 \quad \text{as} \quad t \to \tau, \end{aligned}$$

which means that $\Gamma_n x \in C([0, +\infty); X)$. Further, for any $x \in C_b([0, +\infty); X)$, by the conditions (H_1)-(H_3), we get that

$$\begin{aligned} \|(\Gamma_n x)(t)\| & \leq \|U(t, 0)\|\left[\|x_0\| + \left\|U\left(\frac{1}{n}, 0\right)\right\| \|H(\sigma(x), x)\|\right] + \int_0^t \|U(t, s)\|\|G(s, x(h(s)))\|ds \\ & \leq Me^{-\gamma t}\left[\|x_0\| + Me^{-\frac{\gamma}{n}}L\|x\|_\infty\right] + M\int_0^t e^{-\gamma(t-s)}W_{\|x\|_\infty}(s)ds \\ & \leq M\left[\|x_0\| + ML\|x\|_\infty\right] + M\int_0^t e^{\gamma s}W_{\|x\|_\infty}(s)ds \\ & \leq M\left[\|x_0\| + ML\|x\|_\infty\right] + M\rho \\ & < +\infty, \end{aligned}$$

which implies that $\|\Gamma_n x\|_\infty < +\infty$. Thus, we conclude that the operator Γ_n maps the functions in $C_b([0, +\infty); X)$ into $C_b([0, +\infty); X)$.

Secondly, we show that there exists a $k_0 > 0$ such that $\Gamma_n(B_{k_0}) \subseteq B_{k_0}$ (for all $n \in \mathbb{N}^+$). If it is not true, then for each $k > 0$, there is a function $x_k(\cdot) \in B_k$, but $\Gamma_n x_k \notin B_k$, that is $\|\Gamma_n x_k(t)\| > k$ for some $t(k) \in [0, +\infty)$. On the other hand, however, by the conditions (H_1)-(H_3) again, we get

$$\begin{aligned} k &< \|(\Gamma_n x_k)(t)\| \\ &\leq \|U(t,0)\| \left[\|x_0\| + \left\|U\left(\frac{1}{n},0\right)\right\| \|H(\sigma(x_k),x_k)\|\right] + \int_0^t \|U(t,s)\|\|G(s,x_k(h(s)))\|ds \\ &\leq M\left[\|x_0\| + MLk\right] + M\int_0^t e^{\gamma s} W_k(s)ds \\ &\leq M\left[\|x_0\| + MLk\right] + M\rho, \end{aligned}$$

or

$$\left(k\left(1 - M^2 L\right) - M\rho\right) < M\|x_0\|,$$

which contradicts (8). Hence for some $k_0 > 0$, $\Gamma_n(B_{k_0}) \subseteq B_{k_0}$.

Next we show that Γ_n is completely continuous map. To do this, we first prove that Γ_n is continuous on B_{k_0}. Let $\{x_m\} \subseteq B_{k_0}$ with $x_m \to x$ in B_{k_0}, then by (H_2) and (H_3), we have that

$$G(s, x_m(h(s))) \to G(s, x(h(s))), \quad m \to \infty,$$

$$\sigma(x_m) \to \sigma(x), \quad m \to \infty,$$

and

$$H(\sigma(x_m), x_m) \to H(\sigma(x), x), \quad m \to \infty.$$

Since

$$\|G(s, x_m(h(s))) - G(s, x(h(s)))\| \leq 2W_{k_0}(s) \in L^1,$$

then applying the dominated convergence theorem, we obtain that

$$\begin{aligned} &\|\Gamma_n x_m - \Gamma_n x\|_\infty \\ &\leq \sup_{t\geq 0}\left(M^2 \|H(\sigma(x_m), x_m) - H(\sigma(x), x)\| + M\int_0^t \|G(s, x_m(h(s))) - G(s, x(h(s)))\|\, ds\right) \\ &\to 0 \quad \text{as} \quad m \to \infty, \end{aligned}$$

i.e., Γ_n is continuous.

In the sequel, we prove by using Lemma 2.3 that, for each $n \in \mathbb{N}^+$, $\Gamma_n(B_{k_0}) = \{(\Gamma_n x)(\cdot) : x \in B_{k_0}\}$ is relatively compact in B_{k_0}.

(1) We prove that $\Gamma_n(B_{k_0})$ is a locally equicontinuous family of functions on $[0, T]$ for any $T > 0$.

Let $0 < t_1 < t_2 \leq T$ and $\varepsilon > 0$ be small enough, then

$$\begin{aligned} \|(\Gamma_n x)(t_2) - (\Gamma_n x)(t_1)\| \leq{}& \|U(t_2,0) - U(t_1,0)\| \left[\|x_0\| + \left\|U\left(\frac{1}{n},0\right)\right\| \|H(\sigma(x),x)\|\right] \\ &+ \int_0^{t_1-\varepsilon} \|U(t_2,s) - U(t_1,s)\|\, \|G(s,x(h(s)))\|\, ds \\ &+ \int_{t_1-\varepsilon}^{t_1} \|U(t_2,s) - U(t_1,s)\|\, \|G(s,x(h(s)))\|\, ds \\ &+ \int_{t_1}^{t_2} \|U(t_2,s)\|\|G(s,x(h(s)))\|\, ds \end{aligned}$$

$$
\begin{aligned}
&\leq \|U(t_2,0) - U(t_1,0)\| \left[\|x_0\| + MLk_0\right] \\
&\quad + \sup_{s\in[0,t_1-\varepsilon]} \|U(t_2,s) - U(t_1,s)\| \int_0^{t_1-\varepsilon} W_{k_0}(s)ds \\
&\quad + 2M \int_{t_1-\varepsilon}^{t_1} W_{k_0}(s)ds \\
&\quad + M \int_{t_1}^{t_2} W_{k_0}(s)ds.
\end{aligned}
$$

As $t_2 - t_1 \to 0$ and ε sufficiently small, $\|(\Gamma_n x)(t_2) - (\Gamma_n x)(t_1)\| \to 0$ independently of $x \in B_{k_0}$ since, by Remark 2.1, $U(t,s)$ is continuous in the uniform operators topology for all $t > s$. In a similar way, the functions $\{(\Gamma_n x)(\cdot) : x \in B_{k_0}\}$ are apparently equi-continuous at $t = 0$. In fact, from the properties (b) and (e) of $U(\cdot,\cdot)$, Lemma 2.2 and the compactness of $\overline{U(\frac{1}{n},0)H(\sigma(B_{k_0}),B_{k_0})}$, we have

$$
\begin{aligned}
\|(\Gamma_n x)(t) - (\Gamma_n x)(0)\| &\leq \left\| U(t,0)\left[x_0 - U\left(\frac{1}{n},0\right)H(\sigma(x),x)\right] - U(0,0)\left[x_0 - U\left(\frac{1}{n},0\right)H(\sigma(x),x)\right]\right\| \\
&\quad + \int_0^t \|U(t,s\|\|G(s,x(h(s)))\|ds \\
&\leq \|U(t,0)x_0 - U(0,0)x_0\| + \left\| U(t,0)U\left(\frac{1}{n},0\right)H(\sigma(x),x) - U(0,0)U\left(\frac{1}{n},0\right)H(\sigma(x),x)\right\| \\
&\quad + M\int_0^t W_{k_0}(s)ds \\
&\leq \|U(t,0)x_0 - U(0,0)x_0\| + \sup_{y\in\overline{U(\frac{1}{n},0)H(\sigma(B_{k_0}),B_{k_0})}} \left\|U(t,0)y - U(0,0)y\right\| \\
&\quad + M\int_0^t W_{k_0}(s)ds \\
&\quad \to 0, \quad \text{as} \quad t \to 0.
\end{aligned}
$$

Therefore, the operator Γ_n maps B_{k_0} into a family of locally equicontinuous functions on $[0,T]$ for any $T > 0$.

(2) We verify that for fixed $t \in [0,+\infty)$, the set $\Gamma_n(B_{k_0})(t) = \{(\Gamma_n x)(t) : x \in B_{k_0}\}$ is relatively compact in X.

If $t = 0$, then $\Gamma_n(B_{k_0})(0) = \left\{x_0 - U\left(\frac{1}{n},0\right)H(\sigma(x),x) : x \in B_{k_0}\right\}$, and clearly it is relatively compact in X because $U\left(\frac{1}{n},0\right)$ is compact on X and the set $\{H(\sigma(x),x) : x \in B_{k_0}\}$ is bounded in X.

Let $0 < t < +\infty$ be fixed. For $0 < \epsilon < t$ we define

$$
\begin{aligned}
(\Gamma_n^\epsilon x)(t) &= U(t,0)\left[x_0 - U\left(\frac{1}{n},0\right)H(\sigma(x),x)\right] + \int_0^{t-\epsilon} U(t,s)G(s,x(h(s)))ds \\
&= U(t,0)\left[x_0 - U\left(\frac{1}{n},0\right)H(\sigma(x),x)\right] + U(t,t-\epsilon)\int_0^{t-\epsilon} U(t-\epsilon,s)G(s,x(h(s)))ds.
\end{aligned}
$$

Since $U(t,s)$ is compact for each $t > s$, the set $\Gamma_n^\epsilon(B_{k_0})(t) = \{(\Gamma_n^\epsilon x)(t) : x \in B_{k_0}\}$ is relatively compact X for each ϵ, $0 < \epsilon < t$. In addition, for $x \in B_{k_0}$, we find that

$$
\begin{aligned}
\|(\Gamma_n x)(t) - (\Gamma_n^\epsilon x)(t)\| &\leq \int_{t-\epsilon}^t \|U(t,s)\|\|G(s,x(h(s)))\|ds \\
&\leq M\int_{t-\epsilon}^t W_{k_0}(s)ds \\
&\quad \to 0, \quad \text{as} \quad \epsilon \to 0.
\end{aligned}
$$

Thus there are relatively compact sets arbitrary close to the set $\Gamma_n(B_{k_0})(t)$ and hence the set $\Gamma_n(B_{k_0})(t)$ is also relatively compact in X.

(3) We certify that $\lim_{t,t'\to+\infty} \|(\Gamma_n x)(t) - (\Gamma_n x)(t')\| = 0$ uniformly for $x \in B_{k_0}$.

For any $x \in B_{k_0}$, we have that

$$\begin{aligned}
\|(\Gamma_n x)(t) - (\Gamma_n x)(t')\| &\le \|U(t,0)\| \left[\|x_0\| + \left\|U\left(\frac{1}{n},0\right)\right\| \|H(\sigma(x),x)\|\right] + \|U(t',0)\| \left[\|x_0\| + \left\|U\left(\frac{1}{n},0\right)\right\| \|H(\sigma(x),x)\|\right] \\
&\quad + \int_0^t \|U(t,s)\|\|G(s,x(h(s)))\|ds + \int_0^{t'} \|U(t',s)\|\|G(s,x(h(s)))\|ds \\
&\le Me^{-\gamma t}[\|x_0\| + Me^{-\frac{\gamma}{n}}Lk_0] + Me^{-\gamma t'}[\|x_0\| + Me^{-\frac{\gamma}{n}}Lk_0] \\
&\quad + Me^{-\gamma t}\int_0^t e^{\gamma s}W_{k_0}(s)ds + Me^{-\gamma t'}\int_0^{t'} e^{\gamma s}W_{k_0}(s)ds \\
&\le Me^{-\gamma t}[\|x_0\| + MLk_0] + Me^{-\gamma t'}[\|x_0\| + MLk_0] \\
&\quad + Me^{-\gamma t}\rho + Me^{-\gamma t'}\rho \\
&\quad \to 0, \quad \text{as} \quad t, t' \to +\infty,
\end{aligned}$$

which proves the assertion.

Therefore, from Lemma 2.3, Γ_n is a compact operator on B_{k_0}. This fact combined with the continuity of the operator Γ_n infers that Γ_n is a completely continuous map on B_{k_0}, and there is a fixed point $x_n(\cdot)$ for Γ_n on B_{k_0} via Schauder fixed point theorem. Consequently, the state-dependent nonlocal Cauchy problem (9) has a mild solution $x_n(\cdot)$ on $[0,+\infty)$ for each $n \in \mathbb{N}^+$. This completes the proof of Lemma 3.3. □

Now, we define the solution set D and the set $D(t)$, respectively, by

$$\begin{aligned}
&D = \{x_n \in C_b([0,+\infty);X) : x_n = \Gamma_n x_n, n \ge 1\}, \\
&D(t) = \{x_n(t) : x_n \in D, n \ge 1\}, \quad t \in [0,+\infty).
\end{aligned}$$

Lemma 3.4. *Suppose that the conditions of Theorem 3.2 are satisfied, then D is relatively compact in $C_b([0,+\infty);X)$.*

Proof. Using the same argument as in the proof of Lemma 3.3, it is easy to certify that the functions in D are equi-continuous on $[0,T]$ with any $T > 0$, for each $t \in (0,+\infty)$, $D(t)$ is relatively compact in X, and $\lim_{t,t'\to+\infty} \|(\Gamma_n x)(t) - (\Gamma_n x)(t')\| = 0$ uniformly in $n \in \mathbb{N}^+$. Subsequently, if we can prove $D(0)$ is relatively compact in X then the lemma follows immediately. To do so, for $x_n \in D, n \ge 1$, wet put

$$\overline{x_n}(t) = \begin{cases} x_n(\delta), & t \in [0,\delta), \\ x_n(t), & t \in [\delta,+\infty), \end{cases}$$

where δ comes from the condition (H_3). Then

$$H(\sigma(x_n), x_n) = H(\sigma(\overline{x_n}), \overline{x_n}). \tag{11}$$

Since D is relatively compact in $C_b([\delta,+\infty);X)$, without loss of generality, we may assume that there exists a subsequence of $\{\overline{x_n}\} \subseteq D$, still denote by itself, such that $\overline{x_n} \to x$ in $C_b([\delta,+\infty);X)$, as $n \to \infty$, for some

$x(\cdot)$. Thus, by the continuity of $H(\cdot,\cdot)$, the strong continuity of $U(t,0)$ at $t=0$, $U(0,0)=I$ and (11), we get

$$\begin{aligned}\left\|x_n(0)-\left(x_0-H(\sigma(x),x)\right)\right\| &= \left\|U\left(\frac{1}{n},0\right)H(\sigma(x_n),x_n)-H(\sigma(x),x)\right\| \\ &\leq \left\|U\left(\frac{1}{n},0\right)H(\sigma(x_n),x_n)-U\left(\frac{1}{n},0\right)H(\sigma(x),x)\right\| \\ &\quad+\left\|U\left(\frac{1}{n},0\right)H(\sigma(x),x)-H(\sigma(x),x)\right\| \\ &= \left\|U\left(\frac{1}{n},0\right)\left[H(\sigma(\overline{x_n}),\overline{x_n})-H(\sigma(x),x)\right]\right\| \\ &\quad+\left\|U\left(\frac{1}{n},0\right)H(\sigma(x),x)-U(0,0)H(\sigma(x),x)\right\| \\ &\to 0 \text{ as } n\to\infty,\end{aligned}$$

i.e., $D(0)$ is relatively compact in X. Thus, we obtain that D is relatively compact in $C_b([0,+\infty);X)$ by applying Lemma 2.3 once again. The proof is finished.
□

With the help of the preceding two lemmas we are now in a position to prove Theorem 3.2.

Proof. [Proof of Theorem 3.1.] According to Lemma 3.4, we know that D is relatively compact in $C_b([0,+\infty);X)$, therefore, we may assume, by passing of subsequence if necessary, that $x_n\to x^*\in C_b([0,+\infty);X)$ as $n\to\infty$. By the expression of mild solution $x_n(\cdot)$ for Eq. (9), we have

$$x_n(t)=U(t,0)\left[x_0-U\left(\frac{1}{n},0\right)H(\sigma(x_n),x_n)\right]+\int_0^t U(t,s)G(s,x_n(h(s)))ds,$$

for $0\leq t<+\infty$. Taking the limit as $n\to\infty$ on both sides, we arrive at

$$x^*(t)=U(t,0)\left[x_0-H(\sigma(x^*),x^*)\right]+\int_0^t U(t,s)G(s,x^*(h(s)))ds,$$

for $t\in[0,+\infty)$, which indicates that Eq. (7) has a mild solution $x^*(\cdot)\in C_b([0,+\infty);X)$. The proof is completed. □

In the following, we give a result on the uniqueness of global mild solutions for Eq. (7) by adopting the well-known Banach fixed point theorem. For this purpose, we have the following assumptions:

(H_4) The function $G(\cdot,\cdot):[0,+\infty)\times X\to X$ is continuous and satisfies Lipschitz condition in the second variable, that is, there is $L_1>0$ such that

$$\|G(t,x_1)-G(t,x_2)\|\leq L_1\|x_1-x_2\|,$$

for any $t\in[0,+\infty)$, $x_1,x_2\in X$ and the inequality

$$\|G(t,x)\|\leq L_1(\|x\|+1),$$

holds for any $(t,x)\in[0,+\infty)\times X$.

(H_5) The function $H(\cdot,\cdot):[0,+\infty)\times C_b([0,+\infty);X)\to X$ and function $\sigma(\cdot):C_b([0,+\infty);X)\to[0,+\infty)$ are both continuous, and there exists $L_2>0$ such that

$$\|H(\sigma(u),u)-H(\sigma(v),v)\|\leq L_2\|u-v\|_\infty,$$

for any $u,v\in C_b([0,+\infty);X)$, and

$$\|H(\sigma(u),u)\|\leq L_2(\|u\|_\infty+1)$$

holds for any $u\in C_b([0,+\infty);X)$.

Theorem 3.5. *Let $x(0) \in X$ and suppose that the conditions (H_1), (H_4) and (H_5) hold. Then Eq. (7) has a unique mild solution $x(\cdot) \in C_b([0,+\infty);X)$ if*

$$ML_2 + \frac{ML_1}{\gamma} < 1. \tag{12}$$

Proof. Let the operator Γ on $C_b([0,+\infty);X)$ be defined by

$$(\Gamma x)(t) = U(t,0)\,[x_0 - H(\sigma(x),x)] + \int_0^t U(t,s)G(s,x(h(s)))ds.$$

We will certify that the operator Γ has a unique fixed point in $C_b([0,+\infty);X)$ which is evidently a mild solution to Eq. (7).

Proceeding as in the proof of Lemma 3.3, it is obvious that $\Gamma x \in C_b([0,+\infty);X)$ for any $x \in C_b([0,+\infty);X)$. We next show that Γ is a contraction on $C_b([0,+\infty);X)$. Let $x, y \in C_b([0,+\infty);X)$, applying (H_1), (H_4) and (H_5) we see that

$$\begin{aligned}
\|(\Gamma x)(t) - (\Gamma y)(t)\| &\le \Big\|U(t,0)\,[H(\sigma(x),x) - H(\sigma(y),y)]\Big\| + \left\|\int_0^t U(t,s)[G(s,x(h(s))) - G(s,y(h(s)))]ds\right\| \\
&\le Me^{-\gamma t}L_2\|x-y\|_\infty + ML_1\int_0^t e^{-\gamma(t-s)}\|x(r(s)) - y(r(s))\|ds \\
&\le ML_2\|x-y\|_\infty + \frac{ML_1}{\gamma}\|x-y\|_\infty \\
&= \Big(ML_2 + \frac{ML_1}{\gamma}\Big)\|x-y\|_\infty.
\end{aligned}$$

Due to (12), Γ is contractive on $C_b([0,+\infty);X)$ and thus by Banach fixed point theorem Γ has a unique fixed point $x(\cdot)$ in $C_b([0,+\infty);X)$. This fixed point is the desired solution of Eq. (7). The proof is completed. □

4. Application

In order to show the applicability of the above obtained results, we study in this section the existence and uniqueness properties for the following non-autonomous heat equations with state-dependent nonlocal conditions

$$\begin{cases}
\dfrac{\partial z(t,x)}{\partial t} = a(t)\dfrac{\partial^2 z(t,x)}{\partial x^2} + \dfrac{z(t\cos t, x)}{10e^{2\xi t}}, \;\; t \in [0,+\infty),\; x \in [0,\pi], \\
z(t,0) = z(t,\pi) = 0, \;\; t \in [0,+\infty), \\
z(0,x) + \displaystyle\int_\delta^{+\infty} e^{-4s}[z(s,x) + \sin(z(s,x))]ds = z_0(x), \;\; x \in [0,\pi],
\end{cases} \tag{13}$$

where $z(t,x)$ represents the temperature of the point x at time moment t. $\xi, \delta \in (0,+\infty)$ and $a : [0,+\infty) \to [\xi,+\infty)$ is assumed to be bounded and Hölder continuous function, this is, there are constants $N \ge 1$, $C_a > 0$ and $0 < \theta \le 1$ such that

$$a(t) \le N, \;\; t \in [0,+\infty), \tag{14}$$

and

$$|a(t) - a(s)| < C_a|t-s|^\theta, \;\; t,s \in [0,+\infty). \tag{15}$$

To apply the obtained results to (13), we first need to rewrite this system into the form of Eq. (7). For this purpose, let $X = L^2([0,\pi],\mathbb{R})$ with norm $\|\cdot\|$, and we consider the operator $(A, D(A))$ be defined by

$$Az = z''$$

with the domain

$$D(A) = \{z(\cdot) \in X : z', z'' \in X, \text{ and } z(0) = z(\pi) = 0\}.$$

It is well known that A generates a compact, analytic and self-adjoint C_0-semigroup $(T(t))_{t\geq 0}$. Furthermore, A has a discrete spectrum, and its eigenvalues are $-n^2, n \in \mathbb{N}$, with the corresponding normalized eigenvectors $z_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$, $n = 1, 2, \cdots$. Moreover, we have that

$$Az = \sum_{n=1}^{\infty} -n^2 \langle z, z_n \rangle z_n, \quad z \in D(A),$$

and

$$T(t)z = \sum_{n=1}^{\infty} e^{-n^2 t} \langle z, z_n \rangle z_n, \quad z \in X,$$

from which we see that $\|T(t)\| \leq e^{-t}$.

We now show that under the conditions (P_1)-(P_4), $A(t)$ generates a unique evolution operator $\{U(t,s) : 0 \leq s \leq t < \infty\}$, which is compact for $t > s$. Let the operator family $\{A(t) : 0 \leq t < +\infty\}$ on X be given by

$$D(A(t)) = D(A), \quad t \in [0, +\infty)$$
$$A(t)z = a(t)Az, \quad z \in D(A(t)).$$

Then it is easy to see that $A(t)$ is a closed and the domain $D(A)$ is dense in X, and thus the condition (P_1) is satisfied. In the following, we consider Sturm-Liouville system:

$$\begin{cases} \big(\lambda I - A(t)\big) z(x) = g(x), \ x \in [0, \pi], \\ z(0) = z(\pi) = 0. \end{cases} \tag{16}$$

For $a(t) \geq \xi > 0$, Eq. (16) can be written as

$$\frac{\lambda z(x)}{a(t)} - z''(x) = \frac{g(x)}{a(t)}. \tag{17}$$

Multiplying both sides of formula (17) by z and integrating between 0 to π, we find that

$$\lambda \int_0^{\pi} \frac{|z(x)|^2}{a(t)} dx + \int_0^{\pi} |z'(x)|^2 dx = \int_0^{\pi} \frac{g(x)}{a(t)} z(x) dx.$$

Utilizing (14), Poincaré inequality and Hölder inequality, we have

$$\left(\frac{\lambda}{N} + 1\right) \int_0^{\pi} |z(x)|^2 dx \leq \frac{1}{\xi} \left(\int_0^{\pi} |g(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_0^{\pi} |z(x)|^2 dx\right)^{\frac{1}{2}}.$$

Consequently, we see that

$$\|R(\lambda, A(t)g\| = \|z\| \leq \frac{N}{\xi} \frac{1}{\lambda + N} \|g\|, \quad for\ all\ \lambda > 0,$$

so that we get

$$\|R(\lambda, A(t)\| \leq \frac{C_0}{\lambda + 1}, \quad C_0 = \frac{N}{\xi}, \tag{18}$$

which implies that (18) ensures the condition (P_2). Let us now consider

$$\|\big(A(t)-A(s)\big)A^{-1}(\tau)z\| \le |a(t)-a(s)|\|a^{-1}(\tau)\|\|z\| \le \frac{C_a}{\xi}|t-s|^{\theta}\|z\|, \quad t,s,\tau\in[0,+\infty), \tag{19}$$

where (15) has been used. The inequality (19) indicates that the condition (P_3) holds. Then $A(t)$ generates a unique evolution operator $\{U(t,s): 0\le s\le t<\infty\}$ defined by

$$U(t,s)z=\sum_{n=1}^{\infty} e^{-n^2\int_s^t a(\tau)d\tau}\langle z,e_n\rangle e_n, \quad 0\le s\le t<\infty,\ z\in X. \tag{20}$$

It follows from (20) and $a(t)\ge\xi$ that

$$\|U(t,s)\|\le e^{-\xi(t-s)}, \quad 0\le s\le t<\infty.$$

Then the condition (H_1) holds with $M=1$ and $\gamma=\xi$. Furthermore, using the similar method as in [38], one can easily certify that the resolvent $R(\lambda,A(t))$ is compact, which guarantees the condition (P_4). Therefore, the evolution operator $\{U(t,s): 0\le s\le t<\infty\}$ is compact for $t>s$ by Remark 2.1.

Now, we take $u(t)(x)=z(t,x)$ and define the abstract functions $G:[0,+\infty)\times X\to X$, $\sigma: C_b([0,+\infty);X)\to[0,+\infty)$ and $H(\cdot,\cdot):[0,+\infty)\times C_b([0,+\infty);X)\to X$ by

$$G(t,z)(x)=\frac{z(x)}{10e^{2\xi t}}, \quad z\in X,$$
$$\sigma(u)=e^{-4t}, \quad u\in C_b([0,+\infty);X),$$
$$H(\sigma(u),u)(x)=\int_{\delta}^{+\infty} e^{-4s}[u(s)(x)+\sin(u(s)(x))]ds, \quad u\in C_b([0,+\infty);X).$$

Additionally, we set $h(t)=t\cos t$, Then, non-autonomous heat equation with state-dependent nonlocal conditions (13) can be well reformulated as the abstract form (7) in X.

In the sequel, let us examine that for system (13) the conditions in Theorems 3.2 and 3.5 are all satisfied. First, from the definition of nonlinear term $G(\cdot,\cdot)$, for any $k>0$, we can easily certify that the conditions (H_2) and (H_4) hold with

$$W_k(t)=\frac{k}{10e^{2\xi t}}, \quad \rho=\frac{k}{10\xi}, \quad L_1=1.$$

Meanwhile, the nonlocal function $H(\cdot,\cdot)$ also satisfies the conditions (H_3) and (H_5). In fact, for $u\in C_b([0,+\infty);X)$, in view of Minkowski and Hölder inequalities we find that

$$\begin{aligned}
\|H(\sigma(u),u)\| &\le \left(\int_0^{\pi}\left|\int_{\delta}^{+\infty} e^{-4s}u(s)(x)ds\right|^2 dx\right)^{\frac12} + \left(\int_0^{\pi}\left|\int_{\delta}^{+\infty} e^{-4s}\sin(u(s)(x))ds\right|^2 dx\right)^{\frac12}\\
&\le \left(\int_0^{\pi}\left(\int_{\delta}^{+\infty} e^{-4s}|u(s)(x)|ds\right)^2 dx\right)^{\frac12} + \left(\int_0^{\pi}\left(\int_{\delta}^{+\infty} e^{-4s}|u(s)(x)|ds\right)^2 dx\right)^{\frac12}\\
&\le 2\left(\int_0^{\pi}\left(\int_{\delta}^{+\infty} e^{-4s}ds\right)\left(\int_{\delta}^{+\infty} e^{-4s}|u(s)(x)|^2ds\right)dx\right)^{\frac12}\\
&\le \left(\int_0^{\pi}\int_{\delta}^{+\infty} e^{-4s}|u(s)(x)|^2dsdx\right)^{\frac12}\\
&\le \frac12\|u\|_{\infty},
\end{aligned}$$

which shows (H_3) holds true with $L = \frac{1}{2}$. Similarly, we can also verify (H_5) holds with $L_2 = \frac{1}{2}$. Furthermore, it can easily be checked that the condition (8) is fulfilled by using $M = 1$, $L = \frac{1}{2}$ and $\rho < +\infty$. Therefore, we have the following results.

Proposition 4.1. *Let $z_0(\cdot) \in X$. Then, from Theorem 3.2, non-autonomous heat equation with state-dependent nonlocal conditions* (13) *admits a mild solution on* $[0, +\infty)$.

Proposition 4.2. *Let $z_0(\cdot) \in X$. Then from Theorem 3.5 there is a unique mild solution for non-autonomous heat equation with state-dependent nonlocal conditions* (13) *as long as* $\xi > 2$.

5. Conclusion

In this work, we discuss the existence results of global solutions for non-autonomous evolution equations with state-dependent nonlocal conditions. Firstly, we establish the existence of global solutions for the considered equation by using evolution operators theory and Schauder fixed point theorem. It is worth mentioning that in this case we do not require the compactness of $H(\cdot,\cdot)$. Secondly, we show by using Banach fixed point theorem that these solutions have uniqueness property under the situation that $G(\cdot,\cdot)$ and $H(\cdot,\cdot)$ satisfy Lipschitz condition. Finally, an example is given to illustrate the obtained results. Furthermore, it is interesting to study the global existence and regularity of solutions for finite delay non-autonomous evolution equations with state-dependent nonlocal conditions in the future work.

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References

[1] P. Acquistapace, Evolution operators and strong solutions of abstract linear parabolic equations, Differ. Int. Equ. 1 (1988) 433–457.
[2] H. Amann, Parabolic evolution equations and nonlinear boundary conditions, J. Differ. Equ. 72 (1988) 201–269.
[3] F. Andrade, C. Cuevas, H. R. Henríquez, Existence of asymptotically periodic solutions of partial functional differential equations with state-dependent delay, Appl. Anal. 100 (2021) 2965–2988.
[4] S. Arora, M. T. Mohan, J. Dabas, Approximate controllability of the non-autonomous impulsive evolution equation with state-dependent delay in Banach spaces, Nonl. Anal. Hybrid Syst. 39 (2021) 1–23.
[5] S. Arora, M. T. Mohan, J. Dabas, Existence and approximate controllability of non-autonomous functional impulsive evolution inclusions in Banach spaces, J. Differ. Equ. 307 (2022) 83–113.
[6] S. Arora, S. Singh, J. Dabas, M. T. Mohan, Approximate controllability of semilinear impulsive functional differential systems with non-local conditions, IMA J. Math. Control Inform. 37 (2020) 1070–1088.
[7] M. Benchohra, Z. Bouteffal, J. Henderson, S. Litimein, Measure of noncompactness and fractional integro-differential equations with state-dependent nonlocal conditions in Fréchet spaces, AIMS Math. 5 (2020) 15–25.
[8] L. Byszewski, Theorems about existence and uniqueness of a solution of a semilinear evolution nonlocal Cauchy problem, J. Math. Anal. Appl. 162 (1991) 496–505.
[9] N. Cao, X. Fu, Existence results of solutions for a neutral evolution equation with nonlocal conditions on infinite interval, J. Math. Anal. Appl. 510 (2022) 1–21.
[10] A. N. Carvalho, T. Dlotko, M. J. D. Nascimento, Non-autonomous semilinear evolution equations with almost sectorial operators, J. Evol. Equ. 8 (2008) 631–659.
[11] Y. Chang, R. Ponce, X. Yang, Solvability of fractional differential inclusions with nonlocal initial conditions via resolvent family of operators, Int. J. Nonl. Sci. Numer. Simul. 22 (2021) 33–44.
[12] P. Chen, Non-autonomous stochastic evolution equations with nonlinear noise and nonlocal conditions governed by noncompact evolution families, Discrete Contin. Dyn. Syst. 41 (2021) 2725–2737.
[13] P. Chen, Periodic solutions to non-autonomous evolution equations with multi-delays, Discrete Contin. Dyn. Syst. Ser. B 26 (2021) 2921–2939.
[14] P. Chen, Y. Li, Existence and uniqueness of strong solutions for nonlocal evolution equations, Electron. J. Differ. Equ. 2014 (2014) 1–9.
[15] P. Chen, Y. Li, X. Zhang, Cauchy problem for stochastic non-autonomous evolution equations governed by noncompact evolution families, Discrete Contin. Dyn. Syst. Ser. B 26 (2021) 1531–1547.
[16] K. Deng, Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions, J. Math. Anal. Appl. 179 (1993) 630–637.

[17] M. A. Diop, K. H. Bete, R. Kakpo, C. Ogouyandjou, Existence results for some integro-differential equations with state-dependent nonlocal conditions in Fréchet spaces, Nonauton. Dyn. Syst. 7 (2020) 272–280.
[18] Q. Dong, G. Li, Existence of solutions for semilinear differential equations with nonlocal conditions in Banach spaces, Electron. J. Qual. Theory Differ. Equ. 2009 (2009) 1–13.
[19] K-J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Springer, New York, 2000.
[20] K. Ezzinbi, G. Degla, P. Ndambomve, Controllability for some partial functional integrodifferential equations with nonlocal conditions in Banach spaces, Discuss. Math. Differ. Incl. Control Optim. 35 (2015) 25–46.
[21] W. E. Fitzgibbon, Semilinear functional equations in Banach space, J. Differ. Equ. 29 (1978) 1–14.
[22] A. Friedman, Partial Differential Equations, Holt, Rinehat and Winston, New York, 1969.
[23] X. Fu, X. Liu, Existence of periodic solutions for abstract neutral non-autonomous equations with infinite delay, J. Math. Anal. Appl. 325 (2007) 249–247.
[24] R. B. Guenther, J. W. Lee, Partial Differential Equations of Mathematical Physics and Integral Equations, Dover Publications, New York, 1996.
[25] E. Hernández, On abstract differential equations with state dependent non-local conditions, J. Math. Anal. Appl. 466 (2018) 408–425.
[26] E. Hernández, D. O'Regan, On state dependent non-local conditions, Appl. Math. Lett. 83 (2018) 103–109.
[27] E. Hernández, J. Wu, A. Chadha, Existence, uniqueness and approximate controllability of abstract differential equations with state-dependent delay, J. Differ. Equ. 269 (2020) 8701–8735.
[28] M. A. E. Herzallah, A. H. A. Radwan, Existence and uniqueness of the mild solution of an abstract semilinear fractional differential equation with state dependent nonlocal condition, Kragujevac J. Math. 45 (2021) 909–923.
[29] J. Liang, J. H. Liu, T. Xiao, Nonlocal Cauchy problems for nonautonomous evolution equations, Commun. Pure Appl. Anal. 5 (2006) 529–535.
[30] J. Liang, R. Nagel, T. Xiao, Nonautonomous heat equations with generalized Wentzell boundary conditions, J. Evol. Equ. 3 (2003) 321–331.
[31] M. A. McKibben, Discoving Evolution Equations with Applications: Volume I-Deterministic Equations, CRC Press, New York, 2011.
[32] A. Meraj, D. N. Pandey, Existence and uniqueness of extremal mild solutions for non-autonomous nonlocal integro-differential equations via monotone iterative technique, Filomat 33 (2019) 2985–2993.
[33] I. Mishra, M. Sharma, Approximate controllability of a non-autonomous differential equation, Proc. Indian Acad. Sci. Math. Sci. 128 (2018) 1–13.
[34] M. Necula, M. Popescu, I. I. Vrabie, Viability for delay evolution equations with nonlocal initial conditions, Nonl. Anal. 121 (2015) 164–172.
[35] T. H. Nguyen, T. N. H. Vu, Conditional stability and periodicity of solutions to evolution equations, J. Evol. Equ. 21 (2021) 3797–3812.
[36] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.
[37] S. M. Rankin III, Existence and asymptotic behavior of a functional differential equation in Banach space, J. Math. Anal. Appl. 88 (1982) 531–542.
[38] K. Ravikumar, M. T. Mohan, A. Anguraj, Approximate controllability of a non-autonomous evolution equation in Banach spaces, Numer. Algebra Control Optim. 11 (2021) 461–485.
[39] B. Shklyar, Exact null-controllability of interconnected abstract evolution equations with unbounded input operators, Discrete Contin. Dyn. Syst. 42 (2022) 463–479.
[40] P. Sobolevskii, On equations of parabolic type in Banach space, Amer. Math. Soc. Transl. Ser. 49 (1965) 1–62.
[41] H. Tanabe, Equations of Evolution, Pitman, London, 1979.
[42] H. L. Tidke, M. B. Dhakne, Existence and uniqueness of mild and strong solutions of nonlinear Volterra integrodifferential equations in Banach spaces, Demonstr. Math. 43 (2010) 643–652.
[43] J. Wu, Theory and applications of partial functional differential equations, Springer-Verlag, New York, 1996.
[44] S. Xie, C. Ge, Existence of solutions to nonlocal terminal value problems for nonlinear mixed integro-differential equations in Banach spaces, Acta Math. Sci. Ser. (A) 33 (2013) 551–560.
[45] Z. Yan, L. Han, Optimal mild solutions for a class of nonlocal multi-valued stochastic delay differential equations, J. Optim. Theory Appl. 181 (2019) 1053–1075.
[46] H. Yang, R. P. Agarwal, Y. Liang, Controllability for a class of integro-differential evolution equations involving non-local initial conditions, Int. J. Control. 90 (2017) 2567–2574.
[47] T. Zhu, C. Song, G. Li, Existence of mild solutions for abstract semilinear evolution equations in Banach spaces, Nonl. Anal. 75 (2012) 177–181.