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A time domain characterization of weak Gabor dual frames on the half real line

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Abstract. Due to $\mathbb{R}_+$ not being a group under addition, $L^2(\mathbb{R}_+)$ admits no traditional Gabor system as $L^2(\mathbb{R})$. Observing that $\mathbb{R}_+$ is a group under a new addition "$\oplus$", we in this paper introduce and characterize a class of weak Gabor dual frames in $L^2(\mathbb{R}_+)$ based on this new group structure. Some examples are also provided.

1. Introduction

In the last decades, frame theory has interested many researchers in pure and applied mathematics [2, 28]. The study of structured frames is an important part in the theory of function space frames. Among them, a Gabor frame for $L^2(\mathbb{R})$ is generated by a translation-and-modulation operator system acting on several functions in $L^2(\mathbb{R})$. Constructing Gabor dual frame pairs with desired properties has been attracting much attention of many mathematicians (see[3–8, 11, 12, 15, 18–20, 26, 27, 30] and references therein). Christensen, R. Y. Kim and H. O. Kim in [5, 6] investigated the constructions of the dual window functions of Gabor frames with the "partition of unity" property. Stoeva in [26] characterized Gabor dual frames pairs with compactly supported window functions. Christensen, Janssen, H. O. Kim and R. Y. Kim in [3] investigated a class of window functions for which approximately dual windows can be calculated explicitly, and presented the explicit estimates for the deviation from perfect reconstruction of the Gaussian and two-sided exponential function. In addition, subspace Gabor analysis and Gabor analysis on local fields have also been studied (see [1, 13–17, 21, 24, 25, 30, 31] and references therein). Recently, Li and Jia in [20] generalized "Gabor dual frame" to "weak Gabor dual frame" (also called weak Gabor bi-frame), and characterized weak Gabor dual pairs on periodic subsets of $\mathbb{R}$. Observe that a pair of weak Gabor dual frames is a pair of Gabor dual frames if they are Bessel sequences in addition.

This paper focuses on Gabor analysis on $L^2(\mathbb{R}_+)$ with $\mathbb{R}_+ = [0, \infty)$. In contrast to $\mathbb{R}$, $\mathbb{R}_+$ is not a group under addition. This results in $L^2(\mathbb{R}_+)$ admitting no traditional nontrivial shift invariant system. Thus it does not admit traditional wavelet or Gabor frames. Fortunately, $\mathbb{R}_+$ is an abelian group under a new

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addition "⊕" defined below. Based on this group structure, a class of wavelet frames for $L^2(\mathbb{R}_+)$ were introduced and investigated ([9, 10]). Motivated by the above works, in this paper, we investigate a class of weak Gabor dual frames for $L^2(\mathbb{R}_+)$.

To proceed, let us first review the addition "⊕". We denote by $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ the set of integers, nonnegative integers and positive integers, respectively; by $\mathbb{N}_t$ the set of $\{0, 1, \cdots, t-1\}$ for $t \in \mathbb{N}$; and by $\lfloor y \rfloor$, $\{y\}$ the integer and fractional parts of $y \in \mathbb{R}_+$ respectively. Given $1 < p \in \mathbb{N}$, define *addition* and *subtraction* on $\mathbb{N}_p$ by

$$x_1 \oplus x_2 = (x_1 + x_2)(\text{mod } p) = \begin{cases} x_1 + x_2 & \text{if } x_1 + x_2 < p; \\ x_1 + x_2 - p & \text{if } x_1 + x_2 \geq p \end{cases} \tag{1}$$

and

$$x_1 \ominus x_2 = (x_1 - x_2)(\text{mod } p) = \begin{cases} x_1 - x_2 & \text{if } x_1 \geq x_2; \\ x_1 - x_2 + p & \text{if } x_1 < x_2 \end{cases}$$

for $x_1, x_2 \in \mathbb{N}_p$. Every $y \in \mathbb{R}_+$ corresponds to the unique representation:

$$y = \sum_{j=1}^{\infty} y_{-j} p^{j-1} + \sum_{j=1}^{\infty} y_j p^{-j}, \tag{2}$$

where y_{-j}, $y_j \in \mathbb{N}_p$ are defined by

$$y_{-j} = \lfloor p^{1-j} y \rfloor (\text{ mod } p) \text{ and } y_j = \lfloor p^j y \rfloor (\text{ mod } p) \tag{3}$$

for $j \in \mathbb{N}$. For $\tilde{y} \in \mathbb{R}_+$, we define $\tilde{y}_j$, $\tilde{y}_{-j}$ similarly. Define *addition* "⊕"and *subtraction* "⊖" on $\mathbb{R}_+$ by

$$y \oplus \tilde{y} = \sum_{j=1}^{\infty} \left(y_{-j} \oplus \tilde{y}_{-j} \right) p^{j-1} + \sum_{j=1}^{\infty} \left(y_j \oplus \tilde{y}_j \right) p^{-j} \tag{4}$$

and

$$y \ominus \tilde{y} = \sum_{j=1}^{\infty} \left(y_{-j} \ominus \tilde{y}_{-j} \right) p^{j-1} + \sum_{j=1}^{\infty} \left(y_j \ominus \tilde{y}_j \right) p^{-j} \tag{5}$$

respectively for y, $\tilde{y} \in \mathbb{R}_+$. Then $\mathbb{R}_+$ is a group under "⊕" with the inverse operation "⊖", and the opposite of x is $\ominus x = 0 \ominus x$ for $x \in \mathbb{R}_+$. This makes $L^2(\mathbb{R}_+)$ to be closed under translation based on "⊖", and the Gabor analysis on $L^2(\mathbb{R}_+)$ possible. Define the *quasi-inner product* on $\mathbb{R}_+$ by

$$\langle y, \tilde{y} \rangle_p = \sum_{j=1}^{\infty} (y_j \tilde{y}_{-j} + y_{-j} \tilde{y}_j) \text{ for } y, \tilde{y} \in \mathbb{R}_+, \tag{6}$$

and the binary function

$$\chi(y, \tilde{y}) = e^{\frac{2\pi i}{p} \langle y, \tilde{y} \rangle_p} \text{ for } y, \tilde{y} \in \mathbb{R}_+. \tag{7}$$

And define the *modulation operator* $M_{x_0} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ and *translation operator* $T_{x_0} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ with $x_0 \in \mathbb{R}_+$ respectively by

$$M_{x_0} f(\cdot) = \chi(x_0, \cdot) f(\cdot) \text{ and } T_{x_0} f(\cdot) = f(\cdot \ominus x_0)$$

for $f \in L^2(\mathbb{R}_+)$. It is easy to check that they are both unitary operators on $L^2(\mathbb{R}_+)$, that their adjoint operators are given by

$$M^*_{x_0} f(\cdot) = \overline{\chi(x_0, \cdot)} f(\cdot) \text{ and } T^*_{x_0} f(\cdot) = f(\cdot \oplus x_0) \text{ for } f \in L^2(\mathbb{R}_+),$$

and that

$$M_y T_{\tilde{y}} f(\cdot) = \chi(y, \tilde{y}) T_{\tilde{y}} M_y f(\cdot) \text{ for } f \in L^2(\mathbb{R}_+) \text{ and } y, \tilde{y} \in \mathbb{R}_+.$$

Given $L \in \mathbb{N}$, $\mathbf{g} = \{g_l : 1 \le l \le L\} \subset L^2(\mathbb{R}_+)$, and $a, b > 0$, we define the Gabor system $X(\mathbf{g}, a, b)$ by

$$X(\mathbf{g}, a, b) = \{M_{mb} T_{na} g_l : m, n \in \mathbb{Z}_+, 1 \le l \le L\}. \tag{8}$$

In [23], a necessary condition and two sufficient conditions for such Gabor systems to be frames for $L^2(\mathbb{R}_+)$ are obtained in the time domain. In this paper, we work under the following general setup:

General setup:

Assumption 1. $1 < p \in \mathbb{N}$.

Assumption 2. $a = p^s$, $b = p^t \in \Lambda$ with $s + t \le 0$, where

$$\Lambda = \{p^s : s \in \mathbb{Z}\}. \tag{9}$$

Observe that a and b in this general setup are so special. It is because the equation

$$\chi(\alpha x, y) = \chi(x, \alpha y) \tag{10}$$

will be frequently used. But (10) need not hold for all $x, y, \alpha \in \mathbb{R}_+$ by Lemma 2.6 and Examples 2.7, 2.8 below. Fortunately, (10) holds for $x, y \in \mathbb{R}_+$ if $\alpha \in \Lambda$. Obviously, $e^{2\pi i \alpha x \cdot y} = e^{2\pi i x \cdot \alpha y}$ for $x, y, \alpha \in \mathbb{R}$. This demonstrates that Gabor analysis behaves essentially different between on $\mathbb{R}_+$ and $\mathbb{R}$. This paper is devoted to characterizing weak Gabor dual frame pairs. Let ζ_E denote the characteristic function of E for a measurable subset E of $\mathbb{R}_+$, and write

$$L_c^\infty(\mathbb{R}_+) = \{f \in L^2(\mathbb{R}_+) : f \in L^\infty(\mathbb{R}_+) \text{ and } \mathrm{supp}(f) \text{ is contained in a compact subset of } \mathbb{R}_+\}. \tag{11}$$

Then $L_c^\infty(\mathbb{R}_+)$ is dense in $L^2(\mathbb{R}_+)$. Given $\mathbf{g} = \{g_l : 1 \le l \le L\}$, $\mathbf{h} = \{h_l : 1 \le l \le L\} \subset L^2(\mathbb{R}_+)$, $(X(\mathbf{g}, a, b), X(\mathbf{h}, a, b))$ is said to be *a pair of weak dual frames* for $L^2(\mathbb{R}_+)$ associated with $L_c^2(\mathbb{R}_+)$ if

$$\langle f, \tilde{f} \rangle = \sum_{l=1}^{L} \sum_{m,n \in \mathbb{Z}_+} \langle f, M_{mb} T_{na} g_l \rangle \langle M_{mb} T_{na} h_l, \tilde{f} \rangle \text{ for } f, \tilde{f} \in L_c^\infty(\mathbb{R}_+). \tag{12}$$

Observe that the series in (12) is absolutely convergent by the arguments in Lemma 2.9 below. It is easy to check that $(X(\mathbf{g}, a, b), X(\mathbf{h}, a, b))$ is *a pair of dual frames* for $L^2(\mathbb{R}_+)$ if (12) holds, and $X(\mathbf{g}, a, b)$, $X(\mathbf{h}, a, b)$ are Bessel sequences in $L^2(\mathbb{R}_+)$. Therefore, "weak Gabor dual frames" generalize "Gabor dual frames". Example 3.2 demonstrates that it is a genuine generalization. Our main result is as follows:

Theorem 1.1. *Let p, a, b be as in the general setup and $\mathbf{g}, \mathbf{h} \subset L^2(\mathbb{R}_+)$. Then $(X(\mathbf{g}, a, b), X(\mathbf{h}, a, b))$ is a pair of weak dual frames for $L^2(\mathbb{R}_+)$ associated with $L_c^\infty(\mathbb{R}_+)$ if and only if*

$$\sum_{l=1}^{L} \sum_{n \in \mathbb{Z}_+} \overline{g_l(\cdot \ominus na \oplus \frac{k}{b})} h_l(\cdot \ominus na) = b\delta_{k,0} \tag{13}$$

a.e. on $(0, a)$ for $k \in \mathbb{Z}_+$.

Remark 1.2. *In Theorem 1.1, if $X(\mathbf{g}, a, b)$ and $X(\mathbf{h}, a, b)$ are Bessel sequences in $L^2(\mathbb{R}_+)$ in addition, then, by a standard argument, $(X(\mathbf{g}, a, b), X(\mathbf{h}, a, b))$ is a pair of dual frames if and only if (13) holds.*

Formally, Theorem 1.1 is similar to the "$L^2(\mathbb{R})$-Gabor dual frames" characterization. But its proof is nontrivial due to "$\oplus$" and "$\chi(\cdot, \cdot)$" herein being essentially different from "+" and modulation factor in "$L^2(\mathbb{R})$-Gabor systems". Section 2 focuses on some properties of "$\oplus$" and Gabor systems in $L^2(\mathbb{R}_+)$. In particular, from Lemmas 2.2, 2.6 and Examples 2.4-2.8, we know that the distribution law for "$\oplus$" and multiplication does not hold, and that $\chi(\alpha\cdot, \cdot)$ need not equal to $\chi(\cdot, \alpha\cdot)$ for general $\alpha \in \mathbb{R}_+$. This demonstrates that Gabor analysis behaves different between $\mathbb{R}_+$ and $\mathbb{R}$. Section 3 gives the proof of Theorem 1.1. Some examples are also provided.

2. Preliminaries

This section is devoted to some properties of "$\oplus$" and Gabor systems in $L^2(\mathbb{R}_+)$. By a simple computation, we have the following lemma.

Lemma 2.1. *(i)* $[0, 1) \oplus x = [0, 1) \ominus x = [0, 1)$ *for* $x \in [0, 1)$*;*
(ii) $\{\ominus x : x \in [0, p^J)\} = [0, p^J) = [0, p^J) \oplus [0, p^J) = [0, p^J) \ominus [0, p^J)$ *for* $J \in \mathbb{Z}$*.*

The following lemma shows that the distribution law for "$\oplus$" and multiplication holds if the multipliers belong to Λ.

Lemma 2.2. *Let* Λ *be as in (9), and* $\alpha \in \Lambda$*. Then*

$$\alpha(x \oplus y) = \alpha x \oplus \alpha y \tag{14}$$

and

$$\alpha(x \ominus y) = \alpha x \ominus \alpha y \tag{15}$$

for $x, y \in \mathbb{R}_+$*.*

Proof. Observe that (14) implies (15). Indeed, suppose (14) holds. Then

$$\alpha x = \alpha((x \ominus y) \oplus y) = \alpha(x \ominus y) \oplus \alpha y$$

for $x, y \in \mathbb{R}_+$. This leads to (15). Next we prove (14). First we claim that

$$p(x \oplus y) = px \oplus py \text{ for } x, y \in \mathbb{R}_+ \tag{16}$$

implies

$$\alpha(x \oplus y) = \alpha x \oplus \alpha y \text{ for } x, y \in \mathbb{R}_+ \text{ and } \alpha \in \Lambda. \tag{17}$$

Indeed, suppose (16) holds. Then, given $s \in \mathbb{N}$,

$$\begin{aligned} p^s(x \oplus y) &= p^{s-1}(px \oplus py) \\ &= p^{s-2}(p^2x \oplus p^2y) \\ &= \cdots \\ &= p^sx \oplus p^sy \end{aligned}$$

for $x, y \in \mathbb{R}_+$. Thus

$$x \oplus y = p^s(p^{-s}x \oplus p^{-s}y).$$

This implies that, given $s \in \mathbb{N}$,

$$p^{-s}(x \oplus y) = p^{-s}x \oplus p^{-s}y$$

for $x, y \in \mathbb{R}_+$. Therefore, (17) holds. Next we prove (16). Arbitrarily fix $x, y \in \mathbb{R}_+$. We have

$$\begin{aligned} p(x \oplus y) &= p\left(\sum_{j=1}^{\infty}(x_{-j} \oplus y_{-j})p^{j-1} + \sum_{j=1}^{\infty}(x_j \oplus y_j)p^{-j}\right) \\ &= (x_1 \oplus y_1) + \sum_{j=1}^{\infty}(x_{-j} \oplus y_{-j})p^j + \sum_{j=2}^{\infty}(x_j \oplus y_j)p^{1-j} \\ &= (x_1 \oplus y_1) + \sum_{j=1}^{\infty}(x_{-j} \oplus y_{-j})p^j + \sum_{j=1}^{\infty}(x_{1+j} \oplus y_{1+j})p^{-j}. \end{aligned} \tag{18}$$

Also observe that

$$px = \sum_{j=1}^{\infty} x_{-j}p^{j} + \sum_{j=1}^{\infty} x_{j}p^{1-j} \text{ and } py = \sum_{j=1}^{\infty} y_{-j}p^{j} + \sum_{j=1}^{\infty} y_{j}p^{1-j},$$

equivalently,

$$px = x_1 + \sum_{j=1}^{\infty} x_{-j}p^{j} + \sum_{j=1}^{\infty} x_{1+j}p^{-j} \tag{19}$$

and

$$py = y_1 + \sum_{j=1}^{\infty} y_{-j}p^{j} + \sum_{j=1}^{\infty} y_{1+j}p^{-j}. \tag{20}$$

It leads to (16) by (18). The proof is completed. □

Remark 2.3. *We remark that (14) does not necessarily hold if $\alpha \notin \Lambda$. The following Example 2.4 is a counterexample.*

For $\alpha \in \mathbb{R}_+$, we have

$$\alpha(p \oplus p^{-1}) = (\alpha_1 + \alpha_{-2}) + \sum_{j=1}^{\infty}(\alpha_{-j} + \alpha_{-j-2})p^{j} + (\alpha_{-1} + \alpha_2)p^{-1} + \sum_{j=2}^{\infty}(\alpha_{1+j} + \alpha_{j-1})p^{-j}, \tag{21}$$

$$\alpha p \oplus \alpha p^{-1} = (\alpha_1 \oplus \alpha_{-2}) + \sum_{j=1}^{\infty}(\alpha_{-j} \oplus \alpha_{-j-2})p^{j} + (\alpha_{-1} \oplus \alpha_2)p^{-1} + \sum_{j=2}^{\infty}(\alpha_{j-1} \oplus \alpha_{1+j})p^{-j} \tag{22}$$

by a simple computation. This leads to the following example.

Example 2.4. *$\alpha(p \oplus p^{-1}) \neq \alpha p \oplus \alpha p^{-1}$ for $\alpha \in \mathbb{R}_+$ satisfying*

$$\begin{array}{rl} \textit{either} & \alpha_1 + \alpha_{-2} > p \textit{ or } \alpha_{-1} + \alpha_2 > p \\ \textit{or} & \{j \in \mathbb{N} : \alpha_{-j} + \alpha_{-j-2} > p\} \cup \{j \in \mathbb{N}\backslash\{1\} : \alpha_{j-1} + \alpha_{1+j} > p\} \neq \emptyset. \end{array} \tag{23}$$

The following lemma gives a "$\oplus$"-based partition of $\mathbb{R}_+$.

Lemma 2.5. *Let Λ be as in (9). Then, given $\alpha \in \Lambda$ and $\gamma \in \mathbb{R}_+$, $\{[0, \alpha) \oplus \gamma \oplus \alpha k : k \in \mathbb{Z}_+\}$ is a partition of $\mathbb{R}_+$.*

Proof. By Lemma 2.2 and Lemma 2.1 (*i*), we have

$$\begin{aligned} [0, \alpha) \oplus \gamma \oplus \alpha k &= \alpha\Big([0, 1) \oplus \alpha^{-1}\gamma \oplus k\Big) \\ &= \alpha\Big(([0, 1) \oplus \{\alpha^{-1}\gamma\}) \oplus \lfloor \alpha^{-1}\gamma \rfloor \oplus k\Big) \\ &= \alpha\Big([0, 1) \oplus (\lfloor \alpha^{-1}\gamma \rfloor \oplus k)\Big). \end{aligned}$$

It follows that $\{[0, \alpha) \oplus \gamma \oplus \alpha k : k \in \mathbb{Z}_+\}$ is a partition of $\mathbb{R}_+$ if and only if $\{[0, 1) \oplus (\lfloor \alpha^{-1}\gamma \rfloor \oplus k) : k \in \mathbb{Z}_+\}$ is a partition of $\mathbb{R}_+$. Due to $\lfloor \alpha^{-1}\gamma \rfloor \oplus \mathbb{Z}_+ = \mathbb{Z}_+$ and $\lfloor \alpha^{-1}\gamma \rfloor \oplus k \neq \lfloor \alpha^{-1}\gamma \rfloor \oplus l$ for $k \neq l$ in $\mathbb{Z}_+$, this is equivalent to $\{[0, 1) \oplus k : k \in \mathbb{Z}_+\}$ being a partition of $\mathbb{R}_+$. It holds since $[0, 1) \oplus k = [0, 1) + k$ and $\{[0, 1) + k : k \in \mathbb{Z}_+\}$ is a partition of $\mathbb{R}_+$. The proof is completed. □

The following lemma gives a sufficient condition on $\chi(\alpha\cdot, \cdot) = \chi(\cdot, \alpha\cdot)$.

Lemma 2.6. *Let Λ be as in (9), and $\alpha \in \Lambda$. Then*

$$\langle x, \alpha y\rangle_p = \langle \alpha x, y\rangle_p \tag{24}$$

and

$$\chi(\alpha x, y) = \chi(x, \alpha y) \tag{25}$$

for $x, y \in \mathbb{R}_+$.

Proof. Obviously, (24) implies (25) . Next we prove (24). Similarly to the beginning arguments in Lemma 2.2, we only need to prove

$$\langle x, py\rangle_p = \langle px, y\rangle_p \text{ for } x, y \in \mathbb{R}_+. \tag{26}$$

Now we do this. For $x, y \in \mathbb{R}_+$, we have

$$x = \sum_{j=1}^{\infty} x_{-j}p^{j-1} + \sum_{j=1}^{\infty} x_j p^{-j} \text{ and } y = \sum_{j=1}^{\infty} y_{-j}p^{j-1} + \sum_{j=1}^{\infty} y_j p^{-j}, \tag{27}$$

where $x_j, x_{-j}, y_j, y_{-j} \in \mathbb{N}_p$ for $j \in \mathbb{N}$. This implies that

$$px = \sum_{j=1}^{\infty} x_{-j}p^{j} + \sum_{j=1}^{\infty} x_j p^{1-j}, \qquad py = \sum_{j=1}^{\infty} y_{-j}p^{j} + \sum_{j=1}^{\infty} y_j p^{1-j},$$

equivalently,

$$px = x_1 + \sum_{j=1}^{\infty} x_{-j}p^{j} + \sum_{j=1}^{\infty} x_{1+j} p^{-j} \tag{28}$$

and

$$py = y_1 + \sum_{j=1}^{\infty} y_{-j}p^{j} + \sum_{j=1}^{\infty} y_{1+j} p^{-j}. \tag{29}$$

It follows that

$$\langle px, y\rangle_p \quad = \quad x_1y_1 + \sum_{j=1}^{\infty} x_{-j}y_{1+j} + \sum_{j=1}^{\infty} x_{1+j}y_{-j} \tag{30}$$

by (27) and (28), and

$$\langle x, py\rangle_p \quad = \quad y_1x_1 + \sum_{j=1}^{\infty} y_{-j}x_{1+j} + \sum_{j=1}^{\infty} y_{1+j}x_{-j} \tag{31}$$

by (27) and (29). Combining (30) and (31) leads to (26). The proof is completed. □

The following Examples 2.7 and 2.8 show that (25) need not hold if $\alpha \notin \Lambda$.

Example 2.7. *Let $p = 2$, and $x, y \in \mathbb{R}_+$ satisfy*

$$x_{2j} = y_{2j} = 0 \textit{ for } j \in \mathbb{N}; \quad x_{-2j} = y_{-2j} = 0 \textit{ for } 2 \le j \in \mathbb{N}; \quad x_{-2} = x_{-3} = y_{-2} = y_{-3} = 0; \tag{32}$$

$$x_1 = x_{-1} = y_1 = y_{-1} = 1;\ x_3 \neq y_3. \tag{33}$$

Then $\langle 3x, y\rangle_2 - \langle x, 3y\rangle_2 = y_3 - x_3 \in \{1, -1\}$ by a standard argument. Thus $\chi(x, 3y) \neq \chi(3x, y)$.

Example 2.8. *Let* $2 < p \in \mathbb{N}$ *and* $x, y \in \mathbb{R}_+$ *satisfy*

$$x_j + x_{j+1} < p,\ y_j + y_{j+1} < p \text{ for } j \in \mathbb{N}, \tag{34}$$
$$x_{-j} + x_{-j+1} < p,\ y_{-j} + y_{-j+1} < p \text{ for } 2 \le j \in \mathbb{N}, \tag{35}$$
$$x_{-1} + x_1 \ge p,\ x_{-2} + x_{-1} < p - 1,\ y_2 \ne 0. \tag{36}$$

Then $\langle (p+1)x, y\rangle_p - \langle x, (p+1)y\rangle_p = -py_1 + y_2$. *This implies that* $\chi(x, (p+1)y) \ne \chi((p+1)x, y)$.

The following lemma gives an expression of the inner product of sampling sequences related to two modulation systems.

Lemma 2.9. *Let* Λ *be as in (9), and* $b \in \Lambda$. *Then, given* $g, h \in L^2(\mathbb{R}_+)$,

$$\sum_{m\in\mathbb{Z}_+} \langle f, M_{mb}g\rangle\langle M_{mb}h, f\rangle = \frac{1}{b}\int_{\mathbb{R}_+} \overline{f(x)}h(x) \sum_{k\in\mathbb{Z}_+} \overline{g(x\oplus \frac{k}{b})} f(x\oplus\frac{k}{b})dx$$

for an arbitrary measurable function f *on* $\mathbb{R}_+$ *with* $\sum_{k\in\mathbb{Z}_+} |f(\cdot\oplus\frac{k}{b})|^2 \in L^\infty([0, \frac{1}{b}))$.

Proof. Arbitrarily fix f satisfying $\sum_{k\in\mathbb{Z}_+} |f(\cdot\oplus\frac{k}{b})|^2 \in L^\infty([0, \frac{1}{b}))$. By Lemma 2.5, $\{[0, \frac{1}{b})\oplus\frac{k}{b} : k \in \mathbb{Z}_+\}$ is a partition of $\mathbb{R}_+$. It follows that

$$\begin{aligned}\|f\|^2 &= \int_{[0,\frac{1}{b})}\sum_{k\in\mathbb{Z}_+}|f(x\oplus\frac{k}{b})|^2dx\\ &\le \frac{1}{b}\|\sum_{k\in\mathbb{Z}_+}|f(\cdot\oplus\frac{k}{b})|^2\|_{L^\infty([0,\frac{1}{b}))}\\ &< \infty,\end{aligned} \tag{37}$$

and

$$\begin{aligned}\int_{[0,\frac{1}{b})}(\sum_{k\in\mathbb{Z}_+}|f(x\oplus\frac{k}{b})||g(x\oplus\frac{k}{b})|)^2dx &\le \int_{[0,\frac{1}{b})}(\sum_{k\in\mathbb{Z}_+}|f(x\oplus\frac{k}{b})|^2)(\sum_{k\in\mathbb{Z}_+}|g(x\oplus\frac{k}{b})|^2)dx\\ &\le \|\sum_{k\in\mathbb{Z}_+}|f(\cdot\oplus\frac{k}{b})|^2\|_{L^\infty([0,\frac{1}{b}))}\|g\|^2 < \infty\end{aligned} \tag{38}$$

which leads to $(\sum_{k\in\mathbb{Z}_+}|f(\cdot\oplus\frac{k}{b})||g(\cdot\oplus\frac{k}{b})|) \in L^1([0, \frac{1}{b}))$. Similarly,

$$\int_{[0,\frac{1}{b})}(\sum_{k\in\mathbb{Z}_+}|f(x\oplus\frac{k}{b})||h(x\oplus\frac{k}{b})|)^2dx < \infty. \tag{39}$$

Observe that

$$\begin{aligned}\chi(mb, x\oplus\frac{k}{b}) &= \chi(mb, x)\chi(mb, \frac{k}{b})\\ &= \chi(mb, x)\end{aligned}$$

for $x \in \mathbb{R}_+$ and $m, k \in \mathbb{Z}_+$ by Lemma 2.6. This implies that

$$\begin{aligned}\langle f, M_{mb}g\rangle &= \int_{[0,\frac{1}{b})}\sum_{k\in\mathbb{Z}_+} f(x\oplus\frac{k}{b})\overline{g(x\oplus\frac{k}{b})\chi(mb, x\oplus\frac{k}{b})}dx\\ &= \int_{[0,\frac{1}{b})}\left(\sum_{k\in\mathbb{Z}_+} f(x\oplus\frac{k}{b})\overline{g(x\oplus\frac{k}{b})}\right)\overline{\chi(mb, x)}dx\end{aligned} \tag{40}$$

and

$$\langle f, M_{mb}h\rangle = \int_{[0,\,\frac{1}{b})}\left(\sum_{k\in\mathbb{Z}_+} f(x\oplus\frac{k}{b})\overline{h(x\oplus\frac{k}{b})}\right)\overline{\chi(mb,\,x)}dx. \tag{41}$$

Since $\{\sqrt{b}\chi(mb,\cdot):\ m\in\mathbb{Z}_+\}$ is an orthonormal basis for $L^2([0,\,\frac{1}{b}))$, we have

$$\sum_{k\in\mathbb{Z}_+}\langle f, M_{mb}g\rangle\langle M_{mb}h, f\rangle = \frac{1}{b}\int_{[0,\,\frac{1}{b})}\mathcal{G}(x)\left(\sum_{k\in\mathbb{Z}_+}\overline{f(x\oplus\frac{k}{b})}h(x\oplus\frac{k}{b})\right)dx \tag{42}$$

by (38)-(41), where

$$\mathcal{G}(\cdot) = \sum_{k\in\mathbb{Z}_+} f(\cdot\oplus\frac{k}{b})\overline{g(\cdot\oplus\frac{k}{b})}. \tag{43}$$

From (38) and (39), it follows that

$$|\mathcal{G}(\cdot)|\sum_{k\in\mathbb{Z}_+}|\overline{f(\cdot\oplus\frac{k}{b})}h(\cdot\oplus\frac{k}{b})| \in L^1([0,\,\frac{1}{b})).$$

Thus

$$\begin{aligned}\sum_{m\in\mathbb{Z}_+}\langle f, M_{mb}g\rangle\langle M_{mb}h, f\rangle &= \frac{1}{b}\sum_{k\in\mathbb{Z}_+}\int_{[0,\,\frac{1}{b})}\mathcal{G}(x)\overline{f(x\oplus\frac{k}{b})}h(x\oplus\frac{k}{b})dx\\ &= \frac{1}{b}\int_{\mathbb{R}_+}\overline{f(x)}h(x)\sum_{k\in\mathbb{Z}_+}\overline{g(x\oplus\frac{k}{b})}f(x\oplus\frac{k}{b})dx\end{aligned}$$

by Lemma 2.5 and the $\frac{1}{b}\mathbb{Z}_+$-periodicity of $\mathcal{G}(\cdot)$ according to $\oplus$. The proof is completed. □

Lemma 2.10. *[29, Theorem 2.2] Let p, a, b be as in the general setup and $h\in L^2(\mathbb{R}_+)$. Suppose*

$$B = \frac{1}{b}esssup_{x\in[0,a)}\sum_{k\in\mathbb{Z}_+}|\sum_{n\in\mathbb{Z}_+} h(x\ominus na)\overline{h(x\ominus na\oplus\frac{k}{b})}| < \infty \tag{44}$$

and

$$A = \frac{1}{b}essinf_{x\in[0,a)}\left[\sum_{n\in\mathbb{Z}_+}|h(x\ominus na)|^2 - \sum_{k\in\mathbb{N}}|\sum_{n\in\mathbb{Z}_+} h(x\ominus na)\overline{h(x\ominus na\oplus\frac{k}{b})}|\right] > 0. \tag{45}$$

Then $X(h, a, b)$ is a frame for $L^2(\mathbb{R}_+)$ with bounds A and B. In particular, if (44) is satisfied, then it is a Bessel sequence in $L^2(\mathbb{R}_+)$ with Bessel bound B.

The following lemma reduces the inner product of sampling sequences corresponding to two Gabor systems to an integral.

Lemma 2.11. *Let p, a, b be as in the general setup and $g, h\in L^2(\mathbb{R}_+)$. Then, given $x_0\in\mathbb{R}_+$, we have*

$$\sum_{m,n\in\mathbb{Z}_+}\langle f, M_{mb}T_{na}g\rangle\langle M_{mb}T_{na}h, f\rangle = \frac{1}{b}\int_{[0,\,\frac{1}{b})\oplus x_0}\langle\gamma(x)F(x),\,F(x)\rangle\,dx \tag{46}$$

for $f\in L_c^\infty(\mathbb{R}_+)$, where

$$\gamma(x) = \left(\sum_{n\in\mathbb{Z}_+}\overline{g(x\ominus na\oplus\frac{k}{b})}h(x\ominus na\oplus\frac{j}{b})\right)_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}_+}\quad\text{and } F(x) = \left\{f(x\oplus\frac{k}{b})\right\}_{k\in\mathbb{Z}_+}.$$

Proof. Suppose $\mathrm{supp}(f) \subset [0, p^J)$ for some $J \in \mathbb{Z}$. Write

$$I(x) = \{(n, k) \in \mathbb{Z}_+^2 : x \ominus na, x \ominus na \oplus \frac{k}{b} \in [0, p^J)\}, \tag{47}$$

$$\tilde{I}(x) = \{k \in \mathbb{Z}_+ : x \oplus \frac{k}{b} \in [0, p^J\}) \tag{48}$$

for $x \in \mathbb{R}_+$, and

$$I = \{k \in \mathbb{Z}_+ : y, y \oplus \frac{k}{b} \in [0, p^J) \text{ for some } y \in \mathbb{R}_+\}. \tag{49}$$

Let us estimate (47)-(49). By Lemma 2.1 (*ii*) and the general setup,

$$\begin{aligned} I(x) &\subset \{(n, k) \in \mathbb{Z}_+^2 : na \ominus x \in [0, p^J), \frac{k}{b} \in [0, p^J) \ominus [0, p^J)\} \\ &= \{(n, k) \in \mathbb{Z}_+^2 : na \in [0, p^J) \oplus x, k \in b[0, p^J)\} \\ &\subset \{(n, k) \in \mathbb{Z}_+^2 : n \in p^{-s}([0, p^J) \oplus x), k \in [0, p^{J+t})\}. \end{aligned}$$

This implies that

$$\begin{aligned} I(x) &\subset \{(n, k) \in \mathbb{Z}_+^2 : n \in p^{J-s}([0, 1) \oplus p^{-J}x), k \in [0, p^{J+t})\} \\ &\subset \{(n, k) \in \mathbb{Z}_+^2 : n \in p^{J-s}[\lfloor p^{-J}x \rfloor, \lfloor p^{-J}x \rfloor + 1), k \in [0, p^{J+t})\} \\ &= \{(n, k) \in \mathbb{Z}_+^2 : n \in [p^{J-s}\lfloor p^{-J}x \rfloor, p^{J-s}(\lfloor p^{-J}x \rfloor + 1)), k \in [0, p^{J+t})\} \end{aligned} \tag{50}$$

by Lemma 2.1 (*i*), Lemma 2.2 and the fact that $[0, 1) \oplus k = [0, 1) + k$ for $k \in \mathbb{Z}_+$. Similarly, we have

$$\tilde{I}(x) \subset [p^{J+t}\lfloor p^{-J}x \rfloor, p^{J+t}(\lfloor p^{-J}x \rfloor + 1)), \tag{51}$$

$$I \subset [0, p^{J+t}). \tag{52}$$

It follows that their cardinalities satisfy

$$\mathrm{card}(I(x)) \leq (\lfloor p^{J-s} \rfloor + 1)(\lfloor p^{J+t} \rfloor + 1), \tag{53}$$

$$\mathrm{card}(\tilde{I}(x)) \leq \lfloor p^{J+t} \rfloor + 1, \tag{54}$$

$$\mathrm{card}(I) \leq \lfloor p^{J+t} \rfloor + 1. \tag{55}$$

By (50) and (55),

$$\sum_{k \in \mathbb{Z}_+} |\sum_{n \in \mathbb{Z}_+} \overline{f(\cdot \ominus na)} f(\cdot \ominus na \oplus \frac{k}{b})| \leq (\lfloor p^{J-s} \rfloor + 1)(\lfloor p^{J+t} \rfloor + 1)\|f\|^2_{L^\infty(\mathbb{R}_+)}.$$

This implies that $X(\bar{f}, a, b)$ is a Bessel sequence in $L^2(\mathbb{R}_+)$ by Lemma 2.10. Observe that

$$\langle f, M_{mb}T_{na}g \rangle = \overline{\chi(mb, na)} \langle \bar{g}, M_{mb}T_{\ominus na}\bar{f} \rangle,$$
$$\langle f, M_{mb}T_{na}h \rangle = \overline{\chi(mb, na)} \langle \bar{h}, M_{mb}T_{\ominus na}\bar{f} \rangle.$$

It follows that the left-hand side of (46) is well defined. Also by (51) and (54),

$$\sum_{k \in \mathbb{Z}_+} |f(\cdot \oplus \frac{k}{b})|^2 \leq (\lfloor p^{J+t} \rfloor + 1)\|f\|^2_{L^\infty(\mathbb{R}_+)}. \tag{56}$$

Thus

$$\begin{aligned} &\sum_{m,n \in \mathbb{Z}_+} \langle f, M_{mb}T_{na}g \rangle \langle M_{mb}T_{na}h, f \rangle \\ = \;& \frac{1}{b} \sum_{n \in \mathbb{Z}_+} \int_{\mathbb{R}_+} \overline{f(x)} h(x \ominus na) \sum_{k \in \mathbb{Z}_+} \overline{g(x \ominus na \oplus \frac{k}{b})} f(x \oplus \frac{k}{b}) dx \end{aligned} \tag{57}$$

by Lemma 2.9. By (52), we have

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}_+}\int_{\mathbb{R}_+}|\overline{f(x)}h(x\ominus na)|\sum_{k\in\mathbb{Z}_+}|\overline{g(x\ominus na\oplus \frac{k}{b})}f(x\oplus\frac{k}{b})|dx\\
\leq\ & \|f\|^2_{L^\infty(\mathbb{R}_+)}\sum_{k\in[0,p^{J+t})\cap\mathbb{Z}_+}\sum_{n\in\mathbb{Z}_+}\int_{[0,p^J)}|g(x\ominus na\oplus\frac{k}{b})h(x\ominus na)|dx. \qquad (58)
\end{aligned}$$

Since $[0, p^J)$ is bounded, there exists a finite subset E of $\mathbb{Z}_+$ such that $[0, p^J) \subset \cup_{j\in E}([0, a) + ja)$. By Lemma 2.2,

$$\begin{aligned}
[0, a) + ja &= a([0, 1) + j)\\
&= a([0, 1) \oplus j)\\
&= [0, a) \oplus ja
\end{aligned}$$

for $j \in \mathbb{Z}_+$. So $[0, p^J) \subset \cup_{j\in E}([0, a) \oplus ja)$. Thus (58) implies that

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}_+}\int_{\mathbb{R}_+}|\overline{f(x)}h(x\ominus na)|\sum_{k\in\mathbb{Z}_+}|\overline{g(x\ominus na\oplus \frac{k}{b})}f(x\oplus\frac{k}{b})|dx\\
\leq\ & \|f\|^2_{L^\infty(\mathbb{R}_+)}\sum_{k\in[0,p^{J+t})\cap\mathbb{Z}_+}\sum_{n\in\mathbb{Z}_+}\sum_{j\in E}\int_{[0,a)\oplus ja}|g(x\ominus na\oplus\frac{k}{b})h(x\ominus na)|dx\\
=\ & \|f\|^2_{L^\infty(\mathbb{R}_+)}\sum_{k\in[0,p^{J+t})\cap\mathbb{Z}_+}\sum_{j\in E}\sum_{n\in\mathbb{Z}_+}\int_{[0,a)}|g(x\oplus ja\ominus na\oplus\frac{k}{b})h(x\oplus ja\ominus na)|dx\\
=\ & \|f\|^2_{L^\infty(\mathbb{R}_+)}\sum_{k\in[0,p^{J+t})\cap\mathbb{Z}_+}\sum_{j\in E}\sum_{n\in\mathbb{Z}_+}\int_{[0,a)}|g(x\ominus na\oplus\frac{k}{b})h(x\ominus na)|dx\\
=\ & \mathrm{card}(E)\|f\|^2_{L^\infty(\mathbb{R}_+)}\sum_{k\in[0,p^{J+t})\cap\mathbb{Z}_+}\int_{\mathbb{R}_+}|g(x\oplus\frac{k}{b})h(x)|dx\\
\leq\ & (\lfloor p^{J+t}\rfloor+1)\mathrm{card}(E)\|f\|^2_{L^\infty(\mathbb{R}_+)}\|g\|\|h\|\\
<\ & \infty. \qquad (59)
\end{aligned}$$

Also observe that $\{[0, \frac{1}{b}) \oplus x_0 \oplus \frac{j}{b} : j \in \mathbb{Z}_+\}$ is a partition of $\mathbb{R}_+$ by Lemma 2.5. Collecting (57) and (59) leads to

$$\begin{aligned}
&\sum_{m,n\in\mathbb{Z}_+}\langle f, M_{mb}T_{na}g\rangle\langle M_{mb}T_{na}h, f\rangle\\
=\ & \frac{1}{b}\sum_{n\in\mathbb{Z}_+}\sum_{j\in\mathbb{Z}_+}\int_{[0,\frac{1}{b})\oplus x_0}\overline{f(x\oplus\frac{j}{b})}h(x\ominus na\oplus\frac{j}{b})\sum_{k\in\mathbb{Z}_+}\overline{g(x\ominus na\oplus\frac{k}{b})}f(x\oplus\frac{k}{b})dx\\
=\ & \frac{1}{b}\int_{[0,\frac{1}{b})\oplus x_0}\sum_{j\in\mathbb{Z}_+}\sum_{k\in\mathbb{Z}_+}\sum_{n\in\mathbb{Z}_+}\overline{g(x\ominus na\oplus\frac{k}{b})}h(x\ominus na\oplus\frac{j}{b})f(x\oplus\frac{k}{b})\overline{f(x\oplus\frac{j}{b})}dx\\
=\ & \frac{1}{b}\int_{[0,\frac{1}{b})\oplus x_0}\langle\gamma(x)F(x), F(x)\rangle dx.
\end{aligned}$$

The proof is completed. □

3. Proof of Theorem 1.1 and examples

Proof of Theorem 1.1: By the polarization identity of inner product, $(X(\mathbf{g}, a, b), X(\mathbf{h}, a, b))$ is a pair of weak dual frames for $L^2(\mathbb{R}_+)$ associated with $L_c^\infty(\mathbb{R}_+)$ if and only if

$$\langle f, f\rangle = \sum_{l=1}^{L}\sum_{m,n\in\mathbb{Z}_+}\langle f, M_{mb}T_{na}g_l\rangle\langle M_{mb}T_{na}h_l, f\rangle$$

for $f \in L_c^\infty(\mathbb{R}_+)$. This is equivalent to

$$b\int_{[0,\,\frac{1}{b})\oplus x_0}\langle F(x), F(x)\rangle dx = \int_{[0,\,\frac{1}{b})\oplus x_0}\langle \Gamma(x)F(x), F(x)\rangle dx \text{ for } f \in L_c^\infty(\mathbb{R}_+) \text{ and } x_0 \in \mathbb{R}_+ \tag{60}$$

by Lemma 2.11, where

$$\Gamma(x) = \left(\sum_{l=1}^{L}\sum_{n\in\mathbb{Z}_+}\overline{g_l(x\ominus na\oplus\frac{k}{b})}h_l(x\ominus na\oplus\frac{j}{b})\right)_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}_+} \text{ and } F(x) = \left\{f(x\oplus\frac{k}{b})\right\}_{k\in\mathbb{Z}_+}. \tag{61}$$

Next we prove the equivalence between (60) and (13). To finish the proof, we first show that (60) holds if and only if

$$\Gamma(\cdot) = b\mathcal{I} \text{ a.e. on } (0, \frac{1}{b})\oplus x_0 \text{ for } x_0 \in \mathbb{R}_+, \tag{62}$$

where $\mathcal{I}$ is the identity operator. Obviously, (62) implies (60). Now suppose (60) holds. Arbitrarily fix $c = \{c_j\}_{j\in\mathbb{Z}_+} \in l_0(\mathbb{Z}_+)$, a finitely supported sequence space defined on $\mathbb{Z}_+$, and $E \subset (0, \frac{1}{b})\oplus x_0$ with $|E| > 0$. Take f in (60) by

$$F(\cdot) = \frac{1}{\sqrt{|E|}}\zeta_{\cup_{k\in\mathbb{Z}_+}(E\oplus\frac{k}{b})}(\cdot)c.$$

Then f is well defined, and

$$\frac{b}{|E|}\int_E\langle c, c\rangle dx = \frac{1}{|E|}\int_E\langle\Gamma(x)c, c\rangle dx.$$

It leads to (62) by [22, Theorem 1.39], and the arbitrariness of E and c. Thus (60) and (62) are equivalent. Now we finish the proof by showing the equivalence between (62) and (13). Observe that $\{[0, \frac{1}{b})\oplus x_0\oplus\frac{k}{b} : k \in \mathbb{Z}_+\}$ is a partition of $\mathbb{R}_+$ by Lemma 2.5. By the arbitrariness of x_0 in (62), (62) is equivalent to

$$\Gamma(\cdot) = b\mathcal{I} \text{ a.e. on } \mathbb{R}_+,$$

i.e.

$$\sum_{l=1}^{L}\sum_{n\in\mathbb{Z}_+}\overline{g_l(x\ominus na\oplus\frac{k}{b})}h_l(x\ominus na\oplus\frac{j}{b}) = b\delta_{j,k}$$

for a.e. $x \in \mathbb{R}_+$ and $j, k \in \mathbb{Z}_+$. This is in turn equivalent to

$$\sum_{l=1}^{L}\sum_{n\in\mathbb{Z}_+}\overline{g_l(x\ominus na\oplus\frac{k}{b})}h_l(x\ominus na) = b\delta_{k,0} \text{ for a.e. } x \in \mathbb{R}_+ \text{ and } k \in \mathbb{Z}_+ \tag{63}$$

due to $\mathbb{Z}_+\ominus\mathbb{Z}_+ = \mathbb{Z}_+$. Obviously, (63) is equivalent to (13) due to the $a\mathbb{Z}_+$-periodicity of function $\sum_{l=1}^{L}\sum_{n\in\mathbb{Z}_+}\overline{g_l(x\ominus na\oplus\frac{k}{b})}h_l(x\ominus na)$ with $k \in \mathbb{Z}_+$. The proof is completed. □

The following Examples 3.1 and 3.2 are for Theorem 1.1, and Example 3.2 presents an example of weak dual frame pairs which are not dual frame pairs.

Example 3.1. *Let p, a, b be as in the general setup, and let $L = 2$ and $0 < \lambda < 1$. Choose $\{g_1, g_2\}$ and $\{h_1, h_2\} \subset L^2(\mathbb{R}_+)$ such that*

$$supp(g_1),\ supp(h_1) \subset (0, \lambda a) \text{ and } supp(g_2),\ supp(h_2) \subset (\lambda a, a),$$

and

$$\overline{g_1(\cdot)}h_1(\cdot) + \overline{g_2(\cdot)}h_2(\cdot) = b \tag{64}$$

a.e. on $(0, a)$. Then

$$\sum_{l=1}^{L}\sum_{n\in\mathbb{Z}_+}\overline{g_l(\cdot \ominus na \oplus \frac{k}{b})}h_l(\cdot \ominus na) = b\delta_{0,k} \text{ a.e. on } (0, a) \text{ for } k \in \mathbb{Z}_+.$$

by a simple computation. Therefore, $(X(\{g_1, g_2\}, a, b), X(\{h_1, h_2\}, a, b))$ is a pair of weak dual frames for $L^2(\mathbb{R}_+)$ associated with $L_c^\infty(\mathbb{R}_+)$ by Theorem 1.1.

Example 3.2. *Let p, a, b be as in the general setup, and let $L = 2$ and $0 < \lambda < 1$. Choose $\{g_1, g_2\}$, $\{h_1, h_2\} \subset L^2(\mathbb{R}_+)$ such that*

$$\begin{aligned} &g_1(x) = A_1\zeta_{(0,\lambda a)}(x)x^{\tau_1}, \quad h_1(x) = \tilde{A}_1\zeta_{(0,\lambda a)}(x)x^{-\tau_1} \\ &g_2(x) = A_2\zeta_{(\lambda a,a)(x)}x^{-\tau_2}, \quad h_2(x) = \tilde{A}_2\zeta_{(\lambda a,a)}(x)x^{\tau_2} \end{aligned} \tag{65}$$

on $\mathbb{R}_+$, where $A_l, \tilde{A}_l$ and τ_l with $l = 1, 2$ are constants satisfying $\bar{A}_1\tilde{A}_1 = \bar{A}_2\tilde{A}_2 = b$ and $0 < \tau_1 < \frac{1}{2}$. Then, by Theorem 1.1, $(X(\{g_1, g_2\}, a, b), X(\{h_1, h_2\}, a, b))$ is a pair of weak dual frames for $L^2(\mathbb{R}_+)$ associated with $L_c^2(\mathbb{R}_+)$. But

$$\sum_{n\in\mathbb{Z}_+}|h_1(x \ominus na)|^2 = |\tilde{A}_1|^2\zeta_{[0,\lambda a)}(x)x^{-2\tau_1}$$

for $x \in (0, a)$ by a simple computation, which implies that $\sum_{n\in\mathbb{Z}_+}|h_1(\cdot\ominus na)|^2 \notin L^\infty(\mathbb{R}_+)$. It follows that $X(\{h_1, h_2\}, a, b)$ is not a Bessel sequence in $L^2(\mathbb{R}_+)$ by [29, Lemma 2.2]. Therefore, $(X(\{g_1, g_2\}, a, b), X(\{h_1, h_2\}, a, b))$ is not a pair of dual frames for $L^2(\mathbb{R}_+)$.

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