Oscillation Criteria for Second-Order Half-Linear Differential Equations

J. V. MANOJLOVIĆ
Department of Mathematics
University of Niš, Faculty of Philosophy
Čirila i Metodija 2, 18000 Niš, Yugoslavia
jelenam@archimed.filfak.ni.ac.yu

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Abstract—By using averaging functions, we obtain some criteria for the oscillation of half-linear differential equation

\[ \left[ p(t) |x'(t)|^{\alpha-1} x'(t) \right]' + q(t)|x(t)|^{\alpha-1}x(t) = 0, \quad \alpha > 0, \]

where \( p \in C^1([t_0, \infty); (0, \infty)) \) and \( q \in C([t_0, \infty); \mathbb{R}) \). © 1999 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In this paper, we are concerned with the problem of oscillation of second-order nonlinear differential equation

\[ \left[ p(t) |x'(t)|^{\alpha-1} x'(t) \right]' + q(t)|x(t)|^{\alpha-1}x(t) = 0, \quad (E) \]

where \( p \in C^1([t_0, \infty); (0, \infty)), q \in C([t_0, \infty); \mathbb{R}) \), and \( \alpha > 0 \) is a constant.

By a solution of (E), we mean a function \( x \in C^1[T_x, \infty), T_x \geq t_0 \), which has the property \( |x'(t)|^{\alpha-1}x'(t) \in C^1[T_x, \infty) \) and satisfies (E). We restrict our attention only to the nontrivial solutions \( x(t) \) of (E), i.e., to the solutions \( x(t) \) such that \( \sup\{ |x(t)| : t \geq T \} > 0 \) for all \( T \geq T_x \).

A nontrivial solution of (E) is called oscillatory if it has arbitrarily large zeroes, otherwise, it is said to be nonoscillatory. Equation (E) is called oscillatory if all its solution are oscillatory.

It is well known that equation (E) and the linear differential equation

\[ x''(t) + a(t)x(t) = 0 \quad (E_1) \]

or

\[ (r(t)x'(t))' + c(t)x(t) = 0 \quad (E_2) \]

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have some similar properties. For such results the reader is referred to the papers [1–10]. For example, Sturmian comparison and separation theorems for equation (E₂) have been extended to equation (E) by Elbert [1], Li and Yeh [8]. Thus, the zeroes of two linearly independent solutions of equation (E) separate each other and all nontrivial solutions of (E) are oscillatory or nonoscillatory.

Some of the very important oscillation theorems for second-order linear and nonlinear differential equations involve the average behaviour of the integral of the alternating coefficient \( q(t) \). In 1989, Philos [11] proved the following two oscillation criteria for equation (E₁).

**Theorem A.** Let \( H : \mathcal{D} = \{(t, s) | t \geq s \geq t₀ \} \rightarrow \mathbb{R} \) be a continuous functions, which is such that

\[
H(t, t) = 0, \quad \text{for } t \geq t₀, \quad H(t, s) > 0, \quad \text{for all } (t, s) \in \mathcal{D},
\]

and has a continuous and nonpositive partial derivative on \( \mathcal{D} \) with respect to the second variable. Moreover, let \( h : \mathcal{D} \rightarrow \mathbb{R} \) be a continuous function with

\[
-\frac{\partial H}{\partial s}(t, s) = h(t, s)\sqrt{H(t, s)}, \quad \text{for } (t, s) \in \mathcal{D}.
\]

Then equation (E₁) is oscillatory if

\[
\limsup_{t \to -\infty} \frac{1}{H(t, t₀)} \int_{t₀}^{t} \left[ a(s)H(t, s) - \frac{1}{4} h²(t, s) \right] ds = \infty.
\]

**Theorem B.** Let the functions \( H \) and \( h \) be defined as in Theorem A, and moreover, suppose that

\[
0 < \inf_{s \geq t₀} \left[ \liminf_{t \to -\infty} \frac{H(t, s)}{H(t, t₀)} \right] \leq \infty
\]

and

\[
\limsup_{t \to -\infty} \frac{1}{H(t, t₀)} \int_{t₀}^{t} h²(t, s) ds < \infty.
\]

Then equation (E₁) is oscillatory if there exists a continuous function \( A \) on \([t₀, \infty)\) with

\[
\int_{t₀}^{\infty} A²(s) ds = \infty
\]

and such that

\[
\limsup_{t \to -\infty} \frac{1}{H(t, T)} \int_{T}^{t} \left[ a(s)H(t, s) - \frac{1}{4} h²(t, s) \right] ds \geq A(T), \quad \text{for every } T \geq t₀.
\]

Recently, using a generalized Riccati transformation of linear differential equation, Li [12] gave some extensions to the results of Philos for equation (E₂).

The results of Philos and Li have been extended to the nonlinear differential equation

\[
[a(t)\psi(x(t))x'(t)]' + p(t)x'(t) + q(t)f(x(t)) = 0
\]

by Grace [13]. In this paper, using averaging technique we extend the results of Philos, Li and Grace to the second-order nonlinear differential equations of type (E).
2. MAIN RESULTS

In order to prove our theorems we use the following well-known inequality which is due to Hardy, Little and Polya [14].

**Lemma 1.** If $X$ and $Y$ are nonnegative, then

$$X^q + (q-1)Y^q - qXY^{q-1} \geq 0, \quad q > 1,$$

where equality holds if and only if $X = Y$.

**Theorem 1.** Suppose that there exists a continuous function

$$H : D = \{ (t,s) | t \geq s \geq t_0 \} \rightarrow \mathbb{R}$$

such that

$$H(t,t) = 0, \quad t \geq t_0, \quad H(t,s) > 0, \quad (t,s) \in D. \quad (H_1)$$

$$h(t,s) = -\frac{\partial H(t,s)}{\partial s} \text{ is a nonnegative continuous function on } D. \quad (H_2)$$

If

$$\limsup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[ q(s)H(t,s) - \frac{p(s)h^{\alpha + 1}(t,s)}{(\alpha + 1)^{\alpha + 1}H^\alpha(t,s)} \right] ds = \infty, \quad (C_1)$$

then equation $(E)$ is oscillatory.

**Proof.** Let $x(t)$ be a nonoscillatory solution of equation (1). Assume that $x(t) \neq 0$ for $t \geq t_0$. We define

$$w(t) = \frac{p(t)|x'(t)|^{\alpha - 1}x'(t)}{|x(t)|^{\alpha - 1}x(t)}, \quad t \geq t_0. \quad (1)$$

Then, for every $t \geq t_0$, we obtain

$$w'(t) = -q(t) - \alpha \frac{|w(t)|^{(\alpha + 1)/\alpha}}{p^{1/\alpha}(t)}, \quad (2)$$

and consequently,

$$\int_{t_0}^{t} w'(s)H(t,s) ds = -\int_{t_0}^{t} q(s)H(t,s) ds - \alpha \int_{t_0}^{t} H(t,s) \frac{|w(s)|^{(\alpha + 1)/\alpha}}{p^{1/\alpha}(s)} ds.$$

Since

$$\int_{t_0}^{t} w'(s)\Pi(t,s) ds = -w(t_0)\Pi(t,t_0) - \int_{t_0}^{t} w(s)\frac{\partial H(t,s)}{\partial s} ds, \quad (3)$$

the previous equality becomes

$$\int_{t_0}^{t} q(s)H(t,s) ds \leq w(t_0)H(t,t_0) + \int_{t_0}^{t} |w(s)|h(t,s) ds - \alpha \int_{t_0}^{t} H(t,s) \frac{|w(s)|^{(\alpha + 1)/\alpha}}{p^{1/\alpha}(s)} ds. \quad (4)$$

Taking

$$X = (\alpha H(t,s))^{\alpha/\alpha + 1} \frac{|w(s)|}{p^{1/\alpha + 1}(s)}, \quad q = \frac{\alpha + 1}{\alpha},$$

$$Y = \frac{\alpha^{\alpha/\alpha + 1} h^{\alpha/\alpha + 1}(t,s)}{(\alpha + 1)^{\alpha} H^{(\alpha^2/\alpha + 1)}(t,s)},$$
according to Lemma 1, we obtain for $t > s \geq t_0$

$$|w(s)|h(t, s) - \alpha H(t, s) \frac{|w(s)|^{(\alpha+1)/\alpha}}{p^{1/\alpha}(s)} \leq \frac{p(s)h^{\alpha+1}(t, s)}{(\alpha + 1)^{\alpha+1} H^\alpha(t, s)}.$$ 

Hence, (4) implies

$$\frac{1}{H(t, t_0)} \int_{t_0}^{t} q(s)H(t, s) \, ds \leq w(t_0) + \frac{1}{\lambda H(t, t_0)} \int_{t_0}^{t} p(s) \frac{h^{\alpha+1}(t, s)}{H^\alpha(t, s)} \, ds,$$

for all $t \geq t_0$, where we denote by $\lambda = (\alpha + 1)^{\alpha+1}$. Consequently,

$$\frac{1}{H(t, t_0)} \int_{t_0}^{t} q(s)H(t, s) - \frac{p(s)h^{\alpha+1}(t, s)}{(\alpha + 1)^{\alpha+1} H^\alpha(t, s)} \, ds \leq w(t_0), \quad t \geq t_0.$$ 

Taking the upper limit as $t \to \infty$, we obtain a contradiction, which completes the proof.

REMARK 1. Taking $H(t, s) = (t - s)^\lambda$ for some constant $\lambda > 1$, which obviously satisfies the conditions of Theorem 1, Theorem 1 reduces to the oscillation criteria of Li and Yeh [7].

COROLLARY 1. Let condition $(C_1)$ in Theorem 1 be replaced by

$$\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} p(s)h^{\alpha+1}(t, s) \, ds < \infty,$$

then the conclusion of Theorem 1 holds.

For illustration, we consider the following example.

EXAMPLE 1. Consider the differential equation

$$\left(t^{-\nu}|x'(t)|^{\alpha-1}x'(t)\right)' + t^\lambda \left(\frac{2 - \cos t}{t} + \sin t\right)|x(t)|^{\alpha-1}x(t) = 0, \quad (E_3)$$

for $t \geq t_0$, where $\nu, \lambda, \alpha$ are arbitrary positive constants and $\alpha \neq 2$.

Then, for any $t \geq t_0$, we have

$$\int_{t_0}^{t} q(s) \, ds = \int_{t_0}^{t} d \left[s^\lambda(2 - \cos s)\right] = t^\lambda(2 - \cos t) - t_0^\lambda(2 + \cos t_0) = t^\lambda(2 - \cos t) - k_0 \geq t^\lambda - k_0.$$ 

Taking $H(t, s) = (t - s)^2$, for $t \geq s \geq t_0$ and denoting by $\mu = (\alpha + 1)^{\alpha+1}$, we have

$$\frac{1}{t^2} \int_{t_0}^{t} \left[(t - s)^2 q(s) - \frac{(t - s)^{1-\alpha}}{\mu s^\nu}\right] \, ds$$

$$= \frac{1}{t^2} \int_{t_0}^{t} \left[2(t - s) \left(\int_{t_0}^{s} q(u) \, du\right) - \frac{(t - s)^{1-\alpha}}{\mu s^\nu}\right] \, ds$$

$$\geq \frac{2}{t^2} \int_{t_0}^{t} (t - s) \left(s^\lambda - k_0\right) \, ds - \frac{1}{\mu t_0^\lambda t^2} \int_{t_0}^{t} (t - s)^{1-\alpha} \, ds$$

$$= \frac{2t^\lambda}{(\lambda + 1)(\lambda + 2)} + k_1 + k_2 - k_0 - k_3 \left(1 - \frac{t_0}{t}\right)^{2-\alpha},$$

where

$$k_1 = \frac{2t_0^{\lambda+2}}{\lambda + 2} - k_0 t_0^2, \quad k_2 = 2k_0 t_0 - \frac{2t_0^{\lambda+1}}{\lambda + 1}, \quad k_3 = \frac{1}{t_0^2 \mu (2 - \alpha)}.$$ 

Consequently, condition $(C_1)$ is satisfied. Hence, equation $(E_3)$ is oscillatory by Theorem 1.
THEOREM 2. Let the functions $H$ and $h$ be defined as in Theorem 1 such that conditions $(H_1)$, $(H_2)$,
\[ 0 < \inf_{s \geq t_0} \left[ \liminf_{t \to \infty} \frac{H(t,s)}{H(t,t_0)} \right] \leq \infty, \]  
and
\[ \limsup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} p(s) \frac{H^{\alpha+1}(t,s)}{H^\alpha(t,s)} \, ds < \infty \]  
are satisfied. If there exists a continuous function $\varphi$ on $[t_0, \infty)$ such that for every $T \geq t_0$
\[ \limsup_{t \to \infty} \frac{1}{H(t,T)} \int_{t}^{T} \left[ q(s)H(t,s) - \frac{p(s)H^{\alpha+1}(t,s)}{(\alpha + 1)^{\alpha+1}H^\alpha(t,s)} \right] \, ds \geq \varphi(T) \]  
and
\[ \int_{t_0}^{\infty} \frac{\varphi^{(\alpha+1)}(s)}{p(s)} \, ds = \infty, \]
where $\varphi^+(s) = \max\{\varphi(s), 0\}$, then equation (E) is oscillatory.

PROOF. We suppose that there exists a solution $z(t)$ of equation (E), such that $z(t) \neq 0$ for $t > t_0$. Defining the function $w$ as in the proof of Theorem 1, we get (4) and (5). Then, for $t > T \geq t_0$, we have
\[ \limsup_{t \to \infty} \frac{1}{H(t,T)} \int_{t_0}^{t} \left[ q(s)H(t,s) - \frac{p(s)H^{\alpha+1}(t,s)}{(\alpha + 1)^{\alpha+1}H^\alpha(t,s)} \right] \, ds \leq w(T). \]
Therefore, by condition $(C_3)$, we have
\[ \varphi(T) \leq w(T), \quad \text{for every } T \geq t_0 \]  
and
\[ \limsup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} q(s)H(t,s) \geq \varphi(t_0). \]
We define functions
\[ F(t) = \frac{1}{H(t,t_0)} \int_{t_0}^{t} |w(s)|H(t,s) \, ds, \]
\[ G(t) = \frac{\alpha}{H(t,t_0)} \int_{t_0}^{t} H(t,s) \frac{|w(s)|^{(\alpha+1)/\alpha}}{p^{1/\alpha}(t)} \, ds. \]
Then, by (4) and (7), we see that
\[ \liminf_{t \to \infty} [G(t) - F(t)] \leq w(t_0) - \limsup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} q(s)H(t,s) \, ds \leq w(t_0) - \varphi(t_0) < \infty. \]

We shall next prove that
\[ \int_{t_0}^{\infty} \frac{|w(s)|^{(\alpha+1)/\alpha}(s)}{p^{1/\alpha}(s)} \, ds < \infty. \]
If we suppose that (9) fails, there exists a $t_1 > t_0$ such that
\[ \int_{t_0}^{t} \frac{|w(s)|^{(\alpha+1)/\alpha}(s)}{p^{1/\alpha}(s)} \, ds \geq \frac{\mu}{\alpha \zeta^t}, \quad \text{for all } t \geq t_1, \]
where \( \mu \) is an arbitrary positive number and \( \xi \) is a positive constant such that
\[
\inf_{s \geq t_0} \left[ \liminf_{t \to \infty} \frac{H(t,s)}{H(t,t_0)} \right] > \xi > 0. 
\tag{11}
\]

Using integration by parts and (10), we have for all \( t \geq t_1 \)
\[
G(t) = \frac{\alpha}{H(t,t_0)} \int_{t_0}^{t} H(t,s) d \left( \int_{t_0}^{s} \frac{|w(\tau)|^{(\alpha+1)/\alpha}}{p^{1/\alpha}(\tau)} d\tau \right) \\
= - \frac{\alpha}{H(t,t_0)} \int_{t_0}^{t} \frac{\partial H}{\partial s}(t,s) \left( \int_{t_0}^{s} \frac{|w(\tau)|^{(\alpha+1)/\alpha}}{p^{1/\alpha}(\tau)} d\tau \right) ds \\
\geq - \frac{\alpha}{H(t,t_0)} \int_{t_1}^{t} \frac{\partial H}{\partial s}(t,s) H(s) \left( \int_{t_0}^{s} \frac{|w(\tau)|^{(\alpha+1)/\alpha}}{p^{1/\alpha}(\tau)} d\tau \right) ds \\
\geq - \frac{\mu}{\xi H(t,t_0)} \int_{t_1}^{t} \frac{\partial H}{\partial s}(t,s) H(s) ds = \frac{\mu H(t,t_1)}{\xi H(t,t_0)}. 
\tag{11}
\]

By (11), there is a \( t_2 \geq t_1 \) such that \( H(t,t_1)/H(t,t_0) \geq \xi \) for all \( t \geq t_2 \), and accordingly \( G(t) \geq \mu \) for all \( t \geq t_2 \). Since \( \mu \) is arbitrary,
\[
\lim_{t \to \infty} G(t) = \infty. 
\tag{12}
\]

Further, consider a sequence \( \{T_n\}_{n=1}^{\infty} \) in \((t_0, \infty)\) such that \( \lim_{n \to \infty} T_n = \infty \) and
\[
\lim_{n \to \infty} \left[ G(T_n) - F(T_n) \right] = \liminf_{t \to \infty} [G(t) - F(t)] < \infty. 
\tag{13}
\]

Then, there exists a constant \( M \) such that
\[
G(T_n) - F(T_n) \leq M, 
\tag{14}
\]
for all sufficiently large \( n \). Since (12) ensures that
\[
\lim_{n \to \infty} G(T_n) = \infty, 
\tag{15}
\]
(13) implies
\[
\lim_{n \to \infty} F(T_n) = \infty. 
\tag{16}
\]

By taking into account (14), from (13) we derive for \( n \) sufficiently large
\[
\frac{F(T_n)}{G(T_n)} - 1 \geq - \frac{M}{G(T_n)} > - \frac{1}{2}. 
\tag{17}
\]

Therefore,
\[
\frac{F(T_n)}{G(T_n)} > \frac{1}{2}, \text{ for all large } n, 
\tag{18}
\]

which together with (15) implies
\[
\lim_{n \to \infty} \frac{F^{\alpha+1}(T_n)}{G^{\alpha}(T_n)} = \infty. 
\tag{19}
\]

On the other hand, by Hölder’s inequality, we have for every \( n \in N \)
\[
F(T_n) = \frac{1}{H(T_n,t_0)} \int_{t_0}^{T_n} |w(s)| h(T_n,s) ds, \\
= \int_{t_0}^{T_n} \left( \frac{\alpha^{(\alpha+1)/\alpha}}{H^{(\alpha+1)/\alpha+1}(T_n,t_0) p^{1/\alpha+1}(T_n,s)} \right) \times \left( \frac{\alpha^{-((\alpha+1)/\alpha+1)}}{H^{(\alpha+1)/\alpha+1}(T_n,t_0) H^{(\alpha+1)/\alpha}(T_n,s)} \right) ds \\
\leq \left( \frac{\alpha}{H(T_n,t_0)} \int_{t_0}^{T_n} |w(s)|^{(\alpha+1)/\alpha} H(T_n,s) ds \right)^{\alpha/\alpha+1} \times \left( \frac{1}{\alpha^\alpha H(T_n,t_0)} \int_{t_0}^{T_n} p(s) H^{\alpha+1}(T_n,s) ds \right)^{1/\alpha+1},
\]
and accordingly,
\[
\frac{F^{\alpha+1}(T_n)}{G^\alpha(T_n)} \leq \frac{1}{\alpha^\alpha H(T_n, t_0)} \int_{t_0}^{T_n} p(s) \frac{h^{\alpha+1}(T_n, s)}{H^\alpha(T_n, s)} \, ds.
\]

So, because of (16), we have
\[
\lim_{n \to \infty} \frac{1}{H(T_n, t_0)} \int_{t_0}^{T_n} p(s) \frac{h^{\alpha+1}(T_n, s)}{H^\alpha(T_n, s)} \, ds = \infty,
\]
which gives
\[
\lim_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} p(s) \frac{h^{\alpha+1}(t, s)}{H^\alpha(t, s)} \, ds = \infty,
\]
contradicting condition (C_2). Therefore, (9) holds. Now, from (6) we obtain
\[
\int_{t_0}^{\infty} \frac{\varphi(s)}{p^{1/\alpha}(s)} \, ds \leq \int_{t_0}^{\infty} \frac{|w(s)|[(\alpha/\alpha+1)](s)}{p^{1/\alpha}(s)} \, ds < \infty,
\]
which contradicts (C_4). This completes the proof.

**THEOREM 3.** Let the functions \(H\) and \(h\) be defined as in Theorem 1 such that conditions \((H_1)\), \((H_2)\), \((H_3)\), and
\[
\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} |q(s)|H(t, s) \, ds < \infty
\]
are satisfied. If there exists a continuous function \(\varphi\) on \([t_0, \infty)\) such that for every \(T \geq t_0\)
\[
\liminf_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t}^{T} \left[ q(s)H(t, s) - \frac{p(s)h^{\alpha+1}(t, s)}{(\alpha+1)^{\alpha+1}H^\alpha(t, s)} \right] \, ds \geq \varphi(T),
\]
and condition \((C_4)\) holds, then equation \((E)\) is oscillatory.

**PROOF.** For the nonoscillatory solution \(x(t)\) of equation \((E)\), as in the proof of Theorem 1, (4) and (5) are satisfied. As in proof of Theorem 2, (6) holds for \(t > T > t_0\). Using condition \((C_5)\), we conclude that
\[
\limsup_{t \to \infty} |G(t) - F(t)| \leq w(t_0) - \liminf_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} q(s)H(t, s) \, ds < \infty.
\]
It follows from condition \((C_6)\) that
\[
\varphi(t_0) \leq \liminf_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} q(s)H(t, s) \, ds - \liminf_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \frac{p(s)h^{\alpha+1}(t, s)}{(\alpha+1)^{\alpha+1}H^\alpha(t, s)} \, ds,
\]
so that \((C_5)\) implies
\[
\liminf_{t \to \infty} \frac{1}{(\alpha+1)^{\alpha+1}H(t, t_0)} \int_{t_0}^{t} \frac{p(s)h^{\alpha+1}(t, s)}{H^\alpha(t, s)} \, ds < \infty.
\]
Considering a sequence \(\{T_n\}_{n=1}^{\infty}\) in \((t_0, \infty)\) with \(\lim_{n \to \infty} T_n = \infty,\) such that
\[
\lim_{n \to \infty} |G(T_n) - F(T_n)| = \limsup_{t \to \infty} |G(t) - F(t)|.
\]
Then, using the procedure of the proof of Theorem 2, we conclude that (9) is satisfied. The remainder of the proof proceeds as in the proof of Theorem 2.

We observe that Theorem 2 can be applied in some cases in which Theorem 1 is not applicable. Such a case is described in the following example.
EXAMPLE 2. Consider the differential equation

\[
\left( t^\nu |x'(t)|^{\alpha-1} x'(t) \right)' + t^\lambda \cos t |x(t)|^{\alpha-1} x(t) = 0, \tag{E_4}
\]

for \( t \geq t_0 \), where \( \nu, \lambda, \alpha \) are constants such that \(-1 < \lambda \leq 1, \nu < \alpha, \alpha \neq 2, \) and \( \alpha^2 \lambda \geq (\alpha + 1)(\nu - \alpha) \).

Taking \( H(t, s) = (t-s)^2 \), for \( t \geq s \geq t_0 \), we have

\[
\frac{1}{t^2} \int_{t_0}^t s^\nu (t-s)^{1-\alpha} ds \leq \begin{cases} 
\frac{t^\nu (t-t_0)^{2-\alpha}}{2-\alpha}, & \nu > 0, \\
\frac{t_0^\nu (t-t_0)^{2-\alpha}}{2-\alpha}, & \nu < 0,
\end{cases}
\]

\[
= \begin{cases} 
\frac{t^\nu - \alpha}{2-\alpha} \left( 1 - \frac{t_0}{t} \right)^{2-\alpha}, & \nu > 0, \\
\frac{t_0^\nu - \alpha}{2-\alpha} \left( 1 - \frac{t_0}{t} \right)^{2-\alpha}, & \nu < 0.
\end{cases}
\]

Therefore, condition \( (C_2) \) is satisfied and for arbitrary small constant \( \varepsilon > 0 \), there exists a \( t_1 \geq t_0 \) such that for \( T \geq t_1 \)

\[
\limsup_{t \to \infty} \frac{1}{t^2} \int_T^t \left[ (t-s)^2 s^\lambda \cos s - s^\nu \frac{(t-s)^{1-\alpha}}{(1+\alpha)^{1+\alpha}} \right] ds \geq -T^\lambda \cos T - \varepsilon.
\]

Now, set \( \varphi(T) = -T^\lambda \cos T - \varepsilon. \) Then, there exists an integer \( N \) such that \( (2N+1)\pi - \pi/4 > t_1 \) and if \( n \geq N \)

\[
(2n+1)\pi - \frac{\pi}{4} \leq T \leq (2n+1)\pi + \frac{\pi}{4}, \quad \varphi(T) \geq \delta T^\lambda,
\]

where \( \delta \) is a small constant. Taking into account that \( \alpha^2 \lambda \geq (\alpha + 1)(\nu - \alpha) \), we obtain

\[
\int_{t_0}^\infty \frac{\varphi_\nu^{(\alpha/\alpha+1)}(s)}{a^{\alpha}(s)} ds \geq \sum_{n=N}^{\infty} \delta^{(\alpha/\alpha+1)} \int_{(2n+1)\pi - \pi/4}^{(2n+1)\pi + \pi/4} s^{(\alpha/\alpha+1)} ds = \infty.
\]

Accordingly, all conditions of Theorem 2 are satisfied, and hence, equation \( (E_4) \) is oscillatory.

THEOREM 4. Suppose that the functions \( H \) and \( h \) are defined as in Theorem 1, such that conditions \( (H_1) \) and \( (H_2) \) hold. If there exists a positive, nondecreasing function \( \rho \in C^1([t_0, \infty)) \) and

\[
\limsup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left[ q(s) \rho(s) H(t,s) - \frac{p(s)}{(\alpha + 1)^{\alpha+1} H^\alpha(t,s)} \right] ds = \infty, \tag{C_7}
\]

then equation \( (E) \) is oscillatory.

PROOF. Let \( x(t) \) be a nonoscillatory solution of equation \( (E) \). Without loss of generality, we assume that \( x(t) \neq 0 \) for \( t \geq t_0 \). Now, we define

\[
W(t) = \rho(t) \frac{p(t)|x'(t)|^{\alpha-1} x'(t)}{|x(t)|^{\alpha-1} x(t)}, \quad \text{for all } t \geq t_0.
\]
Then, for every $s \geq t_0$, we obtain

$$W'(s) = -q(s)\rho(s) + \frac{\rho'(s)}{\rho(s)} W(s) - \alpha \frac{|W(s)|^{(\alpha+1)/\alpha}}{(p(s)\rho(s))^{1/\alpha}}. \quad (17)$$

If we multiply (17) by $H(t, s)$ for $t \geq s \geq t_0$, integrate from $t_0$ to $t$ and use (3), we get

$$\int_{t_0}^{t} W'(s) H(t, s) \, ds = - \int_{t_0}^{t} q(s)\rho(s) H(t, s) \, ds + \int_{t_0}^{t} \frac{\rho'(s)}{\rho(s)} W(s) H(t, s) \, ds - \alpha \int_{t_0}^{t} H(t, s) \frac{|W(s)|^{(\alpha+1)/\alpha}}{(p(s)\rho(s))^{1/\alpha}} \, ds.$$

Using (3), we have

$$\int_{t_0}^{t} q(s)\rho(s) H(t, s) \, ds \leq W(t_0)H(t, t_0)$$

$$+ \int_{t_0}^{t} \left( h(t, s) + \frac{\rho'(s)}{\rho(s)} H(t, s) \right) |W(s)| \, ds - \alpha \int_{t_0}^{t} H(t, s) \frac{|W(s)|^{(\alpha+1)/\alpha}}{(p(s)\rho(s))^{1/\alpha}} \, ds. \quad (18)$$

Therefore, according to Lemma 1, with

$$X = (\alpha H(t, s))^{(\alpha+1)/\alpha} \frac{|W(s)|}{(p(s)\rho(s))^{1/\alpha+1}}, \quad q = \frac{\alpha + 1}{\alpha},$$

$$Y = \left( \frac{\alpha}{\alpha + 1} \right)^\alpha \left( \frac{p(s)\rho(s)}{(\alpha H(t, s))^\alpha} \right)^{(\alpha+1)/\alpha} \left( h(t, s) + \frac{\rho'(s)}{\rho(s)} H(t, s) \right) \alpha,$$

we get for $t > s \geq t_0$

$$|W(s)| \left( h(t, s) + \frac{\rho'(s)}{\rho(s)} H(t, s) \right) - \alpha H(t, s) \frac{|W(s)|^{(\alpha+1)/\alpha}}{p^{1/\alpha}(t)} \leq \frac{p(s)\rho(s)}{(\alpha + 1)^{\alpha+1} H(t, s)^\alpha} \left( h(t, s) + \frac{\rho'(s)}{\rho(s)} H(t, s) \right)^{\alpha+1}. \quad (19)$$

From (18) and (19), we obtain

$$\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left[ q(s)\rho(s) H(t, s) - \frac{p(s)\rho(s)}{(\alpha + 1)^{\alpha+1} H(t, s)^\alpha} \times \left( h(t, s) + \frac{\rho'(s)}{\rho(s)} H(t, s) \right)^{\alpha+1} \right] \, ds \leq W(t_0),$$

which contradicts (C8).

**Corollary 2.** Let condition (C7) in Theorem 4 be replaced by

$$\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \frac{p(s)\rho(s)}{H(t, s)^\alpha} \left( h(t, s) + \frac{\rho'(s)}{\rho(s)} H(t, s) \right)^{\alpha+1} \, ds < \infty, \quad (C8)$$

$$\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} q(s)\rho(s) H(t, s) \, ds = \infty, \quad (C9)$$

then the conclusion of Theorem 4 holds.  

**Example 3.** Consider the differential equation

$$\left( t^\nu |x'(t)|^{\alpha-1} x'(t) \right)' + \left[ \lambda t^{\lambda-3}(2 - \cos t) + t^{\lambda-2} \sin t|x(t)|^{\alpha-1} x(t) \right] = 0, \quad (E_0)$$
for \( t \geq t_0 > 0 \), where \( \lambda \) is arbitrary positive constant and \( \nu, \alpha \) are constants such that \( \nu < \alpha - 2 \), \( \alpha < 1 \).

Here, we choose \( \rho(t) = t^2 \) and \( H(t,s) = (t-s)^2 \) for \( t \geq s \geq t_0 \). Then, since \( \rho(t)q(t) = \frac{d}{ds}[s^\lambda(2 - \cos s)] \), as in Example 1, we get

\[
\int_{t_0}^t \rho(s)q(s) \, ds \geq t^\lambda - k_0,
\]

and therefore,

\[
\frac{1}{t^2} \int_{t_0}^t (t-s)^2 \rho(s)q(s) \, ds \geq \frac{2t^\lambda}{(\lambda+1)(\lambda+2)} + \frac{k_1}{t^2} + \frac{k_2}{t} - k_0,
\]

where

\[
k_1 = \frac{2t_0^\lambda + 2}{\lambda + 2} - k_0t_0^2, \quad k_2 = 2k_0t_0 - \frac{2t_0^\lambda + 1}{\lambda + 1}.
\]

Hence, condition (C_9) is satisfied. On the other hand,

\[
\frac{1}{t^2} \int_{t_0}^t \frac{s^\nu+2}{(t-s)^2\alpha} \left( 2(t-s) + \frac{2}{s}(t-s)^2 \right)^{\alpha+1} \, ds
\]

\[
= t^{\alpha-1}2^{\alpha+1} \int_{t_0}^t s^{\nu+1}(t-s)^{1-\alpha} \, ds
\]

\[
\leq 2^{\alpha+1} \left( 1 - \frac{t_0}{t} \right)^{1-\nu+2} 2 t_0^{\nu-\alpha+2} - \frac{t_0^{\nu-\alpha+2}}{\nu - \alpha + 2},
\]

so that condition (C_8) is also satisfied. Consequently, by Corollary 2, equation (E) is oscillatory.

Following the procedure of the proof of the Theorems 2 and 3, we can also prove the following two theorems.

**Theorem 5.** Let the functions \( H, h, \) and \( \rho \) be defined as in Theorem 1 such that conditions (H1), (H2), (H3), and

\[
\lim \sup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \frac{p(s)\rho(s)}{H^\alpha(t,s)} H(t,s) \left( h(t,s) + \frac{\rho'(s)}{\rho(s)} H(t,s) \right)^{\alpha+1} \, ds < \infty
\]

are satisfied. If there exists a continuous function \( \psi \) on \([t_0, \infty)\) such that for every \( T \geq t_0 \)

\[
\lim \sup_{t \to \infty} \frac{1}{H(t,T)} \int_T^t \left[ q(s)H(t,s) - \frac{p(s)\rho(s)}{(\alpha + 1)^{\alpha+1} H^\alpha(t,s)} \times \left( h(t,s) + \frac{\rho'(s)}{\rho(s)} H(t,s) \right)^{\alpha+1} \right] \, ds \geq \psi(T),
\]

and condition (C_4) is satisfied, then equation (E) is oscillatory.

**Theorem 6.** Let the functions \( H, h, \) and \( \rho \) be defined as in Theorem 1 such that conditions (H1), (H2), (H3), and

\[
\lim \sup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^t |q(s)|\rho(s)H(t,s) \, ds < \infty
\]

are satisfied. If there exists a continuous function \( \psi \) on \([t_0, \infty)\) such that for every \( T \geq t_0 \)

\[
\lim \inf_{t \to \infty} \frac{1}{H(t,T)} \int_T^t \left[ q(s)H(t,s) - \frac{p(s)\rho(s)}{(\alpha + 1)^{\alpha+1} H^\alpha(t,s)} \times \left( h(t,s) + \frac{\rho'(s)}{\rho(s)} H(t,s) \right)^{\alpha+1} \right] \, ds \geq \psi(T),
\]

and condition (C_4) holds, then equation (E) is oscillatory.
REFERENCES


