Oscillation criteria for certain fourth order nonlinear functional differential equations

Ravi P. Agarwal\textsuperscript{a}, Said R. Grace\textsuperscript{b}, Jelena V. Manojlovi\textsuperscript{c,∗}

\textsuperscript{a} Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901, USA
\textsuperscript{b} Department of Engineering Mathematics, Faculty of Engineering, Cairo University, Orman, Giza 12221, Egypt
\textsuperscript{c} University of Niš, Faculty of Science and Mathematics, Department of Mathematics and Computer Science, Višegradska 33, 18000 Niš, Serbia and Montenegro

Received 27 October 2005; accepted 7 November 2005

Abstract

Some new criteria for the oscillation of the fourth order functional differential equation

\[
\frac{d}{dt}\left( \frac{1}{a_3(t)} \left( \frac{d}{dt} \frac{1}{a_2(t)} \left( \frac{d}{dt} \frac{1}{a_1(t)} \left( \frac{d}{dt} x(t) \right)^{\alpha_1} \right)^{\alpha_2} \right)^{\alpha_3} \right) + \delta q(t) f \left( x[g(t)] \right) = 0,
\]

where \( \delta = \pm 1 \) are established.
© 2006 Elsevier Ltd. All rights reserved.

Keywords: Oscillation; Nonoscillation; Functional; Nonlinear; Comparison

1. Introduction

Consider the fourth order nonlinear functional differential equation

\[ L_4 x(t) + \delta q(t) f (x[g(t)]) = 0, \]

where \( \delta = \pm 1 \), and

\[ L_0 x(t) = x(t), \quad L_k x(t) = \frac{1}{a_k(t)} \left( \frac{d}{dt} L_{k-1} x(t) \right)^{\alpha_k}, \quad k = 1, 2, 3, \quad L_4 x(t) = \frac{d}{dt} L_3 x(t). \]

In what follows we will assume that

(i) \( a_i(t), q(t) \in C([t_0, \infty), \mathbb{R}^+) = (0, \infty), t_0 \geq 0 \), and

\[ \int_{t_0}^{\infty} a_i^{1/\alpha_i}(s) ds = \infty, \quad i = 1, 2, 3; \]
(ii) \( g(t) \in C(t_0, \mathbb{R} = (-\infty, \infty)), g'(t) \geq 0 \) for \( t \geq t_0 \) and \( \lim_{t \to \infty} g(t) = \infty \);

(iii) \( f \in C(\mathbb{R}, \mathbb{R}), x(f(x)) > 0 \) and \( f'(x) \geq 0 \) for \( x \neq 0 \);

(iv) \( \alpha_i, i = 1, 2, 3 \) are the ratios of positive odd integers, while we denote \( \alpha = \alpha_1 \alpha_2 \alpha_3 \).

The domain \( D(L_4) \) of \( L_4 \) is defined to be the set of all functions \( x : [t_4, \infty) \to \mathbb{R} \) such that \( L_j x(t), 0 \leq j \leq 4 \) exist and are continuous on \([t_4, \infty)\). Our attention is restricted to those solutions \( x \in D(L_4) \) of Eq. (E_3) which satisfy \( \sup \{|x(t)| : t \geq T\} > 0 \) for every \( T \geq t_4 \). We make the standing hypothesis that Eq. (E_3) does possess such solutions.

A solution of Eq. (E_3) is called oscillatory if it has arbitrarily large zeros, otherwise it is said to be nonoscillatory. Eq. (E_3) is called oscillatory if all its solutions are oscillatory.

It is an interesting problem to establish some comparison criteria, by which oscillatory behavior of all solutions of the differential equation (E_3) is inherited from the oscillation of associated second order half-linear differential equations and/or first order delay or advanced differential equations. The results of the present paper are concerned with this problem and are new even in the case of Eq. (E_3) when \( \alpha_i = 1, i = 1, 2, 3 \). By our comparison results, known oscillation criteria for nonlinear equations of second order as well as those of first order can be used to establish sharp results for the oscillatory behavior of equation (E_3). For this, we remark that there is a voluminous literature on oscillation of second order half-linear as well as first order functional differential equations, see for example, Agarwal et al. [1–10], Győri and Ladas [11], Kusano and Lalli [12] and Philos [13] and the references cited therein. Finally, some applications of our results to certain neutral differential equations will be given before the end of both Sections 2 and 3.

2. Properties of solutions of equation (E_1)

We first state the result on the signs of \( L_i x(t), i = 0, 1, 2, 3, 4 \) for a nonoscillatory solution \( x(t) \) of equation (E_1). Namely, one of the following two cases holds:

\[
(-1)^{i+1} L_i x(t) \text{sgn } x(t) > 0, \quad i = 1, 2, 3, \quad \text{and } x(t) L_4 x(t) \leq 0 \text{ eventually}, \tag{2.1}
\]

\[
L_i x(t) \text{sgn } x(t) > 0, \quad i = 1, 2, 3, \quad \text{and } x(t) L_4 x(t) \leq 0 \text{ eventually}. \tag{2.2}
\]

We shall say that the solution \( x(t) \) is of type \( B_1 \) if (2.1) holds and that it is of type \( B_3 \) if (2.2) holds.

In order to prove our main result we will compare the following inequalities

\[
\frac{d}{dt} \left( \frac{1}{a_1(t)} \left( \frac{d}{dt} x(t) \right)^{a_1} \right) + q(t) f(x[g(t)]) \leq 0, \tag{2.3}
\]

\[
\frac{d}{dt} \left( \frac{1}{a_1(t)} \left( \frac{d}{dt} x(t) \right)^{a_1} \right) + q(t) f(x[g(t)]) \geq 0, \tag{2.4}
\]

with the second order half-linear functional differential equation

\[
\frac{d}{dt} \left( \frac{1}{a_1(t)} \left( \frac{d}{dt} x(t) \right)^{a_1} \right) + q(t) f(x[g(t)]) = 0, \tag{2.5}
\]

where \( a_1(t), q(t), g(t), f(x) \) and \( a_1 \) are as in Eq. (E_3) satisfying the conditions (i)–(iv). Namely, the following comparison results was proved in [2,4,5].

**Lemma 2.1.** If the inequality (2.3) (the inequality (2.4)) has an eventually positive (negative) solution, then Eq. (2.5) also has eventually positive (negative) solution.

Moreover, by comparing the inequality

\[
\frac{d}{dt} \left( \frac{1}{a_1(t)} \left( \frac{d}{dt} x(t) \right)^{a_1} \right) - q(t) f(x[g(t)]) \geq 0, \tag{2.6}
\]

and the second order half-linear functional differential equation

\[
\frac{d}{dt} \left( \frac{1}{a_1(t)} \left( \frac{d}{dt} x(t) \right)^{a_1} \right) - q(t) f(x[g(t)]) = 0, \tag{2.7}
\]
where $a_1(t), q(t), g(t), f(x)$ and $\alpha_1$ are as in Eq. $(E_6)$ satisfying the conditions (i)–(iv), we have the following lemma which was proved in [2].

**Lemma 2.2.** If the inequality (2.6) has an eventually positive bounded (unbounded) solution, then Eq. (2.7) also has eventually positive bounded (unbounded) solution.

Also, we can compare the following first order delay differential equations

$$y'(t) + q(t)f(y[g(t)]) \leq 0,$$

$$y'(t) - q(t)f(y[g(t)]) \geq 0,$$  \hspace{1cm} (2.8)\hspace{1cm} (2.9)

with the corresponding differential equationa

$$y'(t) + q(t)f(y[g(t)]) = 0,$$  \hspace{1cm} (2.10)

$$y'(t) - q(t)f(y[g(t)]) = 0.$$  \hspace{1cm} (2.11)

We have the following lemma, which is given in [3,11].

**Lemma 2.3.** If the inequality (2.8) (the inequality (2.9)) has eventually positive solution, then Eq. (2.10) (Eq. (2.11)) also has eventually positive solution.

### 2.1. Nonexistence of solutions of type $B_3$

We shall present some criteria for the nonexistence of solutions of type $B_3$ for equation $(E_1)$.

**Theorem 2.1.** Assume that conditions (i)–(iv) hold and the function $f$ satisfy the condition

$$-f(-xy) \geq f(xy) \geq f(x)f(y) \quad \text{for } xy > 0.$$  \hspace{1cm} (F)

If

$$\int_0^\infty q(s)f\left(\int_0^s \left(a_1(u) \int_0^u a_2^{1/\alpha_2}(\tau)d\tau\right)^{1/\alpha_1} du\right)ds = \infty,$$  \hspace{1cm} (2.12)

then equation $(E_1)$ has no solution of type $B_3$.

**Proof.** Let $x(t)$ be an eventually positive solution of equation $(E_1)$ of type $B_3$. Then there exists a large $t_0 \geq 0$ such that (2.2) holds for $t \geq t_0$. Next, there exist a constant $c > 0$ and a $t_1 \geq t_0$ such that

$$L_2x(t) = \frac{1}{a_2(t)}\left(\frac{d}{dt} L_1x(t)\right)^{\alpha_2} \geq c \quad \text{for } t \geq t_1,$$

or

$$\frac{d}{dt}\left(\frac{1}{a_1(t)}\left(\frac{d}{dt} x(t)\right)^{\alpha_1}\right) \geq (ca_2(t))^{1/\alpha_2} \quad \text{for } t \geq t_1.$$

Integrating this inequality from $t_1$ to $t$, we find

$$x'(t) \geq e^{\frac{1}{\alpha_2}\int_{t_1}^t a_2^{1/\alpha_2}(\tau)d\tau} \left(\frac{1}{a_2(t)}\int_{t_1}^t a_2^{1/\alpha_2}(\tau)d\tau\right)^{1/\alpha_1}, \quad t \geq t_1.$$

Once again, integrating the above inequality from $t_2$ to $g(t) \geq t_2 > t_1$ we get

$$x[g(t)] \geq e^{\frac{1}{\alpha_2}\int_{t_2}^{g(t)} a_2^{1/\alpha_2}(\tau)d\tau} \left(\frac{1}{a_2(t)}\int_{t_1}^{g(t)} a_2^{1/\alpha_2}(\tau)d\tau\right)^{1/\alpha_1}du, \quad t \geq t_2.$$  \hspace{1cm} (2.13)

Using (F) and (2.13) in equation $(E_1)$, we obtain

$$-L_4x(t) = q(t)f(x[g(t)]) \geq f\left(\frac{1}{e^{\frac{1}{\alpha_2}\int_{t_2}^{g(t)} a_2^{1/\alpha_2}(\tau)d\tau}}\right)q(t)f\left(\int_{t_2}^{g(t)} a_2^{1/\alpha_2}(\tau)d\tau\right)^{1/\alpha_1}du.$$
Integrating the above inequality from \( t_2 \) to \( t \), we have
\[
\infty > L_3 x(t_2) - L_3 x(t) \\
g \geq \int_{t_2}^{t} q(s) f \left( \int_{t_2}^{s} a_1(u) \int_{t_2}^{u} a_2^{1/\alpha_2}(\tau) d\tau \right)^{1/\alpha_1} du \, ds \to \infty \quad \text{as} \quad t \to \infty,
\]
contradicting the assumption (2.12). This completes the proof. \( \square \)

**Theorem 2.2.** Let conditions (i)–(iv) and (F) hold, and \( g(t) \leq t \) for \( t \geq t_0 \). If the first order delay equation
\[
y'(t) + Q(t) f \left( y^{1/\alpha}[g(t)] \right) = 0 \tag{2.14}
\]
is oscillatory, where
\[
Q(t) = q(t) f \left( \int_{t_0}^{g(t)} a_1(s) \int_{t_0}^{s} a_2(u) \int_{t_0}^{u} a_3^{1/\alpha_3}(\tau) d\tau \right)^{1/\alpha_2} \frac{1}{\alpha_1} ds,
\]
then equation (E1) has no solution of type B3.

**Proof.** Let \( x(t) \) be an eventually positive solution of equation (E1) of type B3. There exists a large \( t_0 \geq 0 \) such that (2.2) holds for \( t \geq t_0 \). Now,
\[
L_2 x(t) = L_2 x(t_0) + \int_{t_0}^{t} a_3^{1/\alpha_3}(s) a_3^{-1/\alpha_3}(s) L_3 x(s) \frac{d}{ds} L_3 x(s) ds \\
= L_2 x(t_0) + \int_{t_0}^{t} a_3^{1/\alpha_3}(s) L_3^{1/\alpha_3} x(s) ds.
\]
Since \( L_3 x(t) \) is a nonincreasing function on \([t_0, \infty)\), we have
\[
L_2 x(t) \geq \left( \int_{t_0}^{t} a_3^{1/\alpha_3}(s) ds \right) L_3^{1/\alpha_3} x(t) \quad \text{for} \quad t \geq t_0,
\]
or
\[
\frac{d}{dt} L_1 x(t) \geq \left( a_2(t) \int_{t_0}^{t} a_3^{1/\alpha_3}(s) ds \right)^{1/\alpha_2} L_3^{1/\alpha_3} x(t), \quad t \geq t_0.
\]
Integrating the above inequality from \( t_0 \) to \( t \), and using the definition of \( L_1 x(t) \), we have
\[
x'(t) \geq \left( a_1(t) \int_{t_0}^{t} a_2(u) \int_{t_0}^{u} a_3^{1/\alpha_3}(s) ds \right)^{1/\alpha_2} L_3^{1/\alpha_3} x(t), \quad t \geq t_0.
\]
Once again, by integrating the above inequality from \( t_0 \) to \( g(t) > t_0 \), we get
\[
x[g(t)] \geq \left( \int_{t_0}^{g(t)} a_1(s) \int_{t_0}^{s} a_2(u) \int_{t_0}^{u} a_3^{1/\alpha_3}(\tau) d\tau \right)^{1/\alpha_2} L_3^{1/\alpha_3} x(g(t)). \tag{2.15}
\]
Using (F) and (2.15) in equation (E1) and letting \( z(t) = L_3 x(t) \), \( t \geq t_0 \), we obtain
\[
z'(t) + Q(t) f \left( z^{1/\alpha}[g(t)] \right) \leq 0 \quad \text{for} \quad t \geq t_0.
\]
By Lemma 2.3, we see that Eq. (2.14) has an eventually positive solution, which contradicts the hypothesis and completes the proof. \( \square \)

The following corollary is immediate.
Corollary 2.1. Let conditions (i)–(iv) and (F) hold, and \( g(t) \leq t \) for \( t \geq t_0 \). Equation \((E_1)\) has no solution of type \( B_3 \) if one of the following conditions holds:

\[
\frac{f\left(y^{1/\alpha}\right)}{y} \geq 1 \quad \text{for } y \neq 0, \quad \text{and} \quad \liminf_{t \to \infty} \int_{t}^{\infty} Q(s)\,ds > \frac{1}{e}.
\]

(I1)

\[
\int_{t}^{\infty} Q(s)\,ds = \infty, \quad \text{and} \quad \int_{0}^{\infty} \frac{du}{f\left(u^{1/\alpha}\right)} < \infty.
\]

(I2)

where \( Q(t) \) is defined as in Theorem 2.2.

Theorem 2.3. Let conditions (i)–(iv) and (F) hold, and assume that there exists a function \( \eta(t) \in C([t_0, \infty), \mathbb{R}) \) such that \( g(t) \geq \eta(t) \geq t \) for \( t \geq t_0 \). If the half-linear equation

\[
\frac{d}{dt} \left( \frac{1}{a_3(t)} \left( \frac{d}{dt} y(t)^{\beta} \right) \right) + \overline{Q}(t) f\left( y^{1/\alpha_2}(t) \right) = 0
\]

(2.16)

is oscillatory, where

\[
\overline{Q}(t) = q(t) f\left( \left( \int_{t_0}^{t} a_1^{1/\alpha_1}(s)\,ds \right) \left( \int_{t}^{\eta(t)} a_2^{1/\alpha_2}(s)\,ds \right) \right)^{1/\alpha_1},
\]

then equation \((E_1)\) has no solution of type \( B_3 \).

Proof. Let \( x(t) \) be an eventually positive solution of equation \((E_1)\) of type \( B_3 \), so that there exists a large \( t_0 \geq 0 \) such that \((2.2)\) holds for \( t \geq t_0 \). Now for \( t \geq s \geq t_0 \), we have

\[
x(t) = x(s) + \int_{s}^{t} a_1^{1/\alpha_1}(u) L_1^{1/\alpha_1} x(u)\,du,
\]

and since \( L_1 x(t) \) is a nondecreasing function, we have

\[
x(t) \geq \left( \int_{s}^{t} a_1^{1/\alpha_1}(u)\,du \right) L_1^{1/\alpha_1} x(s).
\]

Replacing \( t \) and \( s \) by \( g(t) \) and \( \eta(t) \) respectively, we have

\[
x[g(t)] \geq \left( \int_{\eta(t)}^{g(t)} a_1^{1/\alpha_1}(u)\,du \right) L_1^{1/\alpha_1} x[\eta(t)] \quad \text{for } t \geq t_1 \geq t_0.
\]

(2.17)

Similarly, it can be easily obtained that

\[
L_1 x[\eta(t)] \geq \left( \int_{t}^{\eta(t)} a_2^{1/\alpha_2}(u)\,du \right) L_2^{1/\alpha_2} x(t) \quad \text{for } t \geq t_1.
\]

(2.18)

Combining \((2.17)\) and \((2.18)\), we get

\[
x[g(t)] \geq \left( \int_{\eta(t)}^{g(t)} a_1^{1/\alpha_1}(u)\,du \right) \left( \int_{t}^{\eta(t)} a_2^{1/\alpha_2}(u)\,du \right) L_2^{1/\alpha_2} x(t) \quad \text{for } t \geq t_1 \geq t_2.
\]

(2.19)

Using \((F)\) and \((2.19)\) in equation \((E_1)\) and letting \( z(t) = L_2 x(t) \) for \( t \geq t_2 \), we have

\[
\frac{d}{dt} \left( \frac{1}{a_3(t)} \left( \frac{dz}{dt} \right)^{\beta} \right) + \overline{Q}(t) f\left( z^{1/\alpha_2}(t) \right) \leq 0 \quad \text{for } t \geq t_2.
\]

By applying Lemma 2.1, we see that Eq. \((2.16)\) has an eventually positive solution, which contradicts the hypotheses and completes the proof. \( \square \)
In the last result considering nonexistence of solutions of type $B_3$, instead of the condition $(F)$ for the function $f(x)$ we will introduce the following condition

$$\int_{-\infty}^{\infty} \frac{du}{f^{1/\alpha}(u)} < \infty,$$

($F_1$)

**Theorem 2.4.** Let conditions (i)–(iv) and ($F_1$) hold, and $g(t) > t$ and $g'(t) > 0$ for $t \geq t_0$. If the following two conditions hold

$$\int_{t_0}^{\infty} q(s)ds < \infty, \quad t \geq t_0$$

(2.20)

and

$$\int_{-\infty}^{\infty} \left(a_1(s) \int_{g^{-1}(s)}^{s} \left[a_2(u) \int_{g^{-1}(u)}^{u} \left(a_3(\tau) \int_{g(\tau)}^{\infty} q(v)dv\right)^{1/\alpha_3} d\tau\right]^{1/\alpha_2} du\right)^{1/\alpha_1} ds = \infty,$$

(2.21)

where $g^{-1}$ is the inverse function of $g$, then equation $(E_1)$ has no solution of type $B_3$.

**Proof.** Let $x(t)$ be an eventually positive solution of equation $(E_1)$ of type $B_3$. There exists a $t_0 \geq 0$ sufficiently large such that (2.2) holds for $t \geq t_0$, so that $L_3x(t)$ is a positive and nonincreasing function i.e. $\lim_{t \to \infty} L_3x(t) = \lambda_3 \geq 0$. Accordingly, integrating equation $(E_1)$ from $t \geq t_0$ to $T \geq t$ and letting $T \to \infty$, we get

$$L_3x(t) \geq \int_{t}^{\infty} q(s)f(x[g(s)])ds \geq \left(\int_{g(t)}^{\infty} q(s)ds\right) f(x[g(g(t))]), \quad t \geq t_0$$

or

$$\frac{d}{dt} L_2x(t) \geq \left(a_3(t) \int_{g(t)}^{\infty} q(s)ds\right)^{1/\alpha_3} f^{-1/\alpha_3}(x[g(g(t))]) \quad \text{for } t \geq t_0.$$

Integrating the above inequality from $g^{-1}(t) > t_0$ to $t$, we have

$$\frac{d}{dt} L_1x(t) \geq \left(a_2(t) \int_{g^{-1}(t)}^{t} \left(a_3(s) \int_{g(s)}^{\infty} q(u)du\right)^{1/\alpha_3} ds\right)^{1/\alpha_2} f^{\frac{1}{\alpha_2}}(x[g(g(t))]) \quad \text{for } t \geq t_1 \geq t_0.$$

Once again, integrating the above inequality from $g^{-1}(t)$ to $t$, we get for all $t \geq t_2 \geq t_1$

$$x'(t) \geq \left(a_1(t) \int_{g^{-1}(t)}^{t} \left(a_2(s) \int_{g^{-1}(s)}^{s} \left(a_3(u) \int_{g(u)}^{\infty} q(\tau)d\tau\right)^{1/\alpha_3} du\right)^{1/\alpha_2} ds\right)^{1/\alpha_1} f^{1/\alpha_1}(x(t)).$$

Dividing both sides of this inequality by $f^{1/\alpha_1}(x(t))$, then integrating both sides of the resulting inequality from $t_2$ to $t$, by using ($F_1$) and (2.21), we arrive at the desired contradiction. This completes the proof. \hfill $\square$

**2.2. Nonexistence of solutions of type $B_1$**

Here we will use the following notation

$$Q_1(t) = \left(a_2(t) \int_{t}^{\infty} \left(a_3(s) \int_{s}^{\infty} q(u)du\right)^{1/\alpha_3} ds\right)^{1/\alpha_2}, \quad t \geq t_0$$

$$F(x) = f^{1/\alpha_1}(x), \quad x \in \mathbb{R},$$
**Theorem 2.5.** Let conditions (i)–(iv) and (2.20) hold. If the half-linear equation

$$
\frac{d}{dt} \left( \frac{1}{a_1(t)} \left( \frac{d}{dt} y(t) \right)^{\alpha_1} \right) + Q_1(t) F(y[g(t)]) = 0
$$

(2.22)

is oscillatory, then equation (E₁) has no solution of type B₁.

**Proof.** Let \( x(t) \) be an eventually positive solution of equation (E₁) of type B₁. There exists a large \( t_0 \geq 0 \) such that (2.1) holds for \( t \geq t_0 \). Notice that \( L₃ x(t) \) is a positive and nonincreasing function, so that \( \lim_{t \to \infty} L₃ x(t) = \lambda₃ \geq 0 \) as well as that \( L₂ x(t) \) is a negative and decreasing function, so that \( \lim_{t \to \infty} L₂ x(t) = \lambda₂ \leq 0 \). Now, integrating equation (E₁) from \( t \) to \( t ≥ t_0 \) and letting \( T \to \infty \), we have

$$
-L₃ x(t) ≤ \lambda₃ - L₃ x(t) = - \int^∞_t q(s) f(x(g(s))) \, ds ≤ - f(x[g(t)]) \int^∞_t q(s) \, ds, \quad t ≥ t_0,
$$

or

$$
\frac{d}{dt} L₂ x(t) ≥ \left( a_3(t) \int^∞_t q(u) \, du \right)^{1/\alpha_3} f^{1/\alpha_3}(x[g(t)]) \quad \text{for } t ≥ t_0.
$$

Once again, integrating the above inequality from \( t \) to \( T₁ ≥ t ≥ t_0 \), letting \( T₁ \to \infty \) and using that \( \lambda₂ ≤ 0 \), we find

$$
-L₂ x(t) ≥ \lambda₂ - L₂ x(t) ≥ \left( \int^∞_t \left( a_3(s) \int^∞_s q(u) \, du \right)^{1/\alpha_3} \, ds \right) f^{1/\alpha_3}(x[g(t)]), \quad t ≥ t_0,
$$

or

$$
-\frac{d}{dt} L₁ x(t) ≥ Q₁(t) F(x[g(t)]) \quad \text{for } t ≥ t_0.
$$

(2.23)

By applying Lemma 2.1, we see that Eq. (2.22) has an eventually positive solution, which contradicts the hypothesis and completes the proof. □

**Theorem 2.6.** Let conditions (i)–(iv), (F) and (2.20) hold, and \( g(t) ≤ t \) for \( t ≥ t_0 \). If the first order delay equation

$$
y'(t) + Q_1(t) F \left( \int^g_{t_0} a_1^{1/\alpha_1}(s) \, ds \right) F \left( y^{1/\alpha_1}[g(t)] \right) = 0
$$

(2.24)

is oscillatory, then equation (E₁) has no solution of type B₁.

**Proof.** Let \( x(t) \) be an eventually positive solution of equation (E₁) of type B₁. As in the proof of Theorem 2.5, we obtain the inequality (2.23). Moreover,

$$
x(t) - x(t_0) = \int^{t}_0 x'(s) \, ds = \int^{t}_0 a_1^{1/\alpha_1}(s) L₁^{1/\alpha₁} x(s) \, ds,
$$

and because \( L₁ x(t) \) is a decreasing function, we have

$$
x(t) ≥ L₁^{1/\alpha₁} x(t) \int^{t}_0 a_1^{1/\alpha_1}(s) \, ds, \quad t ≥ t_0,
$$

or letting \( z(t) = L₁ x(t) \) for \( t ≥ t₁ \)

$$
x[g(t)] ≥ \left( \int^g_{t_0} a_1^{1/\alpha_1}(s) \, ds \right) z^{1/\alpha₁}[g(t)] \quad \text{for } t ≥ t₁ ≥ t_0.
$$

(2.25)

Using (2.25) in (2.23), we get

$$
z'(t) + Q_1(t) F \left( \int^g_{t_0} a_1^{1/\alpha_1}(s) \, ds \right) F \left( z^{1/\alpha₁}[g(t)] \right) ≤ 0 \quad \text{for } t ≥ t₁.
$$

By Lemma 2.3, we see that Eq. (2.24) has an eventually positive solution, which contradicts the hypothesis and completes the proof. □
Corollary 2.2. Let conditions (i)–(iv), \((F)\) and \((2.20)\) hold, and \(g(t) \leq t\) for \(t \geq t_0\). Then, equation \((E_1)\) has no solution of type \(B_1\) if one of the following conditions holds:

\[
\frac{F(y^{1/\alpha_1})}{y} \geq 1 \quad \text{for } y \neq 0, \quad \text{and} \quad \liminf_{t \to \infty} \int_{g(t)}^{t} Q_1(s) F \left( \int_{s_0}^{g(s)} a_1^{1/\alpha_1}(u) du \right) ds > \frac{1}{e} \tag{J_1}
\]

\[
\int_{-\infty}^{\infty} \frac{du}{F(u^{1/\alpha_1})} < \infty, \quad \text{and} \quad \int_{-\infty}^{\infty} Q_1(s) F \left( \int_{s_0}^{g(s)} a_1^{1/\alpha_1}(u) du \right) ds = \infty. \tag{J_2}
\]

Theorem 2.7. Let conditions (i)–(iv) hold, where \(g(t) \leq t\) for \(t \geq t_0\) and let \((F_1), \(2.20)\) and

\[
\int_{t_0}^{\infty} Q_1(s) ds < \infty, \quad t \geq t_0
\]

are satisfied. If

\[
\int_{t_0}^{\infty} g'(u) \left[ a_1(g(u)) \right]^{1/\alpha_1} \left( \int_{u}^{\infty} Q_1(s) ds \right)^{1/\alpha_1} du = \infty, \tag{2.27}
\]

then equation \((E_1)\) has no solution of type \(B_1\).

Proof. Let \(x(t)\) be an eventually positive solution of equation \((E_1)\) of type \(B_1\). As in the proof of Theorem 2.5, we obtain \((2.23)\), so that by integration over \([t, \infty)\), using that \(\lim_{t \to \infty} L_1x(t) = \lambda_1 \geq 0\), we find that

\[
\frac{(x'(t))^{\alpha_1}}{a_1(t)} = L_1x(t) \geq \left( \int_{t}^{\infty} Q_1(s) ds \right) F(x[g(t)]), \quad t \geq t_0. \tag{2.28}
\]

Since, \(L_1x(t)\) is a decreasing function and \(g(t) \geq t\) for \(t \geq t_0\), we have

\[
\frac{x'[g(t)]}{a_1(g(t))^{1/\alpha_1}} \geq \frac{x'(t)}{a_1(t)^{1/\alpha_1}} \geq \left( \int_{t}^{\infty} Q_1(s) ds \right)^{1/\alpha_1} F^{1/\alpha_1}(x[g(t)]) \quad \text{for } t \geq t_1 \geq t_0.
\]

Thus,

\[
\frac{x'[g(t)] g'(t)}{F^{1/\alpha_1}(x[g(t)])} \geq g'(t) [a_1(g(t))]^{1/\alpha_1} \left( \int_{t}^{\infty} Q_1(s) ds \right)^{1/\alpha_1}, \quad t \geq t_1.
\]

Integrating the last inequality from \(t_2\) to \(t\) and using the assumptions \((F_1)\) and \((2.27)\), we come to the following contradiction

\[
\infty > \int_{t_2}^{\infty} \frac{du}{f^{1/\alpha_1}(u)} \geq \int_{t_2}^{t} \frac{x'[g(s)] g'(s)}{F^{1/\alpha_1}(x[g(s)])} ds \geq \int_{t_2}^{t} g'(s) [a_1(g(s))]^{1/\alpha_1} \left( \int_{s}^{\infty} Q_1(u) du \right)^{1/\alpha_1} ds \to \infty, \quad \text{as } t \to \infty. \tag*{□}
\]

Theorem 2.8. Let conditions (i)–(iv) and \((2.20)\) hold. If

\[
\int_{\infty}^{\infty} Q_1(u) du = \infty, \tag{2.29}
\]

then equation \((E_1)\) has no solution of type \(B_1\).

Proof. Let \(x(t)\) be an eventually positive solution of equation \((E_1)\) of type \(B_1\). Proceeding as in the proof of Theorem 2.5, we obtain the inequality \((2.23)\) for \(t \geq t_0\). Now, there exist a constant \(b > 0\) and a \(t_1 \geq t_0\) such that

\[
x[g(t)] \geq b \quad \text{for } t \geq t_1. \tag{2.30}
\]
Using (2.30) in (2.23) and integrating from \( t_1 \) to \( t \), we come to the following contradiction
\[
\infty > L_1 x(t_1) \geq \left( \int_{t_1}^t Q_1(s) \, ds \right) F(b) \to \infty \quad \text{as} \; t \to \infty. \tag{2.31}
\]

**Remark 2.1.** In Theorem 2.8 one can easily see that condition (2.29) can be replaced by the stronger one \( \int_0^\infty q(s) \, ds = \infty \). The proof of this case is easy and hence omitted.

### 2.3. Oscillation theorems

We shall combine the obtained results for the nonexistence of solutions of equation (\( E_1 \)) of types \( B_1 \) and \( B_3 \) and establish some oscillation criteria for equation (\( E_1 \)). First we can establish a sufficient condition for all bounded solutions of equation (\( E_1 \)) to be oscillatory.

**Theorem 2.9.** Let conditions (i)–(iv), (2.20) and (2.26) hold. If
\[
\int_{-\infty}^\infty \left( a_1(s) \int_s^\infty Q_1(u) \, du \right)^{1/\alpha_1} \, ds = \infty, \tag{2.31}
\]
then all bounded solutions of equation (\( E_1 \)) are oscillatory.

**Proof.** Let \( x(t) \) be a bounded and eventually positive solution of equation (\( E_1 \)). Clearly \( x(t) \) is of type \( B_1 \). As in the proof of Theorems 2.7 and 2.8, we obtain (2.28) and (2.30) for \( t \geq t_1 \). Now, using (2.30) in (2.28), we easily find
\[
x'(t) \geq \left( a_1(t) \int_t^\infty Q_1(s) \, ds \right)^{1/\alpha_1} F^{1/\alpha_1}(b) \quad \text{for} \; t \geq t_1.
\]
Integrating the above inequality from \( t_1 \) to \( t \) and applying (2.31), we arrive at the desired contradiction. \( \Box \)

Next, by combining Theorems 2.1 and 2.8, we have

**Theorem 2.10.** Let conditions (i)–(iv), \( (F) \) and (2.20) hold. If
\[
\int_{-\infty}^\infty q(s) f \left( \int_0^{g(s)} \left( a_1(u) \int_0^u a_2^{1/\alpha_2}(\tau) \, d\tau \right)^{1/\alpha_1} \, du \right) \, ds = \infty
\]
and
\[
\int_{-\infty}^\infty \left( a_2(s) \int_s^\infty a_3^{1/\alpha_3}(\tau) \, d\tau \right)^{1/\alpha_2} \, ds = \infty,
\]
then equation (\( E_1 \)) is oscillatory.

By combining Theorems 2.2 and 2.6, we get

**Theorem 2.11.** Let conditions (i)–(iv), \((F)\) and (2.20) hold, and \( g(t) \leq t \) for \( t \geq t_0 \). If the first order delay equations
\[
y'(t) + q(t) f \left( \int_0^{g(t)} \left( a_1(s) \int_0^s a_2^{1/\alpha_2}(\tau) \, d\tau \right)^{1/\alpha_1} \, ds \right) f \left( \int_0^{1/\alpha_1}[g(t)] \right) = 0
\]
and
\[
z'(t) + \left( a_2(t) \int_t^\infty \left( a_3(s) \int_s^\infty q(\tau) \, d\tau \right)^{1/\alpha_3} \, ds \right)^{1/\alpha_2} f \left( \int_0^{1/\alpha_3}[a_1^{1/\alpha_1}(s)] \, ds \right) f \left( \int_0^{1/\alpha_3}[g(t)] \right) = 0
\]
are oscillatory, then equation (\( E_1 \)) is oscillatory.
We note that we can also combine Corollaries 2.1 and 2.2 and obtain some sufficient conditions for the oscillation of equation \((E_1)\).

Finally, we combine Theorems 2.3 and 2.5 and obtain

**Theorem 2.12.** Let conditions (i)–(iv), \((F)\) and \((2.20)\) hold, and assume that there exists a function \(\eta(t) \in C([t_0, \infty), \mathbb{R})\) such that \(g(t) \geq \eta(t) \geq t\) for \(t \geq t_0\). If the second order delay half-linear equations

\[
\frac{d}{dt} \left( \frac{1}{a_3(t)} \left( \frac{d}{dt} y(t) \right)^{\alpha_3} \right) + q(t) f \left( \int_{\eta(t)}^{g(t)} a_1^{1/\alpha_1}(s) ds \left( \int_{\eta(t)}^{g(t)} a_2^{1/\alpha_2}(s) ds \right)^{1/\alpha_1} \right) f \left( \frac{1}{y^{\alpha_1} \eta^{\alpha_2}}(t) \right) = 0 \tag{2.32}
\]

and

\[
\frac{d}{dt} \left( \frac{1}{a_3(t)} \left( \frac{d}{dt} z(t) \right)^{\alpha_3} \right) + \left( a_2(t) \int_t^{\infty} (a_3(s) \int_s^{\infty} q(u) du)^{1/\alpha_3} ds \right)^{1/\alpha_2} f \left( \frac{1}{a^{2\alpha_3}}(z[g(t)]) \right) = 0 \tag{2.33}
\]

are oscillatory, then equation \((E_1)\) is oscillatory.

We note that the literature is filled with oscillation results for equations of the second order delay half-linear equations of type \((2.32)\) and \((2.33)\). Therefore, these oscillation results become oscillation criteria for equation \((E_1)\).

### 2.4. Examples and applications

As an illustrative example, we consider a special case of equation \((E_1)\), namely, the equation

\[
\frac{d}{dt} \left( \frac{1}{a_3(t)} \left( \frac{d}{dt} x(t) \right)^{\alpha_3} \right)^{\alpha_2} + q(t) \chi \beta [g(t)] = 0, \tag{\Delta_1}
\]

where \(\beta\) is the ratio of two positive odd integers.

In order to simplify notation in oscillation theorems we will use the notation

\[
\frac{1 + \alpha_2}{\alpha_1 \alpha_2} = \mu, \quad 1 + \frac{1}{\alpha_1} = \eta, \quad 1 + \frac{1}{\alpha_2} = \chi.
\]

From Theorems 2.10–2.12 one can easily see that equation \((\Delta_1)\) is oscillatory if one of the following set of conditions holds:

**\((O_1)\) Condition \((2.20)\),**

\[
\int_0^{\infty} q(s)[g(s)]^\eta \beta ds = \infty
\]

and

\[
\int_0^{\infty} \left( \int_u^{\infty} q(\tau) d\tau \right)^{1/\alpha_3} du = \infty.
\]

**\((O_2)\) Condition \((2.20)\) and \(g(t) \leq t\) for \(t \geq t_0\), and both first order delay equations**

\[
y'(t) + \frac{q(t)}{(1 + \mu)^\beta \chi^{\beta/\alpha_1}} g^{1+\mu}(t) y^\beta [g(t)] = 0
\]

and

\[
z'(t) + \left( \int_t^{\infty} q(\tau) d\tau \right)^{1/\alpha_3} g^{\beta/\alpha_2}(t) z^\beta [g(t)] = 0
\]

are oscillatory.
(O3) Condition (2.20) holds and assume that there exists a nondecreasing function \( \eta(t) \in C([t_0, \infty), \mathbb{R}) \) such that 
\[ g(t) \geq \eta(t) \geq t \quad \text{for} \quad t \geq t_0, \]
and both the second order half-linear equations
\[
\frac{d}{dr} \left( \frac{d}{dr} y(t) \right)^{\alpha_3} + q(t)(g(t) - \eta(t))^{\beta} (\eta(t) - t)^{\beta/\alpha_1} y^{\beta/\alpha_2} (t) = 0
\]
and
\[
\frac{d}{dr} \left( \frac{d}{dr} z(t) \right)^{\alpha_1} + \left( \int_t^\infty \left( \int_s^\infty q(u)du \right)^{1/\alpha_3} ds \right)^{1/\alpha_2} z^{\beta/\alpha_3} (g(t)) = 0
\]
are oscillatory.

We note that all three oscillation results for Eq. (\( \Delta_1 \)) given above are new.

For example, for a special case of Eq. (\( \Delta_1 \)), with \( \beta = \alpha_i = 1, i = 1, 2, 3, 4 \), we consider the equation
\[
\frac{d^4 x}{dr^4} + q(t)x[g(t)] = 0 \quad (\Delta_2)
\]
by applying the set of conditions (\( \Delta_2 \)), we see that this equation is oscillatory if both first order delay differential equations
\[ y'(t) + \frac{q(t)}{6} g^3(t) y[g(t)] = 0 \]
and
\[ z'(t) + \left( \int_t^\infty \left( \int_s^\infty q(u)du \right) ds \right) g(t) z[g(t)] = 0 \]
are oscillatory. As an example, we see that the equation
\[
\frac{d^4 x}{dr^4} + \frac{x[t - \tau]}{t^5} = 0, \quad \tau > 0 \text{ is a constant}
\]
is oscillatory.

Finally, we shall apply our results of this section to neutral equations which include equation (\( E_1 \)) as a special case, i.e., we consider the equation
\[
L_4 (x(t) + p(t)x[\tau(t)]) + q(t)f(x[g(t)]) = 0, \quad (\Delta_3)
\]
where \( \tau(t), p(t) \in C([t_0, \infty), \mathbb{R}) \) satisfy the following conditions:

(P1) \( 0 \leq p(t) \leq 1, p(t) \neq 0, p(t) \neq 1; \)

(T1) \( \tau(t) \) is an increasing function on \([t_0, \infty), \tau(t) \to \infty \text{ as } t \to \infty \) and \( \tau(t) < t \) for \( t \geq t_0. \)

First, observe that if \( x(t) \) is an eventually positive solution of equation (\( E_1 \)) of type \( B_1 \) or \( B_3 \), then
\[ y(t) = x(t) + p(t)x[\tau(t)] \]
is eventually positive and of the same type as \( x(t) \). Then Eq. (\( \Delta_3 \)) reduces to
\[ L_4 y(t) + q(t)f(x[g(t)]) = 0, \quad (2.35) \]
Since \( y(t) \) is of type \( B_1 \) or \( B_3 \), it is \( y'(t) > 0 \) eventually and so,
\[
x(t) = y(t) - p(t)x[\tau(t)]
\]
\[
= y(t) - p(t)[y[\tau(t)] - p[\tau(t)]x[\tau \circ \tau(t)]]
\]
\[
\geq y(t) - p(t)y[\tau(t)] \geq (1 - p(t))y(t) \quad \text{eventually.} \quad (2.36)
\]
Using (F) and (2.31) in Eq. (\( \Delta_3 \)), we obtain for all large \( t \) the following differential inequality
\[
L_4 y(t) + q(t)f(1 - p[g(t)])f(y[g(t)]) \leq 0.
\]
Therefore, now, we see that all results of this section are valid for Eq. (Δ₃) by replacing the function q(t) in the hypotheses of our results by q(t)f(1 − p[g(t)]). The details are left to the reader.

Next, we assume τ(t), p(t) ∈ C([t₀, ∞), ℝ) satisfy the following conditions:

(P₂) p(t) ≥ 1, p(t) ≠ 1, τ(t) > t for t ≥ t₀;
(T₂) τ(t) is an increasing function on [t₀, ∞), τ(t) → ∞ as t → ∞ and τ(t) > t for t ≥ t₀.

As above, we let y(t) be defined by (2.34) and equation (Δ₃) becomes (2.35). Now,

\[ x(τ(t)) = \frac{y(t) - x(t)}{p(t)} \]

or

\[
x(t) = \frac{1}{p[τ^{-1}(t)]} \left( y[τ^{-1}(t)] - x[τ^{-1}(t)] \right)
= \frac{y[τ^{-1}(t)]}{p[τ^{-1}(t)]} - \frac{1}{p[τ^{-1}(t)]} \left[ \frac{y[τ^{-1} \circ τ^{-1}(t)]}{p[τ^{-1} \circ τ^{-1}(t)]} - \frac{x[τ^{-1} \circ τ^{-1}(t)]}{p[τ^{-1} \circ τ^{-1}(t)]} \right]
\geq \frac{y[τ^{-1}(t)]}{p[τ^{-1}(t)]} - \frac{1}{p[τ^{-1}(t)]} \frac{y[τ^{-1} \circ τ^{-1}(t)]}{p[τ^{-1} \circ τ^{-1}(t)]} \text{ eventually,} \tag{2.37}
\]

where \( τ^{-1}(t) \) is the inverse function of \( τ(t) \). Since \( τ(t) > t \) and \( τ^{-1}(t) \), \( y(t) \) are increasing functions, we have that \( y[τ^{-1}(t)] > y[τ^{-1} \circ τ^{-1}(t)] \). Accordingly, from (2.37) we obtain

\[
x(t) \geq \frac{y[τ^{-1}(t)]}{p[τ^{-1}(t)]} \left[ 1 - \frac{1}{p[τ^{-1} \circ τ^{-1}(t)]} \right] := P(t)y[τ^{-1}(t)] \text{ eventually.} \tag{2.38}
\]

Applying assumption (F) in (2.38), from Eq. (2.35), we obtain for all large \( t \) the following differential inequality

\[ L₄y(t) + q(t)f(P[g(t)])f(y[τ^{-1}(g(t))]) \leq 0 \text{ eventually.} \]

Thus, all results of this section remain valid for Eq. (Δ₃) if we replace \( q(t) \) and \( g(t) \) in the hypotheses of these results by \( q(t)f(P[g(t)]) \) and \( τ^{-1} \circ g(t) \) respectively. The details are left to the reader.

For example, by applying the set of conditions (O₁), from the above results, we conclude that the neutral equations

\[
\frac{d}{dt} \left( \frac{d}{dt} \left( \frac{d}{dt} \left( x(t) + \left( 1 - \frac{1}{t} \right) x[\sqrt{t}] \right)^{5/3} \right) \right) + \frac{1}{t^3} x^{11/5}[t - \sqrt{t}] = 0, \quad t \geq 1
\]

and

\[
\frac{d}{dt} \left( \frac{d}{dt} \left( \frac{d}{dt} \left( x(t) + \left( 1 + \frac{1}{t} \right) x[t^2] \right)^{5/3} \right) \right) + \frac{1}{t^3} x^{11/5}[t - \sqrt{t}] = 0, \quad t \geq 1
\]

are oscillatory. The verifications are easy and are left to the reader.

3. Properties of solutions of equation \((E_{-1})\)

For equation \((E_{-1})\) we have the following three cases of the signs of \( L_i x(n), i = 0, 1, 2, 3, 4 \) for a nonoscillatory solution \( x(t) \):

\[ (-1)^i x(t)L_i x(t) > 0, \quad i = 0, 1, 2, 3, \quad \text{and} \quad x(t)L₄ x(t) \geq 0 \text{ eventually}, \tag{3.1} \]

\[ x(t)L_i x(t) > 0, \quad i = 0, 1, 2, \quad x(t)L₃ x(t) < 0, \quad \text{and} \quad x(t)L₄ x(t) \geq 0 \text{ eventually}, \tag{3.2} \]

\[ x(t)L_i x(t) > 0, \quad i = 0, 1, 2, 3, \quad \text{and} \quad x(t)L₄ x(t) \geq 0 \text{ eventually.} \tag{3.3} \]

We shall say that the solution \( x(t) \) is of type \( B₀, B₂ \) and \( B₄ \) if it satisfies respectively (3.1)–(3.3).
3.1. Nonexistence of solutions of type B₂

**Theorem 3.1.** Let conditions (i)–(iv) and (F) hold. If

\[
\int_0^\infty q(s) f \left( \int_{t_0}^{g(s)} a_1^{1/\alpha_1}(u) du \right) ds = \infty, \tag{3.4}
\]

then equation \((E_{-1})\) has no solution of type B₂.

**Proof.** Let \(x(t)\) be an eventually positive solution of equation \((E_{-1})\) of type B₂. There exists a sufficiently large \(t_0 \geq 0\) such that (3.2) holds for \(t \geq t_0\). Therefore, \(L_1x(t)\) is an increasing function, so that there exist a constant \(c > 0\) and a \(t_1 \geq t_0\) such that

\[
L_1x(t) = \frac{1}{a_1(t)} \left( \frac{d}{dt} x(t) \right)^{\alpha_1} \geq c, \quad t \geq t_1,
\]

or

\[
x'(t) \geq c^{1/\alpha_1} a_1^{1/\alpha_1}(t) \quad \text{for } t \geq t_1.
\]

Integrating the above inequality from \(t_2\) to \(g(t) \geq t_2 \geq t_1\) we get

\[
x[g(t)] \geq c^{1/\alpha_1} \int_{t_2}^{g(t)} a_1^{1/\alpha_1}(s) ds, \quad t \geq t_2. \tag{3.5}
\]

Using (3.5) and (F) in equation \((E_{-1})\), we have

\[
L_4x(t) \geq f(c^{1/\alpha_1}) q(t) f \left( \int_{t_2}^{g(t)} a_1^{1/\alpha_1}(s) ds \right) \quad \text{for } t \geq t_2.
\]

Integrating the above inequality from \(t_2\) to \(t\), we obtain

\[
\infty > -L_3x(t_2) \geq L_3x(t) - L_3x(t_2)
\]

\[
\geq f(c^{1/\alpha_1}) \int_{t_2}^{t} q(s) f \left( \int_{t_2}^{g(s)} a_1^{1/\alpha_1}(u) du \right) ds \to \infty \quad \text{as } t \to \infty.
\]

The obtained contradiction completes the proof. \(\square\)

**Theorem 3.2.** Let conditions (i)–(iv) and (F) hold, and \(g(t) \leq t\) for \(t \geq t_0\). If all bounded solutions of the half-linear delay differential equation

\[
\frac{d}{dt} \left( \frac{1}{a_3(t)} \left( \frac{d}{dt} y(t) \right)^{\alpha_3} \right) - q(t) f \left( \int_{t_0}^{g(t)} \left( a_1(s) \int_{t_0}^{s} a_2^{1/\alpha_2}(u) du \right)^{1/\alpha_1} ds \right) f \left( y^{1/\alpha_2} \left[ g(t) \right] \right) = 0 \tag{3.6}
\]

are oscillatory, then equation \((E_{-1})\) has no solution of type B₂.

**Proof.** Let \(x(t)\) be an eventually positive solution of equation \((E_{-1})\) of type B₂, such that (3.2) holds for \(t \geq t_0\). Now

\[
L_1x(t) = L_1x(t_0) + \int_{t_0}^{t} a_2^{1/\alpha_2}(s) L_2^{1/\alpha_2} x(s) ds.
\]

Using the fact that \(L_2x(t)\) is a nonincreasing function on \([t_0, \infty)\), we get

\[
L_1x(t) \geq \left( \int_{t_0}^{t} a_2^{1/\alpha_2}(s) ds \right) L_2^{1/\alpha_2} x(t) \quad \text{for } t \geq t_0
\]

or

\[
x'(t) \geq a_1(t) \int_{t_0}^{t} a_2^{1/\alpha_2}(s) ds \left[ \frac{1}{\alpha_1} \frac{1}{L_2^{1/\alpha_2}} x(t) \right] \quad \text{for } t \geq t_0.
\]
Integrating the above inequality from $t_1$ to $g(t) \geq t_1 \geq t_0$ and letting $y(t) = L_2x(t)$, $t \geq t_1$, we have

$$x[g(t)] \geq \left( \int_{t_1}^{g(t)} \left( a_1(s) \int_{t_0}^{s} a_2^{1/\alpha_2}(u)du \right)^{1/\alpha_1} \frac{1}{\alpha_1} ds \right) y^{1/\alpha_2} \left( g(t) \right) \text{ for } t \geq t_1. \quad (3.7)$$

Using (3.7) and (F) in equation (E\text{--}1), we have for all $t \geq t_1$

$$\frac{d}{dt} \left( \frac{1}{\alpha_3(t)} \left( \frac{d}{dt} y(t) \right)^{\alpha_3} \right) \geq q(t) f \left( \int_{t_1}^{g(t)} \left( a_1(s) \int_{t_0}^{s} a_2^{1/\alpha_2}(u)du \right)^{1/\alpha_1} ds \right) \left( \frac{1}{\alpha_1} \right) \left( y^{1/\alpha_2} \left( g(t) \right) \right). \quad (3.8)$$

Since $L_2x(t)$ is a positive and decreasing function it is bounded. Therefore, $y(t)$ is a positive and bounded solution of differential inequality (3.8), using Lemma 2.2, we see that Eq. (3.6) has an eventually positive and bounded solution, which contradicts the hypothesis and completes the proof. \hfill \Box

**Theorem 3.3.** Let conditions (i)--(iv) and (F) hold, and $g(t) < t$ for $t \geq t_0 \geq 0$. Assume that there exists a nondecreasing function $\xi(t) \in C([t_0, \infty), \mathbb{R})$ such that $g(t) < \xi(t) < t$ for $t \geq t_0$. If the first order delay differential equation

$$z'(t) + q(t) f \left( \int_{t_1}^{g(t)} a_1(s) \int_{t_0}^{s} a_2^{1/\alpha_2}(u)du \right)^{1/\alpha_1} ds \times f \left( \int_{g(t)}^{\xi(t)} a_3^{1/\alpha_3}(u)du \right)^{1/\alpha_3} \left( z^{1/\alpha} \xi(t) \right) = 0 \quad (3.9)$$

is oscillatory then equation (E\text{--}1) has no solution of type $B_2$.

**Proof.** Let $x(t)$ be an eventually positive solution of equation (E\text{--}1) of type $B_2$. As in the proof of Theorem 3.2, we obtain (3.7) for $t \geq t_1$. Besides,

$$-L_2x(s) \leq L_2x(t) - L_2x(s) = \int_{t_0}^{t} a_2^{1/\alpha_2}(u)L_3^{1/\alpha_3}x(u)du \leq L_3^{1/\alpha_3}x(t) \int_{s}^{t} a_3^{1/\alpha_3}(u)du,$$

or

$$y(s) = L_2x(s) \geq \left( \int_{s}^{t} a_3^{1/\alpha_3}(u)du \right) \left( -L_3^{1/\alpha_3}x(t) \right) \text{ for } t \geq s \geq t_1.$$

Replacing $s$ and $t$ by $g(t)$ and $\xi(t)$ respectively, we find

$$y[g(t)] \geq \left( \int_{g(t)}^{\xi(t)} a_3^{1/\alpha_3}(u)du \right) \left( -L_3^{1/\alpha_3}x[\xi(t)] \right) \text{ for } t \geq t_2 \geq t_1. \quad (3.10)$$

Using (3.10) in (3.7), we have for $t \geq t_2$,

$$x[g(t)] \geq \left( \int_{t_1}^{g(t)} a_1(s) \int_{t_0}^{s} a_2^{1/\alpha_2}(u)du \right)^{1/\alpha_1} \left( \int_{g(t)}^{\xi(t)} a_3^{1/\alpha_3}(u)du \right)^{1/\alpha_3} \left( -L_3^{1/\alpha_3}x[\xi(t)] \right). \quad (3.11)$$

Using (F) and (3.11) in equation (E\text{--}1) and letting $z(t) = -L_3x(t) > 0$ for $t \geq t_2$, we obtain

$$z'(t) + q(t) f \left( \int_{t_1}^{g(t)} a_1(s) \int_{t_0}^{s} a_2^{1/\alpha_2}(u)du \right)^{1/\alpha_1} ds \times f \left( \int_{g(t)}^{\xi(t)} a_3^{1/\alpha_3}(u)du \right)^{1/\alpha_3} f \left( z^{1/\alpha} \xi(t) \right) \leq 0.$$ 

By Lemma 2.3, Eq. (3.9) has an eventually positive solution which contradicts the hypothesis and completes the proof. \hfill \Box

In the following result we do not require the existence of the function $\xi(t)$ as needed in Theorem 3.3.
Theorem 3.4. Let conditions (i)–(iv), (F) and (2.20) hold, and \( g(t) \leq t \) for \( t \geq t_0 \). If the first order delay equation

\[
z'(t) + \left(a_3(t) \int_t^\infty q(s)ds\right)^{1/\alpha_3} f^{1/\alpha_3} \left( \int_0^{g(t)} \left(a_1(s) \int_s^{\tilde{s}} a_2^{1/\alpha_2}(u)du\right)^{1/\alpha_1} ds \right) f^{1/\alpha_3} \left( \frac{1}{\alpha^{1/\alpha_2}} [g(t)] \right) = 0 \tag{3.12}
\]

is oscillatory, then equation \((E-1)\) has no solution of type \( B_2 \).

Proof. Let \( x(t) \) be an eventually positive solution of equation \((E-1)\) of type \( B_2 \), so that there exists a large \( t_0 \geq 0 \) such that (3.2) holds for \( t \geq t_0 \). Integrating the equation \((E-1)\) from \( t \) to \( u \geq t \geq t_0 \) we have

\[
L_3x(u) - L_3x(t) = \int_t^u q(s)f(x[g(s)])ds,
\]

and letting \( u \rightarrow \infty \), since \( \lim_{t \rightarrow \infty} L_3x(t) = \lambda_3 \leq 0 \), we get

\[
- \frac{d}{dt} L_2x(t) \geq \left(a_3(t) \int_t^\infty q(s)ds\right)^{1/\alpha_3} f^{1/\alpha_3} (x[g(t)]) \quad \text{for } t \geq t_0.
\tag{3.13}
\]

As in the proof of Theorem 3.2, we obtain (3.7) for \( t \geq t_1 \). Thus, using (F) and (3.7) in the inequality (3.13) and letting \( L_2x(t) = y(t) \) for \( t \geq t_1 \), we obtain

\[
y'(t) + \left(a_3(t) \int_t^\infty q(s)ds\right)^{1/\alpha_3} f^{1/\alpha_3} \left( \int_0^{g(t)} \left(a_1(s) \int_s^{R} a_2^{1/\alpha_2}(u)du\right)^{1/\alpha_1} ds \right) f^{1/\alpha_3} \left( \frac{1}{\alpha^{1/\alpha_2}} [g(t)] \right) \leq 0.
\]

By applying Lemma 2.3, we see that Eq. (3.12) has an eventually positive solution which contradicts the hypothesis and completes the proof. \( \square \)

3.2. Nonexistence of solutions of type \( B_4 \)

Theorem 3.5. Let conditions (i)–(iv) and (F) hold, \( g(t) > t \) for \( t \geq t_0 \), and assume that there exist nondecreasing functions \( \rho(t), \xi(t) \in C([t_0, \infty), \mathbb{R}) \) such that \( g(t) > \rho(t) > \xi(t) \geq t \) for \( t \geq t_0 \). If all unbounded solutions of the second order advanced half-linear equation

\[
\frac{d}{dt} \left( \frac{1}{a_3(t)} \frac{dy}{dt} \right)^{\alpha_3} - q(t)f \left( \int_{\rho(t)}^{g(t)} a_1^{1/\alpha_1}(s)ds \right) f \left( \int_{\xi(t)}^{\rho(t)} a_2^{1/\alpha_2}(s)ds \right) \geq 0,
\tag{3.14}
\]

are oscillatory, then equation \((E-1)\) has no solution of type \( B_4 \).

Proof. Let \( x(t) \) be an eventually positive solution of equation \((E-1)\) of type \( B_4 \). There exists a \( t_0 \geq 0 \) such that (3.3) holds for \( t \geq t_0 \). Now,

\[
x(\sigma) - x(t) = \int_t^\sigma x'(s)ds = \int_t^\sigma a_1^{1/\alpha_1}(s)L_1^{1/\alpha_1}x(s)ds
\]

\[
\geq \left( \int_t^\sigma a_1^{1/\alpha_1}(s)ds \right) L_1^{1/\alpha_1}x(t) \quad \text{for } \sigma \geq t \geq t_0.
\tag{3.15}
\]

Letting \( \sigma = g(t), \tau = \rho(t) \) in (3.15), we have

\[
x[g(t)] \geq \left( \int_{\rho(t)}^{g(t)} a_1^{1/\alpha_1}(s)ds \right) L_1^{1/\alpha_1}x[\rho(t)] \quad \text{for } t \geq t_1 \geq t_0.
\tag{3.16}
\]

Similarly, one can easily find

\[
L_1x[\rho(t)] \geq \left( \int_{\xi(t)}^{\rho(t)} a_2^{1/\alpha_2}(s)ds \right) L_2^{1/\alpha_2}x[\xi(t)] \quad \text{for } t \geq t_1.
\tag{3.17}
\]
Now, combining (3.16) and (3.17), we get
\[ x[g(t)] \geq \left( \int_{\rho(t)}^{g(t)} a_1^{1/\alpha_1}(s)ds \right) \left( \int_{\xi(t)}^{\rho(t)} a_2^{1/\alpha_2}(s)ds \right)^{1/\alpha_1} L_2^{-\frac{1}{\alpha_1\alpha_2}} x[\xi(t)] \quad \text{for } t \geq t_1, \] (3.18)
and by using the assumption (F) from equation (E−1) we obtain
\[ \frac{d}{dt} \left( \frac{1}{a_3(t)} \left( \frac{d}{dt} y(t) \right)^{a_3} \right) \geq q(t) f \left( \int_{\rho(t)}^{g(t)} a_1^{1/\alpha_1}(s)ds \right) f \left( \int_{\xi(t)}^{\rho(t)} a_2^{1/\alpha_2}(s)ds \right)^{1/\alpha_1} f \left( \int_{\eta(t)}^{\xi(t)} a_3^{1/\alpha_3}(s)ds \right)^{1/\alpha_2} y^{\frac{1}{\alpha_1\alpha_2}}[\xi(t)], \] (3.19)
where \( y(t) = L_2 x(t) \) for \( t \geq t_1 \). It is easily verified that for the positive solution of equation E−1 of the type B_4 we have \( \lim_{t \to \infty} L_2 x(t) = \infty \). Therefore, \( y(t) \) is an unbounded solution of the inequality (3.19) and by Lemma 2.3 we conclude that Eq. (3.14) has an eventually unbounded positive solution. The obtained contradiction completes the proof. \( \square \)

**Theorem 3.6.** Let conditions (i)–(iv) and (F) hold, and \( g(t) > t \) for \( t \geq t_0 \), and assume that there exist nondecreasing functions \( \rho(t), \xi(t) \) and \( \eta(t) \in C([t_0, \infty), \mathbb{R}) \) such that \( g(t) > \rho(t) > \xi(t) > \eta(t) > t \) for \( t \geq t_0 \). If the first order advanced equation
\[ z'(t) = q(t) f \left( \int_{\rho(t)}^{g(t)} a_1^{1/\alpha_1}(s)ds \right) f \left( \int_{\xi(t)}^{\rho(t)} a_2^{1/\alpha_2}(s)ds \right)^{1/\alpha_1} \times f \left( \int_{\eta(t)}^{\xi(t)} a_3^{1/\alpha_3}(s)ds \right)^{1/\alpha_2} \] (3.20)
is oscillatory, then equation (E−1) has no solution of type B_4.

**Proof.** Let \( x(t) \) be an eventually positive solution of equation (E−1) of type B_4. As in the proof of Theorem 3.5, we obtain the inequality (3.18). Next, we can easily find
\[ L_2 x[\xi(t)] \geq \left( \int_{\eta(t)}^{\xi(t)} a_3^{1/\alpha_3}(s)ds \right) L_3^{\frac{1}{\alpha_3}} x[\eta(t)] \quad \text{for } t \geq t_2 \geq t_1. \] (3.21)
Combining (3.18) and (3.21), for \( t \geq t_2 \), we get
\[ x[g(t)] \geq \left( \int_{\rho(t)}^{g(t)} a_1^{1/\alpha_1}(s)ds \right) \left( \int_{\xi(t)}^{\rho(t)} a_2^{1/\alpha_2}(s)ds \right)^{1/\alpha_1} \left( \int_{\eta(t)}^{\xi(t)} a_3^{1/\alpha_3}(s)ds \right)^{1/\alpha_2} L_3 \frac{1}{\alpha_3} x[\eta(t)]. \] (3.22)
Using (F) and (3.22) in equation (E−1) and letting \( z(t) = L_3 x(t) \) for \( t \geq t_2 \), we have
\[ z'(t) \geq q(t) f \left( \int_{\rho(t)}^{g(t)} a_1^{1/\alpha_1}(s)ds \right) f \left( \int_{\xi(t)}^{\rho(t)} a_2^{1/\alpha_2}(s)ds \right)^{1/\alpha_1} \times f \left( \int_{\eta(t)}^{\xi(t)} a_3^{1/\alpha_3}(s)ds \right)^{1/\alpha_2} \left( z^{\frac{1}{\alpha_3}}[\eta(t)] \right). \]
By Lemma 2.3 Eq. (3.20) has an eventually positive solution which contradicts the hypothesis and completes the proof. \( \square \)

**Corollary 3.1.** Let conditions (i)–(iv) and (F) hold, and \( g(t) > t \) for \( t \geq t_0 \). Moreover, assume that there exist nondecreasing functions \( \rho(t), \xi(t), \eta(t) \in C([t_0, \infty), \mathbb{R}) \) such that \( g(t) > \rho(t) > \xi(t) > \eta(t) > t \) for \( t \geq t_0 \). Equation (E−1) has no solution of type B_4 if one of the following conditions holds:
(S1) \( \frac{f(u)}{u} \geq 1 \) for \( u \neq 0 \), and

either \( \liminf_{t \to \infty} \int_t^\eta(t) P(s)ds > \frac{1}{e} \), or \( \limsup_{t \to \infty} \int_t^\eta(t) P(s)ds > 1 \),

where

\[
P(s) = q(s)f \left( \int_{\rho(s)}^{g(s)} a_1^{1/\alpha_1}(u)du \right) f \left( \left( \int_{\xi(s)}^{\rho(s)} a_2^{1/\alpha_2}(u)du \right)^{\frac{1}{\alpha_1}} \right) f \left( \left( \int_{\eta(s)}^{\xi(s)} a_3^{1/\alpha_3}(u)du \right)^{\frac{1}{\alpha_2}} \right).
\]

(S2) \( \int \frac{u}{f(u)} \to 0 \) as \( u \to \infty \) and \( \liminf_{t \to \infty} \int_t^\eta(t) P(s)ds > 0 \).

(S3) \( \int_0^\infty \frac{du}{f(u)} < \infty \), and \( \int_0^\infty P(s)ds = \infty \).

**Theorem 3.7.** Let conditions (i)–(iv) and \( F \) hold, \( g(t) > t \) for \( t \geq t_0 \), and assume that there exist nondecreasing functions \( \rho(t), \xi(t) \in C([t_0, \infty), \mathbb{R}) \) such that \( g(t) > \rho(t) > \xi(t) \geq t \) for \( t \geq t_0 \). Equation \((E_{-1})\) has no solution of type \( B_4 \) if one of the following conditions holds:

\[(R_1) \quad \frac{1}{\int_0^\eta(t) f(u)du} \geq 1 \text{ for } u \neq 0, \text{ and } \limsup_{t \to \infty} \int_t^{\xi(t)} P^*(v)dv > 1, \text{ where}
\]

\[
P^*(v) = a_3^{1/\alpha_3}(v) \left[ \int_0^v q(s) f \left( \int_{\rho(s)}^{g(s)} a_1^{1/\alpha_1}(\tau)d\tau \right) f \left( \left( \int_{\xi(s)}^{\rho(s)} a_2^{1/\alpha_2}(\tau)d\tau \right)^{1/\alpha_1} \right) ds \right]^{1/\alpha_3}.
\]

\[(R_2) \quad \frac{1}{\int_0^\eta(t) f(u)du} \to 0 \text{ as } u \to \infty, \text{ and } \limsup_{t \to \infty} \int_t^{\xi(t)} P^*(v)dv > 0.
\]

**Proof.** Let \( x(t) \) be an eventually positive solution of equation \((E_{-1})\) of type \( B_4 \). As in the proof of Theorem 3.5, we obtain the inequality \((3.19)\) for \( t \geq t_1 \geq t_0 \) and then integrating both sides of this inequality from \( t \) to \( v \geq t \geq t_1 \), we have

\[
y'(v) \geq a_3^{1/\alpha_3}(v) \left[ \int_t^v q(s) f \left( \int_{\rho(s)}^{g(s)} a_1^{1/\alpha_1}(\tau)d\tau \right) f \left( \left( \int_{\xi(s)}^{\rho(s)} a_2^{1/\alpha_2}(\tau)d\tau \right)^{1/\alpha_1} \right) f \left( \left( \int_{\eta(s)}^{\xi(s)} a_3^{1/\alpha_3}(\tau)d\tau \right)^{1/\alpha_2} \right) ds \right]^{1/\alpha_3}.
\]  

Combining \((3.23)\) with the relation

\[
y(V) \geq y(V) - y(t) = \int_t^V y'(v)dv \quad \text{for } V \geq v \geq t \geq t_1
\]

and putting \( V = \xi(t) \), we have

\[
\frac{y[\xi(t)]}{f^{1/\alpha_3} \left( \int_0^{\xi(t)} f(u)du \right)} \geq \int_t^{\xi(t)} P^*(v)dv.
\]

Taking \( \limsup \) of both sides of the above inequality as \( t \to \infty \) and applying the hypotheses in \( R_1 \) or \( R_2 \) we arrive at the desired contradiction. This completes the proof. \( \square \)

### 3.3. Nonexistence of solutions of type \( B_0 \)

Now, we shall present some criteria for the nonexistence of solutions of equation \((E_{-1})\) of type \( B_0 \), i.e., the oscillatory behavior of all bounded solutions of equation \((E_{-1})\).
Theorem 3.8. Let conditions (i)–(iv) and (F) hold, and $g(t) < t$ for $t \geq t_0$. Assume that there exist nondecreasing functions $\xi(t), \eta(t) \in C([t_0, \infty), \mathbb{R})$ such that $g(t) < \xi(t) < \eta(t) < t$ for $t \geq t_0$. If all bounded solutions of the second order delay half-linear equation

$$\frac{d}{dt} \left( \frac{1}{a_3(t)} \left( \frac{dy}{dt} \right)^{\alpha_3} \right) - q(t) f \left( \int_{g(t)}^{\xi(t)} a_1^{1/\alpha_1}(s) ds \right) f \left( \int_{\xi(t)}^{\eta(t)} a_2^{1/\alpha_2}(s) ds \right) f \left( \frac{1}{y^{\alpha_2}} \right) \eta(t) = 0.$$  

(3.24)

are oscillatory, then equation $(E_{-1})$ has no solution of type $B_0$.

Proof. Let $x(t)$ be an eventually positive solution of equation $(E_{-1})$ of type $B_0$. Then, there exists a large $t_0 \geq 0$ such that Eq. (3.1) holds for $t \geq t_0$. For $t \geq s \geq t_0$, we obtain

$$x(t) - x(s) = \int_s^t a_1^{1/\alpha_1}(u) L_1^{1/\alpha_1} x(u) du \leq L_1^{1/\alpha_1} x(t) \int_s^t a_1^{1/\alpha_1}(u) du$$

and so we have

$$x(s) \geq \left( -L_1^{1/\alpha_1} x(t) \right) \int_s^t a_1^{1/\alpha_1}(u) du.$$  

Replacing $s$ and $t$ in the above inequality by $g(t)$ and $\xi(t)$ respectively, we find

$$x[g(t)] \geq \left( \int_{g(t)}^{\xi(t)} a_1^{1/\alpha_1}(u) du \right) \left( -L_1^{1/\alpha_1} x[\xi(t)] \right) \quad \text{for } t \geq t_1 \geq t_0.$$  

(3.25)

Similarly, $t \geq s \geq t_0$, we find

$$-L_1 x(s) \geq L_1 x(t) - L_1 x(s) = \int_s^t a_2^{1/\alpha_2}(u) L_2^{1/\alpha_2} x(u) du \leq L_2^{1/\alpha_2} x(t) \int_s^t a_2^{1/\alpha_2}(u) du$$

and then by replacing $s$ and $t$ in the above inequality by $\xi(t)$ and $\eta(t)$ respectively, we get

$$-L_1 x[\xi(t)] \geq L_2^{1/\alpha_2} x[\eta(t)] \int_{\xi(t)}^{\eta(t)} a_2^{1/\alpha_2}(u) du \quad \text{for } t \geq t_1.$$  

(3.26)

Thus, combining (3.25) and (3.26) we have

$$x[g(t)] \geq \left( \int_{g(t)}^{\xi(t)} a_1^{1/\alpha_1}(u) du \right) \left( \int_{\xi(t)}^{\eta(t)} a_2^{1/\alpha_2}(u) du \right) L_2^{1/\alpha_2} \left[ x[\eta(t)] \right] \quad \text{for } t \geq t_1.$$  

(3.27)

Using (3.27) together with the assumption (F) in equation $(E_{-1})$ and letting $y(t) = L_2 x(t)$ for $t \geq t_1$, we have

$$\frac{d}{dt} \left( \frac{1}{a_3(t)} \left( \frac{dy}{dt} \right)^{\alpha_3} \right) \geq q(t) f \left( \int_{g(t)}^{\xi(t)} a_1^{1/\alpha_1}(u) du \right) f \left( \int_{\xi(t)}^{\eta(t)} a_2^{1/\alpha_2}(u) du \right) f \left( \frac{1}{y^{\alpha_2}} \right) \eta(t).$$  

(3.28)

Since $L_2 x(t)$ is a positive and decreasing function and therefore a bounded solution of the inequality (3.28), by Lemma 2.3 we see that Eq. (3.24) has an eventually positive and bounded solution. This contradiction completes the proof. □

Theorem 3.9. Let conditions (i)–(iv) and (F) hold, and $g(t) < t$ for $t \geq t_0$. Assume that there exist nondecreasing functions $\xi(t), \eta(t)$ and $\sigma(t) \in C([t_0, \infty), \mathbb{R})$ such that $g(t) < \xi(t) < \eta(t) < \sigma(t)$ for $t \geq t_0$. If $f(t)$ is a first order delay differential equation
\[ z'(t) + q(t) f \left( \int \xi(t) a_1^{1/a_1}(u) du \right) f \left( \int \eta(t) a_2^{1/a_2}(u) du \right)^{\frac{1}{2\eta_1}} \]
\[ \times f \left( \int \sigma(t) a_3^{1/a_3}(u) du \right)^{\frac{1}{2\sigma_1}} \]
\[ = 0 \]

is oscillatory, then equation (E-1) has no solution of type B₀.

**Proof.** Let \( x(t) \) be an eventually positive solution of equation (E-1) of type B₀. As in the proof of Theorem 3.8, we obtain the inequality (3.27) for \( t \geq t₁ \). Next, using that \( L₂x(t) \) is a positive function and that \( L₃x(t) \) is an increasing function, we find

\[-L₂x[σ(t)] ≤ L₂x[σ(t)] - L₂x[η(t)] = \int_{η(t)}^{σ(t)} a_3^{1/a_3}(u) L₃^{1/a_3} x[σ(u)] du \]
\[ ≤ L₃^{1/a_3} x[σ(t)] \int_{η(t)}^{σ(t)} a_3^{1/a_3}(u) du \quad \text{for } t ≥ t₂ ≥ t₁, \]

or

\[ L₂x[η(t)] ≥ \left( \int_{η(t)}^{σ(t)} a_3^{1/a_3}(u) du \right) \left( -L₃^{1/a_3} x[σ(t)] \right) \quad \text{for } t ≥ t₂ ≥ t₁. \]

From (3.30) and (3.27) for \( t ≥ t₂ \), we have

\[ x[g(t)] ≥ \left( \int \xi(t) a_1^{1/a_1}(u) du \right) \left( \int \eta(t) a_2^{1/a_2}(u) du \right)^{1/a_1} \left( \int \sigma(t) a_3^{1/a_3}(u) du \right)^{\frac{1}{2\sigma_1}} \left( -L₃^{1/a_3} x[σ(t)] \right) \]

From equation (E-1), by applying (F) on (3.31) and letting \( z(t) = -L₃x(t) > 0 \) for \( t ≥ t₂ \), we come to the following differential inequality

\[-z'(t) ≥ q(t) f \left( \int \xi(t) a_1^{1/a_1}(u) du \right) f \left( \int \eta(t) a_2^{1/a_2}(u) du \right)^{\frac{1}{2\eta_1}} \]
\[ \times f \left( \int \sigma(t) a_3^{1/a_3}(u) du \right)^{\frac{1}{2\sigma_1}} \left( z^{\frac{1}{2}}[σ(t)] \right). \]

Accordingly, by Lemma 2.3, we arrive at the desired contradiction that Eq. (3.29) has a positive solution. \( \square \)

**Corollary 3.2.** Let conditions (i)–(iv) and (F) hold, \( g(t) < t \) for \( t ≥ t₀ \), and assume that there exist nondecreasing functions \( ξ(t) \), \( η(t) \) and \( σ(t) \) in \( C([t₀, ∞), \mathbb{R}) \) such that \( g(t) < ξ(t) < η(t) < σ(t) < t \) for \( t ≥ t₀ \). Equation (E-1) has no solution of type B₀ if one of the following conditions holds:

\[ f \left( \frac{u^{\frac{1}{a'}}}{u} \right) ≥ 1 \quad \text{for } u ≠ 0, \]

either

\[ \limsup \int_{σ(t)}^{t} Q₀(s) ds > 1, \quad \text{or} \quad \liminf \int_{σ(t)}^{t} Q₀(s) ds > \frac{1}{e}, \]

where

\[ Q₀(s) = q(s) f \left( \int \xi(s) a_1^{1/a_1}(u) du \right) f \left( \int \eta(s) a_2^{1/a_2}(u) du \right)^{\frac{1}{2\eta_1}} \left( \int \sigma(s) a_3^{1/a_3}(u) du \right)^{\frac{1}{2\sigma_1}}. \]
We may apply the known oscillation criteria for the first delay and advanced differential equations as well as let (3.28) and therefore, establish some sufficient conditions for the oscillation of all bounded and/or unbounded solutions of equation \((3.32)\) for \(u\)(1)

**Theorem 3.10.** Let conditions (i)–(iv) and (F) hold, \(g(t) < t\) for \(t \geq t_0\), and assume that there exist nondecreasing functions \(\xi(t), \eta(t) \in C([t_0, \infty), \mathbb{R})\) such that \(g(t) < \xi(t) < \eta(t) < t\) for \(t \geq t_0\). Equation \((E-1)\) has no solution of type \(B_0\) if one of the following conditions holds:

\[
\left(\frac{u}{f(u^{1/\alpha_3})}\right) \geq 1 \text{ for } u \neq 0, \text{ and } \limsup_{t \to \infty} \int_{\eta(t)}^{t} \overline{Q}_0(v) dv > 1, \text{ where }
\]

\[
\overline{Q}_0(v) = a_3^{1/\alpha_3}(v) \left[ \int_{v}^{t} q(s) f \left( \int_{g(s)}^{\xi(s)} a_1^{1/\alpha_1}(u) du \right) f \left( \int_{\xi(s)}^{\eta(s)} \left( \int_{g(s)}^{u} a_2^{1/\alpha_2}(u) du \right)^{1/\alpha_1} \right) ds \right]^{1/\alpha_3}.
\]

\[
\frac{u}{f(u^{1/\alpha_3})} \to 0 \text{ as } u \to \infty, \text{ and } \limsup_{t \to \infty} \int_{\eta(t)}^{t} \overline{Q}_0(v) dv > 0.
\]

**Proof.** Let \(\chi(t)\) be an eventually positive solution of equation \((E-1)\) of type \(B_0\). As in the proof of Theorem 3.8, we obtain (3.28) for \(t \geq t_1\). Also, we find \(y(t) > 0\) and integrating this inequality from \(v\) to \(t \geq v \geq t_1\), using that \(y'(t) = L_2^\alpha \chi(t) < 0\) for \(t \geq t_1\), we get

\[
\frac{1}{a_3(v)}(-y'(v))^{\alpha_3} \geq \int_{v}^{t} q(s) f \left( \int_{g(s)}^{\xi(s)} a_1^{1/\alpha_1}(u) du \right) f \left( \int_{\xi(s)}^{\eta(s)} \left( \int_{g(s)}^{u} a_2^{1/\alpha_2}(u) du \right)^{1/\alpha_1} \right) f \left( \frac{1}{y^{1/\alpha_2} \eta(s)} \right) ds,
\]

which implies

\[
-y'(v) \geq a_3^{1/\alpha_3}(v) \left[ \int_{v}^{t} q(s) f \left( \int_{g(s)}^{\xi(s)} a_1^{1/\alpha_1}(u) du \right) f \left( \int_{\xi(s)}^{\eta(s)} \left( \int_{g(s)}^{u} a_2^{1/\alpha_2}(u) du \right)^{1/\alpha_1} \right) f \left( \frac{1}{y^{1/\alpha_2} \eta(s)} \right) ds \right]^{1/\alpha_3}.
\]

Combining (3.32) with the relation

\[
y(\tau) - y(t) = -\int_{t}^{\tau} y'(v) dv \quad \text{for } t \geq \tau \geq t_1
\]

where we put \(\tau = \eta(t)\), we have

\[
\frac{y[\eta(t)]}{f^{1/\alpha_3} \left( \frac{1}{y^{1/\alpha_2} \eta(t)} \right)} \geq \int_{\eta(t)}^{t} \overline{Q}_0(v) dv.
\]

Taking \(\limsup\) of both sides of the above inequality as \(t \to \infty\) and applying the hypotheses in (D_1) or (D_2) we arrive at the desired contradiction. This completes the proof. \(\square\)

**Remarks 3.1.** (1) We may note that the hypotheses which imply the nonexistence of solutions of equation \((E-1)\) of type \(B_0\) are equivalent to that which imply the oscillatory behavior of all bounded solutions of equation \((1, 1; -1)\), while the conditions which are concerned with the nonexistence of solutions of equation \((E-1)\) of type \(B_2\) or \(B_4\) are equivalent to that which deal with oscillatory behavior of unbounded solutions of equation \((E-1)\).

(2) We may apply the known oscillation criteria for the first delay and advanced differential equations as well as for the second order half-linear delay and advanced differential equations to such equations which appear in the hypotheses of the results established in Section 3 and therefore, establish some sufficient conditions for the oscillation of all bounded and/or unbounded solutions of equation \((E-1)\). The details are left to the reader.

(3) We note that oscillation criteria for equation \((E-1)\) can be established provided that the function \(g(t)\) is of mixed type argument. The formulation of such results is left to the reader.
3.4. Applications and examples

We shall apply the obtained results of this section to investigate the oscillatory behavior of the mixed type equations of the form

\[ L_4 x(t) = q_1(t) f_1(x[g_1(t)]) + q_2(t) f_2(x[g_2(t)]), \] (Ω)

where the operator \( L_4 \) is as in Eq. \((E_6)\) with \( a_i(t), (i = 1, 2, 3) \) are as in (i) satisfying (1.1), and \( a_i, (i = 1, 2, 3) \) are as in (iv),

(ii) \( q_i(t) \in C([t_0, \infty), \mathbb{R}^+), g_i(t) \in C([t_0, \infty), \mathbb{R}), g_i'(t) \geq 0 \) for \( t \to \infty \) and \( \lim_{t \to \infty} g_i(t) = \infty, i = 1, 2. \)

(iii) \( f_i \in C(\mathbb{R}, \mathbb{R}), x f_i(x) > 0 \) and \( f_i'(x) \geq 0 \) for \( x \neq 0, i = 1, 2. \)

We also assume that functions \( f_i, i = 1, 2 \) satisfy the following condition

\[ -f_i(-xy) \geq f_i(xy) \geq f_i(x)f_i(y), xy > 0 \quad \text{and} \quad i = 1, 2. \] (\( \hat{F} \))

Now, we state some results which insure the oscillation of Eq. \((Ω)\).

**Theorem 3.11.** Let conditions (i), (ii), (iii), (iv) and \( (\hat{F}) \) hold, \( g_1(t) \leq t \) and \( g_2(t) \geq t \) for \( t \geq t_0, \) and assume that there exist nondecreasing functions \( \xi_1(t), \eta_1(t), \) and \( \sigma_1(t) \in C([t_0, \infty), \mathbb{R}), i = 1, 2 \) such that

\[ g_1(t) < \xi_1(t) < \eta_1(t) < \sigma_1(t) \leq t \quad \text{for} \quad t \geq t_0 \] (3.33)

and

\[ g_2(t) > \xi_2(t) > \eta_2(t) > \sigma_2(t) \geq t \quad \text{for} \quad t \geq t_0. \] (3.34)

If the first order delay differential equations

\[ z'(t) + q_1(t) f_1 \left( \int_{t_0}^{t} a_1(s) \int_{t_0}^{s} a_2^{1/\alpha_2}(u) du \right)^{1/\alpha_1} ds \]

\[ \times f_1 \left( \int_{g_1(t)}^{\xi_1(t)} a_3^{1/\alpha_3}(u) du \right)^{1/\alpha_2} f_1 \left( z^{1/\alpha_1} [\xi_1(t)] \right) = 0 \] (3.35)

and

\[ y'(t) + q_1(t) f_1 \left( \int_{t_0}^{t} a_1^{1/\alpha_1}(u) du \right) f_1 \left( \int_{\xi_1(t)}^{\eta_1(t)} a_2^{1/\alpha_2}(u) du \right)^{1/\alpha_1} \]

\[ \times f_1 \left( \int_{g_1(t)}^{\sigma_1(t)} a_3^{1/\alpha_3}(u) du \right)^{1/\alpha_2} f_1 \left( y^{1/\alpha_1} [\sigma_1(t)] \right) = 0 \] (3.36)

and the first order advanced differential equation

\[ x'(t) - q_2(t) f_2 \left( \int_{g_2(t)}^{\xi_2(t)} a_1^{1/\alpha_1}(u) du \right) f_2 \left( \int_{\eta_2(t)}^{\xi_2(t)} a_2^{1/\alpha_2}(u) du \right)^{1/\alpha_1} \]

\[ \times f_2 \left( \int_{g_2(t)}^{\sigma_2(t)} a_3^{1/\alpha_3}(u) du \right)^{1/\alpha_2} f_2 \left( x^{1/\alpha_2} [\sigma_2(t)] \right) = 0 \] (3.37)

are oscillatory, then equation \((Ω)\) is oscillatory.

**Theorem 3.12.** Let conditions (i), (ii), (iii), (iv) and \( (\hat{F}) \) hold, \( g_1(t) \leq t \) and \( g_2(t) \geq t \) for \( t \geq t_0, \) and assume that there exist nondecreasing functions \( \xi_1(t) \) and \( \eta_1(t) \) \( \in C([t_0, \infty), \mathbb{R}), i = 1, 2 \) such that \( g_1(t) < \xi_1(t) < \eta_1(t) \leq t \)
and \( g_2(t) > \xi_2(t) > \eta_2(t) \geq t \) for \( t \geq t_0 \). If all bounded solutions of the second order delay half-linear differential equations
\[
\frac{d}{dt} \left( \frac{1}{a_3(t)} \left( \frac{d}{dt} z(t) \right)^{\alpha_3} \right) - q_1(t) f_1 \left( \int_{t_0}^{g_1(t)} \left( a_1(s) \int_{s}^{\xi_1(t)} a_2^{1/\alpha_2}(u) du \right)^{1/\alpha_1} ds \right) f_1 \left( z^{-\alpha_2 \alpha_3} [g_1(t)] \right) = 0 \quad (3.38)
\]
and
\[
\frac{d}{dt} \left( \frac{1}{a_3(t)} \left( \frac{d}{dt} y(t) \right)^{\alpha_3} \right) - q_1(t) f_1 \left( \int_{g_1(t)}^{\xi_1(t)} a_1^{1/\alpha_1}(u) du \right)
\times f_1 \left( \left( \int_{\xi_1(t)}^{\eta_1(t)} a_2^{1/\alpha_2}(u) du \right)^{1/\alpha_1} \right) f_1 \left( y^{-\alpha_2 \alpha_3} [\eta_1(t)] \right) = 0 \quad (3.39)
\]
and all unbounded solutions of the advanced half-linear differential equation
\[
\frac{d}{dt} \left( \frac{1}{a_3(t)} \left( \frac{d}{dt} x(t) \right)^{\alpha_3} \right) - q_2(t) f_2 \left( \int_{t_0}^{g_2(t)} a_1^{1/\alpha_1}(u) du \right)
\times f_2 \left( \left( \int_{\xi_2(t)}^{\eta_2(t)} a_2^{1/\alpha_2}(u) du \right)^{1/\alpha_1} \right) f_2 \left( x^{-\alpha_2 \alpha_3} [\eta_2(t)] \right) = 0 \quad (3.40)
\]
are oscillatory, then Eq. (\( \Omega \)) is oscillatory.

**Remark 3.2.** We may reduce the hypotheses of Theorem 3.11 by replacing Eqs. (3.35) and (3.36) by
\[
y'(t) + q_1(t) Q_1(t) f_1 \left( y^{1/\alpha} [\sigma_1(t)] \right) = 0,
\]
where
\[
Q_1(t) = \min \left\{ f_1 \left( \int_{t_0}^{g_1(t)} a_1(s) \int_{s}^{\xi_1(t)} a_2^{1/\alpha_2}(u) du \right)^{1/\alpha_1} ds \right\},
\]
\[
f_1 \left( \int_{g_1(t)}^{\xi_1(t)} a_1^{1/\alpha_1}(u) du \right) f_1 \left( \left( \int_{\xi_1(t)}^{\eta_1(t)} a_2^{1/\alpha_2}(u) du \right)^{1/\alpha_1} \right) f_1 \left( \left( \int_{\eta_1(t)}^{g_1(t)} a_3^{1/\alpha_3}(u) du \right)^{1/\alpha_2 \alpha_3} \right)
\]
Also, we can reduce the conditions of Theorem 3.12 by replacing Eqs. (3.38) and (3.39) by
\[
\frac{d}{dt} \left( \frac{1}{a_3(t)} \left( \frac{d}{dt} z(t) \right)^{\alpha_3} \right) - q_1(t) Q_2(t) f_1 \left( z^{-\alpha_2 \alpha_3} [\eta_1(t)] \right) = 0,
\]
where
\[
Q_2(t) = \min \left\{ f_1 \left( \int_{t_0}^{g_1(t)} a_1(s) \int_{s}^{\xi_1(t)} a_2^{1/\alpha_2}(u) du \right)^{1/\alpha_1} ds \right\},
\]
\[
f_1 \left( \int_{g_1(t)}^{\xi_1(t)} a_1^{1/\alpha_1}(u) du \right) f_1 \left( \left( \int_{\xi_1(t)}^{\eta_1(t)} a_2^{1/\alpha_2}(u) du \right)^{1/\alpha_1} \right)
\]
As an illustrative example, we consider a special case of Eq. (\( \Omega \)), namely, equation
\[
\frac{d}{dt} \left( \frac{d}{dt} \left( \frac{d}{dt} x \right)^{\alpha_1} \right)^{\alpha_2} \left( \frac{d}{dt} x \right)^{\alpha_3} = q_1(t)x^\beta [g_1(t)] + q_2(t)x^\gamma [g_2(t)], \quad (\Omega_1)
\]
where \( \beta \) and \( \gamma \) are ratios of positive odd integers.
From Theorems 3.11 and 3.12, one can easily see that Eq. \((\Omega_1)\) is oscillatory if either one of the following conditions holds:

\((G_1)\) Conditions (3.33) and (3.34) hold, and the first order delay differential equation

\[ y'(t) + q_1(t) \hat{Q}_1(t)y^\beta [\sigma_1(t)] = 0, \]

where

\[
\hat{Q}_1(t) = \min \left\{ \left[ \left( \frac{\alpha_1}{1 + \alpha_1} \right) g_1(t) \right]^\beta \frac{1+\alpha_1}{\alpha_1} \right\}
\]

\[
(\xi_1(t) - g_1(t))^\beta (\eta_1(t) - \xi_1(t))^{\beta/\alpha_1} (\sigma_1(t) - \eta_1(t))^{\beta/\alpha_2}
\]

and the advanced first order differential equation

\[ x'(t) - q_2(t) Q^\star(t)x^\gamma [\sigma_2(t)] = 0, \]

where

\[ Q^\star(t) = (g_2(t) - \xi_2(t))^\gamma (\xi_2(t) - \eta_2(t))^\gamma (\eta_2(t) - \sigma_2(t))^\gamma \]

are oscillatory.

\((G_2)\) Conditions (3.33) and (3.34) hold with \(\eta_i(t) = \sigma_i(t), i = 1, 2, t \geq t_0\) and all bounded solutions of the second order delay half-linear differential equation

\[
\frac{d}{dt} \left( \frac{d}{dt} y(t) \right)^{\alpha_3} - q_1(t) \hat{Q}_2(t)y^{\beta} [g_1(t)] = 0,
\]

where

\[
\hat{Q}_2(t) = \min \left\{ \left[ \left( \frac{\alpha_1}{1 + \alpha_1} \right) g_1(t) \right]^\beta \frac{1+\alpha_1}{\alpha_1} \right\}
\]

and all unbounded solutions of the second order advanced half-linear differential equation

\[
\frac{d}{dt} \left( \frac{d}{dt} x(t) \right)^{\alpha_3} - q_2(t) Q^{**}(t)x^{\gamma} [\eta_2(t)] = 0,
\]

where

\[ Q^{**}(t) = (g_2(t) - \xi_2(t))^\gamma (\xi_2(t) - \eta_2(t))^\gamma \]

are oscillatory.

As an example, if \(g_1(t) = t - 4\sqrt{t}, t \geq 16\), we may let \(\xi_1(t) = t - 3\sqrt{t}, \eta_1(t) = t - 2\sqrt{t}\) and \(\sigma_1(t) = t - \sqrt{t}\). Also, if \(g_2(t) = t + 4\sqrt{t}, t \geq 0\), we may take \(\xi_2(t) = t + 3\sqrt{t}, \eta_2(t) = t + 2\sqrt{t}\) and \(\sigma_2(t) = t + \sqrt{t}\).

Next, we shall apply the obtained results to Eq. \((\Delta_3)\), when \(p(t) \in C([t_0, \infty), \mathbb{R}), p(t) \leq 0\) for \(t \geq t_0\) and \(\tau(t) \in C([t_0, \infty), \mathbb{R})\) satisfies condition \((T_1)\). Here, we let \(p_1(t) = -p(t) \geq 0\) for \(t \geq t_0\), and set

\[
y(t) = x(t) - p_1(t)x[\tau(t)]. \tag{3.41}
\]

Now, if \(y(t) > 0\) eventually, then \(x(t) \geq y(t)\) eventually and from Eq. \((\Delta_3)\), we have

\[
L_4 y(t) + q(t) f(y[g(t)]) \leq 0 \quad \text{eventually.} \tag{3.42}
\]

Next, if \(y(t) < 0\) eventually, we let \(z(t) = -y(t) > 0\) eventually and then

\[
x[\tau(t)] = \frac{1}{p_1(t)}[z(t) + x(t)] \geq \frac{z(t)}{p_1(t)}, \quad \text{or} \quad x(t) \geq \frac{z[\tau^{-1}(t)]}{p_1[\tau^{-1}(t)]} \quad \text{eventually.} \tag{3.43}
\]
It follows from Eq. (Δ₃) and the condition (F) that
\[
L₄z(t) - q(t) f \left( \frac{1}{p₁(t)[g(t)]} \right) f \left( z[t⁻¹ \circ g(t)] \right) \geq 0 \quad \text{eventually.} \tag{3.44}
\]

By applying the results of Section 2 to the inequality (3.42) and the results of Section 3 to the inequality (3.44) with \( q(t) \) and \( g(t) \) replaced by \( q(t) f(1/p₁(t)[τ⁻¹(t)]) \) and \( τ⁻¹ \circ g(t) \) respectively, we may obtain appropriate oscillation criteria for the considered Eq. (Δ₃).

Similarly, we can apply our results to the neutral differential equation
\[
L₄(x(t) - p₁(t)x[τ(t)]) = q(t) f(x[g(t)]), \tag{Δ₄}
\]
where \( p₁(t) \in C([t₀, ∞), [0, ∞)), p₁(t) \neq 0 \). Here, we set \( y(t) \) as in (3.41) and when \( y(t) > 0 \) eventually, we see that \( x(t) \geq y(t) \) eventually, and hence
\[
L₄y(t) \geq q(t)f(y[g(t)]) \quad \text{eventually.} \tag{3.45}
\]

Also, if \( y(t) < 0 \) eventually, we let \( z(t) = -y(t) > 0 \) eventually and proceed as above to obtain (3.43), and hence we find
\[
L₄z(t) + q(t) f \left( \frac{1}{p₁(t)[g(t)]} \right) f \left( z[t⁻¹ \circ g(t)] \right) \leq 0 \quad \text{eventually.} \tag{3.46}
\]

As above, we can apply the results of Section 3 to the inequality (3.45) and the results of Section 2 to the inequality (3.46) with \( q(t) \) and \( g(t) \) replaced by \( q(t) f(1/p₁(t)[τ⁻¹(t)]) \) and \( τ⁻¹ \circ g(t) \) respectively, so that we obtain appropriate oscillation criteria for Eq. (Δ₄).

Finally, we will show that the results of Section 3 can be applied to investigate the oscillatory behavior of bounded and unbounded solutions of the neutral Eq.
\[
L₄(x(t) + p(t)x[τ(t)]) - q(t)f(x[g(t)]) = 0, \tag{Δ₅}
\]
where \( p(t) \in C([t₀, ∞), [0, ∞)) \). Here, we set \( y(t) \) as in (2.34), so that equation (Δ₅) becomes
\[
L₄y(t) = q(t)f(x[g(t)]). \tag{3.47}
\]

We will assume that one of the following conditions holds:

(P₁) \( 0 \leq p(t) \leq 1, p(t) \neq 0, p(t) \neq 1, τ(t) < t \) for \( t \geq t₀ \);
(P₂) \( 0 \leq p(t) \leq 1, p(t) \neq 0, p(t) \neq 1, τ(t) > t \) for \( t \geq t₀ \);
(P₃) \( p(t) \geq 1, p(t) \neq 1, τ(t) > t \) for \( t \geq t₀ \);
(P₄) \( p(t) \geq 1, p(t) \neq 1, τ(t) < t \) for \( t \geq t₀ \).

Notice, that if \( x(t) > 0 \) eventually, then \( y(t) > 0 \) eventually. So, we can consider the two cases:

(I) \( Lᵢy(t) > 0 \) eventually, \( i = 1, 2 \) and either (P₁) or (P₃) holds;
(II) \( y'(t) < 0 \) eventually and either (P₂) or (P₄) holds.

Case (I) Assume that \( Lᵢy(t) > 0 \) eventually, \( i = 0, 1, 2 \) and either (P₁) or (P₃) holds. We proceed as in Section 2 and obtain (2.36) or (2.38). Using either of these inequalities in Eq. (3.47) we obtain the inequalities
\[
L₄y(t) \geq q(t)f(1 - p[τ(g(t)]) f(y[g(t)]) \quad \text{eventually} \tag{3.48}
\]
\[
L₄y(t) \geq q(t)f(P[g(t)]) f(y[τ⁻¹ \circ g(t)]) \quad \text{eventually.} \tag{3.49}
\]

By applying some of the results of this section to (3.48) and (3.49) we obtain oscillation results for the unbounded solutions of Eq. (Δ₅).

Case (II) Assume that \( y'(t) < 0 \) eventually and either (P₂) or (P₄) holds. In this case we proceed as above and again obtain the inequalities (3.48) and (3.49). Here we apply some of our results of this section to investigate the oscillatory behavior of all bounded solutions of Eq. (Δ₅).
Remark 3.3. It would be of interest to investigate the oscillatory behavior of Eq. (Δ5) when the arguments τ(t) and g(t) are of mixed type as well as to obtain similar results as those obtained here without the restriction (1.1).

Also, it would be useful to extend our results to more general equations of the form
\[
\frac{d}{dr} \left( \frac{1}{a_{n-1}(t)} \frac{d}{dt} \cdots \frac{d}{dt} \left( \frac{1}{a_2(t)} \frac{d}{dt} \frac{d}{dt} \frac{d}{dt} \cdots \left( \frac{d}{dt} \frac{d}{dt} \frac{d}{dt} x \right)^{\alpha_1} \cdots \right)^{\alpha_{n-2}} \right)^{\alpha_{n-1}} + \delta q(t) f(x[g(t)]) = 0,
\]
where \(\delta = \pm 1\), \(n \geq 1\), and all \(a_i(t), \alpha_i, (i - 1, 2, \ldots, n - 1)\), \(g(t), q(t)\) and \(f(x)\) satisfy conditions (i)–(iv).

It is also of interest to study the oscillatory behavior of \((E_\delta)\) without the restriction \(f'(x) \geq 0\) for \(x \neq 0\) (\(f\) need not be monotonic), in particular see [1,3].

References