Fuzzy generalized closed sets via fuzzy grills

Sumita Das\textsuperscript{a}, M. N. Mukherjee\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Sammilani Mahavidyalaya, E. M. Bypass, Kolkata 700 075, India
\textsuperscript{b}Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kolkata-700019, India

Abstract. In this paper we introduce a kind of generalized fuzzy closed sets in terms of fuzzy grills, termed $G_{gf}$-closed sets, and discuss their basic properties. We also define $G_{gf}$-continuity and characterize it by $G_{gf}$-closed sets. We further define $G_{gf}$-regularity and $G_{gf}$-normality and study their behaviours by means of $G_{gf}$-closed sets.

1. Introduction and preliminaries

It is by now well known that the concept of fuzzy sets, as invented by Zadeh [19], has enormous applications to various fields of present-day science and technology. Now-a-days much research is being conducted on the generalizations of the notion of fuzzy sets. The idea of fuzzy topology was given by Chang [5] and some other mathematicians [15, 18]. After the initiation of grill in general topology by Choquet [7], Azad [2] introduced the concept of fuzzy grill to study fuzzy proximities. Since then a good many research papers have appeared which have encompassed different areas of fuzzy topology by use of fuzzy grill.

The idea of generalized fuzzy closed sets and generalization of fuzzy continuous functions for fuzzy topological spaces and their applications were first given by Balasubramanian and Sundaram [3]. Since then many modifications have been introduced and investigated by others.

In the present paper, we also introduce the concept of a kind of generalized fuzzy closed sets, termed $G_{gf}$-closed sets, in a fuzzy topological space via fuzzy grill, and study it to obtain some information and properties of such sets. It is seen that the class of such sets contains the class of gf-closed sets introduced in [3]. In Section 3, we consider a generalization of fuzzy continuous functions and that too in terms of fuzzy grill and its associated fuzzy topology. In the last section, we take up certain applications of $G_{gf}$-closed sets and the aforesaid ideas. To that end we consider some modified versions of fuzzy regular and fuzzy normal spaces, effected in terms of grills, and try for some preservation theorems.

Throughout this paper, by an fts $X$, we mean a fuzzy topological space $(X, \tau)$, as initiated by Chang [5]. For two fuzzy sets [19] $A, B$ in $X$, i.e., $A, B \in \mathcal{I}^X$ ($I = [0, 1]$), we write $A \leq B$ if $A(x) \leq B(x)$ for each $x \in X$.

2010 Mathematics Subject Classification. Primary 54A40; Secondary 54D10, 54D15

Keywords. Fuzzy grill, $G_{gf}$-closed set, $G_{gf}$-continuity, $G_{gf}$-regularity, $G_{gf}$-normality.

Received: 18 February 2011; Revised: 13 December 2011; Accepted: 13 December 2011

Communicated by Ljubiša D.R. Kočinac

The first author is grateful to the University Grants Commission, India for financial support under the Faculty Development Programme.

Email addresses: das.sumita752@gmail.com (Sumita Das), mukherjeemn@yahoo.co.in (M. N. Mukherjee)
A fuzzy singleton or a fuzzy point [15] with support x and value \( a (0 < \alpha \leq 1) \), is denoted by \( x_\alpha \). A fuzzy point \( x_\alpha \) is said to be \( q \)-coincident with a fuzzy set \( A \) in \( X \), denoted by \( x_\alpha q A \), if there exists \( x \in X \) such that \( A(x) + B(x) > 1 \), whereas the notation \( A q B \) means that \( A \) is \( q \)-coincident[15] with \( B \). i.e., \( A q B \) implies \( A(x)+B(x) > 1 \) for some \( x \in X \). The negations of these statements are denoted by \( A \n q B \) and \( A \overline{\n q} B \) respectively.

The fuzzy sets in \( X \) taking on respectively the constant values 0 and 1 are denoted by \( 0_X \) and \( 1_X \) respectively.

A fuzzy set \( A \) is non-null if \( A \neq 0_X \). For \( A, B \in I^X \), \( A \) is called a \( q \)-nb of \( B \) [15] if for some fuzzy open set \( U \) in \( X \), \( B q U \leq A \); if, in addition, \( A \) itself is fuzzy open then it is called an open-\( q \)-nb of \( B \). The collection of all open \( q \)-nbs of any fuzzy point \( x_\alpha \) is denoted by \( Q(x_\alpha) \). For a fuzzy set \( A \) in an fts \( X \), the fuzzy complement, fuzzy interior and fuzzy closure of \( A \) in \( X \) are written as \((1-A)\) [or sometimes as \((1_X-A)\)], \( \text{int} \, A \) and \( \text{cl} \, A \), respectively.

A non-void collection \( G \) of fuzzy sets in an fts \((X, \tau)\) is called a fuzzy grill [2] on \( X \) if (i) \( 0_X \notin G \) (ii) \( A \in G \), \( B \in I^X \) and \( A \subseteq B \Rightarrow B \in G \) and (iii) \( A, B \in I^X \) and \( A \cup B \in G \Rightarrow A \in G \) or \( B \in G \). Let \((X, \tau)\) be an fts and \( G \) be a fuzzy grill on \( X \). An operator \( \phi_G : I^X \rightarrow I^X \), associated with the fuzzy grill \( G \) and the fuzzy topology \( \tau \), is defined [14] as follows:

For any \( A \in I^X \), \( \phi_G(A) \), simply denoted by \( \phi(A) \), is the union of all fuzzy points \( x_\alpha \) of \( X \) such that if \( U \in Q(x_\alpha) \), then \( A \cup U \in G \), where \( A \cup B \) is the Lukasiewicz conjunction [12] on the powerset \( I^X \), defined by \( A \cup B = \max(0, A + B - 1_X) \), for \( A, B \in I^X \), \( A \n B \) is any fuzzy grill on \( X \), \( A \subseteq B \) and \( A \in G \). Let \((X, \tau)\) be an fts and \( \phi \) be a fuzzy grill on \( X \). An operator \( \phi_G : I^X \rightarrow I^X \), associated with the fuzzy grill \( G \) and the fuzzy topology \( \tau \), is defined [14] as follows:

We need as follows some results about the operator \( \phi \), as pre-requisites.

**Definition 1.2.** (14) A map \( \psi : I^X \rightarrow I^X \) is defined by \( \psi(A) = A \cup \phi(A) \) for all fuzzy set \( A \) in \( X \).

Then we have

**Theorem 1.3.** (14) In an fts \((X, \tau)\), corresponding to a fuzzy grill \( G \), there exists a unique fuzzy topology \( \tau_G \) (say) on \( X \) given by \( \tau_G = \{ U \in I^X / \psi(1_X - U) = 1_X - U \} \), where for any \( A \in I^X \), \( \psi(A) = A \cup \phi(A) = \tau_G\)-cl(A).

**Theorem 1.4.** (14) In an fts \((X, \tau)\),

(i) if \( \mathbb{G}_1 \) and \( \mathbb{G}_2 \) be two fuzzy grills with \( \mathbb{G}_1 \subseteq \mathbb{G}_2 \), then \( \tau_{\mathbb{G}_1} \subseteq \tau_{\mathbb{G}_2} \).

(ii) if \( \mathbb{G} \) be a fuzzy grill and \( B \notin \mathbb{G} \), then \( B \) is closed in \((X, \tau_G)\).

(iii) for any fuzzy set \( A \) and any fuzzy grill \( \mathbb{G} \) on \( X \), \( \phi(A) \) is \( \tau_G\)-closed.

**Theorem 1.5.** (14) For a fuzzy topological space \((X, \tau)\) together with a fuzzy grill \( \mathbb{G} \), \( \tau \subseteq \tau_G \).

**Theorem 1.6.** (13) For any two fuzzy open sets \( A \) and \( B \) in an fts \( X \), \( A \overline{\cap} B \Rightarrow \text{cl} \, A \overline{\cap} B \) and \( A \overline{\cap} \text{cl} \, B \).

From now on by a fuzzy \( \mathbb{G} \)-space we mean an fts \((X, \tau)\) with a fuzzy grill \( \mathbb{G} \) and denote it by \((X, \tau, \mathbb{G})\) (or sometimes simply by \( X \)).

2. \( \mathbb{G}_{gf} \)-closed and \( \mathbb{G}_{gf} \)-open sets in fuzzy \( \mathbb{G} \)-spaces

The concepts of generalized closed and open sets in fuzzy setting were introduced in [3] in the following way:

**Definition 2.1.** ([3]) A fuzzy set \( A \) in an fts \((X, \tau)\) is called \emph{generalized fuzzy closed} (in short, \emph{gf}-closed) if \( \text{cl}(A) \leq U \) whenever \( A \leq U \) and \( U \) is fuzzy open. A fuzzy set \( A \) is called \emph{generalized fuzzy open} (in short, \emph{gf-open}) if its fuzzy complement \((1 - A)\) is \( \mathbb{G}_{gf}\)-closed.
We now initiate below the notion of a class of generalized closed fuzzy sets, larger than the class of $gf$-closed sets, in terms of the operator ‘$\phi$’.

**Definition 2.2.** A fuzzy set $A$ in a fuzzy $G$-space $(X, \tau, G)$ is called a generalized fuzzy closed set with respect to a fuzzy grill $G$ ($G_{gf}$-closed, for short) if $\phi(A) \leq U$ whenever $A \leq U$ and $U$ is fuzzy open with respect to $\tau$. $A$ is called generalized fuzzy open with respect to a fuzzy grill $G$ ($G_{gf}$-open, for short) if its complement $(1 - A)$ is $G_{gf}$-closed.

As every fuzzy closed (respectively fuzzy open) set is $gf$-closed (respectively $gf$-open) [3], we have the following:

**Observation 2.3.** For a fuzzy $G$-space $(X, \tau, G)$, we observe as follows:
(i) For any fuzzy set $A$ in $X$, since $\phi(\phi(A)) \leq \phi(A)$ [by Result 1.1(v)], $\phi(A)$ is $G_{gf}$-closed.
(ii) If $A \not\in G$ then $\phi(A) = 0_x$ [by Result 1.1(iii)]. Thus every non-member of $G$ is $G_{gf}$-closed in $(X, \tau, G)$.
(iii) By the definition of $\tau_G$ and Definition 2.2, we see that a fuzzy set is $G_{gf}$-closed if and only if $(A \leq U \in \tau \Rightarrow \tau_G-cl(A) = A \lor \phi(A) \leq U)$.
(iv) Every fuzzy closed (resp. fuzzy open) set as well as every $gf$-closed (resp. $gf$-open) set is $G_{gf}$-closed (resp. $G_{gf}$-open) [this is clear from Result 1.1(v)].

**Example 2.4.** Let $G = I^X \setminus \{0_x\}$. Then for any non-null fuzzy set $A$ in $X$, we have $A \leq \phi(A)$. Indeed, $x_1 \leq A$ and $U \in Q(x_1) \Rightarrow \lambda + U(x) > 1$ and $\lambda \leq A(x) \Rightarrow A(x) + U(x) - 1 > 0$, i.e., $A + U - 1 \not= 0_x \Rightarrow A + U - 1 \in G \Rightarrow x_1 \leq \phi(A)$. Then by Result 1.1(v) and Theorem 2.5, we have $\phi(A) = cl(A)$ and $\psi(A) = \tau_G-clA = A \lor \phi(A) = clA$, for any fuzzy set $A$ in $X$. So the class of $G_{gf}$-closed sets coincides with that of $gf$-closed sets.

But in general a $G_{gf}$-closed set may not be a $gf$-closed set which we show by the following example:

**Example 2.5.** Let $(X, \tau, G)$ be a fuzzy $G$-space where $X$ is any infinite set, $\tau = \{U \in I^X : 0 \leq U(x) < \frac{1}{2}, x \in X\}$ together with $1_x$ and $G = \{G \in I^X : 0 < G(x), x \in X\}$. Let $x_0$ be a chosen point of $X$ and $A$ be the fuzzy set in $X$ given by $A(x_0) = 0.2$ and $A(x) = 0$ for $x \in X \setminus \{x_0\}$. Then $A$ is not $gf$-closed, since for any $U(\not= 1_x) \in \tau$ with $A \leq U$, we see that $cl(A) \not\leq U$.

But $A$ is $G_{gf}$-closed. In fact, if $A \leq 1_x$, then $\phi(A) \leq 1_x$. So consider the case when $A \leq V(\not= 1_x) \in \tau$. Here for any fuzzy point $x_1$ in $X$ and for any $W \in Q(x_1) \leq \tau, A + W - 1_x \not\in G$. Thus $\phi(A) = 0_x \leq V$ so that $A$ is $G_{gf}$-closed.

**Theorem 2.6.** If $A_1$ and $A_2$ are two $G_{gf}$-closed sets in a fuzzy $G$-space $(X, \tau, G)$, then $A_1 \lor A_2$ is also a $G_{gf}$-closed set.

**Proof.** Let $A_1$ and $A_2$ be two $G_{gf}$-closed sets in a fuzzy $G$-space $(X, \tau, G)$. Suppose $A_1 \lor A_2 \leq U$ for some fuzzy open set $U$. Then $A_1 \leq U$ and $A_2 \leq U$. By hypothesis, $\phi(A_1) \leq U$ and $\phi(A_2) \leq U$ and hence $\phi(A_1 \lor A_2) = \phi(A_1) \lor \phi(A_2) \leq U$ by (iv) of Result 1.1.

But the intersection of two $G_{gf}$-closed sets may not be $G_{gf}$-closed as shown by the following example:

**Example 2.7.** Let $X = [a, b]$ and $\tau = \{0_x, 1_x, P\}$, where $P(a) = 0.6$ and $P(b) = 0.8$, be a fuzzy topology on $X$. Let $G$ be a fuzzy grill on $X$ consisting of those fuzzy sets $G$ on $X$, such that $0.1 \leq G(x) \leq 1$ for all $x \in X$. Let $A$ and $B$ be two fuzzy sets in $X$ defined as $A(a) = 0.9, A(b) = 0.4$ and $B(a) = 0.5, B(b) = 0.9$. Then $A$ and $B$ are both $G_{gf}$-closed sets [indeed, the only fuzzy open set containing $A$ or $B$ is $1_x$]. But $A \land B$ is not a $G_{gf}$-closed set, Here $A \land B \leq P$, but $\phi(A \land B) \not\leq P$. Indeed, $a_{0.7} \not\leq P$ but for any $U \in Q(a_{0.7}), (A \land B) + U - 1 \in G$, so that $a_{0.7} \leq \phi(A \land B)$.

We next investigate as to under which condition a $G_{gf}$-closed set is $gf$-closed. For this we need the following:

**Definition 2.8.** A fuzzy set $A$ in a fuzzy $G$-space $(X, \tau, G)$ is said to be $\tau_G$-dense-in-itself if $A \leq \phi(A)$. 

Theorem 2.9. If $A$ is $\tau_G$-dense-in-itself and a $G_{sf}$-closed set in a fuzzy $G$-space $(X, \tau, G)$, then $A$ is $gf$-closed.

Proof. Let $A$ be $\tau_G$-dense-in-itself and $G_{sf}$-closed in $(X, \tau, G)$, and $U$ be any fuzzy open set in $X$ such that $A \subseteq U$. Then $\tau_G$-cl$(A) = A \cup \phi(A) \subseteq U$ as $A \subseteq \phi(A)$ and $A$ is $G_{sf}$-closed. Since $A$ is $\tau_G$-dense-in-itself, then by (v) of Result 1.1, $cl(1, A) = cl\phi(A) = \phi(A) \subseteq U$ and hence $A$ is $gf$-closed.

Theorem 2.10. For any $G_{sf}$-closed set $A$ in a fuzzy $G$-space $(X, \tau, G)$,
(i) if $A \subseteq B \subseteq \phi(A)$, where $B \in \mathcal{P}^X$, then $B$ is also $G_{sf}$-closed.
(ii) if $A \subseteq B \subseteq \tau_G$-cl$(A)$, then $B$ is also $G_{sf}$-closed.
(iii) $\tau_G$-closure of every $G_{sf}$-closed set is $G_{sf}$-closed

Proof. (i) Suppose $B \subseteq U$ for some fuzzy open set $U$. Then $A \subseteq U$ and since $A$ is $G_{sf}$-closed, $\phi(A) \subseteq U$. Now $B \subseteq \phi(A) \Rightarrow \phi(B) \subseteq \phi(\phi(A)) \subseteq \phi(A) \subseteq U$ [by Result 1.1(i) and (v)] $\Rightarrow B$ is also $G_{sf}$-closed.

(ii) Let $B \subseteq V$ for some $V \in \tau$. Then $A \subseteq V$ and hence $\phi(A) \subseteq V$. Since $B \subseteq \tau_G$-cl$(A) = A \cup \phi(A)$, we have $\phi(B) \subseteq \phi(A \lor \phi(A)) = \phi(A) \lor \phi(\phi(A)) \subseteq \phi(A) \subseteq V \Rightarrow B$ is $G_{sf}$-closed.

(iii) For any $G_{sf}$-closed set $A$, $\tau_G$-cl$(A) \subseteq \tau \Rightarrow \phi(A) \subseteq U \Rightarrow \phi(\tau_G$-cl$(A)) = \phi(A \lor \phi(A)) = \phi(A) \lor \phi(\phi(A)) \subseteq \phi(A) \subseteq U$, which implies that $\tau_G$-cl$(A)$ is $G_{sf}$-closed.

Corollary 2.11. Let $A$ and $B$ be two fuzzy sets in a fuzzy $G$-space $(X, \tau, G)$ such that $A \subseteq B \subseteq \phi(A)$. If $A$ is $G_{sf}$-closed then $A$ and $B$ are $gf$-closed.

Proof. Since $A \subseteq B \subseteq \phi(A)$ and $A$ is $G_{sf}$-closed, then by Theorem 2.10, $B$ is $G_{sf}$-closed. Also by Result 1.1 and $A \subseteq B \subseteq \phi(A)$, we get $\phi(A) \subseteq \phi(B) \subseteq \phi(\phi(A)) \subseteq \phi(A)$ i.e., $\phi(A) = \phi(B)$. Then $A$ and $B$ are $\tau_G$-dense-in-itself and by Theorem 2.9, $A$ and $B$ are $gf$-closed.

Theorem 2.12. Let $(X, \tau, G)$ be a fuzzy $G$-space. Then the following are equivalent:
(a) Every fuzzy set in $X$ is $G_{sf}$-closed.
(b) Every fuzzy open set is $\tau_G$-closed.

Proof. (a) $\Rightarrow$ (b): Suppose every fuzzy set in $(X, \tau, G)$ is $G_{sf}$-closed. Let $U$ be any fuzzy open set in $X$. Then by hypothesis, $U$ is $G_{sf}$-closed and $\tau \subseteq U \Rightarrow \phi(U) \subseteq \tau$ and hence $U$ is $\tau_G$-closed.

(b) $\Rightarrow$ (a): Suppose every fuzzy open set is $\tau_G$-closed. Let $A$ be any fuzzy set in $X$ such that $A \subseteq U$, where $U$ is fuzzy open. Then $\phi(A) \subseteq \phi(U) \subseteq \tau$ (as $U$ is $\tau_G$-closed). So $A$ is $G_{sf}$-closed.

Theorem 2.13. Let $(X, \tau, G)$ be a fuzzy $G$-space, where $G$ is so taken that $\tau \cap G = \{1_X\}$. Then each $A(\neq 1_X) \in \tau$ is a $G_{sf}$-closed set.

Proof. In fact, $A(\neq 1_X)$ is fuzzy open $\Rightarrow A \notin G \Rightarrow A$ is $\tau_G$-closed[by Theorem 1.4(ii) $\Rightarrow \phi(A) \subseteq A$. Then for any fuzzy open set $U$ with $A \subseteq U$ we have $\phi(A) \subseteq U$. Hence $A$ is $G_{sf}$-closed.

We shall now prove some properties of $G_{sf}$-open sets in a fuzzy $G$-space $(X, \tau, G)$.

Theorem 2.14. Let $(X, \tau, G)$ be a fuzzy $G$-space. A fuzzy set $A$ in $X$ is $G_{sf}$-open if and only if $F \subseteq \tau_G$-int$(A)$, whenever $F$ is fuzzy closed and $F \subseteq A$.

Proof. Straightforward.

Theorem 2.15. Let $(X, \tau, G)$ be a fuzzy $G$-space. If $A$ and $B$ are two $G_{sf}$-open sets with $A \lor cl(B) = B \lor cl(A) = 0_X$, then $A \lor B$ is $G_{sf}$-open.

Proof. Let $F$ be a fuzzy closed set such that $F \subseteq A \lor B$. Since $B \lor cl(A) = 0_X$, then $F \lor cl(A) \subseteq A$ and $F \lor cl(A)$ is fuzzy closed. Since $A$ is $G_{sf}$-open, by Theorem 2.14, $F \lor cl(A) \subseteq \tau_G$-int$(A)$.

Similarly we get $F \lor cl(B) \subseteq \tau_G$-int$(B)$.

Now we have, $F = F \lor (A \lor B) \subseteq (F \lor cl(A)) \lor (F \lor cl(B)) \subseteq \tau_G$-int$(A) \lor \tau_G$-int$(B) \subseteq \tau_G$-int$(A \lor B)$. Thus $F \subseteq \tau_G$-int$(A \lor B)$ and then by Theorem 2.14, $(A \lor B)$ is $G_{sf}$-open.
Remark 3.4. Being fuzzy continuous.

Here we give two examples of which the first one shows that a function may be open.

Example 2.19. In a fuzzy G-space (X, τ, G) if τG-int(A) ≤ B ≤ A and A is Gsf-open then B is Gsf-open.

Proof. If τG-int(A) ≤ B ≤ A then (1 − A) ≤ (1 − B) ≤ 1 − τG-int(A) = τG-cl(1 − A). Since (1 − A) is Gsf-closed and (1 − A) ≤ (1 − B) ≤ τG-cl(1 − A) = (1 − A) ∨ ϕ(1 − A), then by Theorem 2.10(iii), (1 − B) is also Gsf-closed and hence B is Gsf-open.

Definition 2.17. A function f : (X, τ, G1) → (Y, σ, G2) is called fuzzy (τG1,σG2)-closed if the image of every fuzzy τG1-closed set in X is σG2-closed in Y.

In what follows, whenever G1 (resp. G2) is a fuzzy grill on an fts (X, τ) (resp. (Y, σ)) and A ⊆ X (resp. Y) is fuzzy closed, then f(A) is fuzzy G2-closed or Gsf-open in X (resp. Y) and hope that the context will not allow any confusion to arise.

Theorem 2.18. If A be a Gsf-closed set in X and f : (X, τ, G1) → (Y, σ, G2) is fuzzy continuous and fuzzy (τG1,σG2)-closed, then f(A) is Gsf-closed in Y.

Proof. Let B be a fuzzy open set in Y such that f(A) ≤ B. Then A ≤ f−1(B) and f−1(B) is fuzzy open in X. Since A is Gsf-closed, ϕ(A) ≤ ϕ−1(B). Thus A ∨ ϕ(A) ≤ ϕ−1(B) ⇒ τG1-cl(A) ≤ f−1(B) ⇒ f(τG1-cl(A)) ≤ B. Since f is fuzzy τG1-closed, f(τG1-cl(A)) is σG2-closed and σG2-cl(f(A)) ≤ f(τG1-cl(A)) ≤ B. Then ϕ(f(A)) ≤ B and so f(A) is Gsf-closed in Y.

However the image of a Gsf-open set need not be Gsf-open under a fuzzy continuous map as is shown below:

Example 2.19. Let X = [a, b], Y = [c, d] be two fuzzy sets. Also let τ = |0X, 1X, P, Q|, where P(a) = 0.5, P(b) = 0.2, Q(a) = 0.6, Q(b) = 0.8 and σ = |0Y, 1Y, R| where R(c) = 0.8, R(d) = 0.6, be two fuzzy topologies on X and Y respectively. Let G1 = |G ∈ lR/0.2 ≤ G(x) ≤ 1; x ∈ X| and G2 = |G ∈ lR/0.1 ≤ G(x) ≤ 1; x ∈ X| be two fuzzy grills on X and Y respectively. Let f : (X, τ, G1) → (Y, σ, G2) be a function such that f(a) = d and f(b) = c. Then we see that f is continuous since f−1(R)(a) = Rf(a) = R(d) = 0.6 = Q(a) and f−1(R)(b) = Rf(b) = R(c) = 0.8 = Q(b). Now P, being fuzzy open in X, is also Gsf-open in X. Also f(P)(c) = sup z∈f−1(c) P(z) = P(b) = 0.2 and f(P)(d) = P(a) = 0.5 imply that (1 − f(P)) ≤ R. But ϕ(1 − f(P)) ≤ R. Indeed, c0.9 ≤ R but for any U ∈ Q(c0.9) ∈ σ, we get (1 − f(P)) + U − 1Y ∈ G2. So c0.9 ≤ ϕ(1 − f(P)). Thus (1 − f(P)) is not Gsf-closed in Y and consequently f(P) is not Gsf-open in Y.

3. Generalized fuzzy continuous functions for fuzzy G-spaces

According to our proposed scheme, in this section we introduce a sort of generalized fuzzy continuous function, termed Gsf-continuous function, in terms of some fuzzy grill G.

Definition 3.1. ([3]) A function f : X → Y is called generalized fuzzy continuous (gf-continuous, for short) if the inverse image of every fuzzy open set in Y is Gsf-continuous in X.

Definition 3.2. A function f : X → Y is called generalized fuzzy continuous with respect to some fuzzy grill G on X (Gsf-continuous, for short) if the inverse image of every fuzzy closed set in Y is Gsf-continuous in X.

Corollary 3.3. A function f : (X, τ, G) → (Y, σ) is Gsf-continuous if and only if the inverse image of each fuzzy open set in Y is Gsf-open in X.

Remark 3.4. Every fuzzy continuous function is gf-continuous [3] and every gf-continuous function is Gsf-continuous. But the converses are false. The first one was done by Balasubramanian and Sundaram [3]. Here we give two examples of which the first one shows that a function may be Gsf-continuous without being fuzzy continuous.
Example 3.5. Let $X = Y = \{a, b, c\}$; $\tau = \{0_X, 1_X, A\}$ and $\sigma = \{0_X, 1_X, B\}$, where $A(a) = 0.2$, $A(b) = 0.4$, $A(c) = 0.6$ and $B(a) = 0.3$, $B(b) = 0.5$, $B(c) = 0.7$, be two fuzzy topologies on $X$ and $Y$ respectively. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then $f$ is not fuzzy continuous since $f^{-1}(B) \notin \tau$. But for any fuzzy grill $G$ on $(X, \tau)$, $f$ is $G_{gf}$-continuous, since for $f^{-1}(1 - B)$ the only $\tau$-open fuzzy set containing $f^{-1}(1 - B)$ is $1_X$ and then obviously $\phi(f^{-1}(1 - B)) \leq 1_X$. Hence $f$ is $G_{gf}$-continuous on $X$.

We next show that a $G_{gf}$-continuous function need not be $gf$-continuous.

Example 3.6. Let $X$ be any set consisting of more than one point and 'a' be any fixed element of $X$. Let $\tau = \{0_X, 1_X, A\}$ and $\tau' = \{0_X, 1_X, B\}$, where $A(a) = 0.6$, $A(x) = 0$ for $x \in X \setminus \{a\}$, and $B(a) = 0.5$, $B(x) = 1$ for $x \in X \setminus \{a\}$. Then $(X, \tau)$ and $(X, \tau')$ are both fuzzy topological spaces. Let $G = \{G \subseteq I^X : 0.6 < G(x) \leq 1; x \in X\}$ be a fuzzy grill on $X$. Consider the identity function $f : (X, \tau) \rightarrow (X, \tau')$. Then $f$ is not $G_{gf}$-continuous on $(X, \tau)$ since for $B \in \tau'$, $f^{-1}(1 - B)$ is not $G_{gf}$-closed in $(X, \tau)$. Indeed, $f^{-1}(1 - B)(a) = (1 - B)(a) = 0.5$ and $f^{-1}(1 - B)(x) = 1 - B(x) = 0$ for any $x \in X \setminus \{a\}$. Thus $f^{-1}(1 - B) \leq A$, but $cl(f^{-1}(1 - B)) = 1_X \notin A$. However, $f$ is $G_{gf}$-continuous, because for any fuzzy point $x_1$ in $X$ and for any $U \in Q(x_1) \subseteq \tau$, $f^{-1}(1 - B) + U - 1 \notin G$ and hence $\phi(f^{-1}(1 - B)) = 0_X \leq A$.

We now show that the inverse image of a $G_{gf}$-closed set under a $G_{gf}$-continuous map need not be $G_{gf}$-closed.

Example 3.7. Let $X = \{a, b\}$. $\tau = \{0_X, 1_X, U\}$ and $\sigma = \{0_X, 1_X, V\}$, where $U(a) = 0.7$, $U(b) = 0.8$ and $V(a) = 0.5$, $V(b) = 0.1$. Then $\tau$ and $\sigma$ are two fuzzy topologies on $X$. Consider the fuzzy grill $G = \{G \subseteq I^X : 0.1 \leq G(x) \leq 1; x \in X\}$ on $X$. Let $f : (X, \tau, G) \rightarrow (X, \sigma, G)$ be the identity function. Then $f$ is $G_{gf}$-continuous since $f^{-1}(1 - V)$ is $G_{gf}$-closed in $X$. In fact, the only fuzzy open set containing $f^{-1}(1 - V)$ is $1_X$ and also $\phi(f^{-1}(1 - V)) \leq 1_X$. Now let us take a fuzzy set $A$ in $(X, \tau, G)$ such that $A(a) = 0.6$ and $A(b) = 0.4$. Then $A$ is $G_{gf}$-closed in $(X, \sigma, G)$. Also, $f^{-1}(A)(a) = A(f(a)) = A(a) = 0.6$ and $f^{-1}(A)(b) = A(f(b)) = A(b) = 0.4$. Now $f^{-1}(A) \leq U$. But $\phi(f^{-1}(A)) \leq U$ as $a_{0.6} \notin U$, although for any $P \in Q(a_{0.8}) \subseteq \tau$, $f^{-1}(A) + P - 1 \in G$ so that $a_{0.8} \leq \phi(f^{-1}(A))$. Thus $f^{-1}(A)$ is not $G_{gf}$-closed in $X$.

4. Some applications of $G_{gf}$-closed sets

In this section we introduce fuzzy generalized regular and normal spaces via $G_{gf}$-closed sets. Before going to that we require the following preliminaries:

Definition 4.1. ([11]) An fts $X$ is said to be fuzzy regular if every fuzzy open set $V$ can be expressed as union of fuzzy open sets $U_a$’s such that $cl(U_a) \subseteq V$, for each $a$.

Definition 4.2. ([16]) An fts $X$ is called fuzzy strongly regular if for any fuzzy open set $A$ in $X$ and each fuzzy point $x_a \leq A$, there exists a fuzzy open set $B$ in $X$ such that $x_a \leq B \subseteq cl(B) \leq A$.

Theorem 4.3. ([16]) A fuzzy strongly regular space is fuzzy regular.

Definition 4.4. A fuzzy $G$-space $(X, \tau, G)$ is said to be $G_{gf}$-regular if for every $G_{gf}$-closed set $F$ and each fuzzy point $x_1$ in $X$ such that $x_1 \notin F$, there exist fuzzy open sets $U$ and $V$ in $X$ for which $x_1 \leq U$, $F \leq V$ and $U \cap V$. 

Theorem 4.5. In a fuzzy $G$-space $(X, \tau, G)$, the following are equivalent:

(a) $(X, \tau, G)$ is $G_{gf}$-regular.
(b) For each fuzzy point $x_1$ in $X$ and each $G_{gf}$-closed set $F$ such that $x_1 \notin F$, there exist two fuzzy open sets $U$ and $V$ such that $x_1 \leq U$, $\tau_G clF \subseteq V \land U \cap V$.
(c) For each $G_{gf}$-closed set $F$ in $X$ and each fuzzy point $x_1$ with $x_1 \notin F$, there exist two fuzzy open sets $U$ and $V$ such that $x_1 \leq U$, $F \leq V$ and $clU \cap clV$.
(d) For each fuzzy point $x_1$ in $X$ and each $G_{gf}$-open set $U$ containing $x_1$, there exists a fuzzy open set $V$ such that $x_1 \leq V \leq clV \subseteq U$. 

Proof. (a) $\Rightarrow$ (b): Let $x_1$ be any fuzzy point and $F$ be any $\mathcal{G}_{sf}$-closed set in $X$ such that $x_1 \in F$. By (a), there exist fuzzy open sets $U$ and $V$ such that $x_1 \in U$, $F \subseteq V$ and $U \cap \overline{V}$. Now $F$ being $\mathcal{G}_{sf}$-closed, $\phi(F) \subseteq V$ i.e., $\tau_{\mathcal{G}}^{-}\text{cl}F \subseteq V$.

(b) $\Rightarrow$ (a): Obvious.

(b) $\Rightarrow$ (c): Let $x_1$ be any fuzzy point and $F$ be any $\mathcal{G}_{sf}$-closed set in $X$ such that $x_1 \in F$. Then by (b), there exist $W, V \in \tau$ such that $x_1 \in W$, $\tau_{\mathcal{G}}^{-}\text{cl}F \subseteq W$ and $V \subseteq \overline{W}$. Thus $W \subseteq \overline{V}$, and hence $x_1 \in \overline{F}$.

Now $clV$, being a fuzzy closed set, is $\mathcal{G}_{sf}$-closed [by Observation 2.3 (iv)] and $x_1 \in \overline{F}$. Then again by (b), there exist two fuzzy open sets $G$ and $H$ in $X$ such that $x_1 \in G$, $\tau_{\mathcal{G}}^{-}\text{cl}(clV) \subseteq H$ and $\overline{G}$ and $\overline{H}$. Hence $\overline{G} \subseteq \overline{H}$ [by Theorem 4.6].

Let us put $U = W \cap G$. Then $U$ and $V$ are fuzzy open sets in $X$ such that $x_1 \in U$, $F \subseteq V$ and $clU \subseteq \overline{V}$. Indeed, $clU = cl(W \cap G) \subseteq clG$ and $clG \subseteq \overline{H} \Rightarrow clU \subseteq \overline{H} \Rightarrow clU \subseteq \overline{V}$ [since $clV \subseteq H$].

(c) $\Rightarrow$ (d) : Let $y_\beta$ be any fuzzy point and $U$ be any $\mathcal{G}_{sf}$-open set in $X$ such that $y_\beta \in U$. Then $y_\beta \in (1 - U)$ and $(1 - U)$ is $\mathcal{G}_{sf}$-closed. Then by (c), there exist $V, W \in \tau$ such that $y_\beta \in V$, $(1 - U) \subseteq W$ and $clV \subseteq clW \Rightarrow y_\beta \subseteq V$, $(1 - W) \subseteq U$ and $clV \subseteq clW \Rightarrow y_\beta \subseteq V$, $(1 - W) \subseteq U$ and $clV \subseteq clW \Rightarrow y_\beta \subseteq V$.

(d) $\Rightarrow$ (a) : Let $x_1$ be any fuzzy point and $F$ be any $\mathcal{G}_{sf}$-closed set in $X$ such that $x_1 \in F$. Then $x_1 \in 1 - F$ and $1 - F$ is a $\mathcal{G}_{sf}$-open set in $X$. By assumption, there exists a $V \in \tau$ such that $x_1 \in V \subseteq clV \subseteq 1 - F$ which gives $F \subseteq 1 - clV$. Thus we have two fuzzy open sets $V$ and $(1 - clV)$ such that $x_1 \in V \subseteq F \subseteq (1 - clV)$ and $V \subseteq (1 - clV)$.

**Theorem 4.6.** Every $\mathcal{G}_{sf}$-regular space is fuzzy strongly regular and hence fuzzy regular.

**Proof.** Let $x_1$ be any fuzzy point in $X$ and $A$ be any fuzzy open set in $X$ such that $x_1 \in A$. Since every fuzzy open set is $\mathcal{G}_{sf}$-open, then by Theorem 4.5(d), there exists a $U \in \tau$ such that $x_1 \in U \subseteq clU \subseteq A$. Thus $X$ is fuzzy strongly regular and then by Theorem 4.3, $X$ is fuzzy regular.

But in general, the reverse implications in the above theorem are not true. The counterexample to show that a fuzzy regular space need not be fuzzy strongly regular is given in [16]. We only show that a fuzzy strongly regular space may not be $\mathcal{G}_{sf}$-regular.

**Example 4.7.** Let $X = \{a, b\}$ and $\tau = \{\emptyset, X, P, Q\}$, where $P(a) = 0.3$, $P(b) = 0.5$ and $Q(a) = 0.7$, $Q(b) = 0.5$. Then $(X, \tau)$ is an fts. Here we see that $P$ and $Q$ are both fuzzy open and fuzzy closed. Thus for any $x_1 \in P$, we have the fuzzy open set $P$ such that $x_1 \in P \subseteq clP = P$. Similarly for any $y_\beta \in Q$, we get $y_\beta \in Q \subseteq clQ = Q$. Thus $X$ is fuzzy strongly regular. But we claim that $X$ is not $\mathcal{G}_{sf}$-regular. Indeed, let us take a fuzzy set $F$ in $X$ where $F(a) = 0.6$, $F(b) = 0.3$. Then $F \subseteq Q$ and $\phi(F) \subseteq clF$ [by Result 1.1(v)] = $Q \Rightarrow F$ is $\mathcal{G}_{sf}$-closed. Now $b_{0.6} \overline{F}$ [since $0.6 + F(b) < 1$], but we cannot find two fuzzy open sets $U$ and $V$ such that $b_{0.6} \subseteq U$, $F \subseteq V$ and $U \subseteq \overline{V}$.

**Definition 4.8.** ([10]) An fts $X$ is called fuzzy normal if for any fuzzy closed set $F$ and a fuzzy open set $U$ in $X$ such that $F \subseteq U$ there exists a fuzzy open set $V$ such that $F \subseteq \text{cl}(V) \subseteq U$.

**Theorem 4.9.** ([16]) An fts $X$ is fuzzy normal if and only if for two fuzzy closed sets $F$ and $G$ with $F \subseteq G$, there exist fuzzy open sets $U$ and $V$ such that $F \subseteq U$, $G \subseteq V$ and $U \subseteq \overline{V}$.

**Definition 4.10.** A fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$ is said to be $\mathcal{G}_{sf}$-normal if for any two $\mathcal{G}_{sf}$-closed sets $F_1$ and $F_2$ in $X$ such that $F_1 \subseteq F_2$, there exist fuzzy open sets $U$ and $V$ in $X$ for which $F_1 \subseteq U$, $F_2 \subseteq V$ and $U \subseteq \overline{V}$.

Evidently every $\mathcal{G}_{sf}$-normal space is fuzzy normal.

**Theorem 4.11.** In a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$, the following are equivalent:

(a) $(X, \tau, \mathcal{G})$ is $\mathcal{G}_{sf}$-normal.
(b) For any two $\mathcal{G}_{sf}$-closed sets $F_1$ and $F_2$ in $X$ such that $F_1 \subseteq F_2$, there exist two fuzzy open sets $U$ and $V$ such that $F_1 \subseteq U$, $F_2 \subseteq V$ and $U \subseteq \overline{V}$.
(c) For each $\mathcal{G}_{sf}$-closed set $F$ and for each $\mathcal{G}_{sf}$-open set $G$ containing $F$, there exists a fuzzy open set $V$ such that $F \subseteq V \subseteq \text{cl}(V) \subseteq G$. 

S. Das, M.N. Mukherjee / Filomat 26:3 (2012), 563-571
Proof. (a) ⇒ (b): Let $F_1 \not\subset F_2$, where $F_1$ and $F_2$ are two $G$-closed sets in $X$. Then by (a), there exist two fuzzy open sets $W$ and $V$ such that $F_1 \subseteq W$, $F_2 \subseteq V$ and $W \not\subset V$. Then $W \not\subset V$ and thus $F_1 \not\subset V$. Now $V$, being fuzzy closed, is $G$-closed also. Thus $F_1$ and $clU$ are two $G$-closed sets in $X$ such that $F_1 \not\subset V$ and $clU \not\subset V$. Again by $G$-normality of $X$, there exist two fuzzy open sets $G$ and $H$ such that $F_1 \not\subset G$, $clV \not\subset H$ and $G \not\subset H$. Hence $clG \not\subset H$. Let us put $U = W \wedge G$. Then $U$ and $V$ are two fuzzy open sets in $X$ such that $V \not\subset U$, $F_2 \subseteq V$ and $clU \not\subset V$.

(b) ⇒ (c): Let $F$ be any $G$-closed set and $G$ be any $G$-open set in $X$ such that $F \subseteq G$. Thus $F \not\subset (1 - G)$ and $(1 - G)$ is $G$-closed. Hence by (b), there exist fuzzy open sets $U$ and $V$ such that $F \subseteq U$, $1 - G \subseteq V$ and $clU \not\subset V$. Now $1 - G \subseteq V$ implies $1 - clV \subseteq G$ and $clU \not\subset V$ implies $clU \not\subset 1 - clV$. Thus $U$ is the required fuzzy open set in $X$ for which $F \subseteq U$.

(c) ⇒ (a): Let $F_1$ and $F_2$ be two $G$-closed sets in $X$ with $F_1 \not\subset F_2$. Then $F_1 \not\subset (1 - F_2)$ and $(1 - F_2)$ is $G$-open. By (c), there exists a fuzzy open set $U$ in $X$ such that $F_1 \subseteq U \subseteq clU \subseteq 1 - F_2$, i.e., $F_1 \subseteq U$, $F_2 \subseteq (1 - clU)$ and $U \not\subset (1 - clU)$.

Proposition 4.12. For any two $G$-closed sets $F_1$ and $F_2$ in a $G$-normal space $(X, \tau, G)$, $F_1 \not\subset F_2$ if and only if $\tau^G$-$clF_1 \not\subset \tau^G$-$clF_2$.

Proof. First let $F_1 \not\subset F_2$. By $G$-normality of $X$, there exist fuzzy open sets $U$ and $V$ in $X$ for which $F_1 \subseteq U$, $F_2 \subseteq V$ and $U \not\subset V$. Since $F_1$ and $F_2$ are $G$-closed sets, $\phi(F_1) \subseteq U$, $\phi(F_2) \subseteq V \Rightarrow \tau^G$-$clF_1 \subseteq U$ and $\tau^G$-$clF_2 \subseteq V$ and thus $\tau^G$-$clF_1 \not\subset \tau^G$-$clF_2$ [since $U \not\subset V$]. The converse part is obvious.

Definition 4.13. A function $f : X \rightarrow Y$ is called $G$-irresolute if the inverse image of every $G$-closed set in $Y$ is $G$-closed in $X$, or equivalently the inverse image of every $G$-open set in $X$ is $G$-open in $X$.


(a) If $X$ is $G$-regular then $Y$ is also $G$-regular.

(b) If $X$ is $G$-normal then $Y$ is also $G$-normal.

Proof. (a) Let $F$ be any $G$-closed fuzzy set in $Y$ and $y_1$ be any fuzzy point in $Y$ such that $y_1 \not\subset F$. Then $f^{-1}(y_1) \not\subset f^{-1}(F)$. Since $F$ is $G$-irresolute, $f^{-1}(F)$ is $G$-closed in $X$. Select some $x \in f^{-1}(y_1)$ (as $f$ is onto) and consider the fuzzy point $x_1$ in $X$. Then $x_1 \not\subset f^{-1}(F)$. Since $X$ is $G$-regular, there exist fuzzy open sets $U$ and $V$ such that $x_1 \subseteq U$, $f^{-1}(F) \subseteq V$ and $U \not\subset V$. Since $f$ is fuzzy open and surjective, $f(U)$ and $f(V)$ are fuzzy open in $Y$ such that $y_1 \subseteq f(U)$, $F \subseteq f(V)$ and $f(U) \not\subset f(V)$. Thus $Y$ is $G$-regular.

(b) Let $F_1$ and $F_2$ be two $G$-closed sets in $Y$ such that $F_1 \not\subset F_2$. Since $f$ is $G$-irresolute, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are $G$-closed sets in $X$ and also $f^{-1}(F_1) \not\subset f^{-1}(F_2)$. Since $X$ is $G$-normal, there exist fuzzy open sets $U$ and $V$ in $X$ for which $f^{-1}(F_1) \subseteq U$, $f^{-1}(F_2) \subseteq V$ and $U \not\subset V$. Now $f$ is surjective, so $F_1 \subseteq f(U)$, $F_2 \subseteq f(V)$ and $f(U) \not\subset f(V)$ and since $f$ is fuzzy open, $f(U)$ and $f(V)$ are fuzzy open sets in $Y$. Thus $Y$ is $G$-normal.

Theorem 4.15. Let $(X, \tau, G)$ and $(X, \sigma, H)$ be two fuzzy $G$-spaces. If $f : X \rightarrow Y$ is a fuzzy continuous, fuzzy $(\tau^G, \sigma^H)$-closed surjective map, then

(a) $Y$ is $G$-regular ⇒ $X$ is $G$-regular.

(b) $Y$ is $G$-normal ⇒ $X$ is also $G$-normal.

Proof. (a) Let $F$ be any $G$-closed fuzzy set in $X$ and $x_1$ be any fuzzy point in $X$ such that $x_1 \not\subset F$. Since $F$ is fuzzy continuous and fuzzy $(\tau^G, \sigma^H)$-closed then by Theorem 2.18, $f(F)$ is $G$-closed in $Y$. Now $f$ being surjective, $f(x_1) \not\subset f(F)$. Also since $Y$ is $G$-regular, there exist fuzzy open sets $U$ and $V$ such that $f(x_1) \subseteq U$, $f(F) \subseteq V$ and $U \not\subset V$. Thus we get $x_1 \subseteq f^{-1}(U)$, $F \subseteq f^{-1}(V)$ and $f^{-1}(U) \not\subset f^{-1}(V)$. By fuzzy continuity of $f$, $f^{-1}(U)$ and $f^{-1}(V)$ are both fuzzy open. So $X$ is $G$-regular.

(b) Let $F_1$ and $F_2$ be two $G$-closed sets in $X$ such that $F_1 \not\subset F_2$. Then by Theorem 2.18, $f(F_1)$ and $f(F_2)$ are $G$-closed in $Y$. Then $f(F_1) \not\subset f(F_2)$ [being surjective]. By $G$-normality of $X$, there exist fuzzy open sets $U$ and $V$ in $X$ for which $f(F_1) \subseteq U$, $f(F_2) \subseteq V$ and $U \not\subset V$, i.e., $F_1 \subseteq f^{-1}(U)$, $F_2 \subseteq f^{-1}(V)$ and $f^{-1}(U) \not\subset f^{-1}(V)$. Now by fuzzy continuity of $f$, $f^{-1}(U)$ and $f^{-1}(V)$ are both fuzzy open in $X$. Hence $X$ is $G$-normal.
Acknowledgement

The authors are grateful to the referee for certain constructive comments towards some improvements of the paper.

References