Advanced results on variational inequality formulation in oligopolistic market equilibrium problem

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Abstract. The aim of the paper is to study the regularity of the solution to the evolutionary variational inequality governing the dynamic oligopolistic market equilibrium problem in presence of production excesses. More precisely, we obtain a Lipschitz continuity result with respect to time for such a solution. Moreover, we introduce a discretization procedure for computing dynamic equilibrium solutions and we provide a numerical example.

1. Introduction

The paper is devoted to the study of advanced results regarding both the regularity of the solution to the variational inequality which expresses the equilibrium conditions for the dynamic oligopolistic market equilibrium problem in presence of production excesses and its computation. Specifically, we prove that the dynamic equilibrium solution is Lipschitz continuous with respect to time. To this aim, we have to estimate the variation rate of projections onto time-dependent constraints set, which is the topical point of this theory. In virtue of a recent paper by P. Falsaperla and F. Raciti (see [17]), the Lipschitz continuity result is enough in order to value the error in a numerical computation based on a discretization procedure with respect to time. In particular, in this paper, we present a numerical scheme by using a discretization procedure. The use of this method requires only the continuity with respect to time of the dynamic equilibrium solution, which we can obtain under mild assumptions on the data. At each step of the discretization, we compute the solution to the static problem by means of a class of projection-contraction methods proposed by M.V. Solodov and P. Tseng in [22]. Then we construct the dynamic numerical solution with a linear interpolation. In literature some similar numerical schemes for the calculation of solutions to dynamic traffic equilibrium problems can be found in [1–5]. Finally, we present the performance of the algorithm on a numerical example.

We recall, now, some contribution about the development of the variational formulation of the dynamic oligopolistic market equilibrium problem. The dynamic framework has been studied in [5] by A. Barbagallo and M.G. Cojocaru who proved the equivalence between the dynamic Cournot-Nash principle and an evolutionary variational inequality. Moreover, they obtained the existence and regularity of the dynamic...
oligopolistic market equilibrium solution. The dynamical approach results fundamental for the study of the model because of “the time-dependent formulation of equilibrium problems allows one to explore the dynamics of adjustment processes in which a delay on time response is operating”, as M.J. Beckmann and J.P. Wallace pointed out in [13]. But, there is another important aspect to take into account, namely, the delay on time response that always happens because the processes have not an infinite speed. For this reason, in [11], such adjustment processes have been studied and represented by means of a memory term, which depends on previous equilibrium solutions according to the Volterra operator. In the subsequent paper [12], applying the infinite-dimensional duality results developed in [16, 20], the existence of the Lagrange variables, which allow to describe the behaviour of the market, is proved. The existence of Lagrange multipliers has been obtained for other equilibrium problems. Moreover, in [12] some sensitivity results have been obtained each of them showing that small changes of the solution happen in correspondence with small changes of the profit function. In [7], an oligopolistic market equilibrium model with an explicit long-term memory has been considered. Then, the Lipschitz continuity of the solution, which depends on the variation rate of projections onto time-dependent constraints set, is shown. In [8] and in [9] the variational formulation of the oligopolistic market equilibrium problems in presence of production excesses and both production and demand excesses are introduced. The equilibrium conditions in terms of the well-known dynamic Cournot-Nash principle are analyzed, and then the equilibrium conditions have been expressed also by using the Lagrange multipliers and the infinite-dimensional duality theory. The two conditions are both expressed by an appropriate evolutionary variational inequality, as a consequence their equivalence is achieved. Moreover, thanks to the variational formulation, existence and continuity results for the equilibrium solutions are shown. In this paper, we consider the dynamic oligopolistic market equilibrium model in which some of the amounts of the commodity available are sold out whereas for a part of the producers can occur an excess of production (see [8]). We remark that the presence of production excesses, during an economic crisis, can be due to a decrease in market demands. For this reason, it is very useful study such a problem.

The paper is organized as follows. In Section 2 we recall the dynamic oligopolistic market equilibrium problem in presence of production excesses. In Section 3 we recall some existence and continuity results for dynamic equilibrium solutions. Section 4 is devoted to prove the Lipschitz continuity result. Then in Section 5 we introduce a numerical scheme in order to compute the dynamic equilibrium solution. At last, in Section 6, we report the computational results.

2. Dynamic oligopolistic market equilibrium model in presence of production excesses

Let us present the dynamic oligopolistic market equilibrium problem in presence of production excesses introduced in [8].

Let us consider, in the time interval [0, T], T > 0, m firms \( P_i \), \( i = 1, \ldots, m \), that produce one only commodity and n demand markets \( Q_j \), \( j = 1, \ldots, n \), that are generally spatially separated. Let \( p_i(t), i = 1, \ldots, m \), be the nonnegative commodity output produced by firm \( P_i \) at the time \( t \in [0, T] \). Let \( q_j(t), j = 1, \ldots, n \), be the nonnegative demand for the commodity at demand market \( Q_j \) at the time \( t \in [0, T] \). Let \( x_i(t), i = 1, \ldots, m, j = 1, \ldots, n \), be the nonnegative commodity shipment between the supply market \( P_i \) and the demand market \( Q_j \) at the time \( t \in [0, T] \). In the following, we set \( x_i(t) = (x_{i1}(t), \ldots, x_{in}(t)), i = 1, \ldots, m, t \in [0, T] \). Let us group the production output into a vector-function \( p : [0, T] \rightarrow \mathbb{R}_+^m \), the demand output into a vector-function \( q : [0, T] \rightarrow \mathbb{R}_+^n \), the commodity shipments into a matrix-function \( x : [0, T] \rightarrow \mathbb{R}_{+}^{mn} \). We assume that the nonnegative commodity shipment belongs to \( L^2([0, T], \mathbb{R}_{+}^{mn}) \). Let \( \epsilon_i(t), i = 1, \ldots, m \), be the nonnegative production excess for the commodity of the firm \( P_i \) at the time \( t \in [0, T] \). Let us group the production excess into a vector-function \( \epsilon : [0, T] \rightarrow \mathbb{R}_+^m \). The following feasibility condition holds, a.e. in \([0, T]\):

\[
p_i(t) = \sum_{j=1}^n x_{ij}(t) + \epsilon_i(t), \quad i = 1, \ldots, m, \tag{1}
\]

namely the quantity produced by each firm \( P_i \) at the time \( t \in [0, T] \) must be equal to the sum of the
commodity shipments from that firm to each demand markets plus the production excess at the same time 
t \in [0, T]. We suppose that \( c \in L^2([0, T], \mathbb{R}_+^n) \), then, it results \( p \in L^2([0, T], \mathbb{R}_+^n) \).

Moreover, let us associate with each firm \( P_i \) a production cost \( f_i \), \( i = 1, \ldots, m \), and assume that the production cost of a firm \( P_i \) may depend upon the entire production pattern, namely,

\[
f_i^* = f_i^*(t, x(t), \varepsilon(t)).
\]

(2)

Similarly, let us associate with each demand market \( Q_j \), a demand price for unity of the commodity \( d_j \), \( j = 1, \ldots, n \), and assume that the demand price of a demand market \( Q_j \) may depend, in general, upon the entire consumption pattern, namely,

\[
d_j = d_j(t, x(l)).
\]

(3)

Let \( g_i \), \( i = 1, \ldots, m \), denote the storage cost of the commodity produced by the firm \( P_i \) and assume that this cost may depend upon the entire production pattern, namely,

\[
g_i^* = g_i^*(t, x(l), \varepsilon(l)).
\]

(4)

Finally, let \( c_{ij} \), \( i = 1, \ldots, m \), \( j = 1, \ldots, n \), denote the transaction cost, which includes the transportation cost associated with trading the commodity between firm \( P_i \) and demand market \( Q_j \). Here we permit the transaction cost to depend upon the entire shipment pattern, namely,

\[
c_{ij} = c_{ij}(t, x(l)).
\]

(5)

Hence, we have the following mappings,

\[
\begin{align*}
f^* & : [0, T] \times L^2([0, T], \mathbb{R}_+^m) \times L^2([0, T], \mathbb{R}_+^n) \to L^2([0, T], \mathbb{R}_+^m), \\
d & : [0, T] \times L^2([0, T], \mathbb{R}_+^m) \to L^2([0, T], \mathbb{R}_+^n), \\
g^* & : [0, T] \times L^2([0, T], \mathbb{R}_+^m) \times L^2([0, T], \mathbb{R}_+^n) \to L^2([0, T], \mathbb{R}_+^m), \\
c & : [0, T] \times L^2([0, T], \mathbb{R}_+^m) \to L^2([0, T], \mathbb{R}_+^n).
\end{align*}
\]

The profit \( \pi_i^*(t, x(t), \varepsilon(t)) \), \( i = 1, \ldots, m \), of the firm \( P_i \) at the time \( t \in [0, T] \) is, then,

\[
\pi_i^*(t, x(t), \varepsilon(t)) = \sum_{j=1}^{n} d_j(t, x(t))x_{ij}(t) - f_i^*(t, x(t), \varepsilon(t)) - g_i^*(t, x(t), \varepsilon(t)) - \sum_{j=1}^{n} c_{ij}(t, x(t))x_{ij}(t),
\]

(6)

namely, it is equal to the price that the demand markets are disposed to pay minus the production costs, the storage costs and the transportation costs.

In virtue of (1), we can express the nonnegative production excess \( \varepsilon_i(t) \) at the time \( t \in [0, T] \) in terms of \( p_i(t) \) and \( x_{ij}(t) \), namely

\[
\varepsilon_i(t) = p_i(t) - \sum_{j=1}^{n} x_{ij}(t), \quad \forall i = 1, \ldots, m, \ a.e. \ [0, T].
\]

(7)

As a consequence, we get

\[
\sum_{j=1}^{n} x_{ij}(t) \leq p_i(t), \quad \forall i = 1, \ldots, m, \ a.e. \ [0, T].
\]

(8)

Then, the production costs and the storage costs in virtue of (2) and (4), respectively, and (7), become

\[
f_i(t, x(l)) = f_i^*(t, x(l), \varepsilon(t)), \quad g_i(t, x(l)) = g_i^*(t, x(l), \varepsilon(t)).
\]
and, analogously, the profit (6) becomes

\[ v_i(t, x(t)) = v'_i(t, x(t), e(t)) = \sum_{j=1}^{n} d_{ij}(t, x(t)) x_j(t) - f_i(t, x(t)) - g_i(t, x(t)) - \sum_{j=1}^{n} c_{ij}(t, x(t)) x_j(t). \]

Let us suppose that the following assumptions on the profit function \( v \) and on the matrix-function

\[ V_D \psi = \left( \frac{\partial v_i}{\partial x_{ij}} \right)_{i=1, \ldots, m} \] hold:

i. \( v(t, x(t)) \) is continuously differentiable a.e. in \([0, T]\),

ii. \( V_D \psi \) is a Carathéodory function such that

\[ \exists h \in L^2([0, T]) : \| V_D \psi(t, x(t)) \|_{\infty} \leq h(t) \| x(t) \|_{\infty}, \text{ a.e. in } [0, T], \]

iii. \( v_i(t, x(t)) \) is pseudoconcave with respect to the variables \( x_i(t), i = 1, \ldots, m, \text{ a.e. in } [0, T] \).

For the readers convenience, we recall that a function \( v \), continuously differentiable, is called pseudoconcave with respect to \( x_i, i = 1, \ldots, m \), (see [18]) if

\[ \left\{ \begin{array}{l}
\frac{\partial v_i}{\partial x_i}(t, x_1, \ldots, x_i, \ldots, x_m) \geq 0 \\
\forall i, \quad x_i(t) \geq 0, \quad \exists h \in L^2([0, T]) : \| V_D \psi(t, x(t)) \|_{\infty} \leq h(t) \| x(t) \|_{\infty}, \text{ a.e. in } [0, T]
\end{array} \right. \]

According to (8), the feasible set for the distribution pattern is:

\[ K = \left\{ x \in L^2([0, T], \mathbb{R}^m) : \ x_i(t) \geq 0, \forall i = 1, \ldots, m, \forall j = 1, \ldots, n, \text{ a.e. in } [0, T], \right. \]

\[ \left. \sum_{j=1}^{n} x_{ij}(t) \leq p_i(t), \forall i = 1, \ldots, m, \text{ a.e. in } [0, T] \right\}. \] (10)

Let us give the dynamic oligopolistic market equilibrium conditions in presence of production excesses by using Lagrange variables associated to the constraints.

**Definition 2.1.** \( x^* \in K \) is a dynamic oligopolistic market problem equilibrium in presence of excesses if and only if for each \( i = 1, \ldots, m, \ j = 1, \ldots, n \) a.e. in \([0, T]\) there exists \( \lambda^*_{ij} \in L^2(0, T), \mu^*_{ij} \in L^2(0, T) \), such that

\[- \frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} + \mu^*_{ij}(t) = \lambda^*_{ij}(t), \]

\[- \frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} + \mu^*_{ij}(t) x_j^*(t) = 0, \]

\[ \lambda^*_{ij}(t) x_j^*(t) = 0, \quad \lambda^*_{ij}(t) \geq 0, \]

\[ \mu^*_{ij}(t) \left( \sum_{j=1}^{n} x_{ij}^*(t) - p_i(t) \right) = 0, \quad \mu^*_{ij}(t) \geq 0. \]

The terms \( \lambda^*_{ij}(t) \) and \( \mu^*_{ij}(t) \) are the Lagrange multipliers associated to the constraints \( x_{ij}^*(t) \geq 0 \) and to \( \sum_{j=1}^{n} x_{ij}^*(t) \leq p_i(t) \), respectively.

The equivalence between the dynamic equilibrium solution and the solution to an evolutionary variational inequality is proved in [8, Theorem 2.2] making use of the infinite-dimensional duality theory (see [15, 16, 20]).
Theorem 2.2. $x^* \in K$ is a dynamic oligopolistic market equilibrium in presence of production excesses if and only if it satisfies the evolutionary variational inequality

$$
\int_0^T \sum_{i=1}^m \sum_{j=1}^n - \frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} (x_{ij}(t) - x^*_{ij}(t)) dt \geq 0, \quad \forall x \in K,
$$

or equivalently

$$
\int_0^T \langle -\nabla Dv(t, x^*(t)), x(t) - x^*(t) \rangle dt \geq 0, \quad \forall x \in K.
$$

We recall that the equivalence between the evolutionary variational inequality (11) and the dynamic Cournot-Nash principle, which we remind in the following, is proved in [5, Theorem 3.1].

Definition 2.3. $x^* \in K$ is a dynamic oligopolistic market equilibrium in presence of production excesses if and only if for each $i = 1, \ldots, m$ and a.e. in $[0, T]$ we have

$$
v_i(t, x^*(t)) \geq v_i(t, x_i(t), x^*_i(t)), \text{ a.e. in } [0, T],
$$

where $x_i(t) = (x_{1i}(t), \ldots, x_{mi}(t))$, a.e. in $[0, T]$ and $x^*_i(t) = (x^*_1(t), \ldots, x^*_i(t), x^*_{i+1}(t), \ldots, x^*_m(t))$.

In other words, we seek to determine a nonnegative commodity distribution matrix function $x$ for which the $m$ firms and the $n$ demand markets will be in a state of equilibrium, such that the $m$ firms try to maximize their own profit at the time $t \in [0, T]$.

Finally, it is worth to point out that (11) is equivalent to the following point-to-point evolutionary variational inequality:

$$
\sum_{i=1}^m \sum_{j=1}^n - \frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} (x_{ij}(t) - x^*_{ij}(t)) \geq 0, \quad \forall x(t) \in K(t), \text{ a.e. in } [0, T],
$$

or equivalently

$$
\langle -\nabla Dv(t, x^*(t)), x(t) - x^*(t) \rangle dt \geq 0, \quad \forall x(t) \in K(t), \text{ a.e. in } [0, T],
$$

where

$$
K(t) = \left\{ x(t) \in \mathbb{R}^{mn} : \begin{array}{l}
    x_{ij}(t) \geq 0, \quad \forall i = 1, \ldots, m, \forall j = 1, \ldots, n, \\
    \sum_{j=1}^n x_{ij}(t) \leq p_i(t), \quad \forall i = 1, \ldots, m
\end{array} \right\}.
$$

3. Existence and regularity results

Now, we recall some results for the existence and the regularity of solutions to the dynamic oligopolistic market equilibrium problem in presence of production excesses.

First of all, we give some definitions (see [19]). Let $X$ be a reflexive Banach space, let $K$ be a subset of $X$ and let $X'$ be the dual space of $X$.

Definition 3.1. A mapping $A : K \rightarrow X'$ is strongly monotone on $K$ if and only if for all $u, v \in K$, $\exists \nu > 0$ such that $\langle Au - Av, u - v \rangle \geq \nu ||u - v||^2$. 
Definition 3.2. A mapping \( A : \mathbb{K} \rightarrow X' \) is pseudomonotone in the sense of Karamardian (K-pseudomonotone) if and only if for all \( u, v \in \mathbb{K} \)

\[
\langle Au, u - v \rangle \geq 0 \Rightarrow \langle Au, u - v \rangle \geq 0.
\]

Definition 3.3. A mapping \( A : \mathbb{K} \rightarrow X' \) is strictly pseudomonotone if and only if for all \( u, v \in \mathbb{K}, u \neq v \)

\[
\langle Au, u - v \rangle \geq 0 \Rightarrow \langle Au, u - v \rangle > 0.
\]

Definition 3.4. A mapping \( A : \mathbb{K} \rightarrow X' \) is pseudomonotone in the sense of Brezis (B-pseudomonotone) if and only if

1. for each sequence \( u_n \) weakly converging to \( u \) (in short \( u_n \rightharpoonup u \)) in \( \mathbb{K} \) and such that \( \limsup_n \langle Au_n, u_n - v \rangle \leq 0 \) it results that
   \[
   \lim \inf_n \langle Au_n, u_n - v \rangle \geq \langle Au, u - v \rangle \quad \forall v \in \mathbb{K}.
   \]
2. for each \( v \in \mathbb{K} \) the function \( u \mapsto \langle Au, u - v \rangle \) is lower bounded on the bounded subset of \( \mathbb{K} \).

Let, now, \( \mathbb{K} \) be a convex subset of \( X \).

Definition 3.5. A mapping \( A : \mathbb{K} \rightarrow X' \) is lower hemicontinuous along line segments, if and only if the function \( \xi \mapsto \langle Ax, u - \xi \rangle \) is lower semicontinuous for all \( u, v \in \mathbb{K} \) on the line segments \([u, v]\).

Definition 3.6. A mapping \( A : \mathbb{K} \rightarrow X' \) is hemicontinuous in the sense of Fan (F-hemicontinuous) if and only if for all \( v \in \mathbb{K} \) the function \( u \mapsto \langle Au, u - v \rangle \) is weakly lower semicontinuous on \( \mathbb{K} \).

Taking into account Theorems 2.6 and 2.7 and Corollary 3.7 in [19], we can establish the following result (see [8]):

Theorem 3.7. Let us set

\[
A = \left[ -\frac{\partial v_j(x')}{\partial x_{ij}} \right]_{i=1,\ldots,m, j=1,\ldots,n}.
\]

\[
A : L^2([0, T], \mathbb{R}^m) \rightarrow L^2([0, T], \mathbb{R}^m),
\]

\[
u = (x_{ij})_{i=1,\ldots,m, j=1,\ldots,n},
\]

\( \mathbb{K} \) the constraint set for the problem in Section 2, if \( A \) is B-pseudomonotone or F-hemicontinuous, or assuming that \( A \) is K-pseudomonotone and lower hemicontinuous along line segments, then the variational inequality

\[
\ll A x', x - x' \gg \geq 0, \quad \forall x \in \mathbb{K},
\]

admits a solution.

We remind that, in the Hilbert space \( L^2([0, T], \mathbb{R}^k) \),

\[
\ll \phi, y \gg = \int_0^T \langle \phi(t), y(t) \rangle dt,
\]

is its duality mapping, where \( \phi \in (L^2([0, T], \mathbb{R}^k))^* = L^2([0, T], \mathbb{R}^k) \) and \( y \in L^2([0, T], \mathbb{R}^k) \).

In the following, we show a continuity result for the solution to the dynamic market equilibrium problem in presence of production excesses. We point out that the property of the set convergence in Kuratowski’s sense satisfied to the feasible sets \( \mathbb{K} \) of dynamic oligopolistic market problems in presence of production excesses (see Lemma 6.1 in [8]) has an important role in order to obtain the continuity results. Moreover, making use of Theorem 4.2 in [6], we get the following results.

Theorem 3.8. Let us assume that the production function \( p \) is continuous on \([0, T]\). Moreover, let us assume that \( -\nabla p \) is a strictly pseudomonotone and continuous function on \([0, T]\). Then, the unique dynamic market equilibrium distribution in presence of production excesses \( x' \in \mathbb{K} \) is continuous on \([0, T]\).
4. Lipschitz continuity result

The aim of this section is to prove a Lipschitz continuity result for the dynamic oligopolistic market equilibrium solution in presence of production excesses. For this reason, we remind a general result obtained in [21] for the solutions to the parameterized variational inequality

\[ \langle A(t, x'(t)), x - x'(t) \rangle \geq 0, \quad \forall x \in K(t), t \in [0, T], \] (15)

where the constraint set \( K(t), t \in [0, T], \) is a closed convex and nonempty subset of \( \mathbb{R}^n, A : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \) is a mapping and \( \langle \cdot, \cdot \rangle \) denotes the scalar product in \( \mathbb{R}^n. \) More precisely, the following result holds (see [21, Theorem 1]):

**Theorem 4.1.** Let \( A \) be strongly monotone, Lipschitz continuous with respect to \( x, \) Lipschitz continuous with respect to \( t, \) and \( \gamma \) there exists \( \kappa \geq 0 \) such that, for \( t_1, t_2 \in [0, T], \)

\[ \|P_{K(t)}(z) - P_{K(t)}(z)\| \leq \kappa |t_1 - t_2|, \quad \forall z \in \mathbb{R}^n, \] (16)

where \( P_{K(t)}(z) = \arg \min_{x \in K(t)} \|z - x\|, \) \( t \in [0, T], \) denotes the projection onto the set \( K(t). \) Then, the unique solution \( x'(t), \) \( t \in [0, T], \) to (15) is Lipschitz continuous in \( [0, T], \) \( t_1 \neq t_2, \) the following estimate holds:

\[ \frac{\|x'(t_2) - x'(t_1)\|}{|t_2 - t_1|} \leq \gamma \left( \|x'|^2_{C([0, T], \mathbb{R}^n)} + \sup_{t_1, t_2 \in [0, T]} \left\| \frac{P_{K(t_2)}(z) - P_{K(t_1)}(z)}{t_2 - t_1} \right\|^2 \right)^{\frac{1}{2}}, \]

where \( \gamma = \gamma(\alpha, \beta, M, T, L). \)

In particular, in our case, from Theorem 4.1 we derive the next result.

**Theorem 4.2.** Let \( -V_\beta D^\alpha \) be strongly monotone (with constant \( \alpha \), Lipschitz continuous with respect to \( x \) (with constant \( \beta \), Lipschitz continuous with respect to \( t \) (with constant \( M \)), and let \( p \) be Lipschitz continuous vector-function. Then, the dynamic oligopolistic market equilibrium solution in presence of production excesses \( x' \) is Lipschitz continuous in \( [0, T] \). Moreover, let \( t_1 \neq t_2, \) the following estimate holds:

\[ \frac{\|x'(t_2) - x'(t_1)\|^2}{|t_2 - t_1|^2} \leq \gamma \left( \|x'|^2 + L^2 \right), \]

where \( \gamma = \gamma(\alpha, \beta, M, T, L). \)

**Proof.** In order to achieve the claim, we need only to show that (16) is verified. In fact, making use of Theorem 4.1 in [14], it results:

\[ \|P_{K(t_2)}(z) - P_{K(t_1)}(z)\| \leq \max \left\{ \|\xi_2 - \xi_1\| : \xi_2 \in K(0)(t_2), \xi_1 \in K(0)(t_1) \right\}, \] (17)

where \( K(0)(t) \) is the set of all the 0-dimensional faces of \( K(t). \) Then, assuming that the production vector-function is Lipschitz continuous, we obtain

\[ \|P_{K(t_2)}(z) - P_{K(t_1)}(z)\| \leq \max \{ ||p_i(t_2) - p_i(t_1)|| : i = 1, \ldots, m \} \]
\[ \leq \max \{ ||p_i(t_2) - p_i(t_1)|| : i = 1, \ldots, m \} \]
\[ \leq \max \{ |L_i| : i = 1, \ldots, m \} |t_2 - t_1|, \] (18)

where \( L_i \) is the Lipschitz constant of the component function \( p_i, \) for every \( i = 1, \ldots, m. \) Taking into account (18), we have:

\[ \|P_{K(t_2)}(z) - P_{K(t_1)}(z)\| \leq L |t_2 - t_1|, \]

where \( L = \max \{ |L_i| : i = 1, \ldots, m \}. \)

This concludes the proof. \( \square \)
5. Solution method

Let us provide an algorithm to compute equilibrium solutions of dynamic oligopolistic market equilibrium problems in presence of production excesses.

In the following, we suppose that the hypotheses of Theorem 3.8 are satisfied. Then, the point-to-point variational inequality (13) is defined for each \( t \in [0, T] \), namely

\[
\langle -\nabla_D v(t, x'(t)), x(t) - x'(t) \rangle \geq 0, \quad \forall x(t) \in \mathbb{K}(t), \; \forall t \in [0, T],
\]

where

\[
\mathbb{K}(t) = \left\{ x(t) \in \mathbb{R}^{mn} : x_{ij}(t) \geq 0, \; \forall i = 1, \ldots, m, \; \forall j = 1, \ldots, n, \; \sum_{j=1}^{n} x_{ij}(t) \leq p_i(t), \; \forall i = 1, \ldots, m \right\}.
\]

Hence, we can consider a partition of \([0, T]\), such that

\[0 = t_0 < t_1 < \ldots < t_r < \ldots < t_N = T.\]

For each point \( t_r \), \( r = 0, 1, \ldots, N \), of the partition, we obtain the finite-dimensional variational inequality

\[
\langle -\nabla_D v(t, x_*(t_r)), x(t_r) - x_*(t_r) \rangle \geq 0, \quad \forall x(t_r) \in \mathbb{K}(t_r),
\]

where

\[
\mathbb{K}(t_r) = \left\{ x(t_r) \in \mathbb{R}^{mn} : x_{ij}(t_r) \geq 0, \; \forall i = 1, \ldots, m, \; \forall j = 1, \ldots, n, \; \sum_{j=1}^{n} x_{ij}(t_r) \leq p_i(t_r), \; \forall i = 1, \ldots, m \right\}.
\]

Now, we can compute the solution to (20) by using a class of projection-contraction methods proposed by Solodov and Tseng in [22] and improved by Tinti in [23].

The idea of these algorithms is to choose a symmetric positive definite matrix \( M \in \mathbb{R}^{pq} \) and a starting point \( x^0(t_r) \in \mathbb{K}(t_r) \), and to iteratively update \( x^k(t_r) \), as follows:

\[
\begin{align*}
\tilde{x}^k(t_r) &= P_{\mathbb{K}(t_r)}(x^k(t_r) + \alpha \nabla_D v(t_r, x^k(t_r))), \\
x^{k+1}(t_r) &= x^k(t_r) - \gamma M^{-1}(T_a(t_r, x^k(t_r)) - T_a(t_r, \tilde{x}^k(t_r))),
\end{align*}
\]

where \( \gamma \in \mathbb{R}_+ \) and \( T_a = (I + \alpha \nabla_D v) \), in which \( I \) is the identity matrix, \( \alpha \in (0, +\infty) \) is chosen dynamically (according to an Armijo type rule), so that \( T_a \) is strongly monotone. These methods converge under condition that a solution exists and the operator is monotone. They have an additional parameter, the scaling matrix \( M \), that must be chosen as a symmetric positive matrix to accelerate the convergence.

After the iterative procedure, we construct the dynamic equilibrium solution by means of a linear interpolation.

With the same technique used in [10], it is possible to prove that, under suitable assumptions, the sequence, generated by the algorithm, converges in \( L^1 \)-sense to the dynamic oligopolistic market equilibrium solution in presence of production excesses.
6. Numerical example

Let us consider a numerical example of the dynamical oligopolistic market equilibrium problem in presence of excesses consisting of three firms and four demand markets, as in Figure 1, in the time interval \([0, 1]\).

Let \(p : [0, 1] \rightarrow \mathbb{R}^3\) be the production function such that, in \([0, 1]\),

\[
p_1(t) = 7t + 2, \quad p_2(t) = 2(2t + 1), \quad p_3(t) = 2t + 5,
\]

As a consequence, the feasible set is

\[
K = \{x \in L^2([0, 1], \mathbb{R}^{3\times 4}) : \quad x_{ij}(t) \geq 0, \quad \forall i = 1, \ldots, 3, \quad j = 1, \ldots, 4, \quad \text{a.e. in } [0, 1],
\]

\[
\sum_{j=1}^{4} x_{ij}(t) \leq p_i(t), \quad \forall i = 1, \ldots, 3, \quad \text{a.e. in } [0, 1]\}.
\]

Let \(p : [0, 1] \times L^2([0, 1], \mathbb{R}^{3\times 4}) \rightarrow \mathbb{R}^3\) be the production cost function defined by, a.e. in \([0, 1]\),

\[
\begin{align*}
f_1(t, x(t)) &= (t + 3)x_{11}(t) + tx_{12}(t) + \frac{3}{4}x_{13}(t) + 2tx_{14}(t), \\
f_2(t, x(t)) &= (t + 3)x_{21}(t) + \frac{3}{4}tx_{22}(t) + x_{23}(t) + 2tx_{24}(t), \\
f_3(t, x(t)) &= (3t + 5)x_{31}(t) + (2t + 1)x_{32}(t) + 2x_{33}(t) + \frac{t}{2}x_{34}(t),
\end{align*}
\]

and let \(g : [0, 1] \times L^2([0, 1], \mathbb{R}^{3\times 4}) \rightarrow \mathbb{R}^3\) be the storage cost given by, a.e. in \([0, 1]\),

\[
\begin{align*}
g_1(t, x(t)) &= (3t + 1)x_{11}(t) + 3tx_{12}(t) + \frac{1}{4}x_{13}(t) + tx_{14}(t), \\
g_2(t, x(t)) &= (3t + 1)x_{21}(t) + \left(\frac{t}{4} + 1\right)x_{22}(t) + 3x_{23}(t) + 3tx_{24}(t), \\
g_3(t, x(t)) &= \left(\frac{t}{2} + 7\right)x_{31}(t) + (t + 2)x_{32}(t) + \frac{2}{3}x_{33}(t) + \frac{t}{4}x_{34}(t).
\end{align*}
\]

Moreover, let us define the demand price function \(d : [0, 1] \times L^2([0, 1], \mathbb{R}^{3\times 4}) \rightarrow \mathbb{R}^4\) as, a.e. in \([0, 1]\),

\[
\begin{align*}
d_1(t, x(t)) &= tx_{11}(t) + x_{12}(t) + 2tx_{13}(t) + 2x_{14}(t) + 10t + 13, \\
d_2(t, x(t)) &= x_{12}(t) + x_{22}(t) + tx_{32}(t) + 8t + 4, \\
d_3(t, x(t)) &= (t + 1)x_{13}(t) + x_{23}(t) + tx_{43}(t) + 6, \\
d_4(t, x(t)) &= x_{14}(t) + 8t.
\end{align*}
\]
Finally, let us assume that the transportation function \( c : [0, 1] \times L^2([0, 1], \mathbb{R}^{3 \times 4}) \rightarrow \mathbb{R}^{3 \times 4} \) is given by, a.e. in 
[0, 1],

\[
\begin{align*}
c_{11}(t, x(t)) &= 2(t + 4)x_{11}(t) + x_{12}(t) + 2tx_{13}(t) + 2x_{14}(t) + \frac{1}{2}(t + 1)x_{24}(t) + 5t + 7, \\
c_{12}(t, x(t)) &= \frac{7}{2}x_{12}(t) + x_{22}(t) + tx_{32}(t) + 3t + 4, \\
c_{13}(t, x(t)) &= (4t + 3)x_{13}(t) + x_{23}(t) + tx_{33}(t) + 3, \\
c_{14}(t, x(t)) &= (8t + 5)x_{14}(t) + 2t, \\
c_{21}(t, x(t)) &= tx_{11}(t) + x_{12}(t) + 2tx_{13}(t) + 2x_{14}(t) + x_{21}(t) + 4t + 5, \\
c_{22}(t, x(t)) &= x_{12}(t) + (3t + 2)x_{22}(t) + tx_{32}(t) + \frac{3}{2}(t + 1)x_{33}(t) + 3, \\
c_{23}(t, x(t)) &= (t + 1)x_{13}(t) + 3x_{23}(t) + tx_{33}(t) + 1 - \frac{t}{5}, \\
c_{24}(t, x(t)) &= x_{14}(t) + (t + 2)x_{24}(t) + 2t, \\
c_{31}(t, x(t)) &= tx_{11}(t) + x_{12}(t) + 2tx_{13}(t) + 2x_{14}(t) + x_{21}(t) + x_{31}(t) + 5t + 1, \\
c_{32}(t, x(t)) &= x_{12}(t) + x_{22}(t) + \left(\frac{3}{2} + t\right)x_{32}(t) + 4t + 1, \\
c_{33}(t, x(t)) &= (t + 1)x_{13}(t) + x_{23}(t) + \frac{5}{2}\left(1 + \frac{3}{5}t\right)x_{33}(t) + tx_{43}(t) + 2 - \frac{2}{3}t, \\
c_{34}(t, x(t)) &= x_{14}(t) + \frac{7}{2}x_{24}(t) + \frac{t}{4}.
\end{align*}
\]

As a consequence, the profit function \( v : [0, 1] \times L^2([0, 1], \mathbb{R}^{3 \times 4}) \rightarrow \mathbb{R}^4 \) is, a.e. in [0, 1],

\[
\begin{align*}
v_1(t, x(t)) &= -(t + 4)x_{11}^2(t) - \frac{5}{2}x_{12}^2(t) - (3t + 2)x_{13}^2(t) - 4(2t + 1)x_{14}^2(t) - \frac{1}{2}(t + 1)x_{11}(t)x_{24}(t) \\
&\quad + (t + 2)x_{11}(t) + 2tx_{12}(t) + 2x_{13}(t) + 3tx_{14}(t), \\
v_2(t, x(t)) &= -x_{21}^2(t) - (3t + 1)x_{22}^2(t) - 2x_{23}^2(t) - (t + 2)x_{24}^2(t) - \frac{3}{2}(t + 1)x_{22}(t)x_{33}(t) \\
&\quad + 4\left(\frac{t}{2} + 1\right)x_{22}(t) + 7tx_{23}(t) + \left(1 + \frac{4}{5}t\right)x_{23}(t) + tx_{24}(t), \\
v_3(t, x(t)) &= -x_{31}^2(t) - \frac{3}{2}x_{32}^2(t) - \frac{5}{2}\left(1 + \frac{3}{5}t\right)x_{33}^2(t) - \frac{7}{2}x_{34}^2(t) - x_{21}(t)x_{31}(t) + \frac{3}{2}tx_{31}(t) \\
&\quad + tx_{32}(t) + \frac{2}{3}(t + 2)x_{34}(t) + 7tx_{34}(t).
\end{align*}
\]

Then, the operator \( V_D : [0, 1] \times L^2([0, 1], \mathbb{R}^{3 \times 4}) \rightarrow \mathbb{R}^{3 \times 4} \) is given by, a.e. in [0, 1],

\[
V_Dv(t, x(t)) = \begin{pmatrix}
-2(t + 4)x_{11}(t) - \frac{1}{2}(t + 1)x_{24}(t) + t + 2 \\
-2x_{21}(t) + 4\left(\frac{t}{2} + 1\right) \\
-2x_{31}(t) - x_{31}(t) + \frac{1}{2}t
\end{pmatrix}
\begin{pmatrix}
-5x_{12}(t) + 2t \\
-2(3t + 1)x_{22}(t) - \frac{3}{2}(t + 1)x_{33}(t) + 7t \\
-3x_{32}(t) + t
\end{pmatrix}
\begin{pmatrix}
-2(3t + 2)x_{13}(t) + 2 \\
-4x_{22}(t) + 1 + \frac{4}{5}t \\
-5\left(1 + \frac{4}{5}t\right)x_{33}(t) + \frac{3}{2}(t + 2)
\end{pmatrix}
+ \begin{pmatrix}
-8(2t + 1)x_{14}(t) + 3t \\
-2(t + 2)x_{24}(t) + t \\
-7x_{34}(t) + 7t
\end{pmatrix}.
\]
Now, we verify that \(-\nabla Dv\) is a strongly monotone operator, in fact, a.e. in \([0, 1]\):
\[
\langle -\nabla Dv(t, x(t)) + \nabla Dv(t, y(t)), x(t) - y(t) \rangle \\
= \left\{ 2(t + 4)[x_{11}(t) - y_{11}(t)] + \frac{1}{2}(t + 1)[x_{24}(t) - y_{24}(t)] \right\} [x_{11}(t) - y_{11}(t)] + 5|x_{12}(t) - y_{12}(t)|^2 \\
+ 2(3t + 2)[x_{13}(t) - y_{13}(t)]^2 + 8(2t + 1)[x_{14}(t) - y_{14}(t)]^2 + 2|x_{21}(t) - y_{21}(t)|^2 \\
+ \left\{ 2(3t + 1)[x_{22}(t) - y_{22}(t)] + \frac{3}{2}(t + 1)[x_{33}(t) - y_{33}(t)] \right\} [x_{22}(t) - y_{22}(t)] \\
+ 4|x_{23}(t) - y_{23}(t)|^2 + 2(t + 2)[x_{24}(t) - y_{24}(t)]^2 + [2|x_{31}(t) - y_{31}(t)| + [x_{21}(t) - y_{21}(t)]] \\
\cdot [x_{31}(t) - y_{31}(t)] + 3|x_{32}(t) - y_{32}(t)|^2 + 5\left( 1 + \frac{3}{5}t \right)[x_{33}(t) - y_{33}(t)]^2 + 7|x_{34}(t) - y_{34}(t)|^2 \\
\geq \frac{1}{2}\|x(t) - y(t)\|_{3\times 4}^2
\]
Hence, every assumptions of Theorem 3.8, so the unique dynamic market equilibrium solution is continuous on \([0, 1]\).

Now, we are able to solve the numerical example making use of the algorithm presented in Section 5, that will determine a sequence \(\{x^k(t)\}_{k\in\mathbb{N}}\) convergent to the dynamic solution. Then, we compute an approximate curve of equilibria, by selecting \(t_\in\{\frac{k}{20} : k \in \{0, 1, \ldots, 20\}\}\). By using a MatLab computation and choosing the initial point
\[
x_0(t_\in) = \begin{pmatrix}
\frac{2}{5}t_\in & \frac{1}{10}t_\in & \frac{1}{10}t_\in & t_\in \\
\frac{5}{4}t_\in & \frac{1}{10}t_\in & \frac{1}{10}t_\in & t_\in \\
\end{pmatrix}
\]
in order to start the iterative method, we obtain the equilibrium solutions for every time instant. The stopping criterion is
\[
\|R(x^k(t_\in))\|_{3\times 4} = \|x^k(t_\in) - x^{k-1}(t_\in)\|_{3\times 4} \leq 10^{-6},
\]
for $r = 0, 1, \ldots, 20$. Then, by interpolating the static equilibrium points we obtain the curves of equilibria (see Figure 2). Finally, Figure 3 shows the curves of production excesses.

References