Exponential spline approach for the solution of fourth order obstacle boundary value problems

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Abstract. An exponential quintic spline technique at mid knots is developed for approximating the solution of system of fourth order boundary value problems associated with obstacle, unilateral and contact problems. The present technique gives a class of methods of order two, four and six. Two numerical examples are considered for the numerical illustration of the proposed method. It is shown that the method developed in this paper is more efficient than the other finite difference, collocation and spline methods.

1. Introduction

We consider the system of fourth order two point boundary value problem of the form

\[ u^{(4)}(x) = \begin{cases} f(x), & a \leq x \leq c, \\ f(x) + g(x)u(x) + r, & c \leq x \leq d, \\ f(x), & d \leq x \leq b, \end{cases} \quad (1) \]

along with the following boundary conditions

\[ u(a) = u(b) = A_1, \quad u(c) = u(d) = A_2, \quad (2) \]
\[ u'(a) = u'(b) = B_1, \quad u'(c) = u'(d) = B_2, \quad (3) \]

where \( f(x) \) and \( g(x) \) are continuous functions on \( [a, b] \) and \( [c, d] \) respectively and \( A_1, A_2, B_1 \) and \( B_2 \) are finite real constants. The mathematical formulation of the obstacle, unilateral, contact and equilibrium problems arising in the field of elasticity, structural analysis, transportation science, economics and optimization results in the form of system (1)-(3). Kikuchi and Oden [15] proved that the problem of equilibrium of linearly elastic bodies in contact with a rigid frictionless foundation can be studied in the framework of variational inequalities. The general variational inequalities can be solved by penalty function and projection function methods [1-2, 6, 8-10, 15-16, 22]. Penalty function methods and Projection function methods are not very efficient, due to the instability created in the penalty function and the difficulty to
find the projections in the projection function methods. In general, it is not possible to obtain the analytical solution of system (1)-(3) for arbitrary choices of \(f(x)\) and \(g(x)\).

Siraj-ul-Islam et al. [12] proposed a quartic non-polynomial spline solution of a system of fourth order boundary value problem at mid knots. In order to develop numerical methods for obtaining smooth, approximate solution of system of fourth order boundary value problem, we apply non-polynomial spline which has the polynomial and exponential parts. Zahra [24, 25] studied such kind of exponential spline for nonlinear fourth order two point boundary value problems. It belongs to quintic non-polynomial spline function space: \(T_5 = \text{span}\{1, x, x^2, x^3, e^{kx}, e^{-kx}\}\), where, \(k\) is the frequency of exponential part of the spline function which can be real or pure imaginary and will be used to raise the accuracy of the method.

As \(k \to 0\), \(T_5\) reduces to \(\text{span}\{1, x, x^2, x^3, x^4, x^5\}\).

\(\text{Equation (5) reduces to quintic spline in} [a, b] \text{ when } k \to 0 \text{ as given in equation (4).} \)

In the present investigation, we have used exponential quintic spline function to develop the new numerical method for the solution of system (1)-(3). The advantage of the new method is its higher accuracy than the other known methods. In Section 2, exponential spline function is presented. Section 3 describes the method for the solution of system (1)-(3). The advantage of the new method is its higher accuracy than other known methods.

2. Exponential quintic spline function

To develop this method, mesh points are taken at off-step points. This approach reduces the error in the solution around the points where the solution satisfies extra continuity conditions. The interval \([a, b]\) is divided into \(N + 1\) subintervals, s.t.

\[\Delta : a = x_0 < x_{1/2} < x_{3/2} < \cdots < x_{N-1/2} < x_N = b,\]

where

\[x_{i-1/2} = a + (i-1/2)h, \quad i = 1, 2, \ldots, N\]

and \(h = \frac{b-a}{N}\). Without loss of generality we take, \(c = \frac{2x+b}{1}\) and \(d = \frac{a+2b}{1}\).

For each of its segment the exponential quintic spline function \(P_i(x)\) has the form:

\[P_i(x) = a_i e^{k(x-x_i)} + b_i e^{-k(x-x_i)} + c_i (x-x_i)^3 + d_i (x-x_i)^2 + e_i (x-x_i) + f_i, \quad i = 0, 1, 2, \ldots, N,\]

where \(a_i, b_i, c_i, d_i, e_i, f_i\) are constants and \(k\) is the free parameter which can be real or purely imaginary. Equation (5) reduces to quintic spline in \([a, b]\) when \(k \to 0\) as given in equation (4).

Let \(u(x)\) be the exact solution of system (1)-(3) and let \(S_i\) be an approximation to \(u_i = u(x_i)\) obtained by a segment \(P_i(x)\) passing through \((x_i, S_i)\) and \((x_{i+1}, S_{i+1})\) with

(i) \(S(x) = P_i(x), \quad x \in [x_i, x_{i+1}], \quad i = 0, 1, 2, \ldots, N,\)

(ii) \(S(x) \in C^4[a, b].\)

To derive the expressions for the coefficients in Equation (5), in terms of the function values \(u_{i-1/2}, u_{i+1/2}\) and second and fourth order spline derivatives, we define

(i) \(P_i(x_{i-1/2}) = u_{i-1/2},\)

(ii) \(P_i(x_{i+1/2}) = u_{i+1/2},\)

(iii) \(P_i'(x_{i-1/2}) = M_{i-1/2},\)

(iv) \(P_i'(x_{i+1/2}) = M_{i+1/2},\)

(v) \(P_i^{(4)}(x_{i-1/2}) = F_{i-1/2},\)
From algebraic manipulations, we get the following expressions:

\[
\begin{align*}
a_i &= \frac{h^4}{32\vartheta^4 \sinh 2\vartheta} \left[ e^\vartheta F_{i+1/2} - e^{-\vartheta} F_{i-1/2} \right], \\
b_i &= \frac{h^4}{32\vartheta^4 \sinh 2\vartheta} \left[ e^\vartheta F_{i-1/2} - e^{-\vartheta} F_{i+1/2} \right], \\
c_i &= \frac{1}{6h} [M_{i+1/2} - M_{i-1/2}] - \frac{h}{24\vartheta^2} [F_{i+1/2} - F_{i-1/2}], \\
d_i &= \frac{1}{4} [M_{i+1/2} + M_{i-1/2}] - \frac{h^2}{16\vartheta^2} [F_{i+1/2} + F_{i-1/2}], \\
e_i &= \frac{1}{h} [u_{i+1/2} - u_{i-1/2}] - \frac{h}{24} [M_{i+1/2} - M_{i-1/2}] + \frac{h^3}{96\vartheta^4} [F_{i+1/2} - F_{i-1/2}] (\vartheta^2 - 6), \\
f_i &= \frac{1}{2} [u_{i+1/2} + u_{i-1/2}] - \frac{h^2}{16} [M_{i+1/2} + M_{i-1/2}] + \frac{h^4}{64\vartheta^4} [F_{i+1/2} + F_{i-1/2}] (\vartheta^2 - 2),
\end{align*}
\]

where \( \vartheta = \frac{\omega}{2} \) and \( i = 0, 1, 2, \ldots, N \).

The continuity of the first derivative implies:

\[
M_{i+1/2} + 22M_{i-1/2} + M_{i+3/2} = 24 \frac{h^2}{\vartheta^2} [u_{i+1/2} - 2u_{i-1/2} + u_{i-3/2}] - \frac{3h^2}{8\vartheta^2} \left[ \left( \frac{1}{\vartheta^2} - \frac{1}{6} - \frac{1}{\vartheta \sinh \vartheta} \right) (F_{i+1/2} + F_{i-3/2}) \right. \\
\left. - 2 \left( \cosh \vartheta \frac{1}{\vartheta \sinh \vartheta} - \frac{1}{\vartheta^2} + \frac{11}{6} \right) F_{i-1/2} \right], \quad i = 2, 3, \ldots, N - 1.
\]

(6)

and the continuity of the third derivative implies:

\[
M_{i+1/2} - 2M_{i-1/2} + M_{i+3/2} = \frac{h^2}{4\vartheta} \left( \frac{1}{\vartheta^2} - \frac{1}{\vartheta \sinh \vartheta} \right) (F_{i+1/2} + F_{i-3/2}) + 2 \left( \cosh \vartheta \frac{1}{\vartheta \sinh \vartheta} - \frac{1}{\vartheta^2} \right) F_{i-1/2}, \quad i = 2, 3, \ldots, N - 1.
\]

(7)

Subtracting equation (7) from equation (6) and then dividing it by 24, we obtain:

\[
M_{i-1/2} = \frac{1}{h^2} [u_{i+1/2} - 2u_{i-1/2} + u_{i-3/2}] - \frac{h^2}{24} \left[ \left( \frac{3}{\vartheta^4} - \frac{3}{2\vartheta^3 \sinh \vartheta} - \frac{1}{4\vartheta \sinh \vartheta} \right) (F_{i+1/2} + F_{i-3/2}) \right. \\
\left. + \left( \cosh \vartheta \left( \frac{3}{\vartheta^4} + \frac{1}{2} \right) - \frac{1}{\vartheta^2} \left( \frac{3}{\vartheta^4} + 6 \right) \right) F_{i-1/2} \right].
\]

(8)

Eliminating \( M_i \)'s between (7) and (8), we get the following relation

\[
u_{i-5/2} - 4u_{i-3/2} + 6u_{i-1/2} - 4u_{i+1/2} + u_{i+3/2} = h^4 \left[ \alpha (F_{i-5/2} + F_{i+3/2}) + \beta (F_{i-3/2} + F_{i+1/2}) + \gamma F_{i-1/2} \right], \quad i = 3, 4, \ldots, N - 2.
\]

(9)

where

\[
\begin{align*}
\alpha &= \frac{1}{96} \left[ 6 \sinh \vartheta - 6 \vartheta - \vartheta^3 \right], \\
\beta &= \frac{1}{24} \left[ 6 \vartheta \cosh^2 \vartheta + \vartheta^3 \cosh^2 \vartheta - 6 \sinh \vartheta - 6 \vartheta^3 \right], \\
\gamma &= \frac{1}{48} \left[ 18 \sinh \vartheta - 6 \vartheta + 12 \vartheta \cosh^2 \vartheta - 6 \vartheta^3 + 22 \vartheta^3 \cosh \vartheta \right].
\end{align*}
\]
3. Numerical method

Now, substituting $u_i^{(4)} = g_i u_i + f_i + r_i$ in the spline relation (9), we obtain $N - 4$ linear algebraic equations in $N$ unknowns, $u_{i-1/2}$, $i = 1, 2, \ldots, N$ as:

$$ [1 - \alpha h^4 g_{i-3/2}] u_{i-3/2} + [4 - \beta h^4 g_{i-1/2}] u_{i-1/2} + [6 - \gamma h^4 g_{i+1/2}] u_{i+1/2} + [-4 - \beta h^4 g_{i+3/2}] u_{i+3/2} ] = [a_{i-3/2} + \beta (f_{i-3/2} + f_{i+1/2}) + \gamma (f_{i+3/2})] $$

$$ + [a (r_{i-3/2} + r_{i+1/2}) + \beta (r_{i-3/2} + r_{i+3/2}) + \gamma (r_{i+3/2})] $$

$$ i = 3, 4, \ldots, N - 2. \quad (10) $$

To obtain the unique solution of system (10), we need four more equations, two at each end of the range of integration for direct computation of $u_{i-1/2}$, $i = 1, 2, \ldots, N$. These boundary equations are obtained by Taylor series and the method of undetermined coefficients. The boundary equations associated with boundary conditions can be determined as follows:

$$ \begin{align*}
- & \frac{5}{3} u_{1/2} + \frac{5}{6} u_{3/2} - \frac{1}{6} u_{5/2} - h^4 d_0 u_0^{(4)} + d_1 u_1^{(4)} + d_2 u_2^{(4)} + d_3 u_3^{(4)} + d_4 u_4^{(4)} + d_5 u_5^{(4)} + d_6 u_6^{(4)} + d_7 u_7^{(4)} + d_8 u_8^{(4)} + d_9 u_9^{(4)} + d_{10} u_{10}^{(4)} ] + t_1 = -u_0 + \frac{5}{24} h^2 u_0 ^{''}, \quad (11) \\
- & \frac{29}{16} u_{1/2} + \frac{139}{16} u_{3/2} - \frac{109}{16} u_{5/2} - u_{7/2} - h^4 d_1 u_1^{(4)} + d_2 u_2^{(4)} + d_3 u_3^{(4)} + d_4 u_4^{(4)} + d_5 u_5^{(4)} + d_6 u_6^{(4)} + d_7 u_7^{(4)} + d_8 u_8^{(4)} + d_9 u_9^{(4)} ] + t_2 = -u_0 - h^2 u_0 ^{''}, \\
- & \frac{29}{16} u_{N-1/2} - \frac{109}{16} u_{N-3/2} - u_{N-5/2} - h^4 d_1 u_{N-1/2}^{(4)} + d_2 u_{N-3/2}^{(4)} + d_3 u_{N-5/2}^{(4)} + d_4 u_{N-7/2}^{(4)} + d_5 u_{N-9/2}^{(4)} + d_{10} u_{N-10}^{(4)} ] + t_{N-1} = -u_N - h^2 u_N ^{''}, \quad (13) \\
- & \frac{5}{3} u_{N-1/2} + \frac{5}{6} u_{N-3/2} - \frac{1}{6} u_{N-5/2} - h^4 d_0 u_N^{(4)} + d_1 u_{N-1/2}^{(4)} + d_2 u_{N-3/2}^{(4)} + d_3 u_{N-5/2}^{(4)} + d_4 u_{N-7/2}^{(4)} + d_5 u_{N-9/2}^{(4)} + d_{10} u_{N-10}^{(4)} ] + t_N = -u_N + \frac{5}{24} h^2 u_N ^{''} \quad (14) 
\end{align*} $$

where $t_1$, $t_2$, $t_{N-1}$ and $t_N$ are local truncation errors associated with the boundary conditions (11)-(14). The local truncation error $t_i$, associated with the scheme (10) is given by

$$ t_i = [1 - (2\alpha + 2\beta + \gamma)] h^4 u_i^{(4)} + \frac{1}{2} [-1 + 2\alpha + 2\beta + \gamma] h^5 u_i^{(5)} + \frac{1}{24} [7 - 3(34\alpha + 10\beta + \gamma)] h^6 u_i^{(6)} $$

$$ + \frac{1}{48} [-5 + 98\alpha + 26\beta + \gamma] h^7 u_i^{(7)} + \frac{1}{1920} [69 - 5(70\alpha + 82\beta + \gamma)] h^8 u_i^{(8)} $$

$$ + \frac{1}{11520} [-115 + 8464\alpha + 726\beta + 3\gamma] h^9 u_i^{(9)} $$

$$ + \frac{1}{967680} [2497 - 21(16354\alpha + 730\beta + \gamma)] h^{10} u_i^{(10)} + O(h^{11}), $$

$$ i = 3, 4, \ldots, N - 2. $$

Now the scheme (10)-(14) gives rise to the class of methods of different orders as follows:

(I) Second Order Method

For any choice of arbitrary $\alpha$ and $\beta$ with $\gamma = 1 - 2(\alpha + \beta)$, and

$$ (d_0, d_1, d_2, d_3, d_4, d_5, d_6) = (-\frac{1}{6}, \frac{77}{1151}, 0, 0, 0, 0, 0), $$

$$ (d_1, d_2, d_3, d_4, d_5, d_6) = (\frac{41}{256}, 1463/768, 0, 0, 0, 0). $$
Then local truncation errors for \((\alpha, \beta, \gamma) = \frac{1}{1730}(1, 236, 1446)\) are

\[
t_i = \begin{cases} 
- \frac{66}{2959} h^6 u_i^{(6)} + O(h^7), & i = 1, N, \\
\frac{1}{24} h^6 u_i^{(6)} + O(h^7), & i = 3, \ldots, N - 2, \\
\frac{560}{1119} h^6 u_i^{(6)} + O(h^7), & i = 2, N - 1.
\end{cases}
\]

(II) Fourth Order Method

For arbitrary \(a\) and \(b = \frac{1-24a}{6}\) with \(y = 1 - 2(a + \beta)\), and

\[
(d_0, d_1, d_2, d_3, d_4, d_5) = (\frac{9}{51176}, -\frac{8}{127}, -\frac{386}{9149}, -\frac{103}{14771}, -\frac{18721}{18721}, \frac{19}{65426}),
\]

\[
(d_1, d_2, d_3, d_4, d_5) = (\frac{647}{43920}, \frac{876}{1487}, \frac{335}{2013}, \frac{218}{220}, \frac{204}{43047}).
\]

Then local truncation errors for \((\alpha, \beta, \gamma) = (-\frac{7}{4900}, \frac{2899}{14550}, \frac{4787}{2757})\) are

\[
t_i = \begin{cases} 
- \frac{64}{124031} h^8 u_i^{(8)} + O(h^9), & i = 1, N, \\
\frac{19}{34920} h^8 u_i^{(8)} + O(h^9), & i = 3, \ldots, N - 2, \\
- \frac{168}{27413} h^8 u_i^{(8)} + O(h^9), & i = 2, N - 1.
\end{cases}
\]

(III) Sixth Order Method

For \((\alpha, \beta, \gamma) = \frac{1}{720}(-1, 124, 474)\) and

\[
(d_0, d_1, d_2, d_3, d_4, d_5) = (\frac{9}{51176}, -\frac{8}{127}, -\frac{386}{9149}, -\frac{103}{14771}, -\frac{18721}{18721}, \frac{19}{65426}),
\]

\[
(d_1, d_2, d_3, d_4, d_5) = (\frac{647}{43920}, \frac{876}{1487}, \frac{335}{2013}, \frac{218}{220}, \frac{204}{43047}).
\]

Then local truncation errors are

\[
t_i = \begin{cases} 
- \frac{28}{185399} h^{10} u_i^{(10)} + O(h^{11}), & i = 1, N, \\
- \frac{1}{3024} h^{10} u_i^{(10)} + O(h^{11}), & i = 3, \ldots, N - 2, \\
- \frac{29}{20138} h^{10} u_i^{(10)} + O(h^{11}), & i = 2, N - 1.
\end{cases}
\]

Remarks:

1. When \((a, \beta, \gamma) \rightarrow \frac{1}{301}(1, 76, 230)\) the relation (9) reduces to quartic polynomial spline [23].
2. When \((a, \beta, \gamma) \rightarrow \frac{1}{1920}(1, 236, 1446)\) the relation (9) reduces to quintic polynomial spline [19].
3. When \((a, \beta, \gamma) \rightarrow \frac{1}{301}(1, 56, 246)\) the relation (9) reduces to sextic polynomial spline at mid points.
4. When \(a = 0, \beta = \frac{6256}{73156}\) and \(\gamma = \frac{25004}{73156}\), then the relation (9) reduces to Al-Said and Noor fourth order finite difference method [5].
5. When \(\alpha = 0, \beta = \frac{1}{24}\) and \(\gamma = \frac{22}{24}\), then the relation (9) reduces to Al-Said and Noor second order finite difference method [6].

4. Spline solutions

The scheme (10) along with the boundary conditions (11) - (14) gives rise to a linear system of order \(NXN\) and may be written in the matrix form as

\[
AU = C + T, \\
A\bar{U} = C, \\
A(U - \bar{U}) = T, \\
AE = T,
\]

where \(U = (u_{i-1/2}), \bar{U} = (\bar{u}_{i-1/2}), T = (t_{i-1/2})\) and \(E = (e_{i-1/2}) = (u_{i-1/2} - \bar{u}_{i-1/2}), i = 1(1)N\), are \(N\) dimensional column vectors.

Also,

\[
A = A_0 - h^4BG, 
\]

where,

\[
A_0 = \begin{bmatrix}
-\frac{5}{3} & \frac{5}{6} & -\frac{1}{6} \\
-\frac{79}{16} & \frac{139}{16} & -\frac{109}{16} & \frac{29}{16} \\
1 & -4 & 6 & -4 & 1 \\
& & & & \\
& & & & \\
1 & -4 & 6 & -4 & 1 \\
\end{bmatrix},
\]

and the matrix \(B\) has the form

\[
B = \begin{bmatrix}
d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\
\alpha & \beta & \gamma & \beta & \alpha \\
\alpha & \beta & \gamma & \beta & \alpha \\
\alpha & \beta & \gamma & \beta & \alpha \\
\alpha & \beta & \gamma & \beta & \alpha \\
\end{bmatrix},
\]

\[
G = diag\left(g_{i-1/2}\right)\) and \(C = [c_1, c_2, c_3, \ldots, c_{N-2}, c_{N-1}, c_N]^T, \)

where \(g_{i-1/2} \neq 0\), for \(\frac{N}{4} < i \leq \frac{3N}{4}\).
We study the variational inequality formulation of the problem (5.1). For this, we define the set \( K \) which is a closed convex set in \( \mathbb{R}^n \) describing the equilibrium configuration of an elastic beam, pulled at the ends and lying over an elastic foundation. Al-Said and Noor [4] considered the linear fourth order boundary value problem describing the equilibrium configuration of an elastic beam, pulled at the ends and lying over an elastic foundation. Now, from Kikuchi and Oden technique [15], we can show that the energy functional associated with the problem

\[
\begin{align*}
    c_i &= \begin{cases} 
        -A_1 + \frac{\alpha}{2} h^2 B_1 + h^4 \int_{j=1}^{5} d_j (f_j - f_{j-1/2}) & \text{if } i = 1, \\
        -A_1 - h^2 B_1 + h^4 \sum_{j=1}^{6} d'_j (f_j - f_{j-1/2}) & \text{if } i = 2, \\
        h^4 (a_i f_i + b_i) + \beta (f_i - f_{i+1}) + \gamma f_i & 3 \leq i \leq N - 2, \\
        h^4 (a_i f_i + b_i) + \beta (f_i - f_{i+1}) + \gamma f_i & i = \frac{N}{4} - 1 \text{ and } i = \frac{3N}{4} + 2, \\
        h^4 (a_i f_i + b_i) + \beta (f_i - f_{i+1}) + \gamma f_i & i = \frac{N}{4} + 1 \text{ and } i = \frac{3N}{4}, \\
        h^4 (a_i f_i + b_i) + \beta (f_i - f_{i+1}) + \gamma f_i & i = \frac{N}{4} + 2 \text{ and } i = \frac{3N}{4} - 1, \\
        h^4 (a_i f_i + b_i) + \beta (f_i - f_{i+1}) + \gamma f_i & i = \frac{N}{4} + 3 \leq i \leq \frac{3N}{4} - 2, \\
        -A_1 - h^2 B_1 + h^4 \sum_{j=1}^{6} d'_j (f_{j+1/2} - f_j) & i = N - 1, \\
        -A_1 + \frac{\alpha}{2} h^2 B_1 + h^4 \int_{j=1}^{5} d_j (f_{j+1/2} - f_j) & i = N, 
    \end{cases}
\end{align*}
\]

where,
\[
r_i = \begin{cases} 
    r, & \frac{N}{4} < i \leq \frac{3N}{4}, \\
    0, & \text{elsewhere.}
\end{cases}
\]

5. Applications

Problem (1) along with the boundary conditions (2) - (3) can be studied under the framework of variational inequalities. Al-Said and Noor [4] considered the linear fourth order boundary value problem describing the equilibrium configuration of an elastic beam, pulled at the ends and lying over an elastic foundation of constant height 1/4 and unit rigidity of the type,

\[
\begin{align*}
    u^{(4)} &\geq f(x), \Omega = [-1, 1], \\
    u &\geq \psi(x), \Omega = [-1, 1], \\
    \{u^{(4)} - f(x)\} &\{u - \psi(x)\} = 0, \Omega = [-1, 1], \\
    u(-1) = u(1) = u''(-1) = u''(1) = 0,
\end{align*}
\]

where, \( f \) is the given force acting on the beam string and \( \psi(x) \) is the obstacle function.

We study the variational inequality formulation of the problem (5.1). For this, we define the set \( K \) as

\[
K = \{v : v \in H^2_0(\Omega) : v \geq \psi \text{ on } \Omega\},
\]

which is a closed convex set in \( H^2_0(\Omega) \), where \( H^2_0(\Omega) \) is a Sobolev space [8, 10, 15], which is basically a Hilbert space. Now, from Kikuchi and Oden technique [15], we can show that the energy functional associated
with the obstacle problem (15) is

\[ I[v] = \int_{-1}^{1} \left( \frac{d^4v}{dx^4} - 2f(x) \right)v(x) \, dx, \quad \text{for all } v \in H^2_0(\Omega), \]

\[ = \int_{-1}^{1} \left( \frac{d^2v}{dx^2} \right)^2 \, dx - 2 \int_{-1}^{1} f(x)v(x) \, dx, \]

\[ = a(v, v) - 2 \langle f, v \rangle, \tag{16} \]

where

\[ a(u, v) = \int_{-1}^{1} \left( \frac{d^2u}{dx^2} \right)^2 \left( \frac{d^2v}{dx^2} \right)^2 \, dx, \tag{17} \]

and

\[ \langle f, v \rangle = \int_{-1}^{1} f(x) v(x) \, dx. \tag{18} \]

It can be proved that the form \( a(u, v) \) defined by (17) is bilinear, positive, and symmetric. Also, the functional \( f \) defined by (18) is linear and continuous. The minimum \( u \) of the functional \( I[v] \) defined by (16) on the closed convex set \( K \) in \( H^2_0(\Omega) \) can be characterized by the variational inequality \([8, 10, 15]\)

\[ a(u, v - u) \geq \langle f, v - u \rangle, \quad \forall v \in K, \tag{19} \]

Hence, the obstacle problem (15) is equivalent to solving the variational inequality problem (19). This equivalence has been used to study the existence of a unique solution of (15); see for example Ref. \([8, 10]\). We can rewrite problem (19) using Lewy and Stampacchia technique \([16]\),

\[ u^{(4)} + \mu(u - \psi)(u - \psi) = f, \quad \text{for } -1 < x < 1, \tag{20} \]

\[ u(-1) = u(1) = 0, \quad u''(-1) = u''(1) = \epsilon, \tag{21} \]

where \( \epsilon \) is a small positive constant and \( \psi \) is the obstacle function and \( \mu(t) \) is the penalty function defined by,

\[ \mu(t) = \begin{cases} 
4, & t \geq 0, \\
0, & t < 0. 
\end{cases} \tag{5.9} \]

Equation (19) describes the equilibrium configuration of an elastic beam, pulled at the ends and lying over an elastic obstacle of constant height \( 1/4 \). Since the obstacle function \( \psi \) is known, it is possible to find the exact solution of the problem in the interval \(-1/2 \leq x \leq 1/2\).

We assume that the obstacle function \( \psi \) is defined by

\[ \psi(x) = \begin{cases} 
-1/4, & -1 \leq x \leq -1/2 \text{ and } 1/2 \leq x \leq 1, \\
1/4, & -1/2 \leq x \leq 1/2. 
\end{cases} \tag{22} \]

From equation (19) - (22), the following system of equations can be obtained as

\[ u^{(4)} = \begin{cases} 
f, & -1 \leq x \leq -1/2 \text{ and } 1/2 \leq x \leq 1, \\
1 - 4u + f, & -1/2 \leq x \leq 1/2. 
\end{cases} \]
with the following boundary conditions
\[ u(-1) = u(-1/2) = u(1/2) = u(1) = 0, \]
\[ u''(-1) = u''(-1/2) = u''(1/2) = u''(1) = 0, \]
and the conditions of continuity of \( u \) and \( u'' \) at \( x = -\frac{1}{2} \) and \( \frac{1}{2} \).

6. Numerical illustrations

We have implemented our method on two examples of the system of fourth order boundary value problems and the maximum absolute error is listed in Tables 1-3. We also compared our results with the existing methods in the references [2-5, 13, 14, 17, 20-22] to illustrate the comparative performance of our method over other existing methods.

A numerical study of the problem in the interval \([-\frac{1}{2}, \frac{1}{2}]\) is given in [18] using Pade’ approximants together with a finite difference technique. Here, we present the numerical study over the whole interval \([-1, 1]\).

Example 6.1. Consider the system of linear fourth order boundary value problem

\[ u^{(4)}(x) = \begin{cases} 1, & -1 \leq x \leq -\frac{1}{2}, \quad \frac{1}{2} \leq x \leq 1, \\ 2 - 4u, & -\frac{1}{2} \leq x \leq \frac{1}{2}. \end{cases} \]

with the boundary conditions
\[ u(-1) = u(-1/2) = u(1/2) = u(1) = 0, \]
\[ u''(-1) = u''(-1/2) = u''(1/2) = u''(1) = 0. \]

The analytical solution of this problem is

\[
u(x) = \begin{cases} \frac{1}{24}x^4 + \frac{1}{8}(x^3 + x^2) + \frac{3}{64}x + \frac{1}{192}, & \text{for } -1 \leq x \leq -\frac{1}{2}, \\ 0.5 - \frac{1}{2}\sin x \sinh x + \beta_2 \cos x \cosh x, & \text{for } -\frac{1}{2} \leq x \leq \frac{1}{2}, \\ \frac{1}{24}x^4 - \frac{1}{8}(x^3 - x^2) - \frac{3}{64}x + \frac{1}{192}, & \text{for } \frac{1}{2} \leq x \leq 1, \\ u(1/2) = u(1) = u''(1/2) = u''(1) = 0, \end{cases}
\]

where,
\[ \beta_1 = \sin(1/2) \sinh(1/2), \quad \beta_2 = \cos(1/2) \cosh(1/2), \quad \beta_3 = \cos(1) + \cosh(1). \]

This was solved by Al-said and Noor [2] using finite difference scheme based on \( h^4u_i = u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2} - \frac{h^2}{8}u_i \) and Khalifa and Noor [13] using collocation method with quintic B-spline as basis function. It is clear from the tables 1-3 that our methods are better than the other existing methods. The results of our methods are better than those of 12th order method of Al-said and Noor [3] based on (0, 8) Pade’ Approximants.
The analytical solution of this problem is

\[ u(x) = \begin{cases} 
-(\frac{3}{2}x^3 + \frac{3}{2}x^2 + \frac{1}{4}x + \frac{1}{2})e, & \text{for } -1 \leq x \leq -\frac{1}{2}, \\
0.25 - \frac{1}{20} \left[ \beta_1 \sin x \sinh x + \beta_2 \cos x \cosh x \right], & \text{for } -\frac{1}{2} \leq x \leq \frac{1}{2}, \\
(\frac{1}{2}x^3 + \frac{3}{2}x^2 - \frac{3}{12}x + \frac{1}{2})e, & \text{for } \frac{1}{2} \leq x \leq 1, \\
u(1) = 0, & \text{for } u''(1) = e, \quad e \to 0.
\end{cases} \]

with the boundary conditions

\[ u(-1) = u(-1/2) = u(1/2) = u(1) = 0, \]
\[ u''(-1) = -u''(-1/2) = u''(1/2) = u''(1) = e, \quad \text{where} \quad e \to 0. \]

The analytical solution of this problem is

Table 1. Observed Maximum absolute errors for Example 1

<table>
<thead>
<tr>
<th>h</th>
<th>1/8</th>
<th>1/16</th>
<th>1/32</th>
<th>1/64</th>
</tr>
</thead>
<tbody>
<tr>
<td>Our Sixth order method</td>
<td>1.04 E - 10</td>
<td>3.88 E - 11</td>
<td>9.82 E - 12</td>
<td>1.35 E - 12</td>
</tr>
<tr>
<td>Our Fourth order method</td>
<td>1.71 E - 08</td>
<td>8.48 E - 10</td>
<td>1.67 E - 10</td>
<td>4.11 E - 11</td>
</tr>
<tr>
<td>Our Second order method</td>
<td>1.35 E - 06</td>
<td>7.69 E - 07</td>
<td>2.87 E - 07</td>
<td>8.53 E - 08</td>
</tr>
<tr>
<td>In [4]</td>
<td>2.53 E - 05</td>
<td>6.38 E - 06</td>
<td>1.66 E - 06</td>
<td>4.29 E - 07</td>
</tr>
<tr>
<td>In [5]</td>
<td>2.4 E - 08</td>
<td>1.5 E - 09</td>
<td>9.5 E - 11</td>
<td>-</td>
</tr>
<tr>
<td>In [17]</td>
<td>1.3 E - 06</td>
<td>8.7 E - 08</td>
<td>6.8 E - 09</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 2. Observed Maximum absolute errors for Example 1

<table>
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<th>1/24</th>
<th>1/48</th>
</tr>
</thead>
<tbody>
<tr>
<td>Our Sixth order method</td>
<td>6.78 E - 11</td>
<td>1.74 E - 11</td>
<td>3.92 E - 12</td>
</tr>
<tr>
<td>Our Fourth order method</td>
<td>2.36 E - 09</td>
<td>3.06 E - 10</td>
<td>7.53 E - 11</td>
</tr>
<tr>
<td>Our Second order method</td>
<td>1.01 E - 06</td>
<td>4.50 E - 07</td>
<td>1.43 E - 07</td>
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<tr>
<td>In [2]</td>
<td>1.83 E - 04</td>
<td>7.17 E - 05</td>
<td>4.59 E - 05</td>
</tr>
<tr>
<td>In [14]</td>
<td>8.42 E - 06</td>
<td>2.16 E - 06</td>
<td>5.40 E - 07</td>
</tr>
<tr>
<td>In [21]</td>
<td>2.19 E - 06</td>
<td>5.47 E - 07</td>
<td>1.48 E - 07</td>
</tr>
<tr>
<td>In [22]</td>
<td>2.19 E - 06</td>
<td>5.47 E - 07</td>
<td>1.48 E - 07</td>
</tr>
<tr>
<td>In [20]</td>
<td>1.16 E - 08</td>
<td>3.31 E - 09</td>
<td>4.55 E - 10</td>
</tr>
</tbody>
</table>

Example 6.2. Consider the system of linear fourth order boundary value problem

\[ u^{(4)}(x) = \begin{cases} 
0, & -1 \leq x \leq -\frac{1}{2}, \quad -\frac{1}{2} \leq x \leq 1, \\
1 - 4u, & -\frac{1}{2} \leq x \leq \frac{1}{2}.
\end{cases} \]
where,

\[ \beta_1 = \sin(1/2) \sinh(1/2), \quad \beta_2 = \cos(1/2) \cosh(1/2), \quad \beta_3 = \cos(1) + \cosh(1). \]

### Table 3. Observed Maximum absolute errors for Example 2 with \( \epsilon = 10^{-16} \)

<table>
<thead>
<tr>
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<th>1/32</th>
<th>1/64</th>
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<td>Our Sixth order method</td>
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<td>3.48 E – 07</td>
<td>1.43 E – 07</td>
<td>4.26 E – 08</td>
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<tr>
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<td>1.4 E – 04</td>
<td>3.6 E – 05</td>
<td>8.9 E – 06</td>
<td>–</td>
</tr>
<tr>
<td>In [3]</td>
<td>1.9 E – 05</td>
<td>4.8 E – 06</td>
<td>1.2 E – 06</td>
<td>–</td>
</tr>
<tr>
<td>In [13]</td>
<td>3.0 E – 04</td>
<td>7.0 E – 05</td>
<td>1.4 E – 05</td>
<td>–</td>
</tr>
<tr>
<td>In [14]</td>
<td>1.89 E – 05</td>
<td>4.78 E – 06</td>
<td>1.18 E – 06</td>
<td>–</td>
</tr>
</tbody>
</table>

### 7. Conclusion

In this paper we study system of fourth order boundary value problem and applied exponential quintic spline method for finding the numerical solution of contact problems in elastostatics, which can be characterized by a sequence of boundary value problems using penalty function technique. Numerical results are obtained for the elastic string in the presence of an elastic obstacle. The results so far obtained are very encouraging and we hope to carry on investigation for other type of unilateral and contact problems arising in science and engineering. The detailed analysis of such methods both analytically and numerically will constitute an interesting subject of future study.

### References