RH-regular Transformations Which Sums a Given Double Sequence

Richard F. Patterson
1 UNF Drive, Department of Mathematics and Statistics, Jacksonville Fl, 32224 USA

Abstract. In 1946 P. Erdos and P. C Rosenbloom presented the following theorem that arose out of discussions they had with R. P. Agnew. Let \{x_n\} be a bounded divergent sequence. Suppose that \{x_n\} is summable by every regular Toeplitz method which sums \{x_n\}. Then \{y_n\} is of the form \{cx_n + d_n\} where \{d_n\} is convergent. The goals of the paper includes the presentation of a multidimensional analog of Erdos and Rosenbloom results in [1].

1. Definitions, Notations and Preliminary Results

Definition 1.1 (Pringsheim, 1900). A double sequence \(x = [x_{k,l}]\) has Pringsheim limit \(L\) (denoted by \(P\)-lim \(x = L\)) provided that given \(\epsilon > 0\) there exists \(N \in \mathbb{N}\) such that \(|x_{k,l} - L| < \epsilon\) whenever \(k, l > N\). Such an \(x\) is describe more briefly as “\(P\)-convergent”.

Definition 1.2 (Patterson, 2000). The double sequence \(y\) is a double subsequence of \(x\) provided that there exist increasing index sequences \(\{n_j\}\) and \(\{k_j\}\) such that if \(x_j = x_{n_j,k_j}\), then \(y\) is formed by

\[
\begin{align*}
x_1 & \quad x_2 & \quad x_5 & \quad x_{10} \\
x_4 & \quad x_3 & \quad x_6 & \quad - \\
x_9 & \quad x_8 & \quad x_7 & \quad - \\
- & \quad - & \quad - & \quad - \\
\end{align*}
\]

In [6] Robison presented the following notion of regular four-dimensional matrix transformation and a Silverman-Toeplitz type characterization of such notion.

Definition 1.3. The four-dimensional matrix \(A\) is said to be RH-regular if it maps every bounded \(P\)-convergent sequence into a \(P\)-convergent sequence with the same \(P\)-limit.

Theorem 1.4. (Hamilton [2], Robison [6]) The four dimensional matrix \(A\) is RH-regular if and only if

\[
\begin{align*}
\text{RH}_1: & \quad P\text{-lim}_{m,n,k,l} a_{m,n,k,l} = 0 \text{ for each } k \text{ and } l; \\
\text{RH}_2: & \quad P\text{-lim}_{m,n} \sum_{k,l=1}^{\infty} a_{m,n,k,l} = 1; \\
\text{RH}_3: & \quad P\text{-lim}_{m,n} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0 \text{ for each } l;
\end{align*}
\]

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Email address: rpatterson@unf.edu (Richard F. Patterson)
2. Main Results

**Theorem 2.1.** Let \( \{x_{k,l}\} \) be a bounded \( P \)-divergent double sequence. Suppose that \( \{y_{k,l}\} \) is \( P \)-summable by every RH-regular summability matrix which sums \( \{x_{k,l}\} \). Then \( \{y_{k,l}\} \) is of the form \( \{c x_{k,l} + a_{k,l}\} \) where \( \{x_{k,l}\} \) is \( P \)-convergent.

**Proof.** Let \( \{x_{m,n}\} \) be a \( P \)-convergent subsequence of \( \{x_{k,l}\} \). Then \( \{x_{m,n}\} \) is summable by

\[
a_{m,n,k,l} \begin{cases} 1, & \text{if } m = m_k \ n = n_l \ \\
0, & \text{otherwise}
\end{cases}
\]

Therefore \( \{y_{m,n}\} \) is also \( P \)-convergent. Let \( \{(m_i' \ n_i') : k, l = 1, 2, 3, \ldots\} \) and \( \{(m''_k \ n''_k) : k, l = 1, 2, 3, \ldots\} \) be index sequences with \( m_i' \neq m_k'' \) and \( n_i' \neq n_k'' \) for all \( (k, l) \) and let

\[
P - \lim_{k,l} x_{m_i',n_i'} = A
\]

and

\[
P - \lim_{k,l} x_{m''_k,n''_k} = B
\]

with \( A \neq B \). The double sequences \( \{y_{m_i',n_i'}\} \) and \( \{y_{m''_k,n''_k}\} \) are also \( P \)-convergent double sequences say to \( \alpha \) and \( \beta \), respectively. Let \( \{x_{m,n}\} \) be any double subsequence of \( \{x_{m,n}\} \) with

\[
P - \lim_{k,l} x_{m,n} = C.
\]

Let \( \lambda \) and \( \rho \) be such that

\[
\lambda + \rho = 1 \text{ and } \lambda A + \rho B = C;
\]

and define \( A \) as follows

\[
da_{m,n,k,l} \begin{cases} \lambda, & \text{if } m = m_k \ n = n_l \ with \ k \ and \ l \ are \ both \ even} \\
\rho, & \text{if } m = m_k \ n = n_l \ with \ k \ and \ l \ are \ both \ odd} \\
1, & \text{if } m = m_k \ n = n_l \ with \ k \ and \ l \ are \ both \ odd} \\
0, & \text{otherwise \ of \ all } m, n, k, \ and \ l\end{cases}
\]

Then \( P - \lim_{k,l} (Ax)_{k,l} = C \). Therefore \( A \) also sums \( \{y_{m,n}\} \) that is

\[
P - \lim_{k,l} y_{m,n} = P - \lim_{k,l} \left( \lambda y_{m',n'} + \rho y_{m'',n''} \right) = \lambda A + \rho A'
\]

where \( \Delta \) and \( \Delta' \) depend only on \( C \). Thus \( P - \lim_{k,l} y_{m,n} \) depend only on \( C \). Infact \( C \) is a linear function.

We now must determine \( R \) and \( T \) such that \( \Delta = RA + T \) and \( \Delta' = RB + T \). Let \( m_k \) and \( n_k \) be positive index sequences and let \( n''_{m_k} \) and \( n''_{n_k} \) be subsequences of \( m_k \) and \( n_k \), respectively, such that

\[
P - \lim_{k,l} x_{m''_{m_k},n''_{n_k}} = C.
\]
Thus by the determination of \( \lambda \) and \( \rho \) above we obtain the following:

\[
\begin{align*}
P - \lim_{k,l} (y_{m_k, n_l} + R x_{m_k, n_l}) &= \lambda \Delta + \rho \Delta' - RC \\
&= \lambda (RA + T) + \rho (RB + T) - RC \\
&= T(\lambda + \rho).
\end{align*}
\]

Therefore every double sequence of \( \{y_{m,n} - Rx_{m,n}\} \) contain a double sequence that is P-converges to \( T \). Therefore

\[
P - \lim_{m,n} (y_{m,n} - Rx_{m,n}) = T.
\]

\( \square \)

The following is clearly a corollary of the above theorem.

**Corollary 2.2.** If \( \{x_{m,n}\} \) and \( \{y_{m,n}\} \) are P-divergent double sequences and \( \{y_{m,n}\} \) is summable by every RH-regular summability matrix which sums \( \{x_{m,n}\} \) then \( \{x_{m,n}\} \) is summable by every RH-regular summability matrix method which sums \( \{y_{m,n}\} \).

By a theorem of Patterson in [4] there are no single four dimensional summability method which has the double sequences of the form \( \{cx_{m,n} + a_{m,n}\} \) as its P-convergence field. Note, however, that the theorem above grant us that this set of double sequences is the common part of the P-convergence field of summability methods that sum \( \{x_{m,n}\} \).

**References**