On some properties of solutions of the \( p \)-harmonic equation

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Abstract. A \( 2p \)-times continuously differentiable complex-valued function \( f = u + iv \) in a simply connected domain \( \Omega \subseteq \mathbb{C} \) is \( p \)-harmonic if \( f \) satisfies the \( p \)-harmonic equation \( \Delta^p f = 0 \). In this paper, we investigate the properties of \( p \)-harmonic mappings in the unit disk \( |z| < 1 \). First, we discuss the convexity, the starlikeness and the region of variability of some classes of \( p \)-harmonic mappings. Then we prove the existence of Landau constant for the class of functions of the form \( Df = zf - z\bar{f} \), where \( f \) is \( p \)-harmonic in \( |z| < 1 \). Also, we discuss the region of variability for certain \( p \)-harmonic mappings. At the end, as a consequence of the earlier results of the authors, we present explicit upper estimates for Bloch norm for bi- and tri-harmonic mappings.

1. Introduction and Preliminaries

A complex-valued function \( f = u + iv \) in a simply connected domain \( \Omega \subseteq \mathbb{C} \) is called \( p \)-harmonic if \( u \) and \( v \) are \( p \)-harmonic in \( \Omega \), i.e. \( f \) satisfies the \( p \)-harmonic equation \( \Delta^p f = 0 \), where

\[
\Delta^p f = \Delta \cdots \Delta f,
\]

where \( p \) is a positive integer and \( \Delta \) represents the Laplacian operator

\[
\Delta := 4 \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.
\]

Throughout this paper we consider \( p \)-harmonic mappings of the unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \). Obviously, when \( p = 1 \) (resp. \( p = 2 \)), \( f \) is harmonic (resp. biharmonic). The properties of harmonic [11, 15] and biharmonic [1–3, 18, 19] mappings have been investigated by many authors. Concerning \( p \)-harmonic mappings, we easily have the following characterization.
Proposition 1.1. A mapping \( f \) is \( p \)-harmonic in \( \mathbb{D} \) if and only if \( f \) has the following representation:

\[
f(z) = \sum_{k=1}^{n} |z|^{2(k-1)} G_{p,k+1}(z),
\]

(1)

where \( G_{p,k+1} \) is harmonic for each \( k \in \{1, \ldots, p\} \).

Proof. We only need to prove the necessity since the proof for the sufficiency part is obvious. Again, as the cases \( p = 1, 2 \) are well-known, it suffices to prove the result for \( p \geq 3 \). We shall prove the proposition by the method of induction. So, we assume that the proposition is true for \( p = n \geq 3 \).

Let \( F \) be an \((n+1)\)-harmonic mapping in \( \mathbb{D} \). By assumption, \( \Delta F \) is \( n \)-harmonic and so can be represented as

\[
\Delta F(z) = \sum_{k=1}^{n} |z|^{2(k-1)} G_{n-k+1}(z),
\]

where \( G_{n-k+1} (1 \leq k \leq n) \) are harmonic functions with

\[
G_{n-k+1}(z) = a_{0,n-k+1} + \sum_{j=1}^{\infty} a_{j,n-k+1} z^j + \sum_{j=1}^{\infty} b_{j,n-k+1} z^j \quad \text{for} \ k \in \{1, \ldots, n\}.
\]

Then

\[
\int_{0}^{\infty} \int_{0}^{\pi} \Delta F \, d\varphi \, dz = \sum_{k=1}^{n} |z|^{2k} T_{p,k+1}(z) + g(z),
\]

where

\[
T_{p,k+1}(z) = \sum_{k=1}^{n} \left( \frac{a_{0,k+1}}{k^2} + \sum_{j=1}^{\infty} a_{j,k+1} z^j + \sum_{j=1}^{\infty} b_{j,k+1} z^j \right)
\]

and \( g \) is a harmonic function in \( \mathbb{D} \). A rearrangement of the series in the sum shows that (1) holds for \( p = n + 1 \). \( \square \)

We remark that the representation (1) continues to hold even if \( f \) is \( p \)-harmonic in a simply connected domain \( \Omega \).

For a sense-preserving \( C^1 \)-mapping (i.e. continuously differentiable), we let

\[
\lambda_f = |f_1| - |f_2| \quad \text{and} \quad \Lambda_f = |f_1| + |f_2|
\]

so that the Jacobian \( J_f \) of \( f \) takes the form

\[
J_f = \lambda_f \Lambda_f = |f_1|^2 - |f_2|^2 > 0.
\]

In [4], the authors obtained sufficient conditions for the univalence of \( C^1 \)-functions. Now we introduce the concepts of starlikeness and convexity of \( C^1 \)-functions.

Definition 1.2. A \( C^1 \)-mapping \( f \) with \( f(0) = 0 \) is called starlike if \( f \) maps \( \mathbb{D} \) univalently onto a domain \( \Omega \) that is starlike with respect to the origin, i.e. for every \( w \in \Omega \) the line segment \([0, w]\) joining \( 0 \) and \( w \) is contained in \( \Omega \). It is known that \( f \) is starlike if it is sense-preserving, \( f(0) = 0 \), \( f(z) \neq 0 \) for all \( z \in \mathbb{D} \setminus \{0\} \) and

\[
\frac{\partial}{\partial t} \left( \arg f(re^{it}) \right) = \text{Re} \left( \frac{Df(z)}{f(z)} \right) > 0 \quad \text{for all} \ z = re^{it} \in \mathbb{D} \setminus \{0\},
\]

where \( Df = zf_z - \bar{z}f_\bar{z} \) (cf. [23, Theorem 1]).
Definition 1.3. Let $f$ and $Df$ belong to $C^1(D)$. Then we say that $f$ is convex in $D$ if it is sense-preserving, $f(0) = 0$, $f(z) \cdot Df(z) \neq 0$ for all $z \in D \setminus \{0\}$ and

$$\text{Re} \left( \frac{D^2f(z)}{Df(z)} \right) > 0 \text{ for all } z \in D \setminus \{0\}.$$

As $\arg Df(re^{i\theta})$ represents the argument of the outer normal to the curve $C_r = \{f(re^{i\theta}) : 0 \leq \theta < 2\pi\}$ at the point $f(re^{i\theta})$, the last condition gives that

$$\frac{\partial}{\partial r} \left( \arg Df(re^{i\theta}) \right) = \text{Re} \left( \frac{D^2f(z)}{Df(z)} \right) > 0 \text{ for all } z = re^{i\theta} \in D \setminus \{0\},$$

showing that the curve $C_r$ is convex for each $r \in (0, 1)$ (see [23, Theorem 2]). Non-analytic starlike and convex functions were studied by Mocanu in [23]. Harmonic starlike and harmonic convex functions were systematically studied by Clunie and Sheil-Small [11], and these two classes of functions have been studied extensively by many authors. See for instance, the book by Duren [15] and the references therein.

The complex differential operator

$$D = z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}}$$

defined by Mocanu [23] on the class of complex-valued $C^1$-functions satisfies the usual product rule:

$$D(af + bg) = aD(f) + bD(g) \text{ and } D(fg) = fD(g) + gD(f),$$

where $a, b$ are complex constants, $f$ and $g$ are $C^1$-functions. The operator $D$ possesses a number of interesting properties. For instance, the operator $D$ preserves both harmonicity and biharmonicity (see also [3]). In the case of $p$-harmonic mappings, we also have the following property of the operator $D$.

Proposition 1.4. $D$ preserves $p$-harmonicity.

Proof. Let $f$ be a $p$-harmonic mapping with the form

$$f(z) = \sum_{k=1}^{p} |z|^{2(k-1)}G_{p-k+1}(z),$$

where each $G_{p-k+1}(z)$ is harmonic in $D$ for $k \in \{1, \ldots, p\}$. As $D(|z|^2) = 0$, the product rule shows that $D(|z|^{2(k-1)}) = 0$ for each $k \in \{1, \ldots, p\}$. In view of this and the fact that $D$ preserves harmonicity gives that

$$D(f(z)) = \sum_{k=1}^{p} |z|^{2(k-1)}D(G_{p-k+1}(z)) + D(|z|^{2(k-1)})G_{p-k+1}(z)$$

$$= \sum_{k=1}^{p} |z|^{2(k-1)}D(G_{p-k+1}(z)).$$

One of the aims of this paper is to generalize the main results of Abdulhadi, et. al. [3] to the case of $p$-harmonic mappings. The corresponding generalizations are Theorems 3.1 and 3.3.

The classical theorem of Landau for bounded analytic functions states that if $f$ is analytic in $D$ with $f(0) = f'(0) - 1 = 0$, and $|f(z)| < M$ for $z \in D$, then $f$ is univalent in the disk $D_\rho := \{z \in \mathbb{C} : |z| < \rho\}$ and in addition, the range $f(D_\rho)$ contains a disk of radius $M\rho^2$ (cf. [20]), where

$$\rho = \frac{1}{M + \sqrt{M^2 - 1}}.$$
Recently, many authors considered Landau’s theorem for planar harmonic mappings (see for example, [6, 8, 9, 13, 16, 22, 28]) and biharmonic mappings (see [1, 7, 8, 21]). In Section 4, we consider Landau’s theorem for \( p \)-harmonic mappings with the form \( D(f) \) when \( f \) belongs to certain classes of \( p \)-harmonic mappings. Our results are Theorems 4.1 and 4.2.

In a series of papers the second author with Yanagihara and Vasudevarao (see [24, 25, 29, 30]) have discussed the regions of variability for certain classes of univalent analytic functions in \( D \). In Section 5 (see Theorem 5.2), we solve a related problem for certain \( p \)-harmonic mappings. Finally, in Section 6, we present explicit upper estimates for Bloch norm for bi- and tri-harmonic mappings (see Corollaries 6.2 and 6.3).

### 2. Lemmas

For the proofs of our main results we require a number of lemmas. We begin to recall the following version of Schwarz lemma due to Heinz ([17, Lemma]) and Colonna [12, Theorem 3], see also [6, 8, 9].

**Lemma 2.1.** Let \( f \) be a harmonic mapping of \( D \) such that \( f(0) = 0 \) and \( f(D) \subset D \). Then

\[
|f(z)| \leq \frac{4}{\pi} \arctan|z| \leq \frac{4}{\pi}|z| \quad \text{for} \quad z \in D
\]

and

\[
\Lambda_f(z) \leq \frac{1}{\pi (1 - |z|^2)} \quad \text{for} \quad z \in D.
\]

**Lemma 2.2.** ([22, Lemma 2.1]) Suppose that \( f(z) = h(z) + g(z) \) is a harmonic mapping of \( D \) with \( h(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) for \( z \in D \). If \( f(0) = 1 \) and \( |f(z)| < M \), then

\[
|a_n|, |b_n| \leq \sqrt{M^2 - 1}, \quad n = 2, 3, \ldots,
\]

\[
|a_n| + |b_n| \leq \sqrt{2M^2 - 2}, \quad n = 2, 3, \ldots
\]

and

\[
\Lambda_f(0) \geq \Lambda_0(M) := \begin{cases} \sqrt{\frac{1}{M^2 - 1} + \frac{1}{M^2 + 1}} & \text{if } 1 \leq M \leq M_0, \\ \frac{\pi}{4M} & \text{if } M > M_0, \end{cases}
\]

where \( M_0 = \frac{n}{2\sqrt{n^2 - 16}} \approx 1.1296 \).

The following lemma concerning coefficient estimates for harmonic mappings is crucial in the proofs of Theorems 3.1 and 3.3. This lemma has been proved by the authors in [10] with an additional assumption that \( f(0) = 0 \). However, for the sake of clarity, we present a slightly different proof than that in [10].

**Lemma 2.3.** Let \( f = h + \overline{g} \) be a harmonic mapping of \( D \) such that \( |f(z)| < M \) with \( h(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \). Then \( |a_0| \leq M \) and for any \( n \geq 1 \)

\[
|a_n| + |b_n| \leq \frac{4M}{\pi},
\]

The estimate (3) is sharp. The extremal functions are \( f(z) \equiv M \) or

\[
f_{\alpha}(z) = \frac{2Ma}{\pi} \arg \left( \frac{1 + \beta z^n}{1 - \beta z^n} \right),
\]

where \( |\alpha| = |\beta| = 1 \).
Proof. Without loss of generality, we assume that $|f(z)| < 1$. For $\theta \in [0, 2\pi)$, let

$$v_\theta(z) = \text{Im} \left( e^{i\theta} f(z) \right)$$

and observe that

$$v_\theta(z) = \text{Im} \left( e^{i\theta} h(z) + e^{-i\theta} g(z) \right) = \text{Im} \left( e^{i\theta} h(z) - e^{-i\theta} g(z) \right).$$

Because $|v_\theta(z)| < 1$, it follows that

$$e^{i\theta} h(z) - e^{-i\theta} g(z) \preceq K(z) = \lambda + \frac{2}{\pi} \log \left( \frac{1 + z \xi}{1 - z} \right),$$

where $\xi = e^{-i\pi} \text{Im}(i)$ and $\lambda = e^{i\theta} h(0) - e^{-i\theta} g(0)$. The superordinate function $K(z)$ maps $\mathbb{D}$ onto a convex domain with $K(0) = \lambda$ and $K'(0) = \frac{2}{\pi} (1 + \xi)$, and therefore, by a theorem of Rogosinski [26, Theorem 2.3] (see also [14, Theorem 6.4]), it follows that

$$|a_n - e^{-2i\theta} b_n| \leq \frac{2}{\pi} |1 + \xi| \leq \frac{4}{\pi}$$

for $n = 1, 2, \ldots$

and the desired inequality (3), with $M = 1$, is a consequence of the arbitrariness of $\theta$ in $[0, 2\pi)$.

For the proof of sharpness part, consider the functions

$$f_n(z) = 2 M \pi \lambda \left( \log \frac{1 + \beta z^n}{1 - \beta z^n} \right), \quad |\alpha| = |\beta| = 1,$$

whose values are confined to a diametral segment of the disk $\mathbb{D}_M$. Also,

$$f_n(z) = 2 M \pi \lambda \left( \sum_{k=1}^{\infty} \frac{1}{2k-1} (\beta z^n)^{2k-1} - \sum_{k=1}^{\infty} \frac{1}{2k-1} (\bar{\beta} z^n)^{2k-1} \right),$$

which gives

$$|a_n| + |b_n| = \frac{4M}{\pi}.$$

The proof of the lemma is complete. \(\square\)

As an immediate consequence of Lemmas 2.2 and 2.3, we have

**Corollary 2.4.** Let $f = h + \bar{g}$ be a harmonic mapping of $\mathbb{D}$ with $h(z) = \sum_{n=1}^{\infty} a_n z^n$, $g(z) = \sum_{n=1}^{\infty} b_n z^n$ and $|f(z)| \leq M$. If $J_f(0) = 1$ and $M \geq \sqrt[2]{\frac{\pi}{4M^2 - 2}}$, then for any $n \geq 2$,

$$|a_n| + |b_n| \leq \frac{2M}{\pi} \leq \sqrt{2M^2 - 2}.$$  

3. The convexity and the starlikeness

The following simple result can be used to generate (harmonic) starlike and convex functions.

**Theorem 3.1.** Let $f$ be a univalent p-harmonic mapping with the form

$$f(z) = G(z) \sum_{k=1}^{p} \lambda_k z^{2(\beta-1)},$$

where $G$ is a locally univalent harmonic mapping and $\lambda_k$ $(k = 1, \ldots, p)$ are complex constants. Then we have the following:
(a) \( \frac{D(f)}{f} = \frac{D(G)}{G} \) and \( \frac{D^2(f)}{D(f)} = \frac{D^2(G)}{D(G)} \).

(b) \( f \) is convex (resp. starlike) if and only if \( G \) is convex (resp. starlike).

**Proof.** (a) The two equalities are immediate consequences of the formula
\[
D(G(z) \sum_{k=1}^{p} \lambda_k |z|^{2(k-1)}) = D(G(z)) \sum_{k=1}^{p} \lambda_k |z|^{2(k-1)}.
\]
So, we omit the details.

(b) It suffices to prove the case of convexity since the proof for the starlikeness is similar.
Let \( z = re^{it} \), where \( 0 < r < 1 \) and \( 0 \leq t < 2\pi \). Then
\[
f(z) = G(z) \sum_{k=1}^{p} \lambda_k |z|^{2(k-1)} = G(re^{it}) \sum_{k=1}^{p} \lambda_k r^{2(k-1)},
\]
so that
\[
\frac{\partial f(re^{it})}{\partial t} = \frac{\partial G(re^{it})}{\partial t} \sum_{k=1}^{p} \lambda_k r^{2(k-1)}
\]
and
\[
\frac{\partial^2 f(re^{it})}{\partial t^2} = \frac{\partial^2 G(re^{it})}{\partial t^2} \sum_{k=1}^{p} \lambda_k r^{2(k-1)}.
\]
Therefore Part (a) yields
\[
\frac{\partial}{\partial t} \left( \arg \frac{\partial f(re^{it})}{\partial t} \right) = \text{Re} \left( \frac{D^2(f)}{D(f)} \right) = \text{Re} \left( \frac{D^2(G)}{D(G)} \right) = \frac{\partial}{\partial t} \left( \arg \frac{\partial G(re^{it})}{\partial t} \right),
\]
from which the proof of Part (b) of this theorem follows. \( \square \)

As an immediate consequence of Theorem 3.1(a), we easily have the following.

**Corollary 3.2.** Let \( f \) be a univalent \( p \)-harmonic mapping defined as in Theorem 3.1. If \( f \) is convex and \( D(f) \) is univalent, then \( D(f) \) is starlike.

Abdulhadi, et. al. [3, Theorem 1] discussed the univalence and the starlikeness of biharmonic mappings in \( D \). A natural question is whether [3, Theorem 1] holds for \( p \)-harmonic mappings. The following result gives a partial answer to this problem.

**Theorem 3.3.** Let \( f \) be a \( p \)-harmonic mapping of \( D \) satisfying \( f(z) = |z|^{2(p-1)}G(z) \), where \( G \) is harmonic, orientation preserving and starlike. Then \( f \) is starlike univalent.

**Proof.** We see that the Jacobian \( J_f \) of \( f \) is
\[
J_f = |f_z|^2 - |f|^2
= |z|^{4(p-1)}(|G_z|^2 - |G|^2) + 2(p-1)|z|^{4p-6}|G|^2 \text{Re} \left( \frac{D(G)}{G} \right)
\geq |z|^{4(p-1)}(|G_z|^2 - |G|^2).
\]
Hence \( J_f(z) > 0 \) when \( 0 < |z| < 1 \) and obviously, \( J_f(0) = 0 \). The univalence of \( f \) follows from a standard argument as in the proof of [3, Theorem 1]. Finally, Theorem 3.1 implies that \( f \) is starlike. \( \square \)
4. The Landau theorem

We now discuss the existence of the Landau constant for two classes of \( p \)-harmonic mappings.

**Theorem 4.1.** Let \( f(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z) \) be a \( p \)-harmonic mapping of \( \mathbb{D} \) satisfying \( \Delta G_{p-k+1}(z) = f(0) = G_p(0) = f_j(0) - 1 = 0 \) and for any \( z \in \mathbb{D} \), \( |G_{p-k+1}(z)| \leq M \), where \( M \geq 1 \). Then there is a constant \( \rho \) (\( 0 < \rho < 1 \)) such that \( D(f) \) is univalent in \( \mathbb{D}_\rho \), where \( \rho \) satisfies the following equation:

\[
\lambda_0(M) - \frac{T(M)}{(1 - \rho)^2} \sum_{k=2}^{p} (2k-1) \rho^{2(k-1)} - \frac{2T(M)\rho^{2k-1}}{(1 - \rho)^3} - \frac{16M}{\pi^2}s_0 \arctan \rho = 0
\]

with

\[
s_0 = \left( \frac{\sqrt{17} - 1}{\sqrt{17} - 3} \right) \sqrt{\frac{2}{5 - \sqrt{17}}} \approx 4.1996,
\]

\[
T(M) = \begin{cases} \sqrt{2M^2 - 2} & \text{if } 1 \leq M \leq M_1 := \frac{\pi}{\sqrt{17} - 8} \approx 2.2976 \\ \frac{4M}{\pi} & \text{if } M > M_1 \end{cases}
\]

and \( \lambda_0(M) \) is given by (2). Moreover, the range \( D(f)(\mathbb{D}_\rho) \) contains a univalent disk \( \mathbb{D}_R \), where

\[
R = \rho \left[ \lambda_0(M) - \frac{T(M)\rho^{2(k-1)}}{(1 - \rho)^2} - \frac{16M}{\pi^2}s_0 \arctan \rho \right].
\]

**Proof.** For each \( k \in \{1, 2, \ldots, p\} \), let

\[
G_{p-k+1}(z) = a_{0,p-k+1} + \sum_{j=1}^{\infty} a_{j,p-k+1}z^j + \sum_{j=1}^{\infty} b_{j,p-k+1} \overline{z}^j,
\]

where \( a_{0,p} = 0 \). We define the function \( H \) as

\[
H = D \left( \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1} \right) = \sum_{k=1}^{p} |z|^{2(k-1)} D(G_{p-k+1}).
\]

Using Lemmas 2.2, 2.3 and Corollary 2.4, we have

\[
|a_{n,p}| + |b_{n,p}| \leq T(M),
\]

where \( T(M) \) is given by (4), and

\[
|a_{j,p-k+1}| + |b_{j,p-k+1}| \leq \frac{4M}{\pi}
\]

for \( j \geq 1 \), \( n \geq 2 \) and \( 2 \leq k \leq p \).

We observe that

\[
f_j(0) = |(G_p)_{j}(0)|^2 - |(G_p)_{j}z(0)|^2 = f_{j_0}(0) = 1
\]

and hence by Lemmas 2.1 and 2.2, we have

\[
\lambda_j(0) \geq \lambda_0(M),
\]

where \( \lambda_0(M) \) is given by (2). Now, we define

\[
q(x) = \frac{2 - x^2}{(1 - x^2)x} \quad (0 < x < 1).
\]
Then there is an $r_0 = \sqrt{\frac{5 - \sqrt{17}}{2}} \approx 0.66$ such that

$$q(r_0) = \min_{0 < x < 1} q(x) = \left(\sqrt{17} - 1\right) \sqrt{\frac{2}{5 - \sqrt{17}}} = s_0.$$

For each $\theta \in [0, 2\pi)$, the function

$$G_\theta(z) = (G_p)_e(z) - (G_p)(0) + ((G_p)_e(z) - (G_p)(0))e^{i(\pi - 2\theta)}$$

is clearly a harmonic mapping of $D$ and satisfies $G_\theta(0) = 0$. Moreover, it follows from Lemma 2.1 that

$$\Lambda_{G_p}(z) \leq \frac{4M}{\pi} \frac{1}{1 - |z|^2}$$

for $z \in D$.

In particular, this observation yields that

$$|G_\theta(z)| \leq \Lambda_{G_p}(z) + \Lambda_{G_p}(0) \leq \frac{4M}{\pi} \left(1 + \frac{1}{1 - |z|^2}\right) = \frac{4M}{\pi} |z|q(|z|)$$

for all $z \in D$.

Since $xq(x) - 1 = \frac{1}{1 - x^2}$ is an increasing function in the interval $(0, 1)$, the inequality (5) shows that for any $z \in D_{r_0}$,

$$|G_\theta(z)| \leq \frac{4M}{\pi} m_0,$$

where $m_0 = (2 - r_0^2)/(1 - r_0^2)$. Next, we consider the mapping $F$ defined on $D$ by

$$F(z) = -\frac{\pi}{4Mm_0} G_\theta(r_0z).$$

Applying Lemma 2.1 to the function $F(z)$ yields that for $z \in D_{r_0}$,

$$|G_\theta(z)| \leq \frac{16M}{\pi^2} m_0 \arctan\left(\frac{|z|}{r_0}\right) \leq \frac{16M}{\pi^2} s_0 \arctan |z|,$$

where $s_0 = m_0/r_0$.

Now, we fix $\rho$ with $\rho \in (0, 1)$. To prove the univalency of $H$, we choose two distinct points $z_1, z_2$ in $D_\rho$. Let $\gamma = \{(z_2 - z_1)t + z_1 : 0 \leq t \leq 1\}$ and $z_2 - z_1 = |z_1 - z_2|e^{i\theta}$. We find that
\[ |H(z_1) - H(z_2)| \]

\[
= \left| \int_{\gamma} H_2(z) \, dz + H_2(z) \, d\bar{z} \right| \\
\geq \left| \int_{\gamma} (G_p)_2(0) \, dz - (G_p)_2(0) \, d\bar{z} \right| \\
- \left| \int_{\gamma} \sum_{k=2}^{p} |z|^{2(k-1)} [z(G_{p-k+1})z^2(z) \, dz - \Xi(G_{p-k+1})z^2(z) \, d\bar{z}] \right| \\
- \left| \int_{\gamma} \sum_{k=2}^{p} (k-1)|z|^{2(k-2)} [z^2(G_{p-k+1})z(z) \, d\bar{z} - \Xi^2(G_{p-k+1})z(z) \, dz] \right| \\
- \left| \int_{\gamma} \sum_{k=2}^{p} k|z|^{2(k-1)} [(G_{p-k+1})z(z) - (G_{p-k+1})z(z) \, d\bar{z}] \right| \\
- \left| \int_{\gamma} [(G_p)_2(z) - (G_p)_2(0)] \, dz - [(G_p)_2(z) - (G_p)_2(0)] \, d\bar{z} \right| \\
\geq |z_1 - z_2| \left[ \lambda_f(0) - |G_p(\rho)| \right. \\
- \sum_{k=1}^{p} \rho^{2(k-1)} \sum_{n=2}^{\infty} n(n-1)(|a_{n,p-k+1}| + |b_{n,p-k+1}|) \rho^{n-1} \\
- \sum_{k=2}^{p} (2k-1)\rho^{2(k-2)} \sum_{n=1}^{\infty} n(|a_{n,p-k+1}| + |b_{n,p-k+1}|) \rho^{n+1} \bigg] \\
> |z_1 - z_2| \left[ \lambda_0(M) - \frac{T(M)}{(1 - \rho)^2} \sum_{k=2}^{p} (2k - 1)\rho^{2(k-1)} \\
- \sum_{k=1}^{p} \frac{2T(M)\rho^{2k-1}}{(1 - \rho)^3} - \frac{16M}{\pi^2 s_0 \arctan \rho} \right].
\]

Let

\[ P(\rho) = \lambda_0(M) - \frac{T(M)}{(1 - \rho)^2} \sum_{k=2}^{p} (2k - 1)\rho^{2(k-1)} - \sum_{k=1}^{p} \frac{2T(M)\rho^{2k-1}}{(1 - \rho)^3} - \frac{16M}{\pi^2 s_0 \arctan \rho}. \]

Then it is easy to verify that \( P(\rho) \) is a decreasing function on the interval \((0, 1)\),

\[ \lim_{\rho \to 0^+} P(\rho) = \lambda_0(M) \quad \text{and} \quad \lim_{\rho \to 1^-} P(\rho) = -\infty. \]

Hence there exists a unique \( \rho_0 \) in \((0, 1)\) satisfying \( P(\rho_0) = 0 \). This observation shows that \( |H(z_1) - H(z_2)| > 0 \) for arbitrary two distinct points \( z_1, z_2 \) in \( |z| < \rho_0 \) which proves the univalency of \( D(F) \) in \( D_{\rho_0} \).
satisfies the following equation:

\[ M \geq [\cdots] \]

for the rest of the values of \( p \).

Theorem 4.1 for cases \( p \) values obtained from [7, Theorem 1.1] for the case \( D \).

Especially, if \( M = 1 \), then \( G(z) = z \), i.e. \( f(z) = |z|^{p-1}z \) which is univalent in \( D \).

\[ \]

<table>
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<th>( M )</th>
<th>( p )</th>
<th>( \rho = \rho(M, p) )</th>
<th>( R = R(M, \rho(M, p)) )</th>
<th>( \rho' )</th>
<th>( R' )</th>
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Table 1: Values of \( \rho \) and \( R \) for Theorem 4.1 for \( p = 2 \), and the corresponding values of \( \rho' \) and \( R' \) of [7, Theorem 1.1] for \( p = 2 \).

For any \( z \) with \( |z| = \rho_0 \), we have

\[
|H(z)| = \left| \sum_{k=1}^{p} [z^{p(k-1)}(G_{p-k+1})z(z) - \bar{z}(G_{p-k+1})\bar{z}(z)] \right|
\]

\[
\geq \left| z(G_p)z(0) - \bar{z}(G_p)\bar{z}(0) \right|
\]

\[
- \left| z(G_p)z(z) - (G_p)z(0) \right| - \left| \bar{z}(G_p)\bar{z}(z) - (G_p)\bar{z}(0) \right|
\]

\[
- \left| \sum_{k=2}^{p} [z^{p(k-1)}(G_{p-k+1})z(z) - \bar{z}(G_{p-k+1})\bar{z}(z)] \right|
\]

\[
\geq \rho_0 \left[ \lambda_0(M) - \sum_{k=2}^{p} T(M)p_0^{2(k-1)} \frac{48M}{\pi^2} s_0 \arctan \rho_0 \right]
\]

\[
= R
\]

and the proof of the theorem is complete. \( \square \)

From Table 1, we see that Theorem 4.1 improves Theorem 1.1 of [7] for the case \( p = 2 \), and the results for the rest of the values of \( p \) are new. In Table 1, third and fourth columns refer to values obtained from Theorem 4.1 for cases \( p = 2, 3, 4 \) for certain choices of \( M \), while the right two columns correspond to the values obtained from [7, Theorem 1.1] for the case \( p = 2 \).

Theorem 4.2. Let \( f(z) = |z|^{p-1}G(z) \) be a \( p \)-harmonic mapping of \( D \) satisfying \( G(0) = f(0) = 1 \) and \( |G(z)| \leq M \), where \( M \geq 1 \) and \( G \) is harmonic. Then there is a constant \( \rho (0 < \rho < 1) \) such that \( D(f) \) is univalent in \( D_\rho \), where \( \rho \) satisfies the following equation:

\[
\lambda_0(M) - \frac{48M}{\pi^2} s_0 \arctan \rho - \frac{2T(M)p_0^{2(p-1)}(1 - \rho^3)}{\rho^3} = 0,
\]

where the constants \( s_0, \lambda_0(M) \) and \( T(M) \) are the same as in Theorem 4.1. Moreover, the range \( D(f)(D_\rho) \) contains a univalent disk \( D_R \), where

\[
R = \rho^{2(p-1)} \left[ \lambda_0(M) - \frac{16M}{\pi^2} s_0 \arctan \rho \right].
\]

Especially, if \( M = 1 \), then \( G(z) = z \), i.e. \( f(z) = |z|^{p-1}z \) which is univalent in \( D \).
Proof. Let \( G(z) = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}_n \). Using Lemmas 2.2, 2.3 and Corollary 2.4, we have
\[
|a_n| + |b_n| \leq T(M) \quad \text{for} \quad n \geq 2.
\]

Note that
\[
I_G(0) = |a_1|^2 - |b_1|^2 = 1
\]
and hence, by Lemmas 2.1 and 2.2, we have
\[
\lambda_G(0) \geq \lambda_0(M).
\]

Next, we set \( H = D(f) = |z|^{2(p-1)}D(G) \) and fix \( \rho \) with \( \rho \in (0,1) \). To prove the univalency of \( f \), we choose two
distinct points \( z_1, z_2 \) in \( D_\rho \). Let \( \gamma = \{ (z_2 - z_1)t + z_1 : 0 \leq t \leq 1 \} \) and \( z_2 - z_1 = |z_1 - z_2|e^{i \theta} \). Then
\[
|H(z_1) - H(z_2)| = \left| \int_{[z_1,z_2]} H_z(z) \, dz + H_{\overline{z}}(z) \, d\overline{z} \right|
\]
\[
= \left| \int_{[z_1,z_2]} p|z|^{2(p-1)}(G_z(z) \, dz - G_{\overline{z}}(z) \, d\overline{z}) \right.
\]
\[
+ \left. |z|^{2(p-1)}(zG_z(z) \, dz - z\overline{G_z(z)} \, d\overline{z}) \right.
\]
\[
+ (p-1)|z|^{2(p-2)}(z^2G_z(z) \, dz - z \overline{G_z(z)} \, d\overline{z}) \right|
\]
\[
\geq \left| \int_{[z_1,z_2]} \left[ G_z(0)(p|z|^{2(p-1)} \, dz + (p-1)|z|^{2(p-2)}z^2 \, d\overline{z}) \right. \right.
\]
\[
- \left. \left. G_{\overline{z}}(0)(p|z|^{2(p-1)} \, d\overline{z} - (p-1)|z|^{2(p-2)}z^2 \, dz) \right| \right.
\]
\[
- \left. \left. p \int_{[z_1,z_2]} |z|^{2(p-1)} \left[ (G_z(z) - G_z(0)) \, dz - (G_{\overline{z}}(z) - G_{\overline{z}}(0)) \, d\overline{z} \right] \right| \right.
\]
\[
- \left. \left. (p-1) \int_{[z_1,z_2]} |z|^{2(p-1)} \left[ \overline{z} \, (G_z(z) - G_z(0)) \, d\overline{z} \right. \right. \right.
\]
\[
- \left. \left. \left. - \frac{\pi}{z_2} (G_z(z) - G_z(0)) \right] \right| \right.
\]
\[
- \int_{[z_1,z_2]} |z|^{2(p-1)}(zG_{\overline{z}}(z) \, dz - z\overline{G_z(z)} \, d\overline{z}) \right|
\]
\[
\geq \left| z_1 - z_2 \right| \left( \int_0^1 |z|^{2(p-1)} \, dt \right) \left[ \lambda_0(M) - \frac{48M}{\pi^2} s_0 \arctan \rho \right.
\]
\[
- \sum_{n=2}^{\infty} n(n-1)(|a_n| + |b_n|) \rho^{n-1} \right]
\]
\[
\geq \left| z_1 - z_2 \right| \left( \int_0^1 |z|^{2(p-1)} \, dt \right) \left[ \lambda_0(M) - \frac{48M}{\pi^2} s_0 \arctan \rho - \frac{2T(M)\rho}{1 - \rho^2} \right].
\]

Since there exists a unique \( \rho \) in (0,1) which satisfies the following equation:
\[
\lambda_0(M) - \frac{48M}{\pi^2} s_0 \arctan \rho - \frac{2T(M)\rho}{(1 - \rho^2)} = 0,
\]
we see that \( H(z_1) \neq H(z_2) \) and so, \( H(z) \) is univalent for \( |z| < \rho_0 \).

Furthermore, we observe that for any \( z \) with \( |z| = \rho_0 \),
\[
|H(z)| = \rho_0^{2(p-1)} |zG_z(0) - z\overline{G_z(0)} + z(G_z(z) - G_z(0)) - \overline{z}(G_{\overline{z}}(z) - G_{\overline{z}}(0))| \geq \rho_0^{2(p-1)} \left[ \lambda_0(M) - \frac{16M}{\pi^2} s_0 \arctan \rho_0 \right] = R.
\]
The proof of the theorem is complete. □

We remark that Theorem 4.2 is an improved version of [7, Theorem 1.2] when \( p = 2 \). In order to be more explicit, we refer to Table 2 in which the third and fourth columns refer to values obtained from Theorem 4.2 for cases \( p = 2, 3 \) for certain choices of \( M \), while the right two columns correspond to the values obtained from [7, Theorem 1.2] for the case \( p = 2 \).

### 5. The Region of Variability

**Definition 5.1.** Let \( \mathcal{H}_p \) denote the set of all \( p \)-harmonic mappings of the unit disk \( \mathbb{D} \) with the normalization \( f_{s-1}(0) = (p - 1)! \) and \( |f(z)| \leq 1 \) for \( |z| < 1 \). Here we prescribe that \( \mathcal{H}_0 = \emptyset \).

For a fixed point \( z_0 \in \mathbb{D} \), let

\[
V_p(z_0) = \{ f(z_0) : f \in \mathcal{H}_p \setminus \mathcal{H}_{p-1} \}.
\]

Now, we have

**Theorem 5.2.** (a) If \( p = 1 \), then \( V_1(z_0) = \{1\} \);

(b) If \( p \geq 2 \), \( V_p(z_0) = \overline{\mathbb{D}} \).

**Proof.** We first prove (a). Let \( f \in \mathcal{H}_1 \) and \( f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \bar{z}^n \). By Parseval’s identity and the hypotheses \( |f(z)| \leq 1 \) and \( f(0) = 1 \), we have

\[
\lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \left( |b_1 r e^{i\theta}|^2 + |g(re^{i\theta})|^2 \right) \, d\theta
\]

\[
= |a_0|^2 + \sum_{n=1}^{\infty} \left( |a_n|^2 + |b_n|^2 \right) \leq 1.
\]

This inequality implies that for any \( n \geq 1 \), \( a_n = b_n = 0 \) which gives that \( f(z) \equiv 1 \) for \( z \in \mathbb{D} \). Thus, we have \( V_1(z_0) = \{1\} \).

In order to prove (b), we consider the function

\[
\phi(z) = \frac{z^{p-1} - w}{1 - w z^{p-1}} = |z|^{2(p-1)} \sum_{n=1}^{\infty} \left( \frac{w^{n+1}}{2} (n-1)(p-1) + z^{p-1} - w - \sum_{n=1}^{\infty} \frac{w^{n+1}}{2} z^{(p-1)n} \right),
\]

where \( w \in \overline{\mathbb{D}} \) and \( p \geq 2 \).

Then \( \phi_{s-1}(0) = (p - 1)! \), \( \Delta^p \phi = 0 \) and therefore, \( \phi \in \mathcal{H}_p \setminus \mathcal{H}_{p-1} \). For each fixed \( a \in \overline{\mathbb{D}} \), \( z \mapsto f_a(z) = (z^{p-1} - a)/(1 - a z^{p-1}) \) is a \( p \)-harmonic mapping and \( f_a(\overline{\mathbb{D}}) \subset \mathbb{D} \).

<table>
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<th>( M )</th>
<th>( p = p(M, p) )</th>
<th>( R = R(M, p(M, p)) )</th>
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Table 2: Values of \( \rho \) and \( R \) for \( p = 2, 3 \), and the corresponding values of \( \rho' \) and \( R' \) of [7, Theorem 1.2] (for \( p = 2 \))
Obviously, \( a \mapsto f_a(z_0) = \frac{z_0^{-1} - a}{1 - az_0} \) is a conformal automorphism of \( D \) and the image of \( \overline{D} \) under \( f_a(z_0) \) is \( \overline{D} \) itself. By hypotheses, we obtain that for any \( g \in \mathcal{H}_p \setminus \mathcal{H}_{p-1} \), \( g(z_0) \in \overline{D} \). Hence \( V_0(z_0) \) coincides with \( \overline{D} \). The proof of this theorem is complete. \( \square \)

By the method of proof used in Theorem 5.2(a), we obtain the following generalization of Cartan’s uniqueness theorem (see [5] or [27, p. 23]) for harmonic mappings.

**Theorem 5.3.** Let \( f \) be a harmonic mapping in \( D \) with \( f(D) \subset D \) and \( f(0) = 1 \). Then \( f(z) = z \) in \( D \).

### 6. Estimates for Bloch norm for bi- and tri-harmonic mappings

In the case of \( p \)-harmonic Bloch mappings, the authors in [10] obtained the following result.

**Theorem 6.1.** Let \( f \) be a \( p \)-harmonic mapping in \( D \) of the form (1) satisfying \( B_f < \infty \), where

\[
B_f := \sup_{z,w \in D, z \neq w} \frac{|f(z) - f(w)|}{\rho(z,w)} < \infty \quad \text{with} \quad \rho(z,w) = \frac{1}{2} \log \left( \frac{1 + |z-w|}{1 - |z-w|} \right).
\]

Then

\[
B_f := \sup_{z \in D} (1 - |z|^2) \left\{ \sum_{k=1}^{p} |z|^{2k-1} |G_{p-k+1}(z)|^2 \right. \\
+ \sum_{k=1}^{p} (k-1)z|z|^{2k-2}G_{p-k+1}(z) \bigg| + \sum_{k=1}^{p} |z|^{2k-1} |G_{p-k+1}(z)| \\
+ \bigg| \sum_{k=1}^{p} (k-1)z|z|^{2k-2}G_{p-k+1}(z) \bigg| \right\} \\
\geq \sup_{z \in D} (1 - |z|^2) \left| \sum_{k=1}^{p} |z|^{2k-1} |G_{p-k+1}(z)| - \sum_{k=1}^{p} |z|^{2k-1} |G_{p-k+1}(z)| \right| \quad (6)
\]

and (6) is sharp. The equality sign in (6) occurs when \( f \) is analytic or anti-analytic.

Furthermore, if for each \( k \in \{1, 2, \ldots, p\} \), the harmonic functions \( G_{p-k+1} \) in (1) are such that \( |G_{p-k+1}(z)| \leq M \), then

\[
B_f \leq 2M\phi_p(y_0). \quad (7)
\]

Here \( y_0 \) is the unique root in \((0, 1)\) of the equation \( \phi'_p(y) = 0 \), where

\[
\phi_p(y) = \frac{2}{\pi} \sum_{k=1}^{p} y^{2k-1} + y(1-y^2) \sum_{k=2}^{p} (k-1)y^{2k-2}. \quad (8)
\]

The bound in (7) is sharp when \( p = 1 \), where \( M \) is a positive constant. The extremal functions are

\[
f(z) = \frac{2M\alpha}{\pi} \log \left( \frac{1 + S(z)}{1 - S(z)} \right),
\]

where \( |\alpha| = 1 \) and \( S(z) \) is a conformal automorphism of \( D \).

In order to emphasize the importance of this result, we recall that, when \( p = 1 \), (6) (resp. (7)) is a generalization of [12, Theorem 1] (resp. [12, Theorem 3]). In the case of \( p = 2 \) of Theorem 6.1, after some computation, one has the following simple formulation for biharmonic mappings.
Corollary 6.2. Let \( f = H + |z|^2 G \) be a biharmonic mapping of \( D \) such that \( B_f < \infty \). Then, we have

\[
B_f \geq \sup_{z \in D} (1 - |z|^2) \left| H_z + |z|^2 G_z \right| - |H_z + |z|^2 G_z|
\]

(9)

and

\[
B_f \leq \frac{4M}{27\pi^3} \left( 8 + 36\pi^2 + (4 + 3\pi^2)^{3/2} \right) \approx 30.7682M.
\]

(10)

Proof. According to our notation, (6) is equivalent to (9). In order to prove (10), we first observe that (7) is equivalent to

\[
B_f \leq 2M \sup_{0 < r < 1} \phi_2(y),
\]

where

\[
\phi_2(y) = \frac{2}{\pi} (1 + y^2) + y(1 - y^2).
\]

Now, to find \( \sup_{0 < r < 1} \phi_2(y) \), we compute the derivative

\[
\phi'_2(y) = 1 + \frac{4}{\pi} y - 3y^2 = -3(y - y_0) \left( y - \frac{2 - \sqrt{4 + 3\pi^2}}{3\pi} \right)
\]

so that \( \phi'_2(y) \geq 0 \) for \( 0 \leq y \leq y_0 \) and \( \phi'_2(y) \leq 0 \) for \( y_0 \leq y < 1 \). Hence

\[
y_0 = \frac{2 + \sqrt{4 + 3\pi^2}}{3\pi} \approx 0.82732
\]

is the critical point of \( \phi_2(y) \). Consequently, \( \phi_2(y) \leq \phi_2(y_0) \). A simple calculation shows that

\[
\begin{align*}
\phi_2(y_0) &= \frac{2}{\pi} (1 + y_0^2) + y_0(1 - y_0^2) \\
&= \frac{2}{\pi} \left( 8 + 12\pi^2 + 4 \sqrt{4 + 3\pi^2} \right) + \frac{2}{3\pi} \sqrt{4 + 3\pi^2} \left( 6\pi^2 - 8 - 4 \sqrt{4 + 3\pi^2} \right) \\
&= \frac{2}{27\pi^3} \left( 16 + 42\pi^2 + 8 \sqrt{4 + 3\pi^2} + \sqrt{4 + 3\pi^2} (3\pi^2 - 4 - 2 \sqrt{3\pi^2 + 4}) \right) \\
&= \frac{2}{27\pi^3} \left( 8 + 36\pi^2 + (4 + 3\pi^2)^{3/2} \right) \approx 15.3841
\end{align*}
\]

and therefore, \( B_f \leq 2M\phi_2(y_0) \) which is the desired inequality (10). The result follows. \( \square \)

In the case of \( p = 3 \) of Theorem 6.1, we have

Corollary 6.3. Let \( f = H + |z|^3 G + |z|^4 K \) be a triharmonic (i.e. 3-harmonic) mapping of the unit disk \( D \) such that \( B_f < \infty \), where \( H, G \) and \( K \) are harmonic in \( D \). Then we have

\[
B_f \geq \sup_{z \in D} (1 - |z|^2) \left| H_z + |z|^3 G_z + |z|^4 K_z \right| - |H_z + |z|^3 G_z + |z|^4 K_z|
\]

(11)

and

\[
B_f \leq 2M\phi_3(y_1) \approx 4.037006M,
\]

(12)

where \( \phi_3(y_1) = \sup_{0 < r < 1} \phi_3(y) \) and

\[
\phi_3(y) = \frac{2}{\pi} (1 + y^2 + y^4) + y(1 + y^2 - 2y^4).
\]
Proof. Set $p = 3$ in Theorem 6.1. Then, (11) is equivalent to (6) and therefore, it suffices to prove (12). The choice $p = 3$ in (7) shows that

$$B_f \leq 2M \sup_{0 < \rho < 1} \phi_3(y),$$

where $\phi_3(y)$ is obtained from (8).

We see that $\phi_3(y)$ has a unique positive root in $(0, 1)$. Also,

$$\phi_3'(y) = 4 \pi (y + 2y^3) + 1 + 3y^2 - 10y^4.$$

Computations show that $\phi_3'(y) \geq 0$ for $0 \leq y \leq y_1$ and $\phi_3'(y) \leq 0$ for $y_1 \leq y < 1$. Hence

$$y_1 \approx 0.891951$$

is the only critical point of $\phi_3(y)$ in the interval $(0, 1)$. It follows that

$$\phi_3(y) \leq \phi_3(y_1) \approx 2.018503.$$

Thus, $B_f \leq 2M\phi_3(y_1)$ which is the desired inequality (12). \hfill \Box

References