A New Note on Almost Increasing and Quasi Monotone Sequences

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Abstract. In [22], we proved a main theorem dealing an application of almost increasing and quasi monotone sequences. In this paper, we prove that theorem under weaker conditions. We also obtained some new and known results.

1. Introduction

A positive sequence \((b_n)\) is said to be an almost increasing sequence if there exists a positive increasing sequence \((c_n)\) and two positive constants \(A\) and \(B\) such that \(Ac_n \leq b_n \leq Bc_n\) (see [1]). A sequence \((d_n)\) is said to be \(δ\)-quasi monotone, if \(d_n \to 0\), \(d_n > 0\) ultimately and \(\Delta d_n \geq -δ_n\), where \(\Delta d_n = d_n - d_{n+1}\) and \(δ=(δ_n)\) is a sequence of positive numbers (see [2]). Let \(\sum a_n\) be a given infinite series with partial sums \((s_n)\). We denote by \(t_n\) the \(nth\) \((C,1)\) mean of the sequence \((a_n)\). A series \(\sum a_n\) is said to be summable \(|C,1|_k\), \(k \geq 1\), if (see [24])

\[
\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty.
\]

Let \((p_n)\) be a sequence of positive numbers such that

\[
P_n = \sum_{i=0}^{n} p_i \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1).
\]

The sequence-to-sequence transformation

\[
R_n = \frac{1}{P_n} \sum_{i=0}^{n} p_i s_i
\]

defines the sequence \((R_n)\) of the Riesz mean or simply the \((N,p_n)\) mean of the sequence \((s_n)\), generated by the sequence of coefficients \((p_n)\) (see [25]). Let \((θ_n)\) be any sequence of positive constants. The series \(\sum a_n\) is said to be summable \(|N,p_n|_k\), \(k \geq 1\), if (see [3])

\[
\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |R_n - R_{n-1}|^k < \infty.
\]

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and it is said to be summable \(|\tilde{N}, p_n, \theta_n|_k\), \(k \geq 1\), if (see [27])

\[
\sum_{n=1}^{\infty} \theta_n^{k-1} |R_n - R_{n-1}|^k < \infty.
\]  

(5)

In the special case \(p_n = 1\) for all values of \(n\), \(|\tilde{N}, p_n|_k\) summability is the same as \(|C, 1|_k\) summability. If we take \(\theta_n = \frac{p_n}{P_n}\), then \(|\tilde{N}, p_n, \theta_n|_k\) summability reduces to \(|\tilde{N}, p_n|_k\) summability. Also, if we take \(\theta_n = n\) and \(p_n = 1\) for all values of \(n\), then we get \(|C, 1|_k\) summability. Furthermore, if we take \(\theta_n = n\), then \(|\tilde{N}, p_n, \theta_n|_k\) summability reduces to \(|R, p_n|_k\) summability (see [4]). Finally, if we take \(p_n = 1\) for all values of \(n\), then we get \(|C, 1, \theta_n|_k\) summability.

2. Known result

Many works dealing with an application of increasing sequences to the some absolute summability methods of infinite series have been done (see [5-23], [26], [29]). Among them, in [22], the following main theorem has been proved.

**Theorem 2.1** Let \((X_n)\) be an almost increasing sequence such that \(|\Delta X_n| = O(X_n/n)\) and let \(\lambda_n \to 0\) as \(n \to \infty\). Suppose that there exists a sequence of numbers \((A_n)\) such that it is \(\delta\)-quasi-monotone with \(\sum n\delta_n X_n < \infty\), \(\sum A_n X_n\) is convergent and \(|\Delta \lambda_n| \leq |A_n|\) for all \(n\). If the conditions

\[
\sum_{n=1}^{m} \frac{1}{n} |\lambda_n| = O(1) \quad \text{as} \quad m \to \infty,
\]

(6)

\[
\sum_{n=1}^{m} \frac{1}{n} |t_n|^k = O(X_m) \quad \text{as} \quad m \to \infty,
\]

(7)

and

\[
\sum_{n=1}^{m} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k |t_n|^k = O(X_m) \quad \text{as} \quad m \to \infty
\]

(8)

are satisfied, then the series \(\sum A_n \lambda_n\) is summable \(|\tilde{N}, p_n, \theta_n|_k\), \(k \geq 1\), where \((\theta_n)\) is any sequence of positive constants such that \(\left(\frac{\theta_n p_n}{P_n}\right)\) is a non-increasing sequence.

3. The main result

The aim of this paper is to prove Theorem 2.1 under weaker conditions. Now we shall prove the following theorem.

**Theorem 3.1** Let \((X_n)\) be an almost increasing sequence such that \(|\Delta X_n| = O(X_n/n)\) and let \(\lambda_n \to 0\) as \(n \to \infty\). Suppose that there exists a sequence of numbers \((A_n)\) such that it is \(\delta\)-quasi-monotone with \(\sum n\delta_n X_n < \infty\), \(\sum A_n X_n\) is convergent and \(|\Delta \lambda_n| \leq |A_n|\) for all \(n\). If the condition (6) of Theorem 2.1 is satisfied and if the conditions

\[
\sum_{n=1}^{m} \frac{|t_n|^k}{n X_n^{k-1}} = O(X_m) \quad \text{as} \quad m \to \infty
\]

(9)

\[
\sum_{n=1}^{m} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{|t_n|^k}{X_n^{k-1}} = O(X_m) \quad \text{as} \quad m \to \infty.
\]

(10)
are satisfied, where \((\theta_n)\) is as in Theorem B, then the series \(\sum_{n=1}^{\infty} a_n \lambda_n\) is summable \(|\lambda_n|, \kappa \geq 1\).

**Remark 3.2** It should be noted that conditions (9) and (10) are the same as conditions (7) and (8), respectively, when \(k=1\). When \(k > 1\), conditions (9) and (10) are weaker than conditions (7) and (8), respectively, but the converses are not true. As in [28] we can show that if (7) is satisfied, then we get that

\[
\sum_{n=1}^{m} \frac{|t_n|^k}{nX_n^{k-1}} = O\left(\frac{1}{X_1^{k-1}}\right) \sum_{n=1}^{m} \frac{|t_n|^k}{n} = O(X_m).
\]

If (9) is satisfied, then for \(k > 1\) we obtain that

\[
\sum_{n=1}^{m} \frac{|t_n|^k}{n} = \sum_{n=1}^{m} \frac{X_n^{k-1}}{nX_n^{k-1}} |t_n|^k = O(X_m^k) \sum_{n=1}^{m} \frac{|t_n|^k}{nX_n^{k-1}} = O(X_m^k) = O(X_m).
\]

The similar argument is also valid for the conditions (8) and (10).

We need following lemmas for the proof of our theorem.

**Lemma 3.4 (5)** Under the conditions of the theorem, we have that

\[
|\lambda_n| X_n = O(1) \quad \text{as} \quad n \to \infty.
\] \hspace{1cm} (11)

**Lemma 3.4 (6)** Let \((X_n)\) be an almost increasing sequence such that \(n | \Delta X_n| = O(X_n)\). If \((A_n)\) is a \(\delta\)-quasi-monotone with \(\sum n \delta_n X_n < \infty\), and \(\sum A_n X_n\) is convergent, then

\[
nA_nX_n = O(1) \quad \text{as} \quad n \to \infty,
\] \hspace{1cm} (12)

\[
\sum_{n=1}^{\infty} nX_n |\Delta A_n| < \infty.
\] \hspace{1cm} (13)

4. **Proof of Theorem 3.1** Let \((T_n)\) be denote the \((N, p_n)\) mean of the series \(\sum a_n \lambda_n\). Then, by definition and changing the order of summation, we have

\[
T_n = \frac{1}{p_n} \sum_{v=0}^{n} p_v \sum_{i=0}^{n} a_i \lambda_i = \frac{1}{p_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v A_v.
\]

Then, for \(n \geq 1\), we have

\[
T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} \lambda_v \frac{v + 1}{v}.
\]

By Abel\’s transformation, we have

\[
T_n - T_{n-1} = \frac{n+1}{nP_n P_{n-1}} p_n t_n \lambda_n - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v t_v \lambda_v + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \Delta \lambda_v t_v \frac{v + 1}{v} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \Delta \lambda_v t_v \frac{v + 1}{v}
\]

\[
= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}.
\]

To complete the proof of the theorem, by Minkowski\’s inequality for \(k > 1\), it is enough to show that

\[
\sum_{n=1}^{\infty} \theta_n^{k-1} |T_{n,r}|^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4.
\]
Firstly, we have that

\[ \sum_{n=1}^{m} \theta_{n}^{k-1} | T_{n,1} |^{k} = \sum_{n=1}^{m} \theta_{n}^{k-1} | \lambda_{n} |^{k-1} | \lambda_{n} | \left( \frac{p_{n}}{P_{n}} \right)^{k} | t_{n} |^{k} \]

\[ = O(1) \sum_{n=1}^{m} | \lambda_{n} | \theta_{n}^{k-1} \left( \frac{1}{X_{0}} \right)^{k-1} \left( \frac{p_{n}}{P_{n}} \right)^{k} | t_{n} |^{k} \]

\[ = O(1) \sum_{n=1}^{m} \Delta | \lambda_{n} | \sum_{v=1}^{n} \theta_{v}^{k-1} \left( \frac{p_{v}}{P_{v}} \right)^{k} | t_{v} |^{k} \]

\[ + O(1) | \lambda_{m} | \sum_{n=1}^{m} \theta_{n}^{k-1} \left( \frac{p_{n}}{P_{n}} \right)^{k} | t_{n} |^{k} \]

\[ = O(1) \sum_{n=1}^{m} | A_{n} | X_{n} + O(1) | \lambda_{m} | X_{m} \]

\[ = O(1) \text{ as } m \to \infty, \]

by virtue of the hypotheses of the theorem and Lemma 3. 3. Now, when \( k > 1 \) applying Hölder’s inequality with indices \( k \) and \( k' \), where \( \frac{1}{k} + \frac{1}{k'} = 1 \), as in \( T_{n,1} \), we have that

\[ \sum_{n=2}^{m+1} \theta_{n}^{k-1} | T_{n,2} |^{k} = O(1) \sum_{n=2}^{m+1} \theta_{n}^{k-1} \left( \frac{p_{n}}{P_{n}} \right)^{k} \left( \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_{v} \right)^{k-1} \]

\[ = O(1) \sum_{v=1}^{m} p_{v} \lambda_{v} |^{k-1} | \lambda_{v} | t_{v} |^{k} \sum_{v=1}^{m} \theta_{v}^{k-1} \left( \frac{p_{v}}{P_{v}} \right)^{k} \left( \frac{1}{X_{0}} \right)^{k-1} \sum_{n=1}^{m+1} p_{n} \]

\[ = O(1) \sum_{v=1}^{m} \theta_{v}^{k-1} \left( \frac{p_{v}}{P_{v}} \right)^{k} | \lambda_{v} | \left( \frac{1}{X_{0}} \right)^{k-1} | t_{v} |^{k} \]

\[ = O(1) \sum_{v=1}^{m} | \lambda_{v} | \theta_{v}^{k-1} \left( \frac{p_{v}}{P_{v}} \right)^{k} | t_{v} |^{k} = O(1) \text{ as } m \to \infty. \]

Again, we have that

\[ \sum_{n=2}^{m+1} \theta_{n}^{k-1} | T_{n,3} |^{k} = O(1) \sum_{n=2}^{m+1} \theta_{n}^{k-1} \left( \frac{p_{n}}{P_{n}} \right)^{k} \left( \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_{v} \right)^{k-1} \]

\[ = O(1) \sum_{v=1}^{m} \theta_{v}^{k-1} \left( \frac{p_{v}}{P_{v}} \right)^{k} \left( \frac{1}{X_{0}} \right)^{k-1} | A_{v} | \sum_{n=1}^{m+1} \theta_{n}^{k-1} p_{n} \left( \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_{v} \right)^{k-1} \]

\[ = O(1) \sum_{v=1}^{m} \theta_{v}^{k-1} t_{v} |^{k} \left( \frac{1}{X_{0}} \right)^{k-1} \left( \frac{1}{X_{0}} \right)^{k-1} | A_{v} | t_{v} |^{k} \]

\[ = (14) \]


$$= O(1) \left( \frac{\theta_1 p_1}{p_1} \right)^{k-1} \sum_{v=1}^{m} v \cdot A_v \cdot \frac{|t_v|^k}{\nu X_v^{k-1}}$$

$$= O(1) \sum_{v=1}^{m-1} \Delta(v \mid A_v) \sum_{i=1}^{v} \frac{|t_i|^k}{X_i^{k-1}} + O(1)m \mid A_m \mid \sum_{v=1}^{m} \frac{|t_v|^k}{\nu X_v^{k-1}}$$

$$= O(1) \sum_{v=1}^{m-1} \frac{\Delta(v \mid A_v)}{X_v} + O(1)m \mid A_m \mid X_m$$

$$= O(1) \sum_{v=1}^{m-1} (v + 1) \mid \Delta A_v \mid - \Delta A_v \mid X_v \mid + O(1)m \mid A_m \mid X_m$$

$$= O(1) \sum_{v=1}^{m-1} v \mid \Delta A_v \mid X_v \mid + O(1) \sum_{v=1}^{m-1} |A_v| X_v \mid + O(1)m \mid A_m \mid X_m$$

$$= O(1) \text{ as } m \to \infty,$$

by virtue of the hypotheses of the theorem and Lemma 3.4. Finally, we have that

$$\sum_{n=2}^{m+1} \theta_n^{-1} \mid T_{n,4} \mid^k \leq \sum_{n=2}^{m+1} \theta_n^{-1} \left( \frac{p_n}{P_n} \right)^{k-1} \sum_{v=1}^{n-1} P_v \mid \lambda_{v+1} \mid^k \cdot t_v \mid^k \cdot \frac{1}{v} \left( \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v \right)^{k-1}$$

$$= O(1) \sum_{v=1}^{m} P_v \mid \lambda_{v+1} \mid^k \cdot t_v \mid^k \cdot \frac{1}{v} \left( \frac{1}{P_{n+1}} \sum_{v=n+1}^{m+1} \theta_n p_n \right)^{k-1}$$

$$= O(1) \sum_{v=1}^{m} P_v \left( \frac{1}{X_v} \right)^{k-1} \mid \lambda_{v+1} \mid^k \cdot t_v \mid^k \cdot \frac{1}{v} \sum_{n=v+1}^{m+1} \left( \frac{\theta_n p_n}{P_n} \right)^{k-1} \frac{P_n}{P_n P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} \left( \frac{\theta_1 p_1}{P_1} \right)^{k-1} \sum_{v=1}^{m} \frac{1}{v} \left( \frac{\theta_1 p_1}{P_1} \right)^{k-1} \mid \lambda_{v+1} \mid^k \cdot t_v \mid^k \cdot \frac{1}{v} \sum_{n=v+1}^{m+1} \left( \frac{\theta_n p_n}{P_n} \right)^{k-1} \frac{P_n}{P_n P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} \Delta \mid \lambda_{v+1} \mid^k \cdot t_v \mid^k \cdot \frac{1}{v} \sum_{n=v+1}^{m+1} \left( \frac{\theta_n p_n}{P_n} \right)^{k-1} \frac{P_n}{P_n P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} \Delta \lambda_v \mid X_v \mid + O(1) \mid \lambda_{m+1} \mid X_{m+1}$$

$$= O(1) \sum_{v=1}^{m} |A_v| X_v \mid + O(1) \mid \lambda_{m+1} \mid X_{m+1}$$

$$= O(1) \text{ as } m \to \infty$$

by virtue of the hypotheses of the theorem and Lemma 3.3. This completes the proof of Theorem 3.1.

If we set $\theta_n = \frac{P_n}{p_n}$, then we obtain the result in [6] under weaker conditions. If we take $p_n = 1$ for all values of $n$ and $\theta_n = n$, then we get a new result concerning the $|C,1|_k$ summability factors of infinite series. Also, if we take $p_n = 1$ for all values of $n$ then we have a new result dealing with the $|C,1|_{\theta_n}$ summability factors of infinite series. Furthermore, if we take $\theta_n = n$, then we have another new result concerning the $|R,p_n|_k$ summability factors of infinite series.
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References