On the Extremal Narumi-Katayama Index of Graphs

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Abstract. The Narumi-Katayama index of a graph \(G\), denoted by \(NK(G)\), is defined as \(\prod_{i=1}^{n} \text{deg}(v_i)\). In this paper, we determine the extremal \(NK(G)\) of trees, unicyclic graphs with given diameter and vertices. Moreover, the second and third minimal \(NK(G)\) of unicyclic graphs with given vertices and the minimal \(NK(G)\) of bicyclic graphs with given vertices are obtained.

1. Introduction

Let \(G\) be a simple graph with the vertex set \(V(G)\) and edge set \(E(G)\). A connected graph \(G\) with \(n\) vertices is a tree (unicyclic or bicyclic graph) if \(|E(G)| = n - 1\) (\(|E(G)| = n\) or \(|E(G)| = n + 1\)). Denote by \(\text{deg}(v_i)\) or \(d_i\) the degree of vertex \(v_i\). The distance between two vertices is defined as the length of a shortest path between them. The diameter of \(G\) is the maximum distance over all pairs of vertices \(u\) and \(v\) of \(G\). In 1984, Narumi and Katayama [1] proposed a definition “simple topological index”:

\[
NK(G) = \prod_{i=1}^{n} \text{deg}(v_i).
\]

On this graph invariant, several works [2,3,4,5,6] are reported and the name “Narumi-Katayama index” is used.

In [6], I. Gutman et al. considered the problem of extremal Narumi-Katayama index and offered a few results filling the gap. For graphs without isolated vertices, I. Gutman et al. [6] presented the minimal, second-minimal and third-minimal (maximal, second-maximal, and third-maximal, resp.) \(NK\)-values and extremal graphs. Moreover, the maximal (second-maximal) Narumi-Katayama index of \(n\)-vertex tree (unicyclic graph) is determined [6]. And the maximal Narumi-Katayama index of \(n\)-vertex bicyclic graphs is given. For connected \(n\)-vertex graphs, the minimal and second minimal Narumi-Katayama index are showed [6]. Consequently, the second-minimal Narumi-Katayama index among \(n\)-vertex trees and the minimal Narumi-Katayama index among \(n\)-vertex unicyclic graphs are presented [6].

In this paper, we determine the extremal \(NK(G)\) of trees, unicyclic graphs with given diameter and vertices. Moreover, the second and third minimal \(NK(G)\) of unicyclic graphs with given vertices and the minimal \(NK(G)\) of bicyclic graphs with given vertices are obtained.

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2. The minimal Narumi-Katayama index of trees and unicyclic graphs with given diameter

**Lemma 2.1** Operation A: For an edge uv of Graph G, let u be a vertex with all adjacency vertices are pendant vertices except a vertex v. If all pendant edges incident with u are grafted to v, then the resulting graph $G^*$ (Fig. 1) satisfies $NK(G) > NK(G^*)$.

![Figure 1: Operation A](image)

**Proof.** By the definition of NK-index,

$$NK(G) - NK(G^*) = \prod_{v \in V(G)} deg(v) \cdot \prod_{v \in V(G) \setminus \{u, v\}} [deg_G(u) \cdot deg_G(v) - 1 \cdot (deg_G(u) + deg_G(v) - 1)]$$

$$= \prod_{v \in V(G) \setminus \{u, v\}} deg(v) \cdot (deg_G(u) - 1) \cdot (deg_G(v) - 1) > 0.$$ 

Hence the result holds. □

**Lemma 2.2** Operation B: Let $G$ be a connected graph. For a cut vertex $v$ of $G$ (we say $v$ is an root of $G$), if $T_1$ is a tree branch of $G$ including $v$ (see Fig. 2), we transform $T_1$ to the star with same order $S_{|T_1|}$ and obtain $G^*$, then $NK(G) \geq NK(G^*)$, with the equality holds if and only if $T_1 \equiv S_{|T_1|}$.

![Figure 2: Operation B](image)

**Proof.** $T_1$ is a tree including vertex $v$. By the definition of NK-index and repeating the operation in Lemma 2.1,

$$NK(G) = \prod_{v \in V(G) \setminus T_1} deg(v) \cdot \prod_{v \in T_1} deg(v) \geq \prod_{v \in V(G) \setminus T_1} deg(v) \cdot \prod_{v \in S_{|T_1|}} deg(v) = NK(G^*).$$

Obviously, the equality holds if and only if $T_1 \equiv S_{|T_1|}$. □

**Lemma 2.3** Operation C: Let $S_{k+1}$ and $S_{l+1}$ be two stars rooted in $u$ and $v$, respectively. If all edges incident to $v$ are grafted to $u$ with $d(u) \geq d(v)$, denoted by the resulting graph $G^*$ (Fig. 3), then $NK(G) > NK(G^*)$.

![Figure 3: Operation C](image)

**Proof.** By the definition of NK-index, then
\[ \text{NK}(G) - \text{NK}(G^*) = \prod_{v \in V(G) \setminus \{u,v\}} \text{deg}(v) \cdot [\text{deg}_G(u) \cdot \text{deg}_G(v) - (\text{deg}_G(u) + l) \cdot (\text{deg}_G(v) - l)] \]
\[ = \prod_{v \in V(G) \setminus \{u,v\}} \text{deg}(v) \cdot l \cdot [\text{deg}_G(u) - \text{deg}_G(v) + l]. \]

Since \( \text{deg}_G(u) \geq \text{deg}_G(v) \) and \( l \geq 1 \), \( \text{NK}(G) - \text{NK}(G^*) > 0 \).

Hence the result follows. \( \square \)

**Theorem 2.4** Let \( T \) be a tree with given diameter \( d \) and \( n \) vertices. Then \( \text{NK}(T) \geq \text{NK}(T_1^*) \), where \( T_1^* \in \mathcal{T}_d^{1*} \) and \( \mathcal{T}_d^{1*} \) (Fig. 4) is the set of trees with given diameter \( d \) and \( S_{n-d} \) rooted in the diametral path excepting the two end vertices.

![Figure 4: A tree \( T_1^* \) in \( \mathcal{T}_d^{1*} \)](image)

**Proof.** For a tree \( T \) with given diameter \( d \), choose a diametral path \( P_{d+1} = v_1 \cdots v_{d+1} \), and replace tree branches rooted in \( v_2, \ldots, v_d \) by stars, then graft two stars to a star. Repeat the operations \( B \) and \( C \). By Lemmas 2.2 and 2.3, the result follows. \( \square \)

**Theorem 2.5** Let \( T \) be a tree with given diameter \( d \), \( n \) vertices and \( T \not\in \mathcal{T}_d^{2*} \). Then \( \text{NK}(T) \geq \text{NK}(T_2^*) \), where \( T_2^* \in \mathcal{T}_d^{2*} \) and \( \mathcal{T}_d^{2*} \) (Fig. 5) is the set of trees with given diameter \( d \) and in the diametral path, \( S_{n-d-1} \) and a vertex are rooted in two different vertices.

![Figure 5: A tree \( T_2^* \) in \( \mathcal{T}_d^{2*} \)](image)

**Proof.** Similar to the proof in Theorem 2.4, by repeating operations in Lemmas 2.2 and 2.3, the \( \text{NK} \)-index of \( T \) is decreasing. Note that \( T \not\in \mathcal{T}_d^{1*} \), in the diametral path \( v_1v_2 \cdots v_{d+1} \), the star \( S_{n-d-1} \) is rooted in a vertex \( v_k \). The only remaining vertex \( u \) is adjacent to \( v_l (i = 2, \ldots, d, i \neq k) \) or one of pendent vertices of \( S_{n-d-1} \)

The resulting graphs are denoted by \( T_2^* \) and \( T_3^* \). By direct calculations, \( \text{NK}(T_2^*) = 2^{d-1} \cdot (n-d) > \text{NK}(T_2^*) = 3 \cdot 2^{d-3} \cdot (n-d) \).

Hence \( \text{NK}(T) \geq \text{NK}(T_2^*) \). \( \square \)

**Lemma 2.6** [7] Let \( G \) be a connected unicyclic graph with at least one pendent vertex, and the diameter of \( G \) be \( D \). If \( d(u, v) = D \), where \( u, v \in V(G) \), then \( u \) or \( v \) should be a pendent vertex.

**Theorem 2.7** Let \( U \not\cong C_v \) be a unicyclic graph with given diameter \( d \) and \( n \) vertices. Then \( \text{NK}(U) \geq \text{NK}(U_1^*) = \text{NK}(U^*) \), where \( U_j^* (j = 2, \ldots, \left\lfloor \frac{d+1}{2} \right\rfloor) \) is a unicyclic graph with diameter \( d \), \( S_{n-d-2} \) and \( C_3 \) rooted in the same vertex of the diametral path \( v_1v_2 \cdots v_{d+1} \) except two end vertices. \( U_2^* \) and \( U^* \) are depicted in Figure 6.

![Figure 6: \( U_2^* \) and \( U^* \) (diameter \( d \))](image)
Proof. By Lemma 2.6, for \( U \neq C_n \), one of endpoints in a diametral path of unicyclic graph is a pendent vertex. There are two cases:

Case 1: A diametral path has an endpoint in the cycle.

For a unicyclic graph \( U \), let a diametral path be \( u_{1}u_{2} \cdots u_{k}v_{1}v_{2} \cdots v_{l} \) (without loss of generation, let \( k \geq l \)), where \( v_{1}, v_{2}, \ldots, v_{l} \) are the vertices in the cycle with \( k+l = d+1 \). \( T_{u_{i}} (i = 2, \ldots, k) \) are the tree branches rooted in \( u_{i} \) with \( \max \{|u_{i}, v_{i}| \mid v_{i} \in V(T_{u_{i}}) \} \leq 1, \ldots, \max \{|u_{i}, v_{i}| \mid v_{i} \in V(T_{u_{i}}) \} \leq d - (k - 1) \).

If \( U_{1} \) is obtained from \( U \) by transforming the branches in Path \( u_{2} \cdots u_{k}v_{1} \) and the cycle to stars, by Lemma 2.2, then \( NK(U) \geq NK(U_{1}) \). If the graph \( U_{1} \) is transformed to \( U_{2} \), where \( U_{2} \) is the unicyclic graph that a star rooted in \( u_{i} \) of Path \( u_{2} \cdots u_{k} \), a star rooted in \( v_{1} \) and a star rooted in \( v_{i} \) of the cycle, by Lemma 2.3, then \( NK(U_{1}) \geq NK(U_{2}) \).

By repeating Operation C in Lemma 2.3, the graph \( U_{3}, U_{4} \) and \( U_{5} \) are obtained.

![Figure 7: U_{i}, i = 2, 3, 4, 5.](image-url)

Note that \( NK(U_{3}) \geq NK(U_{2}) \) and \( NK(U_{3}) \geq NK(U_{4}) = NK(U_{5}) \).

By direct calculation,

\[
NK(U_{3}) - NK(U_{4}) = \prod_{v \in V(U_{3}) \setminus \{u_{1}, v_{1}\}} \deg(v) \cdot [2 \cdot \deg_{U_{3}}(v_{1}) - 3 \cdot (2 + \deg_{U_{3}}(v_{1}) - 3)]
\]

\[
= \prod_{v \in V(U_{3}) \setminus \{u_{1}, v_{1}\}} \deg(v) \cdot (3 - \deg_{U_{3}}(v_{1})) \leq 0.
\]

If \( \deg_{U_{3}}(v_{1}) = 3 \), then \( U_{3} \cong U_{4} \).

If \( \deg_{U_{3}}(v_{1}) > 3 \), then \( NK(U_{3}) \leq NK(U_{4}) \leq NK(U_{2}) \leq NK(U_{1}) \leq NK(U) \).

For even positive integer \( l \), the graph \( U_{1} \) consists of a path with length \( d - \left\lfloor \frac{l}{2} \right\rfloor \), a cycle \( C_{l} \) and a star \( S_{n-d-l+1} \) rooted in the same endpoint of the path.

\[
NK(U_{2}) = 2^{d-2+\frac{l}{2}} \cdot (n - d - \left\lfloor \frac{l}{2} \right\rfloor + 3).
\]

Let \( f(l) = 2^{d-2+\frac{l}{2}}(n - d - \left\lfloor \frac{l}{2} \right\rfloor + 3) \). Then \( f'(l) = \frac{1}{2} \cdot 2^{d-2+\frac{l}{2}}[(n - d - \left\lfloor \frac{l}{2} \right\rfloor + 3)ln2 - 1] > 0 \). \( f(l) \) is an increasing function in \( l \). Then \( f(l) \geq f(4) = 2^{d}(n - d + 1) \) for \( l \geq 4 \), i.e., \( U_{l}^{s'} \) attains the minimal NK-index.

For odd positive integer \( l \), the graph \( U_{1} \) consists of a path with length \( d - \left\lfloor \frac{l-1}{2} \right\rfloor \), a cycle \( C_{l} \) and a star \( S_{n-d-l+1} \) rooted in the same endpoint of the path.

\[
NK(U_{3}) = 2^{d+\frac{l}{2}} \cdot (n - d - \left\lfloor \frac{l}{2} \right\rfloor).
\]

Let \( h(l) = 2^{d+\frac{l}{2}}(n - d - \left\lfloor \frac{l}{2} \right\rfloor) \). Obviously, \( h(l) \) is an increasing function in \( l \). Then \( h(l) \geq h(3) = 2^{d}(n - d + 1) \) for \( l \geq 3 \), i.e., \( U_{l}^{s'} \) attains the minimal NK-index.

Case 2: Two endpoints of each diametral path are pendent vertices.

In order to decrease the NK-value of \( U \), we can transform a unicyclic graph \( U \) to \( U_{6} \), where a star \( S' \) and a unicyclic graph \( U' \) are rooted in \( u \) and \( v \) of a diametrical path. By Lemma 2.2, \( NK(U) \geq NK(U_{6}) \).

By transforming tree branches to stars in \( U' \) and grafting stars to a star, we obtain the graphs \( U_{7} \) and \( U_{8} \).
By Lemmas 2.2 and 2.3, \( NK(U_6) \geq NK(U_7), \ NK(U_8) \geq NK(U_9) \) and

\[
NK(U_7) - NK(U_8) = \prod_{v \in V(U_7)\setminus\{v_k,u\}} \text{deg}(v)(3 \cdot \text{deg}_{U_7}(v_k) - 2 \cdot (\text{deg}_{U_7}(v_k) + 1)) > 0.
\]

Let \( U_6 \) be the unicyclic graph obtained by adding a pendent vertex at \( w \), and identifying \( u \) with \( w \) from \( U_8 \).

Then \( NK(U_6) - NK(U_8) = \prod_{v \in V(U_6)\setminus\{u,w\}} \text{deg}(v)(\text{deg}_{U_6}(u) + 2 - 3 \cdot \text{deg}_{U_6}(u)) < 0. \)

Let \( U_{10} (U_{11}) \) be the unicyclic graph obtained by grafting all pendent edges of vertex \( v \) (\( u \)) to vertex \( u \) (\( v \)) from \( U_9 \). Then

\[
NK(U_{10}) - NK(U_9) = \prod_{v \in V(U_{10})\setminus\{v_k,u\}} \text{deg}(v)(2 \cdot (\text{deg}_{U_9}(v) + \text{deg}_{U_9}(v) - 2) - \text{deg}_{U_9}(v) \cdot \text{deg}_{U_9}(v)) < 0.
\]

Accordingly, \( NK(U_{10}) \leq NK(U_{11}) \).

Hence \( NK(U_{10}) \leq NK(U_6) \leq NK(U_9) \leq NK(U_7) \leq NK(U_8) \leq NK(U) \).

The graph \( U_{10} \) consists of a path with length \( d \), a cycle \( C_l \) with \( l \leq d \) and a star \( S_{n-d-l+1} \) rooted in the same vertex of the path except two end vertices.

And \( NK(U_{10}) = 2^{d-l} \cdot 2^{l-1} \cdot (n - d - l + 4) = 2^{d+l-3} \cdot (n - d - l + 4) \).

Let \( g(l) = 2^{d+l-3}(n - d - l + 4) \). Then \( g'(l) = 2^{d+l-3}[(n - d - l + 4)ln2 - 1] > 0 \). \( g(l) \) is an increasing function in \( l \). Then \( g(l) \geq g(3) = 2^d(n - d + 1) \) for \( l \geq 3 \), i.e., \( U_l \) attains the minimal \( NK \)-index.

By above discussions, \( NK(U) \geq NK(U_l') = NK(U_{l''}) = 2^d(n - d + 1). \quad \square \)

Let \( f(x) = 2^x(n - x + 1) \). Then \( f'(x) = 2^x[ln2 \cdot (n - x + 1) - 1] > 0 \). \( f(x) \) is an increasing function in \( x \). Then \( f(x) \geq f(2) \), i.e., the following corollary holds:

**Corollary 2.8** [6] Among all connected \( n \)-vertex unicyclic graphs, the graph \( Y_n \) (Fig. 11) has minimal Narumi-Katayama index (equal to \( 4(n - 1) \)). This graph is unique.
3. The second and third minimal Narumi-Katayama index of unicyclic graphs

In [6], the minimal Narumi-Katayama index of unicyclic graphs is presented. In this section, we discuss the second and third minimal Narumi-Katayama index of unicyclic graphs.

**Theorem 3.1** Let $U \neq Y_n$. $U'_i$ ($i = 3, 4, 5$) is a unicyclic graph with $n$ vertices and given cycle length $k$, where $U'_i$ ($i = 3, 4, 5$) are depicted in Fig. 12.

Then $NK(U) \geq NK(U'_1) > NK(U'_2) > NK(U'_3)$.

**Proof.** Let $C_k$ be the cycle of unicyclic graph $U$. In order to decrease NK-index, by Lemma 2.2, we can change the tree branches rooted in the cycle $C_k$ to stars. By Operation C of Lemma 2.3, the Narumi-Katayama index is strictly decreasing. Repeated Operations $B$ and $C$, then $U'_3$ is obtained.

Let $U'_3 = U'_3 - uw + vw$, $U'_4 = U'_3 - uw + xw$. $NK(U'_3) - NK(U'_4) = \prod_{v \in V(U'_3) \setminus \{u,v\}} \deg(v)[3 \cdot \deg(u,v) - 2 \cdot (\deg(u,v) + 1)] > 0$. $NK(U'_4) - NK(U'_2) = \prod_{v \in V(U'_3) \setminus \{u,v\}} \deg(v)[3 \cdot 1 - 2 \cdot 2] < 0$.

Then $NK(U'_3) < NK(U'_2) < NK(U'_3) \leq NK(U)$. \(\square\)

**Lemma 3.2** Let $U$ be a unicyclic graph with the cycle $C_k$ and other vertices are pendent vertices. $U'$ is the unicyclic graph obtained by deleting a 2-degree vertex and adding a pendent vertex of $C_k$. Then $NK(U) > NK(U')$.

**Proof.** Let $u$, $v$ be a 2-degree and a vertex of $C_k$.

$NK(U) - NK(U') = \prod_{v \in V(U) \setminus \{u,v\}} \deg(v) \cdot [2 \cdot \deg(u,v) - 1 \cdot (\deg(u,v) + 1)] > 0$.

Hence $NK(U) > NK(U')$. \(\square\)

By Theorem 3.1 and Lemma 3.2, the following result holds:

**Theorem 3.3** Let $U \neq Y_n$. $W_n$ and $M_n$ are the unicyclic graphs $U'_5$ and $U'_3$ in the case $k = 3$. Then $NK(U) > NK(W_n) > NK(M_n)$. 
4. The minimal Narumi-Katayama index of bicyclic graphs

Bicyclic graphs are divided into three types:

![Figure 13: I, II and III-type bicyclic graphs](image)

**Lemma 4.1** Let $B$ be a 1-type bicyclic graph with the cycles $C_p$, $C_q$ and $n$ vertices. Then $NK(B) \geq NK(B'_1)$ ($B'_1$ is depicted in Fig. 14), where $C_p$ and $C_q$ have a common vertex $u$, and the other vertices are pendant vertices attached in $u$.

**Proof.** For a bicyclic graph $B$ with the cycles $C_p$ and $C_q$, the other vertices consist of some tree branches rooted in $C_p$, $C_q$ and vertex $u$. By Lemma 2.2, if these tree branches are transformed into stars, then Narumi-Katayama index is decreasing. Then we can obtain $B'_1$. And $NK(B) \geq NK(B'_1)$.

![Figure 14: $B'_i$ (i = 1, 2, 3, 4)](image)

Let $B'_2$ be the graph obtained by grafting all pendant edges incident with $w$ to $v$ from $B'_1$. Then

$$NK(B'_1) - NK(B'_2) = \prod_{v \in V(B'_1) \setminus \{u, w\}} \text{deg}(v) \cdot [\text{deg}_{B'_1}(v) \cdot \text{deg}_{B'_1}(w) - 2 \cdot (\text{deg}_{B'_1}(v) + \text{deg}_{B'_1}(w) - 2)] > 0$$

and $\text{deg}_{B'_1}(w) > 2$. If $\text{deg}_{B'_1}(v) = 2$ or $\text{deg}_{B'_1}(w) = 2$, then $B'_1 \equiv B'_2$. Let $B'_3$ ($B'_4$) be the graph obtained by grafting all pendant edges incident with $u$ ($v$) to $v$ ($u$) from $B'_2$.

$$NK(B'_2) - NK(B'_3) = \prod_{v \in V(B'_2) \setminus \{u, w\}} \text{deg}(v) \cdot [\text{deg}_{B'_2}(u) \cdot \text{deg}_{B'_2}(v) - 2 \cdot (\text{deg}_{B'_2}(v) + \text{deg}_{B'_2}(u) - 2)]$$

and $\text{deg}_{B'_2}(w) > 2$. If $\text{deg}_{B'_2}(v) = 2$, then $B'_2 \equiv B'_3$. If $\text{deg}_{B'_2}(v) > 2$, then $NK(B'_2) > NK(B'_3)$.

$$NK(B'_3) - NK(B'_4) = \prod_{v \in V(B'_3) \setminus \{u, v\}} \text{deg}(v) \cdot [\text{deg}_{B'_3}(v) - 2 \cdot (\text{deg}_{B'_3}(v) - 2)]$$

If $\text{deg}_{B'_3}(v) = 2$, then $B'_3 \equiv B'_4$. If $\text{deg}_{B'_3}(v) > 2$, then $NK(B'_3) > NK(B'_4)$.

Combining above discussions, we have:

1. If $\text{deg}_{B'_1}(v) = 2$, then $B'_1 \equiv B'_2 \equiv B'_3$.
2. If $\text{deg}_{B'_2}(v) > 2$, then $NK(B'_2) > NK(B'_3)$ and $NK(B'_3) > NK(B'_4)$.

Hence $NK(B) \geq NK(B'_1)$.

**Lemma 4.2** For a bicyclic graph $B'_n$, the minimal Narumi-Katayama index is attained when there are $n - 5$ pendant vertices, denoted by $B'_n(3, 3, n - 5)$. 

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Lemma 4.3 Let $B$ be a II-type bicyclic graph. Then $NK(B) \geq NK(B_3^*)$, where $B_3^*$ is depicted in Figure 15.

![Figure 15: $B_3^{1*}$ and $B_3^*$](image)

Proof. Let $B$ be a II-type bicyclic graph. In order to decrease $NK(B)$, by repeating operations in Lemmas 2.2 and 2.3, we can obtain the bicyclic graph $B_3^{1*}$ and $NK(B) \geq NK(B_3^{1*})$.

Let $B_3^{1*}$ ($B_3^*$) be the graph obtained by grafting all pendant vertices of vertex $u$ ($v$) from $B_3^{1*}$. By Lemma 2.2, $NK(B_3^{1*}) \geq NK(B_3^{1*})$ and $NK(B_3^{1*}) \geq NK(B_3^*)$.

$NK(B_3^*) - NK(B_3^{1*}) = \prod_{v \in V(B_3^{1*}) \cup \{u,v\}} \text{deg}(v) \cdot \left[2 \cdot (\text{deg}_{B_3^*}(u) + \text{deg}_{B_3^*}(v) - 2) - 3 \cdot (\text{deg}_{B_3^*}(v) + \text{deg}_{B_3^*}(u) - 3)\right]$.

If $\text{deg}_{B_3^*}(u) = 3$ and $\text{deg}_{B_3^*}(v) = 2$, then $B_3^{2*} \equiv B_3^*$.

Otherwise, $NK(B_3^*) \leq NK(B_3^{2*}) \leq NK(B_3^{1*}) \leq NK(B)$.

Lemma 4.4 For a bicyclic graph $B_5$, the minimal Narumi-Katayama index is attained when there are $n - 4$ pendant vertices, denoted by $B_5^*(n - 4)$.

Proof. Suppose there are $k$ 2-degree vertices in $B_5^*$. Then $NK(B_5^*) = (n - k + 1) \cdot 2^k \cdot 3$. Let $f(k) = 3 \cdot 2^k(n - k + 1)$. Since $f'(k) = 3 \cdot 2^k[(n - k + 1) \ln 2 - 1] > 0$, $f(k)$ is an increasing function in $k$. Then $f(k) \geq f(2)$ for $k \geq 2$, i.e., when $B_5^* \equiv B_5^*(n - 4)$, $NK(B_5^*(n - 4))$ attains the minimal value.

Lemma 4.5 Let $B$ be a III-type bicyclic graph with $n$ vertices. Then $NK(B) \geq NK(B_6^*)$, where $B_6^*$ is depicted in Figure 16.

![Figure 16: $B_6^{1*}$, $B_6^{4*}$ and $B_6^*$](image)

Proof. For a III-type bicyclic graph $B$, similar to the proof of Lemma 4.3, and repeating the operations in Lemmas 2.2 and 2.3, we can obtain the bicyclic graph $B_6^{1*}$ with $NK(B) \geq NK(B_6^{1*})$.

Let $B_6^{1*}$ ($B_6^{4*}$) be the graph obtained by grafting all pendant vertices of vertex $u$ ($v$) from $B_6^{1*}$.

Since $\text{deg}_{B_6^{1*}}(u) \geq 3$ and $\text{deg}_{B_6^{1*}}(v) \geq 3$,

$NK(B_6^{1*}) - NK(B_6^{4*}) = \prod_{v \in V(B_6^{1*}) \cup \{u,v\}} \text{deg}(v) \cdot \left[\text{deg}_{B_6^{1*}}(u)\text{deg}_{B_6^{1*}}(v) - 3 \cdot (\text{deg}_{B_6^{1*}}(v) + \text{deg}_{B_6^{1*}}(u) - 3)\right] \geq 0$;
\[ \text{NK}(B_6^{3^*}) - \text{NK}(B_6^{3^*}) = \prod_{v \in V(B_6^{3^*})[u,v]} \text{deg}(v) \cdot [\text{deg}_{B_6^k}(v) \cdot 3 - 2 \cdot (\text{deg}_{B_6^k}(v) - 2 + 3)] \geq 0. \]

Then \( \text{NK}(B_6^{1^*}) \geq \text{NK}(B_6^{2^*}) \geq \text{NK}(B_6^{3^*}). \)

Similarly, by grafting all pendent vertices of vertex \( y \) to \( x \) from \( B_6^{3^*} \), we obtain the graph \( B_6^{4^*} \) and \( \text{NK}(B_6^{4^*}) \geq \text{NK}(B_6^{5^*}). \)

Let \( B_6^{5^*} \) be the graph obtained by grafting all pendent vertices of vertex \( x \) to \( u \) from \( B_6^{4^*} \).

\[ \text{NK}(B_6^{4^*}) - \text{NK}(B_6^{5^*}) = \prod_{v \in V(B_6^{5^*})[u,v]} \text{deg}(v) \cdot [\text{deg}_{B_6^k}(u)\text{deg}_{B_6^k}(x) - 3 \cdot (\text{deg}_{B_6^k}(u) + \text{deg}_{B_6^k}(x) - 3)] \geq 0. \]

Then \( \text{NK}(B_6^{5^*}) \geq \text{NK}(B_6^{6^*}). \)

Let \( B_6^{6^*} \) be the graph obtained by grafting all pendent vertices of vertex \( u \) to \( w \) from \( B_6^{5^*} \).

\[ \text{NK}(B_6^{5^*}) - \text{NK}(B_6^{6^*}) = \prod_{v \in V(B_6^{6^*})[u,v]} \text{deg}(v) \cdot [\text{deg}_{B_6^k}(u)\text{deg}_{B_6^k}(w) - 3 \cdot (\text{deg}_{B_6^k}(u) + \text{deg}_{B_6^k}(w) - 3)] \geq 0; \]

\[ \text{NK}(B_6^{6^*}) - \text{NK}(B_6^*) = \prod_{v \in V(B_6^*)[u,v]} \text{deg}(v) \cdot [3 \cdot \text{deg}_{B_6^k}(w) - 2 \cdot (\text{deg}_{B_6^k}(w) - 2 + 3)] \geq 0. \]

Then \( \text{NK}(B_6^{6^*}) \geq \text{NK}(B_6^{k^*}) \geq \text{NK}(B_6^*). \)

Hence \( \text{NK}(B) \geq \text{NK}(B_6^{1^*}) \geq \text{NK}(B_6^{2^*}) \geq \text{NK}(B_6^{3^*}) \geq \text{NK}(B_6^{4^*}) \geq \text{NK}(B_6^{5^*}) \geq \text{NK}(B_6^{k^*}) \geq \text{NK}(B_6^*). \)

**Lemma 4.6** For a bicyclic graph \( B_6^* \), the minimal Narumi-Katayama index is attained when \( p = 3, q = 3 \) and other vertices are pendent vertices, denoted by \( B_6^*(3,3,n-6) \).

**Proof.** Suppose there are \( k \) 2-degree vertices in \( B_6^* \). Then \( \text{NK}(B_6^*) = (n-k+1) \cdot 2^k \cdot 3 \). By the proof of Lemma 4.4, \( \text{NK}(B_6^*) \) is increasing in \( k \). For \( k \geq 4 \), i.e., when \( p = 3, q = 3 \) and \( n-6 \) vertices are pendent vertices, i.e., \( \text{NK}(B_6^*(3,3,n-6)) \) attains the minimal value. \( \square \)

**Theorem 4.7** Let \( B \) be a bicyclic graph with \( n \) vertices. Then \( \text{NK}(B) = \text{NK}(B_6^*(n-4)). \) The equality holds if and only if \( B \equiv B_6^*(n-4) \).

**Proof.** For a bicyclic graph \( B, B \) belongs to one of three types of bicyclic graphs. By Lemmas 4.1-4.6, \( B \) attains the minimum NK-value in \( B_6^*(3,3,n-5), B_6^*(n-4) \) or \( B_6^*(3,3,n-6) \). By direct calculations, \( \text{NK}(B_6^*(3,3,n-5)) = 2^4 \cdot (n-1), \text{NK}(B_6^*(n-4)) = 2^2 \cdot 3 \cdot (n-1), \) and \( \text{NK}(B_6^*(3,3,n-6)) = 3 \cdot 2^4 \cdot (n-3). \)

Then \( \text{NK}(B_6^*(3,3,n-5)) = \text{NK}(B_6^*(n-4)) \) and \( \text{NK}(B_6^*(3,3,n-6)) = \text{NK}(B_6^*(n-4)). \)

Then \( \text{NK}(B) = \text{NK}(B_6^*(n-4)) \) if \( B \equiv B_6^*(n-4) \).

Hence \( \text{NK}(B) \geq \text{NK}(B_6^*(n-4)). \) \( \square \)

**References**


