On Equitorsion Concircular Tensors of Generalized Riemannian Spaces

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Abstract. In this paper we consider concircular vector fields of manifolds with non-symmetric metric tensor. The subject of our paper is an equitorsion concircular mapping. A mapping $f : \mathbf{GR}_N \rightarrow \mathbf{GR}_N$ is an equitorsion if the torsion tensors of the spaces $\mathbf{GR}_N$ and $\mathbf{GR}_N$ are equal.

For an equitorsion concircular mapping of two generalized Riemannian spaces $\mathbf{GR}_N$ and $\mathbf{GR}_N$, we obtain some invariant curvature tensors of this mapping $Z_{\theta}$, $\theta = 1, 2, \ldots, 5$, given by equations (3.14, 3.21, 3.28, 3.31, 3.38). These quantities are generalizations of the concircular tensor $Z$ given by equation (2.5).

1. Introduction

The use of non-symmetric basic tensors and non-symmetric connection became especially actual after appearance of the works of A. Einstein \cite{2}-\cite{4} related to the Unified Field Theory (UFT). Remark that in the UFT the symmetric part $g_{ij}$ of the basic tensor $g_{ij}$ is related to gravitation, and antisymmetric one $\bar{g}_{ij}$ to electromagnetism.

A generalized Riemannian space $\mathbf{GR}_N$ in the sense of Eisenhart’s definition \cite{5} is a differentiable $N$-dimensional manifold, equipped with non-symmetric basic tensor $g_{ij}$.

Let us consider two $N$-dimensional generalized Riemannian spaces $\mathbf{GR}_N$ and $\mathbf{GR}_N$ with basic tensors $g_{ij}$ and $\bar{g}_{ij}$ respectively. Generalized Christoffel symbols of the first kind of the spaces $\mathbf{GR}_N$ and $\mathbf{GR}_N$ are given by

\begin{equation}
\Gamma_{i,jk} = \frac{1}{2}(g_{jk,i} - g_{ik,j} + g_{ik,j}), \quad \Gamma_{i,jk} = \frac{1}{2}((\bar{g}_{jk,i} - \bar{g}_{ik,j} + \bar{g}_{ik,j}))\end{equation}

where, for example, $g_{ijk} = \partial g_{ij}/\partial x^k$. Connection coefficients of these spaces are generalized Christoffel symbols of the second kind $\tilde{\Gamma}_{jk} = g^{il} \Gamma_{p,jk}$ and $\tilde{\Gamma}_{jk} = \bar{g}^{il} \Gamma_{p,jk}$ respectively, where $(g^{ij}) = (g_{ij})^{-1}$ and $ij$ denotes symmetrization with division of the indices $i$ and $j$. Generally the generalized Christoffel symbols

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are not symmetric, i.e. $\Gamma_{jk}^i \neq \Gamma_{kj}^i$. We suppose that $g = \det(g_{ij}) \neq 0$, $\overline{g} = \det(\overline{g}_{ij}) \neq 0$, $\psi = \det(\psi_{ij}) \neq 0$, $\overline{\psi} = \det(\overline{\psi}_{ij}) \neq 0$.

A diffeomorphism $f : \text{GR}_N \to \overline{\text{GR}}_N$ is a conformal mapping if for the basic tensors $g_{ij}$ and $\overline{g}_{ij}$ of these spaces the condition
$$\overline{g}_{ij} = e^{2\psi} g_{ij}$$
is satisfied, where $\psi$ is an arbitrary function of $x$, and the spaces are considered in the common system of local coordinates $x^\ell$.

In this case for the Christoffel symbols of the first kind of the spaces $\text{GR}_N$ and $\overline{\text{GR}}_N$ the relation
$$\overline{\Gamma}_{i,j,k} = e^{2\psi}(\Gamma_{i,j,k} + g_{ij}\psi_k - g_{jk}\psi_i + g_{ik}\psi_j)$$
is satisfied and for the Christoffel symbols of the second kind we have
$$\overline{\Gamma}^{i}_{jk} = \Gamma^{i}_{jk} + g^{ip}(g_{jp}\psi_k - g_{jk}\psi_p + g_{pk}\psi_j),$$
where $\psi_{jk} = \partial \psi / \partial x^k$. Let us denote $\psi_{hk} = \psi_{hk}$ and $\psi_{ij} = g^{ip}\psi_{pj}$. Now, from (1.4) we have
$$\overline{\Gamma}^{i}_{jk} = \Gamma^{i}_{jk} + g^{ip}(g_{jp}\psi_k - g_{jk}\psi_p + g_{pk}\psi_j) + g^{ip}(g_{jp}\psi_k - g_{jk}\psi_p + g_{pk}\psi_j),$$
i.e.
$$\overline{\Gamma}^{i}_{jk} = \Gamma^{i}_{jk} + \delta^{i}_{j} \psi_{k} + \delta^{i}_{k} \psi_{j} - \psi^{l}g_{lp} + \xi^{i}_{jk},$$
where
$$\xi^{i}_{jk} = g^{ip}(g_{jp}\psi_k - g_{jk}\psi_p + g_{pk}\psi_j) = -\xi^{i}_{kj}, \quad \psi^{i} = \frac{1}{N}(\overline{\Gamma}^{i}_{pp} - \Gamma^{i}_{pp}).$$
and $ij$ denotes an antisymmetrisation with division. In the corresponding points $M(x)$ and $\overline{M}(x)$ of a conformal mapping we can put
$$\overline{\Gamma}^{i}_{jk} = \Gamma^{i}_{jk} + P_{jk}^{i} \quad (i, j, k = 1, \ldots, N),$$
where $P_{jk}^{i}$ is the deformation tensor of the connection $\Gamma$ of $\text{GR}_N$ according to the conformal mapping $f : \text{GR}_N \to \overline{\text{GR}}_N$.

Notice that in $\text{GR}_N$ we have
$$\overline{\Gamma}^{p}_{p} = 0, \quad (\text{eq. (2.10) in [14]}).$$

Based on the non-symmetry of the connection in a generalized Riemannian space one can define four kinds of covariant derivatives. For example, for a tensor $a_{ij}^{d}$ in $\text{GR}_N$ we have
$$a^{d}_{ij,m} = a^{d}_{jm} + \Gamma^{p}_{jm} a^{d}_{ip} - \Gamma^{p}_{jm} a^{d}_{ip}, \quad a^{d}_{ij,m} = a^{d}_{jm} + \Gamma^{p}_{jm} a^{d}_{ip} - \Gamma^{p}_{jm} a^{d}_{ip},$$
$$a^{d}_{ij,m} = a^{d}_{jm} + \Gamma^{p}_{jm} a^{d}_{ip} - \Gamma^{p}_{jm} a^{d}_{ip}, \quad a^{d}_{ij,m} = a^{d}_{jm} + \Gamma^{p}_{jm} a^{d}_{ip} - \Gamma^{p}_{jm} a^{d}_{ip}.$$
In the case of the space $\mathcal{GR}_N$ we have five independent curvature tensors [24]:

\[
\begin{align*}
K^i_{\frac{1}{2}jmn} &= \Gamma^i_{jmn} - \Gamma^i_{jm,n} + \Gamma^j_{mni} - \Gamma^j_{mni} + \Gamma^j_{mni} - \Gamma^j_{mni}, \\
K^i_{\frac{3}{2}jmn} &= \frac{1}{2}(\Gamma^i_{jmn} - \Gamma^i_{jm,n} + \Gamma^m_{jni} - \Gamma^m_{jni}), \\
K^i_{\frac{4}{2}jmn} &= \frac{1}{2}(\Gamma^i_{jmn} - \Gamma^i_{jm,n} + \Gamma^m_{jni} - \Gamma^m_{jni}), \\
K^i_{\frac{5}{2}jmn} &= \frac{1}{2}(\Gamma^i_{jmn} - \Gamma^i_{jm,n} + \Gamma^m_{jni} - \Gamma^m_{jni}).
\end{align*}
\]

We use the conformal mapping $f : \mathcal{GR}_N \to \mathcal{GR}_{\bar{N}}$ to obtain the tensors $\overline{K}^i_{\theta jmn}$ ($\theta = 1, ..., 5$), where for example

\[
\overline{K}^i_{\theta jmn} = \bar{\Gamma}^i_{jmn} - \bar{\Gamma}^i_{jm,n} + \bar{\Gamma}^m_{jni} - \bar{\Gamma}^m_{jni}.
\]  

(1.9)

2. Concircular vector field

In 1940. K. Yano [23] considered the conformal mapping $\bar{g}_{ij} = \psi^2 g_{ij}$ of two Riemannian spaces. In this case, he proved that geodesics are invariant under this mapping if and only if

\[
\psi^2 g_{ij} = \omega g_{ij},
\]

(2.1)

where $;\psi$ is a covariant derivative, $g_{ij}$ a symmetric metric tensor, $\omega$ an invariant and $\psi_i$ is a gradient vector.

When N. S. Sinyukov studied geodesic mappings of symmetric spaces [18], he wrote this condition in terms of $\xi = e^{-\psi}$. It is easy to see that the formula (2.1) transforms to

\[
\xi_{ij} = \rho g_{ij},
\]

(2.2)

where $\rho = -\omega e^{-\psi}$, $\xi_{ij} = \xi_i$. The vector field $\xi_i$, was called concircular vector field by K. Yano [23]. In the case when $\rho = \text{const.}$, $\xi$ is called convergent, and in the case $\rho = B\xi + C$, $(B, C = \text{const.})$, $\xi$ is called special concircular. A space with concircular vector field was called equidistant space by N.S. Sinyukov.

**Definition 2.1.** [1] A generalized Riemannian space $\mathcal{GR}_N$ with a non-symmetric metric tensor $g_{ij}$ is called an equidistant space, if its adjoint Riemannian space $\mathcal{R}_N$ is an equidistant space, i.e. if there exists a non-vanishing one-form $\psi \in \mathcal{GR}_N$, $\psi_i \neq 0$ satisfying

\[
\psi_{ij} = \rho g_{ij},
\]

(2.3)

where $;\psi$ denotes the covariant derivative with respect to the symmetric part of the connection of the space $\mathcal{GR}_N$. For $\rho \neq 0$ equidistant spaces belong to the primary type, and for $\rho \equiv 0$ to the particular.

The following definition is a consequence of the previous definition

**Definition 2.2.** A Concircular mapping $f : \mathcal{GR}_N \to \mathcal{GR}_{\bar{N}}$ is a conformal mapping if the following equation is valid

\[
\psi_{ij} = \psi_{ij} - \psi_i \psi_j = \omega g_{ij},
\]

(2.4)

where $\psi_i = \frac{1}{N}(\Gamma^i_{\theta jmn} - \Gamma^i_{\theta jm,n})$, $\omega$ is an invariant, and $;\psi$ is the covariant derivative with respect to the connection $\Gamma^i_{\theta jmn}$. 

In the case of a concircular mapping \( f : \mathbb{R}_N \rightarrow \overline{\mathbb{R}}_N \) of two Riemannian spaces \( \mathbb{R}_N \) and \( \overline{\mathbb{R}}_N \), we have an invariant geometric object

\[
Z^i_{jmn} = R^i_{jmn} - \frac{R}{N(N-1)}(\delta^i_m g_{jn} - \delta^i_n g_{jm}),
\]

where \( R^i_{jmn} \) is the Riemann-Christoffel curvature tensor of the space \( \mathbb{R}_N \), \( R_{jm} \) the Ricci tensor and \( R \) the scalar curvature. The object \( Z^i_{jmn} \) is called the concircular curvature tensor.

\[3. \text{ Equitorsion concircular curvature tensors}\]

For a concircular mapping \( f : GR_N \rightarrow \overline{GR}_N \), it is not possible to find a generalization of the concircular curvature tensor. For that reason, we define a special concircular mapping.

**Definition 3.1.** A concircular mapping \( f : GR_N \rightarrow \overline{GR}_N \) is **equitorsion** if the torsion tensors of the spaces \( GR_N \) and \( \overline{GR}_N \) are equal at corresponding points.

According to (1.7), this means that

\[
\Gamma^i_{jk} - \Gamma^i_{kj} = \sigma^i_{jk} = 0.
\]

\[3.1. \text{ Equitorsion concircular curvature tensor of the first kind}\]

Using (1.7), we get a relation between the first kind curvature tensors of the spaces \( GR_N \) and \( \overline{GR}_N \):

\[
\overline{R}^i_{jmn} = K^i_{jmn} + P^i_{jm} - P^i_{jn} + P^i_{mn} - P^p_{jm} P^i_{pn} - 2 \delta^i_m g^p_{jn} + (\delta^p_m g^i_{jn} - \delta^i_n g^p_{jm}) \Delta \psi + \psi_p \delta^i_m \Gamma^p_{jm} - \psi_p \delta^i_n \Gamma^p_{jn} - 2 \psi^p g^i_{jm} \Gamma^p_{jn} - 2 \psi^p g^i_{jn} \Gamma^p_{jm} + \psi^p g^m_{jn} \Gamma^p_{jm} - \psi^p g^m_{jm} \Gamma^p_{jn},
\]

where we denoted

\[
\psi^p = g^{p1} \psi_{1p}, \quad \Delta \psi = g^{m1} \psi_m = \psi_p \psi^p.
\]

Contracting with respect to the indices \( i \) and \( n \) in (3.3) we get

\[
\overline{R}^i_{jm} = K^i_{jm} - 2(N-1) \omega g^j_{jm} - (N-1) \Delta \psi g^j_{jm} + (N-2) \psi^p g^i_{jm} \Gamma^p_{jm} + 2 \psi^p \Gamma^p_{jm},
\]

In case of concircular mappings, it is easy to prove the following formula

\[
\overline{\rho}^i_{j1} = e^{-2\psi} g^{ij}.
\]

In (3.5) multiplying by \( g^m_{jm} \) and contracting with respect to the indices \( j \) and then \( m \) we get

\[
\omega = \frac{1}{2N(1-N)}(e^{2\psi} \overline{K} - K) - \frac{1}{2} \Delta \psi,
\]

where \( \overline{K} = \overline{\rho}^i_{j1} \overline{K}^j_{1p} \), and \( K = \rho^i_{j1} \psi_{1p} \) are scalar curvatures of the first kind of the spaces \( GR_N \) and \( \overline{GR}_N \) respectively. From (3.7), we have
It is easy to see that for concircular mappings the following formula is valid
\[ g^{\mu}_{\nu} g_{\mu \nu} = \tilde{g}^{\mu}_{\nu} \tilde{g}_{\mu \nu}. \]  

(3.9)

From (1.2) follows
\[ \psi_i = \frac{1}{2N} \left( \frac{\partial}{\partial x^i} \ln \tilde{g} - \frac{\partial}{\partial x^j} \ln g \right), \]

\[ \text{where } g = \det (g_{ij}), \quad \tilde{g} = \det (\tilde{g}_{ij}). \]  

(3.10)

From (3.1) and (3.10) we obtain
\[ \Gamma_{j, mn} \psi^j = \frac{1}{2N} \Gamma_{j, mn} \tilde{g}^{\mu}_{\nu} \frac{\partial}{\partial x^\rho} \ln \tilde{g} - \frac{1}{2N} \Gamma_{j, mn} g^{\mu}_{\nu} \frac{\partial}{\partial x^\rho} \ln g \]

and
\[ \Gamma_{j, mn} \psi^j = \frac{1}{2N} \Gamma_{j, mn} \tilde{g}^{\mu}_{\nu} \frac{\partial}{\partial x^\rho} \ln \tilde{g} - \frac{1}{2N} \Gamma_{j, mn} g^{\mu}_{\nu} \frac{\partial}{\partial x^\rho} \ln g. \]

(3.11)

(3.12)

Taking into account (3.10), (3.11), (3.12), we can write the relation (3.3) in the form
\[ \overline{Z}_{1, jmn} = Z_{1, jmn}, \]

(3.13)

where
\[ Z_{1, jmn} = \frac{K_{1, jmn}}{N(N-1)} \left( \delta_{ij} g_{mn} - \delta_{im} g_{jn} \right) \]
\[ + \frac{1}{2N} \left( -\delta_{ij} \Gamma_{mn}^{\rho} + 2\delta_{ij} \Gamma_{mn}^{\rho} + \delta_{ij} \Gamma_{mn}^{\rho} + 2\delta_{ij} g_{mn} \Gamma_{ij}^{\rho} - \delta_{ij} g_{mn} \Gamma_{ij}^{\rho} + g_{mn} \Gamma_{ij}^{\rho} \right) \frac{\partial}{\partial x^\rho} \ln g, \]

(3.14)

and analogously for the geometrical object \( \overline{Z}_{1, jmn} \in G\overline{R}_N \). The tensor \( Z_{1, jmn} \) is an invariant of equitorsion concircular mappings, and one can call it the equitorsion concircular curvature tensor of the first kind. So, the following theorem is proved:

**Theorem 3.1.** Let the generalized Riemannian spaces GR\(_N\) and G\(\overline{R}_N\) be defined by virtue of their non-symmetric basic tensors \( g_{ij} \) and \( \tilde{g}_{ij} \), respectively. The equitorsion concircular curvature tensor of the first kind \( Z_{1, jmn} \) (3.14) is an invariant of the equitorsion concircular mapping \( f : GR_N \to G\overline{R}_N \).

### 3.2. Equitorsion concircular curvature tensor of the second kind

For the second kind curvature tensors of the spaces GR\(_N\) and G\(\overline{R}_N\) we get the relation
\[ K_{2, jmn} = K_{2, jmn}^i + p_{j, in}^i - p_{j, in}^m - p_{i, jm}^n - p_{i, jm}^m - p_{j, in}^m \]

\[ + 2\delta_{ij} \omega g_{mn} - 2\delta_{ij} \omega g_{mn} + (\delta_{im} g_{jn} - \delta_{jn} g_{im}) \Delta \psi. \]

(3.15)

i.e., using (1.5, 1.7, 2.4) one obtains
\[ K_{2, jmn} = K_{2, jmn}^i + 2\delta_{ij} \omega g_{mn} - (N-1) \Delta \psi g_{mn}. \]

(3.16)

Contracting with respect to the indices \( i \) and \( n \) in (3.16) we get
\[ K_{2, jmn} = K_{2, jmn} - 2(N-1) \omega g_{mn} - (N-1) \Delta \psi g_{mn}. \]

(3.17)

In the previous equation multiplying by \( g^{mn} \) and contracting with respect to \( j \) and then to \( m \), we get
\[ \Delta \psi K_{2} = \frac{K}{2} + 2N(1 - N) \omega + N(1 - N) \Delta \psi, \]

(3.18)
where $\overline{K} = \frac{2}{3}K_{n^p}$, and $\overline{K} = g^{\overline{p}q}\overline{K}_{n^q}$ are scalar curvatures of the second kind of the spaces $\text{GR}_N$ and $\text{GR}_N$ respectively. From (3.18), we have

$$\omega = \frac{1}{2N(1 - N)} \left( e^{2\omega} \overline{K} - \frac{1}{2} K \right) - \frac{1}{2} \Delta \psi.$$

(3.19)

And finally, taking into account (3.10, 3.11, 3.12), we can write the relation (3.16) in the form

$$\overline{Z}_{ijmn}^i = Z_{ijmn}^i,$$

(3.20)

where

$$Z_{ijmn}^i = \frac{K_{ijmn}}{2} - \frac{1}{N(N - 1)} K(\delta^i_n g_{jm} - \delta^i_m g_{jn})$$

(3.21)

and analogously for $\overline{Z}_{ijmn}^i \in \text{GR}_N$. The tensor $Z_{ijmn}^i$ is an invariant of equitorsion concircular mappings, and one can call it the equitorsion concircular curvature tensor of the second kind. So, we have:

**Theorem 3.2.** Starting from the curvature tensor $K^i_{jn^p}$, one obtains an invariant tensor $Z^i_{ijmn}$ with respect to the equitorsion concircular mapping $f : \text{GR}_N \rightarrow \text{GR}_N$ in the form (3.21).

### 3.3. Equitorsion concircular curvature tensor of the third kind

In the case of the third kind curvature tensors of the spaces $\text{GR}_N$ and $\text{GR}_N$ we get the relation

$$\overline{K}_{ijmn}^i = K_{ijmn}^i + P^{i}_{jmnp} - P^{i}_{jmnp} + P^{i}_{jmnp} - P^{i}_{jmnp},$$

$$+ P^{i}_{jmnp} \Gamma^{i}_{jmnp} - P^{i}_{jmnp} \Gamma^{i}_{jmnp} + P^{i}_{jmnp} \Gamma^{i}_{jmnp} - 2 P^{i}_{jmnp} \Gamma^{i}_{jmnp},$$

(3.22)

i.e., using (1.5, 1.7, 2.4) one obtains

$$\overline{K}_{ijmn}^i = K_{ijmn}^i + 2 \delta^i_n \omega g_{jm} - 2 \delta^i_m \omega g_{jm} + (\delta^i_n g_{jm} - \delta^i_m g_{jm}) \Delta \psi$$

$$- 2 \psi^i_{jmnp} \Gamma^{i}_{jmnp} + \psi^i_{jmnp} \Gamma^{i}_{jmnp} - 2 \psi^i_{jmnp} \Gamma^{i}_{jmnp} + \psi^i_{jmnp} \Gamma^{i}_{jmnp} + \psi^i_{jmnp} \Gamma^{i}_{jmnp} + 2 \psi^i_{jmnp} \Gamma^{i}_{jmnp} + 2 \psi^i_{jmnp} \Gamma^{i}_{jmnp},$$

(3.23)

Contracting (3.23) with respect to the indices $i$ and $n$, the previous equation becomes

$$\overline{K}_{ijmn}^i = K_{ijmn}^i - 2(1 - N) \omega g_{jm} - (1 - N) \Delta \psi g_{jm} + (N - 1) \psi^i_{jmnp} \Gamma^{i}_{jmnp} + 2 \psi^i_{jmnp} \Gamma^{i}_{jmnp},$$

(3.24)

Multiplying (3.24) by $g^{jm}_{n^p}$ and contracting we get

$$e^{2\omega} K = K + 2N(1 - N) \omega + N(1 - N) \Delta \psi,$$

(3.25)

where $\overline{K} = g^{mn}_{n^q}$, and $K = g^{mn}_{n^q} K_{n^q}$ are scalar curvatures of the third kind of the spaces $\text{GR}_N$ and $\text{GR}_N$ respectively. From (3.25), we have

$$\omega = \frac{1}{2N(1 - N)} \left( e^{2\omega} \overline{K} - \frac{1}{3} R \right) - \frac{1}{2} \Delta \psi,$$

(3.26)

Finally,

$$\overline{Z}_{ijmn}^i = Z_{ijmn}^i$$

(3.27)
where
\[ Z^{i}_{jmn} = K^{i}_{jmn} - \frac{1}{N(N-1)} K(C_{i}^{j} g_{jm} - C_{i}^{m} g_{jn}) \]
\[ + \frac{1}{2N} (2\delta_{i}^{j} \Gamma_{jm}^{\nu} - \delta_{i}^{j} \Gamma_{jm}^{\nu} + 2\delta_{m}^{j} \Gamma_{jm}^{ir} - \delta_{m}^{j} \Gamma_{jm}^{ir} - g_{jm}^{\nu} \Gamma_{mn}^{\nu} - 2g_{jm}^{\nu} g_{mn}^{\nu} \Gamma_{ij}^{\nu} - g_{jm}^{\nu} g_{mn}^{\nu}) \frac{\partial}{\partial x^{\nu}} \ln g. \] (3.28)

And analogously for \( Z^{i}_{jmn} \) of the space \( \mathcal{GR}_{N} \). The tensor \( Z^{i}_{jmn} \) is an invariant of equitorsion concircular mappings, and one can call it the **equitorsion concircular curvature tensor of the third kind**. Now we have proved

**Theorem 3.3.** From the curvature tensor \( K^{i}_{jmn} \) we obtain an invariant tensor \( Z^{i}_{jmn} \) according to the equitorsion concircular mapping \( f : \mathcal{GR}_{N} \rightarrow \mathcal{GR}_{N} \) in the form (3.28).

### 3.4. Equitorsion concircular curvature tensor of the fourth kind

For curvature tensors of the fourth kind we get
\[ K^{i}_{jmn} = \frac{1}{N(N-1)} K(C_{i}^{j} g_{jm} - C_{i}^{m} g_{jn}) \] (3.29)
i.e.
\[ K^{i}_{jmn} = K^{i}_{jmn} + 2\delta_{i}^{j} \omega g_{jn} - 2\delta_{i}^{j} \omega g_{jm} + (\delta_{i}^{j} g_{jm} - \delta_{i}^{j} g_{jm}) \Delta \psi. \] (3.30)

Using the same procedure like in the previous cases, in this case an invariant object of the equitorsion concircular mapping is in the form
\[ Z^{i}_{jmn} = K^{i}_{jmn} - \frac{1}{N(N-1)} K(C_{i}^{j} g_{jm} - C_{i}^{m} g_{jn}) \] (3.31)
where \( K^{i}_{jmn} \) is the Ricci curvature tensor of the fourth kind and \( K \) a scalar curvature of the fourth kind. The object \( Z^{i}_{jmn} \) is a tensor and we call it **equitorsion concircular curvature tensor of the fourth kind** of the equitorsion mapping. So, the next theorem is valid:

**Theorem 3.4.** From the curvature tensor \( K^{i}_{jmn} \) one obtains an invariant tensor \( Z^{i}_{jmn} \) (3.31) of the equitorsion mapping of generalized Riemannian spaces.

### 3.5. Equitorsion concircular curvature tensor of the fifth kind

For the curvature tensors of the fifth kind of the spaces \( \mathcal{GR}_{N} \) and \( \mathcal{GR}_{N} \) we have
\[ K^{i}_{jmn} = \frac{1}{N(N-1)} K(C_{i}^{j} g_{jm} - C_{i}^{m} g_{jn}) \] (3.32)
i.e.
\[ K^{i}_{jmn} = K^{i}_{jmn} + 2\delta_{i}^{j} \omega g_{jn} - 2\delta_{i}^{j} \omega g_{jm} + (\delta_{i}^{j} g_{jm} - \delta_{i}^{j} g_{jm}) \Delta \psi. \] (3.33)

Contracting with respect to the indices \( i, n \) and denoting
\[ K^{i}_{jmn} = K^{i}_{jmn}, \quad K^{i}_{jmn} = K^{i}_{jmn}, \] (3.34)
we obtain
\[ K^{i}_{jmn} = K^{i}_{jmn} - 2(N - 1) \omega g_{jm} - (N - 1) \Delta \psi g_{jm}. \] (3.35)
wherefrom, multiplying by $\overline{g}^{jm} = e^{-2\psi} g_{jm}$ and contracting with respect to the indices $j$ and $m$ one obtains
\[
\omega = \frac{1}{2N(1 - N)} (e^{2\psi} \overline{K} - \overline{K}) - \frac{1}{2} \Delta \psi.
\] (3.36)

After eliminating $\omega$ from (3.33) we can write
\[
\overline{Z}^{i}_{jmn} = Z^{i}_{jmn},
\] (3.37)

where
\[
Z^{i}_{jmn} = K^{i}_{jmn} - \frac{1}{N(N - 1)} R^{i}_{jm} (\delta^{j}_{n} - \delta^{j}_{m}).
\] (3.38)

The object $Z^{i}_{jmn}$ is an invariant of the concircular equitorsion mapping. We call it \textbf{equitorsion concircular curvature tensor of the fifth kind}. So, the following theorem is proved:

**Theorem 3.5.** Starting from the curvature tensor $K^{i}_{jmn}$, we obtain an invariant tensor $Z^{i}_{jmn}$ (3.38) of the equitorsion concircular mapping $f : \text{GR}_{N} \rightarrow \text{G\textbar{R}}_{N}$.

4. Concluding remarks

For $g_{ij}(x) = g_{ji}(x)$ the space $\text{GR}_{N}$ reduces to the Riemannian space $\text{R}_{N}$. The curvature tensors $K_{\theta}$, $\theta = 1, \ldots, 5$ in a generalized Riemannian space reduce to the single curvature tensor $R$ in Riemannian space (in the symmetric case).

In the case of equitorsion concircular mapping of the Riemannian spaces (in the symmetric case) $Z^{i}_{\theta}$, ($\theta = 1, \ldots, 5$), given by the formulas (3.14, 3.21, 3.28, 3.31, 3.38) reduce to the concircular curvature tensor \cite{18, 23}
\[
Z^{i}_{jmn} = R^{i}_{jmn} - \frac{R}{N(N - 1)} (\delta^{j}_{n} g_{jm} - \delta^{j}_{m} g_{jn}).
\] (4.1)

All these new quantities can be quite interesting for further investigation.

References


\[11\] Minčić, S. M., Ricci identities in the space of non-symmetric affine connection, Mat. Vesnik, 10(25), (1973), 161–172.


