Tauberian Theorems for Statistically \((C,1)\)-Convergent Sequences of Fuzzy Numbers

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Abstract. In this paper, we have determined necessary and sufficient Tauberian conditions under which statistically convergence follows from statistically \((C,1)\)-convergence of sequences of fuzzy numbers. Our conditions are satisfied if a sequence of fuzzy numbers is statistically slowly oscillating. Also, under additional conditions it is proved that a bounded sequence of fuzzy numbers which is \((C,1)\)-level-convergent to its statistical limit superior is statistically convergent.

1. Introduction and Preliminaries

Let \(K\) be a subset of natural numbers \(\mathbb{N}\) and \(K_n = \{k \leq n : k \in K\}\), The natural density of \(K\) is given by

\[
\delta(K) = \lim_{n \to \infty} \frac{1}{n} |K_n|
\]

if this limit exists, where \(|A|\) denotes the number of elements in \(A\). The concept of statistical convergence was introduced by Fast [9]. A sequence \((x_k)_{k \in \mathbb{N}}\) of (real or complex) numbers is said to be statistically convergent to some number \(l\) if for every \(\epsilon > 0\) we have

\[
\lim_{n \to \infty} \frac{1}{n+1} \left| \{k \leq n : |x_k - l| \geq \epsilon \} \right| = 0.
\]

In this case, we write \(\text{st-} \lim_{k \to \infty} x_k = l\).

A sequence \((x_k)\) is \((C,1)\) convergent to \(\ell\) if \(\lim \sigma_n = \ell\), where

\[
\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} x_k \quad (n = 0, 1, 2, ...),
\]

(1)
is the first arithmetic mean, also called Cesàro mean (of first order). In this case we write \((C,1)\)-\(\lim_{k \to \infty} x_k = \ell\).

The idea of statistical \((C,1)\)-convergence was introduced in [15] by Moricz. A sequence \((x_k)\) is statistically \((C,1)\)-convergent to \(\ell\) if \(st- \lim \sigma_n = \ell\). In addition to, necessary and sufficient conditions, under which \(st- \lim_{k \to \infty} x_k = \ell\) follows from \(st- \lim \sigma_n = \ell\) are presented in [15] by Moricz.
In this paper, our primary interest is to obtain the results in [15] for sequences of fuzzy numbers. We recall the basic definitions dealing with fuzzy numbers.

A fuzzy number is a fuzzy set on the real axis, i.e. a mapping \( u : \mathbb{R} \rightarrow [0,1] \) which satisfies the following four conditions:

(i) \( u \) is normal, i.e., there exists an \( x_0 \in \mathbb{R} \) such that \( u(x_0) = 1 \).

(ii) \( u \) is fuzzy convex, i.e. \( u[\lambda x + (1 - \lambda)y] \geq \min(u(x), u(y)) \) for all \( x, y \in \mathbb{R} \) and for all \( \lambda \in [0,1] \).

(iii) \( u \) is upper semi-continuous.

(iv) The set \( [u]_0 := \{ x \in \mathbb{R} : u(x) > 0 \} \) is compact, where \( \{ x \in \mathbb{R} : u(x) > 0 \} \) denotes the closure of the set \( \{ x \in \mathbb{R} : u(x) > 0 \} \) in the usual topology of \( \mathbb{R} \).

We denote the set of all fuzzy numbers on \( \mathbb{R} \) by \( E^1 \) and called it as the space of fuzzy numbers. \( \alpha \)-level set \( [u]_\alpha \) of \( u \in E^1 \) is defined by

\[
[u]_\alpha := \begin{cases} 
\{ x \in \mathbb{R} : u(x) \geq \alpha \}, & \text{if } 0 < \alpha \leq 1, \\
\{ x \in \mathbb{R} : u(x) > \alpha \}, & \text{if } \alpha = 0.
\end{cases}
\]

The set \( [u]_\alpha \) is closed, bounded and non-empty interval for each \( \alpha \in [0,1] \) which is defined by \( [u]_\alpha := [u^- (\alpha), u^+ (\alpha)] \). \( \mathbb{R} \) can be embedded in \( E^1 \), since each \( r \in \mathbb{R} \) can be regarded as a fuzzy number \( \tilde{r} \) defined by

\[
\tilde{r}(x) := \begin{cases} 
1, & \text{if } x = r, \\
0, & \text{if } x \neq r.
\end{cases}
\]

Let \( u, v, w \in E^1 \) and \( k \in \mathbb{R} \). Then the operations addition and scalar multiplication are defined on \( E^1 \) as

\[
 u + v = w \iff [w]_\alpha = [u]_\alpha + [v]_\alpha \text{ for all } \alpha \in [0,1] \\
 w^{-\alpha} = u^{-\alpha} + v^{-\alpha} \text{ and } w^{+\alpha} = u^{+\alpha} + v^{+\alpha} \text{ for all } \alpha \in [0,1],
\]

\[
[ku]_\alpha = k[u]_\alpha \text{ for all } \alpha \in [0,1].
\]

**Lemma 1.1.** [6]

(i) If \( \overline{0} \in E^1 \) is neutral element with respect to \(+\), i.e., \( u + \overline{0} = \overline{0} + u = u \), for all \( u \in E^1 \).

(ii) With respect to \( \overline{0} \), none of \( u \neq \overline{r}, r \in \mathbb{R} \) has opposite in \( E^1 \).

(iii) For any \( a, b \in \mathbb{R} \) with \( a, b \geq 0 \) or \( a, b \leq 0 \), and any \( u \in E^1 \), we have \((a + b)u = au + bu\).

For general \( a, b \in \mathbb{R} \), the above property does not hold.

(iv) For any \( a \in \mathbb{R} \) and any \( u, v \in E^1 \), we have \( au + av = au + av \).

(v) For any \( a, b \in \mathbb{R} \) and any \( u \in E^1 \), we have \( a(bu) = (ab)u \).

Let \( W \) be the set of all closed bounded intervals \( A \) of real numbers with endpoints \( A \) and \( \overline{A} \), i.e., \( A := [\underline{A}, \overline{A}] \). Define the relation \( d \) on \( W \) by

\[
d(A, B) := \max(|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|).
\]

Then it can be easily observed that \( d \) is a metric on \( W \) and \( (W, d) \) is a complete metric space, (cf. Nanda [16]). Now, we may define the metric \( D \) on \( E^1 \) by means of the Hausdorff metric \( d \) as follows

\[
D(u,v) := \sup_{\alpha \in [0,1]} d([u]_\alpha, [v]_\alpha) := \sup_{\alpha \in [0,1]} \max(|[u^- \alpha] - [v^- \alpha]|, |[u^+ \alpha] - [v^+ \alpha]|).
\]

One can see that

\[
D(u, \overline{0}) = \sup_{\alpha \in [0,1]} \max(|u^- \alpha|, |u^+ \alpha|) = \max(|u^- (0)|, |u^+ (0)|).
\]

(2)

Now, we may give:
Proposition 1.2. [6] Let \( u, v, w, z \in E^1 \) and \( k \in \mathbb{R} \). Then,

(i) \( (E^1, D) \) is a complete metric space.

(ii) \( D(ku, kv) = |k|D(u, v) \).

(iii) \( D(u + v, w + v) = D(u, w) \).

(iv) \( D(u + v, w + z) \leq D(u, w) + D(v, z) \).

(v) \[ |D(u, 0) - D(v, 0)| \leq D(u, v) \leq D(u, 0) + D(v, 0) \].

One can extend the natural order relation on the real line to intervals as follows:

\[ A \leq B \text{ if and only if } \bar{A} \leq \bar{B} \text{ and } \overline{A} \leq \overline{B} \]

Also, the partial ordering relation on \( E^1 \) is defined as follows:

\[ u \leq v \iff [u]_0 \leq [v]_0 \iff u^- (\alpha) \leq v^- (\alpha) \text{ and } u^+(\alpha) \leq v^+(\alpha) \text{ for all } \alpha \in [0, 1]. \]

We say that \( u < v \) if \( u \leq v \) and there exists \( a_0 \in [0, 1] \) such that \( u^- (a_0) < v^- (a_0) \) or \( u^+(a_0) < v^+(a_0) \). Two fuzzy numbers \( u \) and \( v \) are said to be incomparable if neither \( u \leq v \) nor \( v \leq u \) holds. In this case we use the notation \( u \neq v \).

Following Matloka [14], we give some definitions concerning with the sequences of fuzzy numbers. Nanda [16] introduced Cauchy sequences of fuzzy numbers and showed that every Cauchy sequence of fuzzy numbers is convergent.

A sequence \( u = (u_k) \) of fuzzy numbers is a function \( u \) from the set \( \mathbb{N} \), the set of natural numbers, into the set \( E^1 \). The fuzzy number \( u_k \) denotes the value of the function at \( k \in \mathbb{N} \) and is called as the \( k \)-th term of the sequence. By \( \omega(F) \), we denote the set of all sequences of fuzzy numbers.

A sequence \( (u_n) \in \omega(F) \) is said to be convergent to \( u \in E^1 \), if for every \( \varepsilon > 0 \) there exists an \( n_0 = n_0(\varepsilon) \in \mathbb{N} \) such that

\[ D(u_n, u) < \varepsilon \text{ for all } n \geq n_0. \]

By \( c(F) \), we denote the set of all convergent sequences of fuzzy numbers.

A sequence \( u = (u_k) \) of fuzzy numbers is said to be Cauchy if for every \( \varepsilon > 0 \) there exists a positive integer \( n_0 \) such that

\[ D(u_k, u_m) < \varepsilon \text{ for all } k, m \geq n_0. \]

By \( C(F) \), we denote the set of all Cauchy sequences of fuzzy numbers.

A sequence \( (u_n) \in \omega(F) \) is said to be bounded if the set of its terms is a bounded set. That is to say that a sequence \( (u_n) \in \omega(F) \) is said to be bounded if there exists \( M > 0 \) such that \( D(u_n, 0) \leq M \) for all \( n \in \mathbb{N} \). By \( \ell_{\infty}(F) \), we denote the set of all bounded sequences of fuzzy numbers.

The level convergence of a sequence of fuzzy numbers given by Fang and Huang [8], as follows: A sequence \( (u_n) \) is level-converges to \( \mu \in E^1 \), written as \( (l) - \lim_{n \to \infty} u_n = \mu \) if

\[ \lim_{n \to \infty} u^- (\alpha) = \mu^- (\alpha) \text{ and } \lim_{n \to \infty} u^+ (\alpha) = \mu^+ (\alpha) \]

for all \( \alpha \in [0, 1] \). Obviously, \( u_n \to \mu \) implies that \( (l) - \lim_{n \to \infty} u_n = \mu \) and the converse is not true in general.

Statistical convergence of sequences of fuzzy numbers was introduced by Nuray and Savas [17]. A sequence \( (u_k) : k = 0, 1, 2, \ldots \) of fuzzy numbers is said to be statistically convergent to some fuzzy number \( \mu_0 \) if for every \( \varepsilon > 0 \) we have

\[ \lim_{n \to \infty} \frac{1}{n+1} \left| \{ k \leq n : D(u_k, \mu_0) \geq \varepsilon \} \right| = 0. \]
In this case we write
\[ \text{st-} \lim_{k \to \infty} u_k = \mu_0. \] (3)

The statistical boundedness of a sequence of fuzzy numbers was introduced and studied by Aytar and Pehlivan [4]. The sequence \( u = (u_k) \) is said to be statistically bounded if there exists a real number \( M \) such that the set
\[ \{ k \in \mathbb{N} : D(u_k, \overline{0}) > M \} \]
has natural density zero.

Aytar et al. [2] defined the concept of statistical limit superior and limit inferior of statistically bounded sequences of fuzzy numbers. Given \( u = (u_k) \in \mathcal{W}(F) \), define the following sets:
\[ A_u = \left\{ \mu \in E^1 : \delta \left( \{ k \in \mathbb{N} : u_k < \mu \} \right) \neq 0 \right\}, \]
\[ \overline{A}_u = \left\{ \mu \in E^1 : \delta \left( \{ k \in \mathbb{N} : u_k > \mu \} \right) = 1 \right\}, \]
\[ B_u = \left\{ \mu \in E^1 : \delta \left( \{ k \in \mathbb{N} : u_k > \mu \} \right) \neq 0 \right\}, \]
\[ \overline{B}_u = \left\{ \mu \in E^1 : \delta \left( \{ k \in \mathbb{N} : u_k < \mu \} \right) = 1 \right\}. \]

If \( u = (u_k) \) is a statistically bounded sequence of fuzzy numbers, then, the notions st- \( \lim \inf \) and st- \( \lim \sup \) is defined as follows:
\[ \text{st-} \lim \inf u_k = \inf A_u = \sup \overline{A}_u, \]
\[ \text{st-} \lim \sup u_k = \sup B_u = \inf \overline{B}_u. \]

Lemma 1.3. [2] Let \( u = (u_k) \) be a statistically bounded sequence of fuzzy numbers, \( \nu = \text{st-} \lim \inf u_k \) and \( \mu = \text{st-} \lim \sup u_k \). Then, for every \( \varepsilon > 0 \)
\[ \delta \left( \{ k \in \mathbb{N} : u_k < \nu - \varepsilon \} \right) = 0 \quad \text{and} \quad \delta \left( \{ k \in \mathbb{N} : u_k > \mu + \varepsilon \} \right) = 0. \]

Basic results on statistical convergence of sequences of fuzzy numbers may be found in [3–5, 13, 18].

The statistical Cesàro convergence of a sequence of fuzzy numbers has been defined in [1]. We say that \( (u_k) \) is statistically Cesàro convergent (written statistically (C,1)-convergent) to a fuzzy number \( \mu_0 \) if
\[ \text{st-} \lim_{n \to \infty} \sigma_n = \mu_0. \] (4)

Kwon [13] proved that if a sequence \( (u_k) \) is bounded, then
\[ \text{st-} \lim_{k \to \infty} u_k = \mu_0 \quad \text{implies} \quad \text{st-} \lim_{n \to \infty} \sigma_n = \mu_0. \]

Example 1.4. Let \( (u_k) = (u_0, v_0, u_0, v_0, \ldots) \) where
\[ u_0(t) = \begin{cases} \frac{2t^2}{3}, & \text{if } t \in [0, 2] \\ 0, & \text{otherwise} \end{cases} \]
and
\[ v_0(t) = \begin{cases} \frac{2t^2}{3}, & \text{if } t \in [-2, 0] \\ 0, & \text{otherwise} \end{cases} \]

Then \( \alpha \)-level set of arithmetic means \( \sigma_n \) of \( (u_k) \)
\[ [\sigma_{2n}]_\alpha = \left[ \frac{2n}{2n+1} (\alpha - 1), \frac{2(n+1)}{2n+1} (1-\alpha) \right] \quad \text{and} \quad [\sigma_{2n-1}]_\alpha = [\alpha - 1, 1 - \alpha]. \]

So, \( (\sigma_n) \) is convergent to \( w_0 = (u_0 + v_0)/2 \) and hence it is statistically (C,1)-convergent to \( w_0 \). But \( (u_n) \) is not statistically convergent.

Our goal is to find (so-called Tauberian) conditions under which the converse implication holds.
2. The Main Results

Firstly we need two lemmas.

**Lemma 2.1.** Let \((u_n)\) be a sequence of fuzzy numbers which is statistically \((C,1)\)-convergent to a fuzzy number \(\mu_0\). Then for every \(\lambda > 0\),
\[
\text{st-} \lim_{n} \sigma_{\lambda_n} = \mu_0
\]
where by \(\lambda_n\) we denote the integral part of the product \(\lambda n\), in symbol \(\lambda_n := [\lambda n]\).

\[\text{Proof. Case } \lambda > 1.\]

For all \(\varepsilon > 0\),
\[
\{n \leq N : D(\sigma_{\lambda_n}, \mu_0) \geq \varepsilon\} \subseteq \{n \leq \lambda N : D(\sigma_n, \mu_0) \geq \varepsilon\},
\]
whence we find
\[
\frac{1}{N + 1} \left| \{n \leq N : D(\sigma_{\lambda_n}, \mu_0) \geq \varepsilon\} \right| \leq \frac{\lambda}{\lambda N + 1} \left| \{n \leq \lambda N : D(\sigma_n, \mu_0) \geq \varepsilon\} \right|.
\]

Now, (5) follows from the statistical convergence of \((\sigma_n)\) to \(\mu_0\).

Case \(0 < \lambda < 1\). We claim that the same term \(\sigma_m\) can not occur more than \(1 + \lambda^{-1}\) times in the sequence \((\sigma_{\lambda_n})_{n=0,1,2,...}\). In fact, if for some integers \(k\) and \(t\), we have
\[
m = \lambda_k = \lambda_{k+1} = ... = \lambda_{k+t-1} < \lambda_{k+t}
\]
or equivalently
\[
m \leq \lambda k < \lambda(k+1) < ... < \lambda(k+t-1) < m+1 \leq \lambda(k+t)
\]
then
\[
m + \lambda(t-1) \leq \lambda(k+t-1) < m + 1
\]
whence \(\lambda(t-1) < 1\), that is \(t < 1 + \lambda^{-1}\). Consequently,
\[
\frac{1}{N + 1} \left| \{n \leq N : D(\sigma_{\lambda_n}, \mu_0) \geq \varepsilon\} \right| \leq \left(1 + \frac{1}{\lambda}\right) \frac{\lambda N + 1}{\lambda N + 1} \left| \{n \leq \lambda N : D(\sigma_n, \mu_0) \geq \varepsilon\} \right|
\]
\[
\leq \frac{2(\lambda + 1)}{\lambda N + 1} \left| \{n \leq \lambda N : D(\sigma_n, \mu_0) \geq \varepsilon\} \right|
\]
provided \((\lambda N + 1)/(N + 1) \leq 2\lambda\), which is the case if \(N\) is large enough. So, (5) follows from the statistical convergence of \((\sigma_n)\) to \(\mu_0\). \(\square\)

**Lemma 2.2.** Let \((u_n)\) be a sequence of fuzzy numbers which is statistically \((C,1)\)-convergent to a fuzzy number \(\mu_0\). Then, for every \(\lambda > 1\),
\[
\text{st-} \lim_{n} \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} u_k = \mu_0
\]
and for every \(0 < \lambda < 1\),
\[
\text{st-} \lim_{n} \frac{1}{n - \lambda_n} \sum_{k=n+1}^{n} u_k = \mu_0.
\]
Proof. Case \( \lambda > 1 \). If \( \lambda > 1 \) and \( n \) is large enough in the sense that \( \lambda_n > n \), then
\[
D\left( \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} u_k, \mu_0 \right) \leq \frac{\lambda_n + 1}{\lambda_n - n} D(\sigma_{\lambda_n}, \sigma_n) + D(\sigma_n, \mu_0)
\] (7)

Now (6) follows from (7), Lemma 2.1, the statistical convergence of \( (\sigma_n) \) and the fact that for large enough \( n \)
\[
\frac{\lambda - 1}{\lambda - 1} = \frac{\lambda n - 1}{\lambda n - n} < \frac{\lambda n + 1}{\lambda n - n} < \frac{\lambda n + 1}{\lambda n - n - 1} \leq \frac{2\lambda - 1}{\lambda - 1}.
\] (8)

Case \( 0 < \lambda < 1 \). This time, we use the following inequality:
\[
D\left( \frac{1}{n - \lambda_n} \sum_{k=\lambda_n+1}^{n} u_k, \mu_0 \right) \leq \frac{\lambda_n + 1}{n - \lambda_n} D(\sigma_{\lambda_n}, \sigma_n) + D(\sigma_n, \mu_0)
\]
provided \( n \) is large enough in the sense that \( \lambda_n < n \); and the following inequality for large enough \( n \),
\[
\frac{\lambda_n + 1}{n - \lambda_n} \leq \frac{2\lambda}{1 - \lambda}.
\] (9)

Now we are ready to give our main results.

**Theorem 2.3.** Let \( (u_k) \) be a sequence of fuzzy numbers which is statistically \((C, 1)\)-convergent to a fuzzy number \( \mu_0 \). Then \( (u_k) \) is statistically convergent to \( \mu_0 \) if and only if one of the following two conditions holds: For every \( \varepsilon > 0 \),
\[
\inf \lim \sup_{N \to \infty} \frac{1}{N + 1} \left| \left\{ n \leq N : D\left( \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} u_k, \mu_0 \right) \geq \varepsilon \right\} \right| = 0
\] (10)
or
\[
\inf \lim \sup_{N \to \infty} \frac{1}{N + 1} \left| \left\{ n \leq N : D\left( \frac{1}{n - \lambda_n} \sum_{k=\lambda_n+1}^{n} u_k, \mu_0 \right) \geq \varepsilon \right\} \right| = 0.
\] (11)

Proof. The necessity follows from Lemma 2.2.

Sufficiency. Assume that conditions (4) and one of (10) and (11) are satisfied. In order to prove (3), it is enough to prove that
\[
st- \lim_D (u_n, \sigma_n) = 0.
\]

First, we consider the case \( \lambda > 1 \). Since
\[
D(\sigma_n, u_n) \leq D\left( \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} u_k, \mu_0 \right) + \frac{\lambda_n + 1}{\lambda_n - n} D(\sigma_{\lambda_n}, \sigma_n)
\]
for any \( \varepsilon > 0 \) we have
\[
\left\{ n \leq N : D(\sigma_n, u_n) \geq \varepsilon \right\} \subseteq \left\{ n \leq N : D\left( \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} u_k, \mu_0 \right) \geq \frac{\varepsilon}{2} \right\}
\] (12)
\[
\cup \left\{ n \leq N : D\left( \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} u_k, \mu_0 \right) \geq \frac{\varepsilon}{2} \right\}.
\]

Given any \( \delta > 0 \), by (10) there exists some \( \lambda > 1 \) such that
\[
\lim \sup_{N \to \infty} \frac{1}{N + 1} \left| \left\{ n \leq N : D\left( \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} u_k, \mu_0 \right) \geq \frac{\varepsilon}{2} \right\} \right| \leq \delta.
\] (13)
On the other hand, by Lemma 2.1 and (8), we have
\[
\limsup_{N \to \infty} \frac{1}{N+1} \left| \left\{ n \leq N : \frac{\lambda_n + 1}{\lambda_n - n} D(u_{n}, u_n) \geq \frac{\epsilon}{2} \right\} \right| = 0. \tag{14}
\]

Combining (12) with (14) we get that
\[
\limsup_{N \to \infty} \frac{1}{N+1} \left| \left\{ n \leq N : D(u_{n}, u_n) \geq \epsilon \right\} \right| \leq \delta.
\]
Since \( \delta > 0 \) is arbitrary, we conclude that for every \( \epsilon > 0 \),
\[
\lim_{N \to \infty} \frac{1}{N+1} \left| \left\{ n \leq N : D(u_{n}, u_n) \geq \epsilon \right\} \right| = 0.
\]

Secondly, we consider the case \( 0 < \lambda < 1 \). Since
\[
D(u_{n}, u_n) \leq D(\lambda_n u_n) + \frac{\lambda_n + 1}{n - \lambda_n} D(u_{n}, u_n)
\]
for any \( \epsilon > 0 \), we have
\[
\left\{ n \leq N : D(u_{n}, u_n) \geq \epsilon \right\} \subseteq \left\{ n \leq N : \frac{\lambda_n + 1}{n - \lambda_n} D(u_{n}, u_n) \geq \frac{\epsilon}{2} \right\}
\]
\[
\cup \left\{ n \leq N : D\left(\frac{1}{n - \lambda_n} \sum_{k=1}^{n} u_k, u_n\right) \geq \frac{\epsilon}{2} \right\}.
\]
Given any \( \delta > 0 \), by (11) there exist some \( 0 < \lambda < 1 \) such that
\[
\limsup_{N \to \infty} \frac{1}{N+1} \left| \left\{ n \leq N : D\left(\frac{1}{n - \lambda_n} \sum_{k=1}^{n} u_k, u_n\right) \geq \frac{\epsilon}{2} \right\} \right| \leq \delta.
\]
Using a similar argument as in the case \( \lambda > 1 \), by Lemma 2.1 and condition (9), we conclude that
\[
\lim_{N \to \infty} \frac{1}{N+1} \left| \left\{ n \leq N : D(u_{n}, u_n) \geq \epsilon \right\} \right| = 0.
\]

A sequence \((u_k)\) of fuzzy numbers is said to be \textit{statistically slowly oscillating} if for every \( \epsilon > 0 \)
\[
\inf_{\lambda > 1} \limsup_{N \to \infty} \frac{1}{N+1} \left| \left\{ n \leq N : \max_{n-k \leq \lambda_n} D(u_k, u_n) \geq \epsilon \right\} \right| = 0 \tag{15}
\]
or equivalently,
\[
\inf_{0 < \lambda < 1} \limsup_{N \to \infty} \frac{1}{N+1} \left| \left\{ n \leq N : \max_{\lambda_n \times \delta \leq n} D(u_k, u_n) \geq \epsilon \right\} \right| = 0. \tag{16}
\]

\textbf{Example 2.4.} The sequence \((u_k)\) where
\[
u_k(t) = \begin{cases} 1 - \frac{r}{\log(k+1)}, & \text{if } 0 \leq t \leq 1 + \log(k+1), \\ 0, & \text{otherwise}, \end{cases}
\]
is statistically slowly oscillating. Because for every \( \epsilon > 0 \) there exist \( n_0 = n_0(\epsilon) \) and \( \lambda = 10^c \) such that for all \( n_0 < n < k \leq \lambda_n \)
\[
D(u_k, u_n) = |\log(k+1) - \log(n+1)| = \log \frac{k+1}{n+1} < \log \lambda = \epsilon.
\]
Therefore, for \( n_0 < N \) the set
\[
\left\{ n_0 < n \leq N : \max_{n < k \leq l} D(u_k, u_n) \geq \varepsilon \right\}
\]
is empty. Consequently, condition (15) is satisfied.

The conditions (15) and (16) are clearly imply the conditions (10) and (11), respectively. This gives rise to the following corollary of Theorem 2.3.

**Corollary 2.5.** Let \((u_k)\) be a statistically slowly oscillating sequence of fuzzy numbers. Then
\[
\text{st–lim}_n \sigma_n = \mu_0 \text{ implies st–lim}_n u_n = \mu_0. \tag{17}
\]

Condition (15) is satisfied if there exists a constant \( H \) such that
\[
kD(u_k, u_{k-1}) \leq H \tag{18}
\]
for all large enough \( k \), say \( k > N_1 \). In fact, given \( \varepsilon > 0 \), chose \( 1 < \lambda < 1 + \varepsilon / H \). Since \( N_1 < n < k \leq \lambda n \), by (18) we have
\[
D(u_k, u_n) \leq \sum_{j=n+1}^{k} D(u_j, u_{j-1}) \leq \sum_{j=n+1}^{k} \frac{H}{j} \leq H \left( \frac{k-n}{n} \right) = H \left( \frac{k}{n} - 1 \right) < H(\lambda - 1) < \varepsilon.
\]
But, for \( N_1 < N \), the set
\[
\left\{ N_1 < n \leq N : \max_{n < k \leq l} D(u_k, u_n) \geq \varepsilon \right\}
\]
is empty. Consequently, condition (15) is satisfied.

**Remark 2.6.** Kwon [13] proved that if condition (18) is satisfied, then implication (17) holds as well as
\[
\text{st–lim}_n u_n = \mu_0 \text{ implies } \lim_{n \to \infty} u_n = \mu_0.
\]

**Definition 2.7.** \((u_k)\) is Cesàro level-convergent (written \((C,1)\)-level-convergent) to a fuzzy number \( \mu_0 \), if \( \lim_{n \to \infty} \sigma_n = \mu_0 \) i.e.,
\[
\lim_{n \to \infty} \sigma_n^+(\alpha) = \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} u_k^+(\alpha) = \mu_0^+(\alpha), \quad \lim_{n \to \infty} \sigma_n^-(\alpha) = \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} u_k^-(\alpha) = \mu_0^-(\alpha)
\]
for all \( \alpha \in [0, 1] \).

Note that level-convergence implies \((C,1)\)-level-convergence. But the converse is not true in general.

**Example 2.8.** [21] Let
\[
\nu_n^+(\alpha) = 1, \quad \nu_n^-\alpha) = \left\{ \begin{array}{ll}
(\alpha - \frac{1}{2})^{1/(n+1)}, & \text{if } \frac{1}{2} < \alpha \leq 1, \\
0, & \text{if } 0 \leq \alpha \leq \frac{1}{2}
\end{array} \right.
\]
\[
\mu_n^+(\alpha) = \frac{1}{2}, \quad \mu_n^-\alpha) = \left\{ \begin{array}{ll}
\frac{1}{2}, & \text{if } \frac{1}{2} < \alpha \leq 1, \\
0, & \text{if } 0 \leq \alpha \leq \frac{1}{2}
\end{array} \right.
\]
There exists a unique fuzzy number \( \nu_n \in E^1 \) and a unique fuzzy number \( u_0 \in E^1 \) such that \([\nu_n]_E = [\nu_n^+(\alpha), \nu_n^-(\alpha)]\) and \([\mu_0]_E = [\mu_0^+(\alpha), \mu_0^-(\alpha)]\) for all \( \alpha \in [0, 1] \). Let \( u = (u_n) = (\nu_0, \overline{\nu}_1, \overline{\nu}_2, \ldots) \). Since
\[
\lim_{n \to \infty} \sigma_n^+(\alpha) = \mu_0^+(\alpha), \quad \lim_{n \to \infty} \sigma_n^-(\alpha) = \mu_0^-(\alpha)
\]
\((u_n)\) is \((C,1)\)-level-convergent to \( \mu_0 \) but \((u_n)\) is not level-convergent to \( \mu_0 \). Furthermore, since \( D(\sigma_n, \mu_0) = \frac{1}{2} \), \((u_n)\) is not \((C,1)\) convergent to \( \mu_0 \).
In [11, Theorem 5] Fridy and Orhan proved that a sequence of real numbers which is \((C, 1)\)-convergent to its statistical limit superior is statistically convergent. The next theorem is an analogue of that result for sequences of fuzzy numbers.

**Theorem 2.9.** Let \((u_k)\) be a bounded sequence of fuzzy numbers. Assume that \((u_k)\) is \((C, 1)\)-level-convergent to \(\mu = st - \lim sup u_k\) and there is a number \(\varepsilon_0 > 0\) such that for each \(\varepsilon \in (0, \varepsilon_0)\),

\[
\delta((k \in \mathbb{N} : u_k = \mu + \tau)) = 0, \quad \delta((k \in \mathbb{N} : u_k = \mu - \tau)) = 0.
\]

Then \((u_k)\) is statistically convergent to \(\mu\).

**Proof.** Since \(\mu = st - \lim sup u_k\), then

\[
\delta((k \in \mathbb{N} : u_k > \mu - \tau)) = 0
\]

for each \(\varepsilon > 0\). Suppose that \((u_k)\) is not statistically convergent to \(\mu\). Then, there is a \(\varepsilon_1 \in (0, \varepsilon_0)\) such that

\[
\delta((k \in \mathbb{N} : u_k < \mu - \tau)) \neq 0.
\]

Define \(m = \mu - \tau\), \(B = \sup_{n} u_n\) and

\[
K' = \{k \in \mathbb{N} : u_k < m\},
K'' = \{k \in \mathbb{N} : m \leq u_k \leq \mu + \tau\},
K''' = \{k \in \mathbb{N} : \mu + \tau < u_k \} \cup \{k \in \mathbb{N} : u_k > m\} \cup \{k \in \mathbb{N} : u_k = \mu + \tau\}.
\]

Since \(\delta(K'') = 0\), \(\delta(K') \neq 0\) and \(\delta(K') = 1 - \delta(K)\) there are infinitely many \(n\) such that

\[
\frac{1}{n+1} |K'_n| \geq d > 0
\]

and for each such \(n\) we have

\[
\sigma_n = \frac{1}{n+1} \sum_{k \in K'_n} u_k + \frac{1}{n+1} \sum_{k \in K''_n} u_k + \frac{1}{n+1} \sum_{k \in K'''_n} u_k
\]

\[
\quad < \frac{m}{n+1} |K'_n| + \frac{\mu + \tau}{n+1} |K''_n| + \frac{B}{n+1} |K'''_n|.
\]

So there is an \(a_0 \in [0, 1]\) such that

\[
\sigma^*_{a_0} < \frac{m^-(a_0)}{n+1} |K'_n| + \frac{\mu^-(a_0) + \varepsilon}{n+1} |K''_n| + \frac{B^-(a_0)}{n+1} |K'''_n| + o(1)
\]

\[
\quad = m^-(a_0) \frac{|K'_n|}{n+1} + (\mu^-(a_0) + \varepsilon) \left(1 - \frac{|K''_n|}{n+1}\right) + o(1)
\]

\[
\quad \leq \mu^-(a_0) - d (\mu^-(a_0) - m^-(a_0)) + \varepsilon (1 - d) + o(1).
\]

Since \(\varepsilon \in (0, \varepsilon_0)\) is arbitrary it follows that

\[
\lim \inf \sigma^*_{a_0} \leq \mu^-(a_0) - d (\mu^-(a_0) - m^-(a_0)) < \mu^-(a_0).
\]

Hence, \((u_k)\) is not \((C, 1)\)-level-convergent to \(\mu\) and this completes the proof. Note that the result can be found also by using the following inequality

\[
\sigma^*_{a_0} < \frac{m^+(a_0)}{n+1} |K'_n| + \frac{\mu^+(a_0) + \varepsilon}{n+1} |K''_n| + \frac{B^+(a_0)}{n+1} |K'''_n|.
\]

The following is a dual result for \(st - \lim inf\).
Theorem 2.10. Let \((u_k)\) be a bounded sequence of fuzzy numbers. Assume that \((u_k)\) is \((C,1)\)-level-convergent to \(v = sl - \liminf u_k\) and there is a number \(\varepsilon_0 > 0\) such that for each \(\varepsilon \in (0, \varepsilon_0)\),
\[
\delta(\{k \in \mathbb{N} : u_k < v - \varepsilon\}) = 0, \quad \delta(\{k \in \mathbb{N} : u_k > v + \varepsilon\}) = 0.
\]
Then, \((u_k)\) is statistically convergent to \(v\).

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References