A New Factor Theorem for Generalized Cesàro Summability

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Abstract. In [6], we proved a theorem dealing with an application of quasi-\(f\)-power increasing sequences. In this paper, we prove that theorem under less and weaker conditions. This theorem also includes several new results.

1. Introduction

A positive \(X = (X_n)\) is said to be a quasi-\(f\)-power increasing sequence if there exists a constant \(K = K(X,f) \geq 1\) such that \(K f_n X_n \geq f_m X_m\) for all \(n \geq m \geq 1\), where \(f = (f_n) = (n^\eta (\log n)^\sigma)\), \(\eta \geq 0\), \(0 < \sigma < 1\) (see [13]). If we take \(\eta = 0\), then we get a quasi-\(\sigma\)-power increasing sequence (see [12]). We write 

\[
\BV = \BV_0 \cap \CV,
\]

where 

\[
\BV_0 = \{ x = (x_k) \in \Omega : \lim_{k} |x_k| = 0 \}, \quad \BV = \{ x = (x_k) \in \Omega : \sum_{k} |x_k - x_{k+1}| < \infty \}
\]

and \(\Omega\) being the space of all real or complex-valued sequences. Let \(\sum a_n\) be a given infinite series with partial sums \((s_n)\).

We denote by \(u_{n,\alpha,\delta}\) and \(t_{n,\alpha,\delta}\) the \(n\)th Cesàro means of order \((\alpha,\delta)\), with \(\alpha + \delta > -1\), of the sequence \((s_n)\) and \((na_n)\), respectively, that is (see [8])

\[
u_{n,\alpha,\delta} = \frac{1}{A_{n+\delta}} \sum_{v=0}^{n} A_{n+\alpha}^{-1} A_{n+\alpha}^{\delta} s_v, \tag{1}
\]

\[
u_{t,\alpha,\delta} = \frac{1}{A_{n+\delta}} \sum_{v=1}^{n} A_{n+\alpha}^{-1} A_{n+\alpha}^{\delta} v a_v, \tag{2}
\]

where

\[
A_{n+\alpha} = O(n^{\alpha+\delta}), \quad \alpha + \delta > -1, \quad A_0 = 1 \quad \text{and} \quad A_{-\alpha} = 0 \quad \text{for} \quad n > 0.
\]

Let \((\varphi_n)\) be a sequence of complex numbers. The series \(\sum a_n\) is said to be summable \(\varphi - \{ C, \alpha, \delta \}, \, k \geq 1 \) and \(\alpha + \delta > -1\), if (see [4],[9])

\[
\sum_{n=1}^{\infty} | \varphi_n (u_{n,\alpha,\delta} - u_{n-1,\alpha,\delta}) |^p = \sum_{n=1}^{\infty} n^{-k} | \varphi_n t_{n,\alpha,\delta} |^p < \infty. \tag{4}
\]
In the special case when \( \varphi_n = n^{1-\frac{1}{2}} \), \( \varphi - |C, \alpha, \delta|_k \) summability is the same as \( |C, \alpha, \delta|_k \) summability (see [9]). Also, if we set \( \varphi_n = n^{\gamma+1-\frac{1}{2}} \), then \( \varphi - |C, \alpha, \delta|_k \) summability reduces to \( |C, \alpha, \delta; \gamma|_k \) summability (see [4]). If we take \( \delta = 0 \), then we have \( \varphi - |C, \alpha|_k \) summability (see [1]). Furthermore, if we take \( \varphi_n = n^{1-\frac{1}{2}} \) and \( \beta = 0 \), then we get \( |C, \alpha|_k \) summability (see [10]). Finally, if we take \( \varphi_n = n^{\gamma+1-\frac{1}{2}} \) and \( \delta = 0 \), then we obtain \( |C, \alpha, \gamma|_k \) summability (see [11]).

### 2. The known result

**Theorem A (6).** Let \((\lambda_n) \in \mathbb{B} V_{\sigma}\) and let \((X_n)\) be a quasi-f-power increasing sequence for some \(\sigma (0 < \sigma < 1)\). Suppose also that there exist sequences \((\beta_n)\) and \((\lambda_n)\) such that

\[
|\Delta \lambda_n| \leq \beta_n, \quad \beta_n \rightarrow 0 \quad as \quad n \rightarrow \infty, 
\]

\[
\sum_{n=1}^\infty n |\Delta \beta_n| X_n < \infty, 
\]

\[
|\lambda_n| X_n = O(1) \quad as \quad n \rightarrow \infty, 
\]

If there exists an \(\epsilon > 0\) such that the sequence \(n^{r-k} |\varphi_n|^{1}\) is non-increasing and if the sequence \(\theta_n^{\alpha, \beta}\) is defined by

\[
\theta_n^{\alpha, \beta} = \begin{cases} 
\max_{1 \leq v \leq n} |\alpha, \beta|_{rv}^{\alpha, \beta}, & \alpha = 1, \beta > -1 \\
|\alpha, \beta|_{nv}^{\alpha, \beta}, & 0 < \alpha < 1, \beta > -1
\end{cases}
\]

satisfies the condition

\[
\sum_{n=1}^m \left( \frac{|\varphi_n| \theta_n^{\alpha, \beta}}{n^k} \right)^k = O(X_m) \quad as \quad m \rightarrow \infty,
\]

then the series \(\sum a_n \lambda_n\) is summable \(\varphi - |C, \alpha, \delta|_k\), \(k \geq 1, 0 < \alpha \leq 1, \delta > -1, \eta \geq 0\) and \((\alpha + \delta)k + \epsilon > 1\).

**Remark 2.2** It should be noted that in the statement of Theorem 2.1, a different notation has been used for the quasi-f-power increasing sequences. If we take \(\eta = 0\), then we obtain a known theorem (see [5]).

**3. The main result**

The aim of this paper is to prove Theorem 2.1 under weaker conditions. Now, we shall prove the following theorem:

**Theorem 3.1** Let \((X_n)\) be a quasi-f-power increasing sequence and the sequences \((\lambda_n)\) and \((\beta_n)\) such that conditions (5)-(8) are satisfied. If there exists an \(\epsilon > 0\) such that the sequence \(n^{r-k} |\varphi_n|^{1}\) is non-increasing and if the condition

\[
\sum_{n=1}^m \left( \frac{|\varphi_n| \theta_n^{\alpha, \beta}}{n^k X_n^{k-1}} \right) = O(X_m) \quad as \quad m \rightarrow \infty,
\]

satisfies, then the series \(\sum a_n \lambda_n\) is summable \(\varphi - |C, \alpha, \delta|_k\), \(k \geq 1, 0 < \alpha \leq 1, \delta > -1, \eta \geq 0\) and \((\alpha + \delta - 1)k + \epsilon > 0\).

**Remark 3.2** It should be noted that condition (11) is the same as condition (10) when \(k=1\). When \(k > 1\), condition (11) is weaker than condition (10), but the converse is not true. As in [14] we can show that if (10) is satisfied, then we get that

\[
\sum_{n=1}^m \left( \frac{|\varphi_n| \theta_n^{\alpha, \beta}}{n^k X_n^{k-1}} \right) = O\left(\frac{1}{X_1^{k-1}}\right) \sum_{n=1}^m \left( \frac{|\varphi_n| \theta_n^{\alpha, \beta}}{n^k} \right) = O(X_m).
\]

If (11) is satisfied, then for \(k > 1\) we obtain that

\[
\sum_{n=1}^m \left( \frac{|\varphi_n| \theta_n^{\alpha, \beta}}{n^k} \right) = m X_1^{k-1} \sum_{n=1}^m \left( \frac{|\varphi_n| \theta_n^{\alpha, \beta}}{n^k X_n^{k-1}} \right) = O(X_m) \sum_{n=1}^m \left( \frac{|\varphi_n| \theta_n^{\alpha, \beta}}{n^k X_n^{k-1}} \right) = O(X_m) \neq O(X_m).
\]
Also it should be noted that the condition "\((\lambda_n) \in BV\)" has been removed. We need the following lemmas for the proof of our theorem.

**Lemma 3.3 ([3])** If \(0 < \alpha \leq 1, \delta > -1\) and \(1 \leq v \leq n\), then

\[
| \sum_{p=0}^{n} A_{n-p}^{\alpha-1} \lambda_n \delta p | \leq \max_{1 \leq m \leq v} \left| \sum_{p=0}^{n} A_{n-p}^{\alpha-1} \lambda_n \delta p \right|.
\] (12)

**Lemma 3.4 ([7])** Under the conditions on \((X_n), (\beta_n)\) and \((\lambda_n)\) as expressed in the statement of the theorem, we have the following:

\[
nX_n\beta_n = O(1),
\] (13)

\[
\sum_{n=1}^{\infty} \beta_n X_n < \infty.
\] (14)

4. Proof of Theorem 3.1 Let \((T_n^{a,\delta})\) be the \(n\)th \((C, \alpha, \delta)\) mean of the sequence \((n\alpha, \lambda_n)\). Then, by (2), we have that

\[
T_n^{a,\delta} = \frac{1}{A_n^{\alpha+\delta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} \lambda_n \delta v \alpha \lambda_v.
\]

First applying Abel’s transformation and then use of Lemma 3.3, we have that

\[
|T_n^{a,\delta}| \leq \frac{1}{A_n^{\alpha+\delta}} \sum_{v=1}^{n-1} |\Delta \lambda_v| \left( \left| \sum_{p=1}^{n} A_{n-p}^{\alpha-1} \lambda_n \delta p \right| + \left| \frac{\lambda_n}{A_n^{\alpha+\delta}} \right| \left| \sum_{p=1}^{n} A_{n-p}^{\alpha-1} \lambda_n \delta p \right| \right).
\]

\[
\leq \frac{1}{A_n^{\alpha+\delta}} \sum_{v=1}^{n-1} A_v^{\alpha} \theta^{a,\delta}_v |\Delta \lambda_v| + \left| \frac{\lambda_n}{A_n^{\alpha+\delta}} \right| \left| \sum_{p=1}^{n} A_{n-p}^{\alpha-1} \lambda_n \delta p \right|
\]

\[
= T_{n,1}^{a,\delta} + T_{n,2}^{a,\delta}.
\]

To complete the proof of the theorem, by Minkowski’s inequality, it is sufficient to show that

\[
\sum_{n=1}^{\infty} n^{-k} \left| \rho_n T_{n,r}^{a,\delta} \right| \phi^n < \infty \quad \text{for} \quad r = 1, 2.
\]
Now, when $k > 1$, applying Hölder’s inequality with indices $k$ and $k'$, where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that
by virtue of the hypotheses of the theorem and Lemma 3.4. This completes the proof of Theorem 3.1. If we set $\epsilon = 1$ and $\varphi_n = n^{1-k}$, then we obtain a new result concerning the $|C, \alpha, \delta|_k$ summability. Also, if we take $\varphi_n = n^{r+1-k}$, then we obtain another new result dealing with the $|C, \alpha, \delta, \gamma|_k$ summability factors. If we take $\eta = 0$, then we obtain Theorem A under weaker conditions. If we set $\eta = 0$ and $\delta = 0$, then we obtain the result of Bor and Özarslan under weaker conditions (see [2]). Furthermore, if we take $\epsilon = 1$, $\delta = 0$ and $\varphi_n = n^{r+1-k}$, then we get the result of Bor under weaker conditions (see [7]).

References