A New Theorem on the Absolute Riesz Summability Factors

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Abstract. In [5], we proved a main theorem dealing with absolute Riesz summability factors of infinite series using a quasi-$\delta$-power increasing sequence. In this paper, we generalize that theorem by using a general class of power increasing sequences instead of a quasi-$\delta$-power increasing sequence. This theorem also includes some new and known results.

1. Introduction

A positive sequence $(b_n)$ is said to be an almost increasing sequence if there exists a positive increasing sequence $(c_n)$ and two positive constants $A$ and $B$ such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). We write $BV \cap CO = BV \cap C$, where $C = \{ x = (x_k) \in \Omega : \lim_{k} |x_k| = 0 \}$, $BV = \{ x = (x_k) \in \Omega : \sum_{k} |x_k - x_{k+1}| < \infty \}$ and $\Omega$ being the space of all real-valued sequences. A positive sequence $X = (X_n)$ is said to be a quasi-$f$-power increasing sequence if there exists a constant $K = K(X, f) \geq 1$ such that $K f_m X_m \geq f_m X_m$ for all $n \geq m \geq 1$, where $f = (f_n) = \{ n^\sigma (\log n)^\delta \}, \sigma \geq 0, 0 < \delta < 1$ (see [10]). If we take $\sigma=0$, then we get a quasi-$\delta$-power increasing sequence (see [9]). Let $\sum a_n$ be a given infinite series with partial sums $(s_n)$. We denote by $u_n$ and $t_n$ the $n$th $(C, 1)$ means of the sequence $(s_n)$ and $(na_n)$, respectively. The series $\sum a_n$ is said to be summable $| C, 1 |, k \geq 1$, if (see [7])

$$\sum_{n=1}^{\infty} n^{k-1} | u_n - u_{n-1} |^k = \sum_{n=1}^{\infty} n | t_n |^k < \infty. $$

(1)

Let $(p_n)$ be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-1} = P_{-i} = 0, i \geq 1).$$

(2)

The sequence-to-sequence transformation

$$V_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v S_v$$

(3)

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defines the sequence \((V_n)\) of the Riesz mean or simply the \((\bar{N}, \delta)\) mean of the sequence \((s_n)\), generated by the sequence of coefficients \((p_n)\) (see [8]). The series \(\sum a_n\) is said to be summable \(\mid N, p_n \mid_k, k \geq 1\), if (see [2])
\[
\sum_{n=1}^{\infty} (p_n/p_n^\lambda)^{k-1} \mid V_n - V_{n-1} \mid^k < \infty. \tag{4}
\]
In the special case \(p_n = 1\) for all values of \(n\), \(\mid \bar{N}, p_n \mid_k\) summability reduces to \(\mid C, 1 \mid_k\) summability.

2. Known result
In [5], we have proved the following theorem dealing with \(\mid \bar{N}, p_n \mid_k\) summability factors of infinite series.

**Theorem 2.1** Let \((\lambda_n) \in \mathcal{B}'\mathcal{V}_O\) and let \((X_n)\) be a quasi-\(\delta\)-power increasing sequence for some \(\delta (0 < \delta < 1)\) and let there be sequences \((\beta_n)\) and \((\lambda_n)\) such that
\[
|\Delta \lambda_n| \leq \beta_n, \tag{5}
\]
\[
\beta_n \to 0 \quad \text{as} \quad n \to \infty, \tag{6}
\]
\[
\sum_{n=1}^{\infty} n |\Delta \beta_n| X_n < \infty, \tag{7}
\]
\[
|\lambda_n| X_n = O(1). \tag{8}
\]
If
\[
\sum_{n=1}^{\infty} \frac{|f_n|^k}{\nu} = O(X_n) \quad \text{as} \quad n \to \infty \tag{9}
\]
and \((p_n)\) is a sequence such that
\[
P_n = O(np_n), \tag{10}
\]
\[
P_n \Delta p_n = O(p_n p_{n+1}), \tag{11}
\]
then the series \(\sum_{n=1}^{\infty} a_n \frac{P_{\lambda_n}}{mp_n}\) is summable \(\mid \bar{N}, p_n \mid_k, k \geq 1\).

It should be noted that if we take \((X_n)\) as an almost increasing sequence, then we get a result which was proved in [4]. Also it should be remarked that we can take \((\lambda_n) \in \mathcal{B}'\mathcal{V}\) instead of \((\lambda_n) \in \mathcal{B}'\mathcal{V}_O\) and it is sufficient to prove Theorem 2.1.

3. Main result
The aim of this paper is to generalize Theorem 2.1 by using a quasi-\(f\)-power increasing sequence instead of a quasi-\(\delta\)-power increasing sequence. Now we shall prove the following theorem.

**Theorem 3.1** Let \((\lambda_n) \in \mathcal{B}'\mathcal{V}\) and let \((X_n)\) be a quasi-\(f\)-power increasing sequence for some \(\delta (0 < \delta < 1)\) and \(\sigma \geq 0\). If the conditions (5)-(11) are satisfied, then the series \(\sum_{n=1}^{\infty} a_n \frac{P_{\lambda_n}}{m p_n}\) is summable \(\mid \bar{N}, p_n \mid_k, k \geq 1\).

It should be noted that if we take \(\sigma = 0\), then we get Theorem 2.1.

We require the following lemmas for the proof of our theorem.

**Lemma 3.2** \((\mathbf{[3]})\) If the conditions (10) and (11) are satisfied, then \(\Delta (P_{n}/p_n n^2) = O(1/n^2)\).

**Lemma 3.3** \((\mathbf{[6]})\) Except for the condition \((\lambda_n) \in \mathcal{B}'\mathcal{V}\) under the conditions on \((X_n)\), \((\beta_n)\) and \((\lambda_n)\) as expressed in the statement of the theorem, we have the following:
\[
nX_n \beta_n = O(1), \tag{12}
\]
\[
\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{13}
\]

4. Proof of Theorem 3.1
Let \((T_n)\) be the sequence of \((\bar{N}, p_n)\) mean of the series \(\sum_{n=1}^{\infty} a_n \frac{P_{\lambda_n}}{m p_n}\). Then, by definition, we have
\[
T_n = \frac{1}{p_n} \sum_{n=1}^{n} p_r \sum_{r=1}^{n} a_r P_{\lambda_r} \frac{\lambda_r}{p_r} = \frac{1}{p_n} \sum_{n=1}^{n} (p_n - p_{n-1}) a_r P_{\lambda_r} \frac{\lambda_r}{vp_r}. \tag{14}
\]
Then, for \( n \geq 1 \) we have that

\[
T_n - T_{n-1} = \frac{p_n}{P_n^2} \sum_{v=1}^{n} \frac{P_{v-1}P_v \lambda_v}{\nu^2_p} = \frac{p_n}{P_n^2} \sum_{v=1}^{n} \frac{P_{v-1}P_v \nu \lambda_v}{\nu^2_p}.
\]

Using Abel’s transformation, we get that

\[
T_n - T_{n-1} = \frac{p_n}{P_n^2} \sum_{v=1}^{n-1} \Delta \left( \frac{P_{v-1}P_v \lambda_v}{\nu^2_p} \right) \sum_{r=1}^{n} r \lambda_v + \frac{\lambda_n}{n^2} \sum_{v=1}^{n} \nu \lambda_v.
\]

\[
T_n - T_{n-1} = \frac{p_n}{P_n^2} \sum_{v=1}^{n-1} \frac{P_v}{\nu_p} (v + 1) t \nu \lambda_v - \frac{p_n}{P_n^2} \sum_{v=1}^{n-1} \frac{P_v \lambda_v}{\nu_p} \sum_{v=1}^{n} \frac{P_v}{\nu_p} - T_{n-1} + T_{n-2} + T_{n-3} + T_{n-4}.
\]

To complete the proof of Theorem 3.1, by Minkowski’s inequality it is sufficient to show that

\[
\sum_{n=1}^{\infty} \left( \frac{p_n}{P_n} \right)^{k-1} | T_{n,r} |^{\frac{k}{r}} < \infty, \quad \text{for } \ r = 1, 2, 3, 4.
\]

When \( k > 1 \), we can apply Hölder’s inequality with indices \( k \) and \( k' \), where \( \frac{1}{k} + \frac{1}{k'} = 1 \), and so we get that

\[
\sum_{n=1}^{\infty} \left( \frac{p_n}{P_n} \right)^{k-1} | T_{n,1} |^{k} = O(1) \sum_{n=2}^{\infty} \left( \frac{p_n}{P_n} \right)^{k} \left( \sum_{v=1}^{n-1} \frac{P_v}{\nu_p} t \nu \lambda_v \left| \frac{1}{\nu^2_p} \right|^{\frac{k}{k}} \right) = O(1) \sum_{n=2}^{\infty} \left( \frac{p_n}{P_n} \right)^{k} \sum_{v=1}^{n-1} \frac{P_v}{\nu_p} t \nu \lambda_v \left| \frac{1}{\nu^2_p} \right|^{\frac{k}{k'}} \times \left\{ \frac{1}{\nu^2_p} \sum_{v=1}^{n-1} \frac{P_v}{\nu_p} \right\}^{k-1} = O(1) \sum_{n=1}^{m} \frac{p_n}{\nu_p} t \nu \lambda_v \left| \frac{1}{\nu^2_p} \right|^{\frac{k}{k'}} \sum_{n=m+1}^{\infty} \frac{p_n}{P_n} \left| t \nu \lambda_v \left| \frac{1}{\nu^2_p} \right|^{\frac{k}{k'}} \right) = O(1) \sum_{n=1}^{m} \frac{p_n}{\nu_p} t \nu \lambda_v \left| \frac{1}{\nu^2_p} \right|^{\frac{k}{k'}} \left( \frac{1}{\nu^2_p} \right)^{k} = O(1) \sum_{n=1}^{m} \left( \frac{1}{\nu^2_p} \right)^{k-1} \frac{1}{\nu^2_p} t \nu \lambda_v \left| \frac{1}{\nu^2_p} \right|^{\frac{k}{k'}}
\]
as $m \to \infty$, by virtue of the hypotheses of Theorem 3.1 and Lemma 3.3.

Now, by using (10), we have that

$$\sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{k-1} |T_{n,2}|^k = O(1) \sum_{n=2}^{m+1} \frac{p_n}{p_n p_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} \left| \Delta \lambda_v | p_v \right| |t_v| \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{p_n p_{n-1}^k} \sum_{v=1}^{n-1} \left( \frac{P_v}{p_v} \right)^k |t_v|^k p_v$$

$$\times \left\{ \frac{1}{p_n} \sum_{v=1}^{n-1} p_v \right\}$$

$$= O(1) \sum_{n=2}^{m+1} \left( \frac{P_v}{p_v} \right)^k (\beta_v)^k |t_v|^k p_v \sum_{n=n+1}^{m+1} \frac{p_n}{p_n p_{n-1}^k}$$

$$= O(1) \sum_{n=2}^{m+1} \left( \frac{P_v}{p_v} \right)^{k-1} (\beta_v)^k |t_v|$$

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$$= O(1) \sum_{n=2}^{m+1} |t_v|^k$$

$$= O(1) \sum_{n=2}^{m+1} \Delta(\beta_v) \sum_{v=1}^{n} |t_v|^k + O(1)m\beta_m \sum_{v=1}^{m} |t_v|^k$$

$$= O(1) \sum_{n=2}^{m+1} |\Delta \beta_v| |X_v| + O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1)m\beta_m X_m$$

$$= O(1) \text{ as } m \to \infty,$$
by virtue of the hypotheses of Theorem 3.1 and Lemma 3.3. The other parts of the proof can be done similar as in [5] by using Lemma 3.2 and Lemma 3.3 and therefore we omitted it. If we take $p_n = 1$ for all values of $n$, then we get a new result dealing with $|C, 1|_k$ summability factors of infinite series.

References