On Weak and Strong Convergence of an Explicit Iteration Process for a Total Asymptotically Quasi-I-Nonexpansive Mapping in Banach Space

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Abstract. In this paper, we introduce a new class of Lipschitzian maps and prove some weak and strong convergence results for explicit iterative process using a more satisfactory definition of self mappings. Our results approximate common fixed point of a total asymptotically quasi-I-nonexpansive mapping $T$ and a total asymptotically quasi-nonexpansive mapping $I$, defined on a nonempty closed convex subset of a Banach space.

1. Introduction

Let $E$ be a real normed linear space, $K$ a nonempty subset of $E$ and $T : K \to K$ a mapping. Denote by $F(T)$ the set of fixed points of $T$, that is, $F(T) = \{x \in K : Tx = x\}$ and we denote by $D(T)$ the domain of a mapping $T$. Throughout this paper, we always assume that $E$ is a real Banach space and $F(T) \neq \emptyset$. Now, we recall the well-known concept and results. A mapping $T : K \to K$ is called asymptotically nonexpansive if there exists a sequence \( \{k_n\} \subset [1, \infty) \) with \( k_n \to 1 \) such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all \( x, y \in K \) and \( n \geq 1 \). A mapping $T : K \to K$ is said asymptotically quasi-nonexpansive if there exists a sequence \( \{k_n\} \subset [1, \infty) \) with $\lim_{n \to \infty} k_n = 1$ such that

$$\|T^n x - p\| \leq k_n \|x - p\|$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$. Let $T : K \to K$, $I : K \to K$ be two mappings of nonempty subset $K$ of a real normed linear space $E$. Then $T$ is said asymptotically $I$-nonexpansive if there exists a sequence \( \{k_n\} \subset [1, \infty) \)
Remark 1.1. If \( F(T) \cap F(I) \neq \emptyset \) then an asymptotically \( I \)-nonexpansive mapping is asymptotically quasi \( I \)-nonexpansive. But, there exists a nonlinear continuous asymptotically quasi \( I \)-nonexpansive mappings which is not asymptotically \( I \)-nonexpansive.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [3]. They proved that if \( K \) is a nonempty closed convex bounded subset of a real uniformly convex Banach space and \( T : K \rightarrow K \) is an asymptotically nonexpansive mappings, then \( T \) has a fixed point. Liu [5] studied iterative sequences for asymptotically quasi-nonexpansive mappings. The weak and strong convergence of implicit iteration process to a common fixed point of a finite family of \( I \)-asymptotically nonexpansive mappings were studied by Temir [10]. Temir and Gul [11] defined \( I \)-asymptotically quasi-nonexpansive mapping in Hilbert space and they proved convergence theorem for \( I \)-asymptotically quasi-nonexpansive mapping defined in Hilbert space.

A mapping \( T : K \rightarrow K \) is called a total asymptotically nonexpansive mapping (see [1]) if there exist nonnegative real sequences \( \{\mu_n\}, \{l_n\} \) with \( \mu_n, l_n \to 0 \) as \( n \to \infty \) and strictly increasing continuous function \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) with \( \phi(0) = 0 \) such that for all \( x, y \in K \),

\[
\|T^n x - T^n y\| \leq \|x - y\| + \mu_n \phi(\|x - y\|) + l_n, \quad n \geq 1. \tag{1}
\]

Let \( T : K \rightarrow K, I : K \rightarrow K \) be two mappings of a nonempty subset \( K \) of a real normed space \( E \). \( T \) is said to be total asymptotically \( I \)-nonexpansive mapping (see [6]) if there exist nonnegative real sequences \( \{\mu_n\}, \{l_n\} \) with \( \mu_n, l_n \to 0 \) as \( n \to \infty \) and strictly increasing continuous function \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) with \( \phi(0) = 0 \) such that for all \( x, y \in K \),

\[
\|T^n x - T^n y\| \leq \|I^n x - I^n y\| + \mu_n \phi(\|I^n x - I^n y\|) + l_n, \quad n \geq 1. \tag{2}
\]

Note that if \( I = Id \) (\( Id \) is the identity mapping), then (2) reduces to (1). One can see that if \( \phi(\xi) = \xi \), then (1) reduces to \( \|T^n x - T^n y\| \leq (1 + \mu_n) \|x - y\| + l_n, \quad n \geq 1 \). In addition, if \( l_n = 0 \) for all \( n \geq 1 \), then total asymptotically nonexpansive mappings coincide with asymptotically nonexpansive mappings.
Let $K$ be a nonempty closed subset of a real Banach space $E$. Then a mapping $T : K \to K$ is called a uniformly $L$-Lipschitzian mapping if there exists a constant $L > 0$ such that
\[ \|T^n x - T^n y\| \leq L \|x - y\| \]  \hspace{1cm} (3)
for all $x, y \in K$ and $n \geq 1$.

The class of a total asymptotically nonexpansive mappings was introduced by Alber et al. [1] to unify various definitions of asymptotically nonexpansive mappings. They constructed a scheme which convergences strongly to a fixed point of a total asymptotically nonexpansive mappings. Mukhamedov and Saburov [6] studied strong convergence of an explicit iteration process for a totally asymptotically $I$-nonexpansive mapping in Banach spaces.

**Definition 1.2.** [2] Let $K$ be a nonempty closed subset of a real normed linear space $E$. A mapping $T : K \to K$ is said to be total asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{\mu_n\}, \{l_n\}$ with $\mu_n, l_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x \in K, p \in F(T)$,
\[ \|T^n x - p\| \leq \|x - p\| + \mu_n \phi(\|x - p\|) + l_n, \quad n \geq 1. \]  \hspace{1cm} (4)

**Definition 1.3.** Let $T : K \to K, I : K \to K$ be two mappings of a nonempty closed subset $K$ of a real normed space $E$. $T$ is said to be total asymptotically quasi-$I$-nonexpansive if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{\mu_n\}, \{l_n\}$ with $\mu_n, l_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x \in K, p \in F(T)$,
\[ \|T^n x - p\| \leq \|T^n x - p\| + \mu_n \phi(\|T^n x - p\|) + l_n, \quad n \geq 1. \]  \hspace{1cm} (5)

Note that if $I = \text{Id}$ ($\text{Id}$ is the identity mapping), then (5) reduces to (4). One can see that if $\phi(\xi) = \xi$, then (4) reduces to $\|T^n x - p\| \leq (1 + \mu_n) \|x - p\| + l_n, \quad n \geq 1$. In addition, if $l_n = 0$ for all $n \geq 1$, then total asymptotically quasi-nonexpansive mappings coincide with asymptotically quasi-nonexpansive mappings.

**Definition 1.4.** Let $K$ be a nonempty closed subset of a real normed linear space $E$. A mapping $T : K \to K$ is said to be total uniformly $L$-Lipschitzian if there exist a constant $L > 0$ such that
\[ \|T^n x - T^n y\| \leq L \|x - y\| + \mu_n \phi(\|x - y\|) + l_n, \quad n \geq 1. \]  \hspace{1cm} (6)

One can see that if $\mu_n = 0$ and $l_n = 0$ for all $n \geq 1$, then (6) reduces to (3).

**Example 1.5.** Let us consider that $\mathbb{R}$, the set of real numbers, endowed with the usual topology. Let $K = [0, 1] \subset \mathbb{R}$. The mapping $T : K \to K$ is defined by
\[ Tx = \begin{cases} \frac{1}{2}, & x \in \left[0, \frac{3}{4}\right] \\ \frac{x + 1}{\sqrt{2}}, & x \in \left[\frac{1}{2}, 1\right] \end{cases} \]
for all $x \in K$. Let $\phi$ be a strictly increasing continuous function such that $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$. Let $\{\mu_n\}_{n \geq 1}$ and $\{l_n\}_{n \geq 1}$ in $\mathbb{R}$ be two sequences defined by $\mu_n = \frac{1}{n}$ and $l_n = \frac{1}{n+1}$, for all $n \geq 1$ ($\lim_{n \to \infty} \mu_n = \lim_{n \to \infty} \frac{1}{n} = 0$, $\lim_{n \to \infty} l_n = \frac{1}{n+1} = 0$). Note that $T^n x = \frac{x}{2}$ for all $x \in K$ and $n \geq 2$ and $F(T) = \{\frac{1}{2}\}$. Clearly, $T$ is both uniformly continuous and total asymptotically nonexpansive mapping on $K$. Also, for all $x, y \in K$ and $L > 0$, we obtain

$$
|T^n x - T^n y| \leq L |x - y|.
$$

(7)

for all $n \geq 1$.

In fact, if $x \in \left[0, \frac{1}{2}\right]$, then $|x - \frac{1}{2}| = |x - Tx|$. Similarly, if $x \in \left[\frac{1}{2}, 1\right]$, then $|x - \frac{1}{2}| = x - \frac{1}{2} \leq x - \frac{\sqrt{2x - x^2}}{\sqrt{3}} = |x - Tx|$. Hence, we get $d(x, F(T)) = |x - \frac{1}{2}| \leq |x - Tx|$. But, $T$ is not Lipschitzian. Indeed, suppose not, i.e., there exists $L > 0$ such that

$$
|Tx - Ty| \leq L |x - y|
$$

for all $x, y \in K$. If we take $x = 1 - \frac{1}{2(1+L)} > \frac{1}{2}$ and $y = 1$, then

$$
\frac{\sqrt{2x - x^2}}{\sqrt{3}} \leq L |1 - x| \iff \frac{\sqrt{2x - x^2}}{\sqrt{3}} \leq \frac{1}{1 + L} = \frac{1}{4 + \sqrt{6} + \sqrt{3}}.
$$

This is a contradiction.

Also, since $\phi$ is strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ and $\mu_n = \frac{1}{n}$, $l_n = \frac{1}{n+1}$, for all $n \geq 1$ and $L > 0$, it follows that we have

$$
L \left(\frac{1}{n} \phi\left(|x - y|\right) + \frac{1}{n+1}\right) \geq 0
$$

(8)

for all $x, y \in K$. Due to (7) and (8), there exists $L > 0$ such that for all $x, y \in K$,

$$
|T^n x - T^n y| \leq L \left[|x - y| + \frac{1}{n} \phi\left(|x - y|\right) + \frac{1}{n+1}\right], \quad n \geq 1.
$$

Then, $T$ is a total uniformly $L$-Lipschitzian mapping on $K$.

Mukhamedov and Saburov [6] studied strong convergence of an explicit iteration process for a totally asymptotically $L$-nonexpansive mapping in Banach spaces. This iteration scheme is defined as follows.

Let $K$ be a nonempty closed convex subset of a real Banach space $E$. Consider $T : K \to K$ is a total asymptotically quasi $L$-nonexpansive mapping, where $I : K \to K$ is a total asymptotically quasi-nonexpansive mapping. Then for two given sequences $\{x_n\}, \{\beta_n\}$ in $[0, 1]$ we shall consider the following iteration scheme:

$$
\begin{align*}
\begin{cases}
x_0 \in K, \\
x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n y_n, \quad n \geq 0, \\
y_n = (1 - \beta_n) x_n + \beta_n I^n x_n.
\end{cases}
\end{align*}
$$

(9)

Inspired and motivated by this facts, we study the convergence theorems of the explicit iterative scheme involving a total asymptotically quasi-$l$-nonexpansive mapping in a nonempty closed convex subset of uniformly convex Banach spaces.

In this paper, we will prove the weak and strong convergences of the explicit iterative process (9) to a common fixed point of $T$ and $I$. 
2. Preliminaries

Recall that a Banach space $E$ is said to satisfy Opial condition [7] if, for each sequence $\{x_n\}$ in $E$ such that $\{x_n\}$ converges weakly to $x$ implies that

$$\liminf_{n \to \infty} \|x_n - x\| < \lim_{n \to \infty} \inf \|x_n - y\|$$

for all $y \in E$ with $y \neq x$. It is well known that (see [4]) inequality (10) is equivalent to

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|.$$  

**Definition 2.1.** Let $K$ be a closed subset of a real Banach space $E$ and let $T : K \to K$ be a mapping. $T$ is said to be semiclosed (demisected) at zero, if for each bounded sequence $\{x_n\}$ in $K$, the conditions $x_n$ converges weakly to $x \in K$ and $Tx_n$ converges strongly to $0$ imply $Tx = 0$.

**Definition 2.2.** Let $K$ be a closed subset of a real Banach space $X$ and let $T : K \to K$ be a mapping. $T$ is said to be semicompact, if for any bounded sequence $\{x_n\}$ in $K$ such that $\|x_n - Tx_n\| \to 0$, $n \to \infty$, then there exists a subsequence $\{x_{n_l}\} \subset \{x_n\}$ such that $x_{n_l} \to x^* \in K$ strongly.

**Lemma 2.3.** [8] Let $X$ be a uniformly convex Banach space and let $b$, $c$ be two constant with $0 < b < c < 1$. Suppose that $\{t_n\}$ is a sequence in $[b, c]$ and $\{x_n\}$, $\{y_n\}$ are two sequence in $X$ such that

$$\lim_{n \to \infty} \|t_n x_n + (1 - t_n) y_n\| = d, \quad \limsup_{n \to \infty} \|x_n\| \leq d, \quad \limsup_{n \to \infty} \|y_n\| \leq d,$$

holds some $d \geq 0$. Then $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

**Lemma 2.4.** [9] Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three sequences of nonnegative real numbers with $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$. If the following conditions is satisfied:

$$a_{n+1} \leq (1 + b_n) a_n + c_n, \quad n \geq 1,$$

then the limit $\lim_{n \to \infty} a_n$ exists.

3. Main Results

In this section, we prove the convergence theorems of an explicit iterative scheme (9) for a total asymptotically quasi-$I$-nonexpansive mapping in Banach spaces. In order to prove our main results, the following lemmas are needed.

**Lemma 3.1.** Let $E$ be real Banach space and $K$ be a nonempty closed convex subset of $E$. Let $T : K \to K$ be a total asymptotically quasi-$I$-nonexpansive mapping with sequences $\{\mu_n\}$, $\{l_n\}$ and $I : K \to K$ be a total asymptotically quasi-nonexpansive mapping with sequences $\{\tilde{\mu}_n\}$, $\{\tilde{l}_n\}$ such that $F = F(T) \cap F(I) \neq \emptyset$. Suppose that there exist $M_i$, $N_i > 0$, $i = 1, 2$, such that $\phi(\zeta) \leq M_2 \zeta$ for all $\zeta \geq M_1$ and $\phi(\zeta) \leq N_2 \zeta$ for all $\zeta \geq N_1$. Then for any $x, y \in K$ we have

$$\|T^p x - p\| \leq (1 + N_2 \mu_n) \|x - p\| + \phi(N_1) \tilde{\mu}_n + \tilde{l}_n$$

(11)
\[ \|T^nx - p\| \leq (1 + M_2\mu_n)(1 + N_2\tilde{\mu}_n)\|x - p\| + (1 + M_2\mu_n)(\phi(N_1)\tilde{\mu}_n + I_n) + \phi(M_1)\mu_n + l_n. \]  

(12)

**Proof.** Since \( \phi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) are strictly increasing continuous functions, it follows that \( \phi(\xi) \leq \phi(M_1) \), \( \varphi(\zeta) \leq \varphi(N_1) \) whenever \( \xi \leq M_1, \zeta \leq N_1 \), respectively. By the hypothesis of lemma we get

\[ \phi(\xi) \leq \phi(M_1) + M_2\xi, \quad \varphi(\zeta) \leq \varphi(N_1) + N_2\zeta, \]  

(13)

for all \( \xi, \zeta \geq 0 \). Since \( T : K \to K, I : K \to K \) are a total asymptotically quasi-I-nonexpansive mapping and a total asymptotically quasi-nonexpansive mapping, respectively, then from (13) we obtain

\[ \|T^nx - p\| \leq (1 + N_2\tilde{\mu}_n)\|x - p\| + \phi(N_1)\tilde{\mu}_n + I_n. \]

Similarly, from (11) and (13) we obtain

\[ \|T^nx - p\| \leq (1 + M_2\mu_n)(\|T^nx - p\| + \phi(M_1)\mu_n + l_n) \]

\[ \leq (1 + M_2\mu_n)(1 + N_2\tilde{\mu}_n)\|x - p\| + (1 + M_2\mu_n)(\phi(N_1)\tilde{\mu}_n + I_n) + \phi(M_1)\mu_n + l_n. \]

This completes the proof. \( \square \)

**Lemma 3.2.** Let \( E \) be real Banach space and \( K \) be a nonempty closed convex subset of \( E \). Let \( T : K \to K \) be a total asymptotically quasi-l-nonexpansive mapping with sequences \( \{\mu_n\}, \{l_n\} \) and \( I : K \to K \) be a total asymptotically quasi-nonexpansive mapping with sequences \( \{\tilde{\mu}_n\}, \{\tilde{l}_n\} \) such that \( F = F(T) \cap F(I) \neq \emptyset \). Also, let \( \{\alpha_n\} \) and \( \{\beta_n\} \) are sequences in \([0, 1]\). Suppose that \( \sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} l_n < \infty, \sum_{n=1}^{\infty} \tilde{\mu}_n < \infty, \sum_{n=1}^{\infty} \tilde{l}_n < \infty \) and there exist \( M_i, N_i > 0, i = 1, 2, \) such that \( \phi(\xi) \leq M_2\xi \) for all \( \xi \geq M_1 \) and \( \varphi(\zeta) \leq N_2\zeta \) for all \( \zeta \geq N_1 \). Then sequence \( \{x_n\} \) by (9) is bounded and for each \( p \in F = F(T) \cap F(I) \) the limit \( \lim_{n \to \infty} \|x_n - p\| \) exists.

**Proof.** Since \( F = F(T) \cap F(I) \neq \emptyset \), for any given \( p \in F \), it follows from (9) and (12) that

\[ \|y_n - p\| \leq (1 + N_2\beta_n\tilde{\mu}_n)\|x_n - p\| + \beta_n\left(\phi(N_1)\tilde{\mu}_n + I_n\right). \]  

(14)

Using a similar method, from (9), (11) and (14), we have

\[ \|x_{n+1} - p\| \leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|T^ny_n - p\| \]

\[ \leq (1 - \alpha_n)\|x_n - p\| + \alpha_n(1 + M_2\mu_n)(1 + N_2\tilde{\mu}_n)\|y_n - p\| + \alpha_n(1 + M_2\mu_n)(\phi(N_1)\tilde{\mu}_n + I_n) + \alpha_n\left(\phi(M_1)\mu_n + l_n\right) \]

\[ \leq \left(1 + \alpha_n\left(1 + M_2\mu_n\right)\left(1 + N_2\tilde{\mu}_n\right) - 1\right)\|x_n - p\| + \alpha_n\left(1 + M_2\mu_n\right)\left(\phi(N_1)\tilde{\mu}_n + I_n\right)\left(\beta_n\left(1 + N_2\tilde{\mu}_n\right) + 1\right) + \phi(M_1)\mu_n + l_n. \]  

(15)
Defining
\[ a_n = \|x_n - p\| \]
\[ b_n = a_n \left( 1 + M_2 \mu_n \right) \left( 1 + N_2 \mu_n \right) \left( 1 + N_2 \beta_n \mu_n \right) - 1 \]
\[ c_n = a_n \left( 1 + M_2 \mu_n \right) \left( \phi(N_1) \mu_n + l_n \right) \left( 1 + N_2 \beta_n \mu_n \right) + 1 + \phi(M_1) \mu_n + l_n \]
in (15) we have \( a_{n+1} \leq (1 + b_n) a_n + c_n \). Since \( \sum_{n=1}^{\infty} b_n < \infty \), \( \sum_{n=1}^{\infty} c_n < \infty \), Lemma 2.4 implies the existence of

the limit \( \lim_{n \to \infty} a_n \). This completes the proof.

**Theorem 3.3.** Let \( E \) be a real Banach space and \( K \) be a nonempty closed convex subset of \( E \). Let \( T : K \to K \) be a total uniformly \( L_1 \)-Lipschitzian asymptotically quasi-I-nonexpansive mapping with sequences \( \{ \mu_n \} \) and \( I : K \to K \) be a total uniformly \( L_2 \)-Lipschitzian asymptotically quasi-nonexpansive mapping with sequences \( \{ \mu_n \} \), \( \{ l_n \} \) such that

\( F = F(T) \cap F(I) \neq \emptyset \). Suppose that \( \sum_{n=1}^{\infty} \mu_n < \infty \), \( \sum_{n=1}^{\infty} l_n < \infty \), \( \sum_{n=1}^{\infty} \mu_n \phi(M_1) < \infty \), \( \sum_{n=1}^{\infty} l_n \phi(M_1) < \infty \) and there exist \( M_i, N_i > 0 \), \( i = 1, 2 \), such that \( \phi(\xi) \leq M_2 \xi \) for all \( \xi \geq M_1 \) and \( \phi(\xi) \leq N_2 \xi \) for all \( \xi \geq N_1 \). Then the sequence \( \{ x_n \} \) by \( (9) \), converges strongly to a common fixed point of \( F = F(T) \cap F(I) \) if and only if

\[ \lim_{n \to \infty} d(x_n, F) = 0. \]  

**Proof.** For any given \( p \in F \), we have (see (15))

\[ \|x_{n+1} - p\| \leq (1 + b_n) \|x_n - p\| + c_n, \quad n \geq 1. \]  

(17)

It suffices to show that \( \lim_{n \to \infty} d(x_n, F) = 0 \) implies that \( \{ x_n \} \) converges to a common fixed point of \( T \) and \( I \).

Necessity. Since (17) holds for all \( p \in F \), we obtain from it that

\[ d(x_{n+1}, F) \leq (1 + b_n) d(x_n, F) + c_n, \quad n \geq 1. \]

Lemma 2.4 implies that \( \lim_{n \to \infty} d(x_n, F) \) exists. But, \( \lim_{n \to \infty} d(x_n, F) = 0 \). Hence, \( \lim_{n \to \infty} d(x_n, F) = 0 \).

Sufficiency. Let us prove that the sequence \( \{ x_n \} \) converges to a common fixed point of \( T \) and \( I \). Firstly, we show that \( \{ x_n \} \) is a Cauchy sequence in \( E \). In fact, as \( 1 + t \leq \exp(t) \) for all \( t > 0 \). For all integer \( m \geq 1 \), we obtain from inequality (17) that

\[ \|x_{n+m} - p\| \leq \exp \left( \sum_{i=n}^{n+m-1} b_i \right) \|x_n - p\| + \left( \sum_{i=n}^{n+m-1} c_i \right) \exp \left( \sum_{i=n}^{n+m-1} b_i \right), \]

so that for all integers \( m \geq 1 \) and all \( p \in F \),

\[ \|x_{n+m} - x_n\| \leq \|x_{n+m} - p\| + \|x_n - p\| \]

\[ \leq \left( 1 + \exp \left( \sum_{i=n}^{\infty} b_i \right) \right) \|x_n - p\| + \exp \left( \sum_{i=n}^{\infty} b_i \right) \sum_{i=n}^{\infty} c_i \]

\[ \leq A \left( \|x_n - p\| + \sum_{i=n}^{\infty} c_i \right), \]

(18)
for all \( p \in F \), where \( 0 < A - 1 = \exp(\sum_{i=n}^{\infty} b_i) < \infty \). Taking the infimum over \( p \in F \) in (18) gives
\[
\|x_{n+m} - x_n\| \leq A \left( d(x_n, F) + \sum_{i=n}^{\infty} c_i \right),
\]
(19)

Now, since \( \lim_{n \to \infty} d(x_n, F) = 0 \) and \( \sum_{i=n}^{\infty} c_i < \infty \), given \( \varepsilon > 0 \), there exists an integer \( n_0 > 0 \) such that for all \( n > n_0 \) we have \( d(x_n, F) < \frac{\varepsilon}{2} \) and \( \sum_{i=n}^{\infty} c_i < \frac{\varepsilon}{2} \). So, for all integers \( n > n_0 \) and \( m \geq 1 \), we obtain (19) that
\[
\|x_{n+m} - x_n\| \leq \varepsilon
\]
which means that \( \{x_n\} \) is a Cauchy sequence in \( E \), and completeness of \( E \) yields the existence of \( x^* \in E \) such that \( x_n \to x^* \) strongly.

Now, we show that \( x^* \) is a common fixed point of \( T \) and \( I \). Suppose that \( x^* \notin F \). Since \( F \) is closed subset of \( E \), one has \( d(x^*, F) > 0 \). However, for all \( p \in F \), we have
\[
\|x^* - p\| \leq \|x_n - x^*\| + \|x_n - p\|.
\]
This implies that
\[
d(x^*, F) \leq \|x_n - x^*\| + d(x_n, F),
\]
so, we obtain \( d(x^*, F) = 0 \) as \( n \to \infty \), which contradicts \( d(x^*, F) > 0 \). Hence, \( x^* \) is a common fixed point of \( T \) and \( I \). This completes the proof.

**Lemma 3.4.** Let \( E \) be a real uniformly Banach space and \( K \) be a nonempty closed convex subset of \( E \). Let \( T : K \to K \) be a total uniformly \( L_1 \)-Lipschitzian asymptotically quasi-\( I \)-nonexpansive mapping with sequences \( \{\mu_n\}, \{l_n\} \) and \( I : K \to K \) be a total uniformly \( L_2 \)-Lipschitzian asymptotically quasi-\( I \)-nonexpansive mapping with sequences \( \{\mu_n\}, \{l_n\} \) such that \( F = F(T) \cap F(I) \neq \emptyset \). Suppose that \( \sum_{n=1}^{\infty} \mu_n < \infty \), \( \sum_{n=1}^{\infty} l_n < \infty \), \( \sum_{n=1}^{\infty} \tilde{\mu}_n < \infty \), \( \sum_{n=1}^{\infty} \tilde{l}_n < \infty \) and there exist \( M_i, N_i > 0 \), \( i = 1, 2 \), such that \( \phi(\xi) \leq M_2 \xi \) for all \( \xi \geq M_1 \) and \( \phi(\zeta) \leq N_2 \zeta \) for all \( \zeta \geq N_1 \). Assume that \( \{\alpha_n\} \) and \( \{\beta_n\} \) are two sequences in \([0, 1 - l] \), where \( 0 < l < 1 \). Then the sequence \( \{x_n\} \) by (9) satisfies the following:
\[
\lim_{n \to \infty} \|x_n - T x_n\| = 0, \tag{20}
\]
\[
\lim_{n \to \infty} \|x_n - I x_n\| = 0. \tag{21}
\]

**Proof.** By Lemma 3.2, \( \lim_{n \to \infty} \|x_n - p\| \) exists. Assume that, for any \( p \in F = F(T) \cap F(I), \lim_{n \to \infty} \|x_n - p\| = r \). If \( r = 0 \), the conclusion is obvious. Suppose \( r > 0 \).

First, we will prove that
\[
\lim_{n \to \infty} \|x_n - T^nx_n\| = 0, \quad \lim_{n \to \infty} \|x_n - P^nx_n\| = 0. \tag{22}
\]
It follows from (9) that
\[
\|x_{n+1} - p\| = \|(1 - \alpha_n)(x_n - p) + \alpha_n(T^ny_n - p)\| \to r, \tag{23}
\]
as \( n \to \infty \). By means of \( \sum_{n=1}^{\infty} \mu_n < \infty \), \( \sum_{n=1}^{\infty} l_n < \infty \), \( \sum_{n=1}^{\infty} \tilde{\mu}_n < \infty \), \( \sum_{n=1}^{\infty} \tilde{l}_n < \infty \), from (12) and (14) we get

\[
\lim_{n \to \infty} \sup \left\| T^n y_n - p \right\| \leq \lim_{n \to \infty} \sup \left\| y_n - p \right\| \leq \lim_{n \to \infty} \sup \left\| x_n - p \right\| = r. \tag{24}
\]

Hence, using (23), (24) and Lemma 2.3, we obtain

\[
\lim_{n \to \infty} \left\| x_n - T^n y_n \right\| = 0. \tag{25}
\]

From (9) and (25) we have

\[
\lim_{n \to \infty} \left\| x_{n+1} - x_n \right\| = 0. \tag{26}
\]

From (25) and (26) we get

\[
\lim_{n \to \infty} \left\| x_{n+1} - T^n y_n \right\| \leq \lim_{n \to \infty} \left\| x_{n+1} - x_n \right\| + \lim_{n \to \infty} \left\| x_n - T^n y_n \right\| = 0. \tag{27}
\]

On the other hand, from (12) and (14) we have

\[
\left\| x_n - p \right\| \leq \left\| x_n - T^n y_n \right\| + (1 + M_2 \mu_n) \left( 1 + N_2 \tilde{\mu}_n \right) \left\| y_n - p \right\| \\
+ (1 + M_2 \mu_n) (\phi(N_1) \tilde{\mu}_n + \tilde{l}_n) + \phi(M_1) \mu_n + l_n \\
\leq \left\| x_n - T^n y_n \right\| + (1 + M_2 \mu_n) \left( 1 + N_2 \tilde{\mu}_n \right) \left( 1 + N_2 \beta_n \tilde{\mu}_n \right) \left\| x_n - p \right\| \\
+ (1 + M_2 \mu_n) (\phi(N_1) \tilde{\mu}_n + \tilde{l}_n) \left( \beta_n \left( 1 + N_2 \tilde{\mu}_n \right) + 1 \right) + \phi(M_1) \mu_n + l_n. \tag{28}
\]

From (28) we obtain

\[
\lim_{n \to \infty} \left\| x_n - p \right\| \leq \lim_{n \to \infty} \left\| x_n - T^n y_n \right\| + \lim_{n \to \infty} \left\| y_n - p \right\| \\
\leq \lim_{n \to \infty} \left\| x_n - T^n y_n \right\| + \lim_{n \to \infty} \left\| x_n - p \right\|. \tag{29}
\]

Then (29) with the squeeze theorem, imply that

\[
\lim_{n \to \infty} \left\| y_n - p \right\| = r.
\]

From (9) we can see that

\[
\left\| y_n - p \right\| = \left\| (1 - \beta_n) (x_n - p) + \beta_n (T^n x_n - p) \right\| \to r, \quad n \to \infty. \tag{30}
\]

Furthermore, from (11) we get

\[
\lim_{n \to \infty} \sup \left\| T^n x_n - p \right\| \leq \lim_{n \to \infty} \sup \left\| x_n - p \right\| = r. \tag{31}
\]

Now, applying Lemma 2.3 to (30) we obtain

\[
\lim_{n \to \infty} \left\| x_n - T^n x_n \right\| = 0. \tag{32}
\]
From (26) and (32) we have
\[
\lim_{n \to \infty} \|x_{n+1} - I^nx_n\| \leq \lim_{n \to \infty} \|x_{n+1} - x_n\| + \lim_{n \to \infty} \|x_n - I^nx_n\| = 0.
\] (33)

It follows from (9) that
\[
\|y_n - x_n\| = \beta_n \|x_n - I^nx_n\|.
\] (34)

Hence, from (32) and (34) we obtain
\[
\lim_{n \to \infty} \|y_n - x_n\| = 0.
\] (35)

Consider
\[
\|x_n - T^nx_n\| \leq \|x_n - T^ny_n\| + L_1 \|y_n - x_n\| + L_1 \left( \mu_n \phi \left( \|x_n - y_n\| \right) \right) + l_n.
\] (36)

Then, from (25), (35) and (36) we obtain
\[
\lim_{n \to \infty} \|x_n - T^nx_n\| = 0.
\] (37)

From (26) and (35) we have
\[
\lim_{n \to \infty} \|x_{n+1} - y_n\| \leq \lim_{n \to \infty} \|x_{n+1} - x_n\| + \lim_{n \to \infty} \|y_n - x_n\| = 0.
\] (38)

Finally, from
\[
\|x_n - Tx_n\| \leq \|x_n - T^nx_n\| + L_1 \|x_n - y_{n-1}\| + L_1 \left( \mu_n \phi \left( \|x_n - y_{n-1}\| \right) \right) + l_n
\]
\[
+ L_1 \|T^{n-1}y_{n-1} - x_n\| + L_1 \left( \mu_n \phi \left( \|T^{n-1}y_{n-1} - x_n\| \right) \right) + l_n,
\] (39)

which with (27), (37) and (38) we get
\[
\lim_{n \to \infty} \|x_n - Tx_n\| = 0.
\] (40)

Similarly, we obtain
\[
\|x_n - Ix_n\| \leq \|x_n - T^nx_n\| + L_2 \|x_n - x_{n-1}\| + L_2 \left( \mu_n \phi \left( \|x_n - x_{n-1}\| \right) \right) + l_n
\]
\[
+ L_2 \|P^{n-1}x_{n-1} - x_n\| + L_2 \left( \mu_n \phi \left( \|P^{n-1}x_{n-1} - x_n\| \right) \right) + l_n,
\] (41)

which with (26), (32) and (33) implies
\[
\lim_{n \to \infty} \|x_n - Ix_n\| = 0.
\] (42)

This completes the proof.
Theorem 3.5. Let $E$ be a real uniformly Banach space satisfying Opial condition and let $K$ be a nonempty closed convex subset of $E$. Let $C : E \to E$ be an identity mapping. Let $T : K \to K$ be a total uniformly $L_1$-Lipschitzian asymptotically quasi-I-nonexpansive mapping with sequences $\{\mu_n\}, \{l_n\}$ and $I : K \to K$ be a total uniformly $L_2$-Lipschitzian asymptotically quasi-nonexpansive mapping with sequences $\{\tilde{\mu}_n\}, \{\tilde{l}_n\}$ such that $F = F(T) \cap F(I) \neq \emptyset$.

Suppose that $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} l_n < \infty$, $\sum_{n=1}^{\infty} \tilde{\mu}_n < \infty$, $\sum_{n=1}^{\infty} \tilde{l}_n < \infty$ and there exist $M_i, N_i > 0, i = 1, 2$, such that $\phi(\xi) \leq M_2 \xi$ for all $\xi \geq M_1$ and $\phi(\zeta) \leq N_2 \zeta$ for all $\zeta \geq N_1$. Assume that $\{\alpha_n\}, \{\beta_n\}$ are two sequences in $[t, 1 - t]$, where $0 < t < 1$. If the mappings $C - T$ and $C - I$ are semiclosed at zero, then the explicit iterative sequence $\{x_n\}$ defined by (9) converges weakly to a common fixed point of $T$ and $I$.

Proof. Let $p \in F = F(T) \cap F(I)$. By Lemma 3.2, we know that $\lim_{n \to \infty} \|x_n - p\|$ exists and $\{x_n\}$ is bounded. Since $E$ is uniformly convex, then every bounded subset of $E$ is weakly compact. Since $\{x_n\}$ is a bounded sequence in $K$, then there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $q_1 \in K$. Thus, from (40) and (42) it follows that

$$\lim_{n_j \to \infty} \|x_{n_j} - Tx_{n_j}\| = 0, \quad \lim_{n_j \to \infty} \|x_{n_j} - Ix_{n_j}\| = 0.$$  \hspace{1cm} (43)

Since the mappings $C - T$ and $C - I$ are semiclosed at zero, we find $Tq_1 = q_1$ and $Iq_1 = q_1$. Namely, $q_1 \in F = F(T) \cap F(I)$.

Finally, let us prove that $\{x_n\}$ converges weakly to $q_1$. Actually, suppose the contrary, that is, there exists some subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $q_2 \in K$ and $q_1 \neq q_2$. Then by the same method as given above, we can also prove that $q_2 \in F = F(T) \cap F(I)$.

Since $q_1, q_2 \in F = F(T) \cap F(I)$, according to Lemma 3.2 $\lim_{n \to \infty} \|x_n - q_1\|$ and $\lim_{n \to \infty} \|x_n - q_2\|$ exist, we have

$$\lim_{n \to \infty} \|x_n - q_1\| = r_1, \quad \lim_{n \to \infty} \|x_n - q_2\| = r_2,$$  \hspace{1cm} (44)

where $d_1, d_2 \geq 0$. Because of the Opial condition of $E$, we obtain

$$r_1 = \lim_{n \to \infty} \sup_{n_j \to \infty} \|x_{n_j} - q_1\| < \lim_{n \to \infty} \sup_{n \to \infty} \|x_n - q_1\| = r_2$$

$$r_1 = \lim_{n \to \infty} \sup_{n_j \to \infty} \|x_{n_j} - q_1\| < \lim_{n \to \infty} \sup_{n \to \infty} \|x_n - q_2\|.  \hspace{1cm} (45)$$

This is a contradiction. Hence $q_1 = q_2$. This implies that $\{x_n\}$ converges weakly to $q$. This completes the proof. \hfill \Box

Theorem 3.6. Let $E$ be a real uniformly Banach space and $K$ be a nonempty closed convex subset of $E$. Let $T : K \to K$ be a total uniformly $L_1$-Lipschitzian asymptotically quasi-I-nonexpansive mapping with sequences $\{\mu_n\}, \{l_n\}$ and $I : K \to K$ be a total uniformly $L_2$-Lipschitzian asymptotically quasi-nonexpansive mapping with sequences $\{\tilde{\mu}_n\}, \{\tilde{l}_n\}$ such that $F = F(T) \cap F(I) \neq \emptyset$. Suppose that $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} l_n < \infty$, $\sum_{n=1}^{\infty} \tilde{\mu}_n < \infty$, $\sum_{n=1}^{\infty} \tilde{l}_n < \infty$ and there exist $M_i, N_i > 0, i = 1, 2$, such that $\phi(\xi) \leq M_2 \xi$ for all $\xi \geq M_1$ and $\phi(\zeta) \leq N_2 \zeta$ for all $\zeta \geq N_1$. Assume that $\{\alpha_n\}, \{\beta_n\}$ are two sequences in $[t, 1 - t]$, where $0 < t < 1$. If at least one mapping of the mappings $T$ and $I$ is semicompact, then the explicit iterative sequence $\{x_n\}$ defined by (9) converges strongly to a common fixed point of $T$ and $I$. 

\hfill \Box
Proof. Without any loss of generality, we may assume that $T$ is semicompact. This with (40) means that there exists a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ such that $x_{n_k} \to x^*$ strongly and $x^* \in K$. Since $T, I$ are continuous, then from (40) and (42) we find

$$\|x^* - Tx^*\| = \lim_{n_k \to \infty} \|x_{n_k} - Tx_{n_k}\| = 0, \quad \|x^* - Ix^*\| = \lim_{n_k \to \infty} \|x_{n_k} - Ix_{n_k}\| = 0. \tag{46}$$

This shows that $x^* \in F = F(T) \cap F(I)$. According to Lemma 3.2 the limit $\lim_{n \to \infty} \|x_n - x^*\|$ exists. Then

$$\lim_{n \to \infty} \|x_n - x^*\| = \lim_{n_k \to \infty} \|x_{n_k} - x^*\| = 0,$$

which means that $\{x_n\}$ converges to $x^* \in F$. This completes the proof. \(\square\)

References


