Some Equalities on $q$-Gamma and $q$-Digamma Functions

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Abstract. In this paper, we give some equalities on $q$-gamma and $q$-digamma functions for negative integer values of $x$ by aid of using the concepts of neutrix and neutrix limit.

1. Introduction and Preliminaries

$Q$-calculus; the $q$-analogue of the classical calculus; is a quite popular subject today. The deformed calculus has found a lot of applications in mathematics, statistics and physics. These are many $q$-functions of the ordinary special functions such as $q$-beta, $q$-gamma and $q$-zeta functions. In a rude way, the definition of a $q$-analogue $M_q$ of a mathematical object $M$ is such that the limit of $M_q$ as $q$ tends to 1 is $M$. In this paper we aim to give some equalities on $q$-gamma and $q$-digamma functions by using their $q$-integral representations and the neutrix calculus developed by van der Corput.

Definition 1.1. (Neutrix) Let $N'$ be a nonempty set and let $N$ be a commutative, additive group of functions mapping $N'$ into a commutative, additive group $N''$. The group $N$ is called neutrix if the function which is identically equal to zero is the only constant function occurring in $N$. The function which belongs to $N$ is called “negligible function” in $N$.

Let $N'$ be a domain lying in a topological space with a limit point $b$ not belonging to $N'$ and $N$ be a commutative additive group of functions defined on $N'$ with the following property:

"$f \in N$, $\lim_{\epsilon \to b} f(\epsilon) = c$ (constant) for $\epsilon \in N'$ then $c = 0$".

Then this group $N$ is a neutrix.

Definition 1.2. (Neutrix limit) Let $f$ be a real valued function defined on $N'$ and suppose that it is possible to find a constant $c$ such that $f(x) - c$ is negligible in $N$. Then $c$ is called the neutrix limit of $f(x)$ as $x$ tends to $y$ and denoted by

$$N\text{-}\lim_{x \to y} f(x) = c.$$
Let \( f \) be a function defined on a subject of real or complex plane. The \( q \)-gamma function is defined by the neutrix limit as

\[
\Gamma_q(x) = \int_0^1 t^{x-1} E_q^{-\alpha} dt,
\]

and its derivatives are defined by

\[
\Gamma_q^{(r)} = \int_0^1 t^{x-1} \ln^r E_q^{-\alpha} dt, \quad r = 0, 1, 2, \ldots
\]

Using the regularization technique, it has been shown in [6] that for \( x > -n, n = 1, 2, \ldots, x \neq 0, -1, -2, \ldots \), the \( q \)-gamma function is defined by the neutrix limit as

\[
\Gamma_q(x) = N^{-\lim}_{\epsilon \to 0} \int_{e^{-\epsilon}}^0 t^{x-1} E_q^{-\alpha} dt
\]
and in [9] the authors give an equation for the function $\Gamma_q(x)$ for negative integer values of $x$ with using the Heaviside’s function $H(x)$; which is equal to zero for $x < 0$ and to 1 for $x > 0$. That is

$$
\Gamma_q(-n) = \int_0^{1/1-q} t^{-n-1} \ln t \left[ E_q^{-qt} - \sum_{j=0}^{n-1} \frac{(-1)^j q^{\frac{j+1}{q}}}{[j]!} t^j - \frac{(-1)^n q^{\frac{n+1}{q}}}{[n]!} t^n H(1-t) \right] dt
$$

$$
+ (1-q)^{n+1} \sum_{j=0}^{n-1} \frac{(-1)^j q^{\frac{j+1}{q}}}{(q, q_j) (1-q^{-n})}.
$$

(4)

2. Main Results

In this section, using neutrix calculus, we give some results on $q$-gamma function with its first derivative and then show that $q$-digamma function can be defined at negative integers. At first, we need the following lemmas.

Lemma 2.1. We have

$$
\Gamma_q(0) = \frac{q-1}{lnq} \Gamma_q'(1).
$$

(5)

Proof. By taking $g(t) = E_q^{-qt}$, $d_q(f(t)) = t^{-1}$ and using $q$-integration by parts given in (2), then we obtain

$$
\Gamma_q(0) = N \lim_{\epsilon \to 0} \int_{\epsilon}^{1/1-q} t^{-1} E_q^{-qt} dt
$$

$$
= \frac{q-1}{lnq} \ln (1-q) E_q^{-qt} - \frac{q-1}{lnq} \ln \epsilon E_q^{-\epsilon} + \frac{q-1}{lnq} \int_{\epsilon}^{1/1-q} \ln t E_q^{-qt} dt
$$

Since $E_q^{(0)} = (1 + (1-q)t)^{\frac{1}{q}}$, the first term on the right side is equal to zero and second one includes $\ln \epsilon$, which is negligible function, so the neutrix limit of the last term collides with ordinary one which is equal to the equation (3) at $x = 1$, then we get the desired result.

Lemma 2.2. For $n = 1, 2, \ldots$,

$$
\Gamma_q(-n) = \frac{1}{[-n]} \Gamma_q(-n + 1) + \frac{(-1)^n q^{\frac{n+1}{q}}}{[n][n]!}.
$$

(6)

Proof. On $q$-integrating by parts to equation (4) we have

$$
\Gamma_q(-n) = \frac{1}{[-n]} \left\{ \int_{1}^{1/1-q} t^{-n} E_q^{-qt} dt + \int_{0}^{1} t^{-n} E_q^{-qt} - \sum_{j=0}^{n-1} \frac{(-1)^j q^{\frac{j+1}{q}}}{[j]!} t^j dt \right\}
$$

$$
- E_q^{-1} \frac{1}{[-n]} \left[ E_q^{-1} - \sum_{j=0}^{n-1} \frac{(-1)^j q^{\frac{j+1}{q}}}{[j]!} \right] + \sum_{j=0}^{n-1} \frac{(-1)^j q^{\frac{j+1}{q}}}{[j]! [-n + j]}
$$

$$
= \frac{1}{[-n]} \left\{ \int_{1}^{1/1-q} t^{-n} E_q^{-qt} dt + \int_{0}^{1} t^{-n} E_q^{-qt} - \sum_{j=0}^{n-1} \frac{(-1)^j q^{\frac{j+1}{q}}}{[j]!} t^j dt \right\}
$$

$$
+ \sum_{j=0}^{n-2} \frac{(-1)^j q^{\frac{j+1}{q}}}{[j][n-j+1]} - \frac{(-1)^n q^{\frac{n+1}{q}}}{[n][n]!}
$$

$$
= \frac{1}{[-n]} \Gamma_q(-n + 1) + \frac{(-1)^n q^{\frac{n+1}{q}}}{[n][n]!}.
$$
as desired.

Note that by using the equations (5) and (6) and mathematical induction, \( q \)-gamma function satisfies the equation

\[
\Gamma_q(-n) = \left(\frac{(-1)^n q^{-\frac{n+1}{2}}}{[n]!}\right) (\varphi_q(n) + \Gamma_q(0))
\]  

(7)

where (for example, see [10] and references therein)

\[
\varphi_q(n) = \sum_{j=1}^{n} \frac{1}{j}
\]

for \( n = 0, -1, -2, \ldots \) and this result tends to the equation

\[
\Gamma(-n) = (-1)^n (\varphi(n) + \Gamma(0))
\]

where

\[
\varphi(n) = \sum_{j=0}^{n} \frac{1}{j}
\]

shown in [4] and [11] as \( q \to 1 \).

**Theorem 2.3.** Let \( H \) denotes Heaviside’s function. Then for \( n \in \mathbb{N} \),

\[
\Gamma_q(-n) = \int_{0}^{1/q^{1-q}} t^{-n-1} \ln \left[t \sum_{j=0}^{n-1} (-1)^j q^{\frac{j+1}{2}} \left[\frac{1}{[j]! t^{j}} - \frac{(-1)^n q^{\frac{n+1}{2}}}{[n]! t^{n}H(1-t)}\right] \right] dt
\]

\[
+ \ln q^{-1} (1-q)^{-1} \sum_{j=0}^{n-1} \frac{(-1)^j q^{\frac{j+1}{2}}}{[j]![n-j]} (1-q)^{n-j} + \ln q^{-1} \sum_{j=0}^{n-1} \frac{(-1)^j q^{\frac{j+1}{2}}}{[j]!(n-j)} (1-q)^{n-j}.
\]  

(8)

**Proof.** By definitions, we have

\[
\int_{e}^{1} t^{-n-1} \ln t E_q^{-\theta q} dt = \int_{e}^{1/q^{1-q}} t^{-n-1} \ln \left[t \sum_{j=0}^{n-1} (-1)^j q^{\frac{j+1}{2}} \left[\frac{1}{[j]! t^{j}} - \frac{(-1)^n q^{\frac{n+1}{2}}}{[n]! t^{n}H(1-t)}\right] \right] dt
\]

\[
+ \ln q^{-1} (1-q)^{-1} \sum_{j=0}^{n-1} \frac{(-1)^j q^{\frac{j+1}{2}}}{[j]![n-j]} (1-q)^{n-j} + \ln q^{-1} \sum_{j=0}^{n-1} \frac{(-1)^j q^{\frac{j+1}{2}}}{[j]!(n-j)} (1-q)^{n-j}.
\]

Now calculating the last two integrals on the right side of the equation and then taking the neutrix limit of the both sides of the equation we get

\[
N \lim_{\varepsilon \to 0} \int_{e}^{1} t^{-n-1} \ln t E_q^{-\theta q} dt = \int_{e}^{1/q^{1-q}} t^{-n-1} \ln \left[t \sum_{j=0}^{n-1} (-1)^j q^{\frac{j+1}{2}} \left[\frac{1}{[j]! t^{j}} - \frac{(-1)^n q^{\frac{n+1}{2}}}{[n]! t^{n}H(1-t)}\right] \right] dt
\]

\[
+ \ln q^{-1} (1-q)^{-1} \sum_{j=0}^{n-1} \frac{(1-q)^{n-j}}{[n-j]} + \ln q^{-1} (1-q)^{n-j} (q-1)[n+j]^{-2}
\]

as desired. □
Theorem 2.4. For \( n = 1, 2, \ldots \)

\[
\Gamma_q'(-n) = \frac{\ln q^{-1}}{(q-1)[n]} \Gamma_q(-n) + \frac{\ln q^{-1}}{[n]} \Gamma_q(-n + 1) + \frac{1}{[n]} \Gamma_q'(-n + 1).
\]

(9)

Proof. On \( q \)-integrating by parts to equation (8) we have

\[
\Gamma_q'(-n) = \frac{\ln q^{-1}}{(q-1)[n]} \int_0^{1/q} t^{-n-1} E_q^{\#} - \sum_{j=0}^{n-1} \frac{(-1)^n q^{(n+1)} \Gamma}{[j]!} t^j - \frac{(-1)^n q^{(n+1)} t^n H(1-t)}{[n-1]} dt
\]

\[
+ \frac{\ln q^{-1}}{[n]} \int_0^{1/q} t^{-n} \ln E_q^{\#} - \sum_{j=0}^{n-2} \frac{(-1)^n q^{(n+1)} \Gamma}{[j]!} t^j - \frac{(-1)^n q^{(n+1)} t^n H(1-t)}{[n-1]} dt
\]

\[
+ \frac{1}{[n]} \int_0^{1/q} t^{-n} \ln \left[ E_q^{\#} - \sum_{j=0}^{n-1} \frac{(-1)^n q^{(n+1)} \Gamma}{[j]!} t^j - \frac{(-1)^n q^{(n+1)} t^n H(1-t)}{[n-1]} \right] dt
\]

\[
- \frac{\ln q^{-1}}{(q-1)[n]} \sum_{j=0}^{n} \frac{(-1)^n q^{(n+1)} \Gamma}{[j]!} (1 - q)^{n-j} - \ln (1 - q)^{-1} \sum_{j=0}^{n} \frac{(-1)^n q^{(n+1)} \Gamma}{[j]!} (1 - q)^{n-j}.
\]

The first three terms on the right side of the equation are the integral parts of the definitions of \( \Gamma_q(-n) \), \( \Gamma_q(-n + 1) \) and \( \Gamma_q'(-n + 1) \) respectively, because of that, adding and extracting the missing series of these definitions one can see that sums of the remaining series are equal to zero and this completes the proof. \( \square \)

Theorem 2.5. For all real values of \( x \),

\[
\Gamma_q'(x) = N_{\varepsilon \to 0} \Gamma_q'(x + \varepsilon).
\]

(10)

Proof. Since \( \Gamma_q'(x) \) is a continuous function for \( x \neq 0, -1, -2, \ldots \) its neutrally limit becomes normal limit as \( \varepsilon \) tends to zero and the result follows for \( x \neq 0, -1, -2, \ldots \). Now we will consider \( \Gamma_q'(x) \) at the point \( x = -n, n = 1, 2, \ldots \) For \( 0 < \varepsilon < 1 \), we have from equation (8) that

\[
\Gamma_q'(-n + \varepsilon) = \int_0^{1/q} t^{-n+\varepsilon-1} \ln E_q^{\#} - \sum_{j=0}^{n-1} \frac{(-1)^n q^{(n+1)} \Gamma}{[j]!} t^j \] \[dt\]

\[
+ \ln (1 - q)^{-1} \sum_{j=0}^{n-1} \frac{(-1)^n q^{(n+1)} \Gamma}{[j]!} (1 - q)^{n-j} + \ln (1 - q)^{-1} \sum_{j=0}^{n-1} \frac{(-1)^n q^{(n+1)} \Gamma}{[j]!} (1 - q)^{n-j}
\]

\[
= \int_0^{1/q} t^{-n+\varepsilon-1} \ln E_q^{\#} - \sum_{j=0}^{n-1} \frac{(-1)^n q^{(n+1)} \Gamma}{[j]!} t^j \] \[dt\]

\[
+ \ln (1 - q)^{-1} \sum_{j=0}^{n-1} \frac{(-1)^n q^{(n+1)} \Gamma}{[j]!} (1 - q)^{n-j} + \ln (1 - q)^{-1} \sum_{j=0}^{n-1} \frac{(-1)^n q^{(n+1)} \Gamma}{[j]!} (1 - q)^{n-j}
\]

\[
+ \frac{(-1)^n q^{(n+1)} \Gamma}{[n]!} \int_\varepsilon^1 t^{-1} \ln tdq.t.
\]
Note that the neutrix limit is unique and its precisely the same as the ordinary one, if it exists. Then taking neutrix limit of both sides, we obtain

\[
\begin{align*}
N \lim_{\epsilon \to 0} \Gamma_{q}^\epsilon(-n + \epsilon) &= \int_{0}^{1} \frac{1}{q^{x-1}} - \sum_{j=0}^{n-1} \frac{(1-q^j)}{[j]!} q^{\frac{x-1}{q^j}} t^j - \frac{(1-q^\epsilon)^{n+1}}{[n]!} t^n H(1-t) d_q t \\
&+ \ln q^{-1} (1-q)^{-1} \sum_{j=0}^{n-1} \frac{(1-q^j)}{[j]!} (1-q)^{n-j} + \frac{\ln q^{-1} \sum_{j=0}^{n-1} (1-q^j)}{q-1} \sum_{j=0}^{n} [j]! (1-q)^{n-j} \\
&= \Gamma_{q}(-n).
\end{align*}
\]

The case of \( x = -n - \epsilon \) for \( 0 < \epsilon < 1 \) can be proved similarly.

For any \( x > 0 \), we have

\[
\Gamma_{q}(x + 1) = [x] \Gamma_{q}(x). \tag{11}
\]

It has been shown in [9] that equation (11) can be given for all real numbers by the neutrix limit such that

\[
\Gamma_{q}(x + 1) = N \lim_{\epsilon \to 0} [x + \epsilon] \Gamma_{q}(x + \epsilon). \tag{12}
\]

Differentiating equation (11), we get

\[
\Gamma_{q}^\epsilon(x + 1) = -q^{x} \ln q \Gamma_{q}(x) + [x] \Gamma_{q}^\epsilon(x) \tag{13}
\]

for \( x \neq 0, -1, -2, \ldots \).

Now we give that, equation (13) can be extended for all real values of \( x \).

**Theorem 2.6.** For all \( x \) we have

\[
\Gamma_{q}^\epsilon(x + 1) = N \lim_{\epsilon \to 0} \frac{q^{x+\epsilon} \ln q}{q-1} \Gamma_{q}(x + \epsilon) + [x + \epsilon] \Gamma_{q}^\epsilon(x + \epsilon).
\]

**Proof.** The result can easily be obtained because of the continuity of \( \Gamma_{q}^\epsilon(x) \) for \( x \neq 0, -1, -2, \ldots \). Equation (8) also satisfies for all real values of \( x \). By rewriting this equation for \( n = 1, 2, \ldots \) and \( 0 < \epsilon < 1 \) as

\[
\Gamma_{q}^\epsilon(-n + \epsilon + 1) = [-n] \Gamma_{q}^\epsilon(-n + \epsilon) - \frac{\ln q^{-1}}{q-1} \Gamma_{q}(-n + \epsilon) - \ln q^{-1} \Gamma_{q}(-n + \epsilon + 1)
\]

then we get from taking the neutrix limit of both sides that

\[
N \lim_{\epsilon \to 0} \Gamma_{q}^\epsilon(-n + \epsilon + 1) = N \lim_{\epsilon \to 0} [-n + \epsilon] \Gamma_{q}^\epsilon(-n + \epsilon) - \frac{\ln q^{-1}}{q-1} \Gamma_{q}(-n + \epsilon) - \ln q^{-1} \Gamma_{q}(-n + \epsilon + 1).
\]

Hence with the previous theorem and equation (2) we have

\[
\begin{align*}
&= N \lim_{\epsilon \to 0} [-n + \epsilon] \Gamma_{q}^\epsilon(-n + \epsilon) + \frac{\ln q}{q-1} N \lim_{\epsilon \to 0} \Gamma_{q}(-n + \epsilon) + \ln q N \lim_{\epsilon \to 0} [-n + \epsilon] \Gamma_{q}(-n + \epsilon) \\
&= N \lim_{\epsilon \to 0} [-n + \epsilon] \Gamma_{q}^\epsilon(-n + \epsilon) + \frac{\ln q}{q-1} \Gamma_{q}(-n + \epsilon) + \ln q \Gamma_{q}^\epsilon(-n + \epsilon) \\
&= N \lim_{\epsilon \to 0} [-n + \epsilon] \Gamma_{q}^\epsilon(-n + \epsilon) + \frac{\ln q}{q-1} \Gamma_{q}(-n + \epsilon) + \ln q \Gamma_{q}^\epsilon(-n + \epsilon) \\
&= N \lim_{\epsilon \to 0} \frac{-q^{x+\epsilon} \ln q}{1-q} \Gamma_{q}(-n + \epsilon) + [-n + \epsilon] \Gamma_{q}^\epsilon(-n + \epsilon)
\end{align*}
\]
as desired. □

The $\psi_q(x)$ function is defined by

$$\psi_q(x) = \left( \ln \Gamma_q(x) \right)' = \frac{\Gamma_q'(x)}{\Gamma_q(x)}$$

for $n \neq 0, -1, -2, \ldots$. By the previous results, it gives the idea that we define the $q$-digamma function $\psi_q(-n)$ by

$$\psi_q(-n) = \lim_{c \to 0} \frac{\Gamma_q(-n + e)}{\Gamma_q(-n + e)}$$

for $n = 0, -1, -2, \ldots$, provided the neutrix limit exists. Now we will show the existence of the function $\psi_q(-n)$ for $n = 0, 1, 2, \ldots$.

**Theorem 2.7.** For $n = 0, 1, 2, \ldots$

$$\psi_q(-n) = \psi_q(1) + \frac{\ln q}{q - 1} q \phi_q(n).$$

**Proof.** If we take $0 < |e| < 1$ and use equation (11) and (13), we get

$$\frac{\Gamma_q'(e)}{\Gamma_q(e)} = \frac{\Gamma_q'(e + 1) + \frac{q^e \ln q}{1 - q} \Gamma_q'(e)}{[e] \Gamma_q(e)} = \frac{\Gamma_q'(e + 1) + \frac{q^e \ln q}{1 - q} \Gamma_q'(e)}{\Gamma_q(e + 1) + \frac{1}{(1 - q)[e]}}.$$ 

Now taking neutrix limit of both sides, it follows that

$$\lim_{c \to 0} \frac{\Gamma_q'(e)}{\Gamma_q(e)} = \lim_{c \to 0} \frac{\Gamma_q'(e + 1) + \frac{q^e \ln q}{1 - q} \Gamma_q'(e)}{\Gamma_q(e + 1) + \frac{1}{(1 - q)[e]}}.$$

providing that $\psi_q(0)$ exists and $\psi_q(0) = \psi_q(1)$. For the case of $n = 1, 2, \ldots$, assuming the existence of $\psi_q(-n + 1)$, we have

$$\frac{\Gamma_q(-n + e)}{\Gamma_q(-n + e)} = \frac{\Gamma_q'(-n + e + 1) + \frac{q^{-n \cdot e} \ln q}{1 - q} \Gamma_q'(-n + e)}{[-n + e] \Gamma_q'(-n + e)} = \frac{\Gamma_q(-n + e + 1) + \frac{q^{-n \cdot e} \ln q}{1 - q} \Gamma_q'(-n + e)}{\Gamma_q'(-n + e + 1) + \frac{1}{(1 - q)[-n + e]}}.$$

Then we get

$$\lim_{c \to 0} \frac{\Gamma_q(-n + e)}{\Gamma_q(-n + e)} = \lim_{c \to 0} \frac{\Gamma_q'(-n + e + 1) + \frac{q^{-n \cdot e} \ln q}{1 - q} \Gamma_q'(-n + e)}{\Gamma_q'(-n + e + 1) + \frac{1}{(1 - q)[-n + e]}}.$$

By induction, it follows that

$$\psi_q(-n) = \psi_q(1) + \frac{\ln q}{q - 1} q \phi_q(n)$$

for $n = 1, 2, \ldots$. Hence the proof is completed. □

Note that all results that we obtain in this paper, tends to the results in [4] and [5] as $q \to 1$.

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References