Soliton Solutions for (2+1) and (3+1)-Dimensional Kadomtsev-Petviashvili-Benjamin-Bona-Mahony Model Equations and their Applications

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Abstract. The Kadomtsev-Petviashvili-Benjamin-Bona-Mahony (KP-BBM) model equations as a water wave model, are governing equations, for fluid flows, describes bidirectional propagating water wave surface. The soliton solutions for (2+1) and (3+1)-Dimensional Kadomtsev-Petviashvili-Benjamin-Bona-Mahony (KP-BBM) equations have been extracted. The solitary wave ansatz method are adopted to approximate the solutions. The corresponding integrability criteria, also known as constraint conditions, naturally emerge from the analysis of the problem.

1. Introduction

The propagation of nonlinear wave is one of the key phenomenon of nature and a growing interest has been drawn to the study of nonlinear waves in the dynamical system. The nonlinear equations have plenty of applications in sciences and engineering like electrochemistry, electromagnetic theory, fluid dynamics, acoustics, cosmology, astrophysics and plasma physics etc., see for references [1-6].

In the last few eras great improvement have been made in the progress of methods for finding the exact solutions of nonlinear equations but the advancement achieved is inadequate. Taking into account the merits and demerits of analytic methods, it is observed that there is no single outstanding preferable method which can be applied to any kind of nonlinear problems to obtain exact solutions. Consequently, it is apprehended that all of these methods are problem dependent, viz. some approaches work well with certain problems but not the others. Therefore, it is rather substantial to relate some established techniques in the literature to nonlinear partial differential equations; for details see also [7-14].

The solitary wave Ansatz method [15-33] have been adopted to present the solutions of (2+1) and (3+1)-Dimensional Kadomtsev-Petviashvili-Benjamin-Bona-Mahony (KP-BBM) equations, are respectively

\textbf{2010 Mathematics Subject Classification.} 02.30.Jr; 47.10.A-; 52.25.Xz; 52.35.Fp.
\textbf{Keywords.} Solitary wave soliton; shock wave soliton; singular solitons; Exact solutions
Received: 18 January 2017; Accepted: 09 April 2017
Communicated by Mića Stanković
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defined by

\[ (u_t + \mu_1 u_x - \mu_2 (u^2)_x - \mu_3 u_{xx})_t + \mu_4 u_{yy} = 0 \]  
\[ (u_t + \mu_1 u_x - \mu_2 (u^2)_x - \mu_3 u_{xx})_t + \mu_4 u_{yy} + \mu_5 u_{zz} = 0 \]  

Where \( \mu_1, \mu_2, \mu_3, \mu_4 \) and \( \mu_5 \) are real parameters. This article is organized as follows. In section 2, the solitary wave solution has been found, while in section 3, the shock wave solutions have been established and the singular forms are discussed in section 4. In last section, the conclusion have been drawn.

2. Solitary wave solitons

2.1. (2+1)-D KP-BBM equation

To calculate the solitary wave solitons for (1.1), suppose

\[ u(x, y, t) = \frac{A}{\cosh^3 \psi} \text{ where } \psi = \alpha x + \beta y - vt \]  

A is the amplitude; \( \alpha, \beta \) are the inverse widths and \( v \) is the velocity of the solitary wave. \( \lambda \) is to be determined later. By using (2.3)

\[ u_t = \frac{A(\lambda + 1)\alpha \nu}{\cosh^{1+2} \psi} - \frac{A\lambda^2 \nu}{\cosh^4 \psi} \]  
\[ u_x = \frac{A\lambda^2 \alpha^2}{\cosh^4 \psi} - \frac{A(\lambda + 1)\alpha^2}{\cosh^{1+2} \psi} \]  
\[ (u^2)_x = \frac{4\alpha^2 \lambda^2 \alpha^2}{\cosh^{2+2} \psi} - \frac{2\lambda^2 \lambda (2\lambda + 1)\alpha^2}{\cosh^{2+4} \psi} \]  
\[ u_{tt} = \frac{-A\lambda^4 \alpha \nu}{\cosh^4 \psi} + \frac{2\lambda(\lambda + 1)(\lambda^2 + 2\lambda + 2)\alpha^2 \nu}{\cosh^{1+2} \psi} - \frac{A(\lambda + 1)(\lambda + 2)\lambda \alpha^3 \nu}{\cosh^{1+4} \psi} \]  
\[ u_{yy} = \frac{A\lambda^2 \beta^2}{\cosh^4 \psi} - \frac{A(\lambda + 1)\beta^2}{\cosh^{1+2} \psi} \]  

substituting (2.4)-(2.8) into (1.1)

\[ \frac{A(\lambda + 1)\alpha \nu}{\cosh^{1+2} \psi} - \frac{A\lambda^2 \nu}{\cosh^4 \psi} + \mu_1 \frac{A\lambda^2 \alpha^2}{\cosh^4 \psi} - \mu_1 \frac{A(\lambda + 1)\alpha^2}{\cosh^{1+2} \psi} - \frac{4\mu_2 A^2 \lambda^2 \alpha^2}{\cosh^{2+2} \psi} + \mu_2 \frac{2\lambda^2 (2\lambda + 1)\alpha^2}{\cosh^{2+2} \psi} + \mu_3 \frac{A\lambda^4 \alpha \nu}{\cosh^4 \psi} - \frac{2\mu_3 A\lambda (\lambda + 1)(\lambda^2 - 2\lambda \lambda + 2)\alpha^3 \nu}{\cosh^{1+2} \psi} + \mu_3 \frac{A(\lambda + 1)(\lambda + 2)(\lambda + 3)\alpha^3 \nu}{\cosh^{1+4} \psi} + \mu_4 \frac{A\lambda^2 \beta^2}{\cosh^4 \psi} - \frac{A(\lambda + 1)\beta^2}{\cosh^{1+2} \psi} = 0 \]

by comparing the powers \( 2\lambda, \lambda + 2 \) and \( \lambda + 4, 2\lambda + 2 \)

\[ A\lambda(\lambda + 1)\alpha \nu - \mu_1 A\lambda(\lambda + 1)\alpha^2 - 4\mu_2 A^2 \lambda^2 \alpha^2 \]  
\[ -2\mu_3 A\lambda (\lambda + 1)(\lambda^2 + 2\lambda + 2)\alpha^3 \nu - \mu_4 A(\lambda + 1)\beta^2 = 0 \]  
\[ 2\mu_2 A^2 (2\lambda + 1)\alpha^2 + \mu_3 A\lambda (\lambda + 1)(\lambda + 2)\lambda \alpha^3 \nu = 0 \]

set \( \lambda = 2 \)

\[ A = \frac{6\mu_3 (\alpha^2 \mu_1 + \beta^2 \mu_4)}{4\alpha^2 \mu_2 \mu_3 - \mu_2}, \quad \nu = \frac{\alpha^2 \mu_1 + \beta^2 \mu_4}{\alpha - 4\alpha^2 \mu_3} \]

thus

\[ u(x, y, t) = \frac{A}{\cosh^2(\alpha x + \beta y - vt)} \]
2.2. (3+1)-D KP-BBM equation

To calculate the solitary wave solitons for (1.2), suppose

\[ u(x, y, z, t) = \frac{A}{\cosh^4 \psi} \quad \text{where} \quad \psi = \alpha x + \beta y + \gamma z - vt \tag{10} \]

\( A \) is the amplitude; \( \alpha, \beta, \gamma \) are the inverse widths and \( \nu \) is the velocity of the solitary wave. \( \lambda \) is to be determined later. By using (2.10)

\[
\begin{align*}
    u_{tx} &= \frac{A(\lambda + 1)\alpha \nu}{\cosh^{4+2} \psi} - \frac{A\lambda^2 \nu}{\cosh^{4} \psi} \tag{11} \\
    u_{xx} &= \frac{A\lambda^2 \alpha^2}{\cosh^4 \psi} - \frac{A\lambda(\lambda + 1)\alpha^2}{\cosh^{4+2} \psi} \tag{12} \\
    (u^2)_{xx} &= \frac{4A^2 \lambda^2 \alpha^2}{\cosh^{4+1} \psi} - \frac{2A^2 \lambda(2\lambda + 1)\alpha^2}{\cosh^{4+2} \psi} \tag{13} \\
    u_{xtx} &= \frac{-A\lambda^4 \alpha^3 \nu}{\cosh^{4} \psi} + \frac{2A\lambda(\lambda + 1)(\lambda^2 + 2\lambda + 2)\alpha^3 \nu}{\cosh^{4+2} \psi} - \frac{A\lambda(\lambda + 1)(\lambda + 2)(\lambda + 3)\alpha^3 \nu}{\cosh^{4+4} \psi} \tag{14} \\
    u_{yy} &= \frac{A\lambda_2 \beta^2}{\cosh^4 \psi} - \frac{A\lambda(\lambda + 1)\beta^2}{\cosh^{4+2} \psi} \tag{15} \\
    u_{zz} &= \frac{A\lambda^2 \gamma^2}{\cosh^4 \psi} - \frac{A\lambda(\lambda + 1)\gamma^2}{\cosh^{4+2} \psi} \tag{16} \\
\end{align*}
\]

substituting (2.4)-(2.9) into (1.2)

\[
\begin{align*}
    &\frac{A\lambda(\lambda + 1)\alpha \nu}{\cosh^{4+2} \psi} - \frac{A\lambda^2 \nu}{\cosh^{4} \psi} + \frac{A\lambda^2 \alpha^2}{\cosh^{4} \psi} - \frac{A\lambda(\lambda + 1)\alpha^2}{\cosh^{4+2} \psi} - \frac{4\mu_2}{\cosh^{4+1} \psi} - \frac{2A^2 \lambda(2\lambda + 1)\alpha^2}{\cosh^{4+2} \psi} \\
    &+ \frac{2A^2 \lambda(2\lambda + 1)\alpha^2}{\cosh^{4+2} \psi} + \frac{A\lambda^4 \alpha^3 \nu}{\cosh^{4} \psi} - \frac{2\mu_3}{\cosh^{4+2} \psi} - \frac{A\lambda(\lambda + 1)(\lambda^2 - 2\mu_3 \lambda + 2)\alpha^3 \nu}{\cosh^{4+2} \psi} \\
    &+ \frac{A\lambda(\lambda + 1)(\lambda + 2)(\lambda + 3)\alpha^3 \nu}{\cosh^{4+2} \psi} + \frac{A\lambda^2 \beta^2}{\cosh^{4} \psi} - \frac{A\lambda(\lambda + 1)\beta^2}{\cosh^{4+2} \psi} \\
    &+ \frac{A\lambda^2 \gamma^2}{\cosh^{4} \psi} - \frac{A\lambda(\lambda + 1)\gamma^2}{\cosh^{4+2} \psi} = 0
\end{align*}
\]

by comparing the powers \( 2\lambda, \lambda + 2 \) and \( \lambda + 4, 2\lambda + 2 \)

\[ A\lambda(\lambda + 1)\alpha \nu - \mu_1 A\lambda(\lambda + 1)\alpha^2 - 4\mu_2 A^2 \lambda^2 \alpha^2 \\
-2\mu_3 A\lambda(\lambda + 1)(\lambda^2 + 2\lambda + 2)\alpha^3 \nu \\
-\mu_4 A\lambda(\lambda + 1)\beta^2 - \mu_5 A\lambda(\lambda + 1)\gamma^2 = 0 \]

\[ 2\mu_2 A^2 \lambda(2\lambda + 1)\alpha^2 + \mu_3 A\lambda(\lambda + 1)(\lambda + 2)(\lambda + 3)\alpha^3 \nu = 0 \]

set \( \lambda = 2 \)

\[
A = \frac{6\mu_3(\alpha^2 \mu_1 + \beta^2 \mu_4 + \gamma^2 \mu_5)}{4\alpha^2 \mu_2 \mu_3 - \mu_2} \quad \nu = \frac{\alpha^2 \mu_1 + \beta^2 \mu_4 + \gamma^2 \mu_5}{\alpha - 4\alpha^3 \mu_3}
\]

thus

\[ u(x, y, z, t) = \frac{A}{\cosh^2(\alpha x + \beta y + \gamma z - vt)} \tag{17} \]
3. Shock wave solitons

3.1. (2+1)-D KP-BBM equation

To calculate the shock wave solitons for (1.1), suppose

\[ u(x, y, t) = A \tanh^3 \psi \quad \text{where} \quad \psi = ax + \beta y - vt \quad \text{and} \quad \lambda > 0 \]  \hspace{1cm} (18)

from (3.18)

\[
\begin{align*}
\text{(u)}_{xt} &= -A \lambda (1-\alpha)\alpha \tanh^{\lambda-2} \psi + 2A \lambda^2 \alpha \tanh^{\lambda} \psi - A \lambda (1-\alpha)\alpha \tanh^{\lambda} \psi \\
\text{(u)}_{xx} &= A \lambda (1-\alpha)\alpha \tanh^{\lambda-2} \psi - 2A \lambda^2 \alpha \tanh^{\lambda} \psi + A \lambda (1-\alpha)\alpha \tanh^{\lambda} \psi \\
(\mu^2)_{xx} &= 2A^2 \lambda (2\lambda - 1) \alpha \tanh^{2\lambda-2} \psi - 8A^2 \lambda^2 \alpha \tanh^{2\lambda} \psi + 2A^2 \lambda (2\lambda + 1) \alpha \tanh^{2\lambda} \psi \\
\text{(u)}_{xxt} &= -A \lambda (1-\alpha)\alpha \tanh^{\lambda} \psi + 4A \lambda (1-\alpha)\alpha \tanh^{\lambda} \psi \\
&\quad - 2A \lambda^2 (3\lambda^2 + 5\lambda) \alpha \tanh^{\lambda} \psi + 4A \lambda (1-\alpha)\alpha \tanh^{\lambda} \psi \\
&\quad - A \lambda (1-\alpha)\alpha \tanh^{\lambda} \psi \\
\text{(u)}_{yy} &= A \lambda (1-\alpha)\alpha \tanh^{\lambda} \psi - 2A \lambda^2 \alpha \tanh^{\lambda} \psi + A \lambda (1-\alpha)\alpha \tanh^{\lambda} \psi \\
&\quad - 2A \lambda^2 (3\lambda^2 + 5\lambda) \alpha \tanh^{\lambda} \psi + 4A \lambda (1-\alpha)\alpha \tanh^{\lambda} \psi \\
&\quad - A \lambda (1-\alpha)\alpha \tanh^{\lambda} \psi \\
\end{align*}
\]

substituting (3.19)-(3.23) into (1.1)

\[
\begin{align*}
&-A \lambda (1-\alpha)\alpha \tanh^{\lambda} \psi + 2A \lambda^2 \alpha \tanh^{\lambda} \psi - A \lambda (1-\alpha)\alpha \tanh^{\lambda} \psi \\
&+ \mu_1 A \lambda (\lambda - 1) \alpha \tanh^{\lambda} \psi - 2\mu_1 A \lambda \alpha \tanh^{\lambda} \psi + \mu_1 A \lambda (1-\alpha)\alpha \tanh^{\lambda} \psi \\
&- \mu_2 2A^2 \lambda (2\lambda - 1) \alpha \tanh^{2\lambda} \psi + 8\mu_2 A^2 \lambda \alpha \tanh^{2\lambda} \psi - 2\mu_2 A^2 \lambda (2\lambda + 1) \alpha \tanh^{2\lambda} \psi \\
&+ \mu_3 A \lambda (\lambda - 1) \alpha (2\lambda - 1) \alpha \tanh^{2\lambda-2} \psi - 4\mu_3 A \lambda (1-\alpha)\alpha \tanh^{2\lambda-2} \psi \\
&+ 2\mu_3 A \lambda^2 (3\lambda^2 + 5) \alpha \tanh^{3\lambda} \psi - 4\mu_3 A \lambda (1-\alpha)\alpha \tanh^{3\lambda} \psi \\
&+ \mu_3 A \lambda (1-\alpha)\alpha \tanh^{3\lambda} \psi + \mu_4 A \lambda (\lambda - 1) \beta \tanh^{\lambda\nu} \psi \\
&- 2\mu_4 A \lambda^2 \alpha \tanh^{\lambda} \psi + \mu_4 A \lambda (1-\alpha)\alpha \tanh^{\lambda} \psi = 0
\end{align*}
\]

by comparing the powers \(2\lambda, \lambda + 2\) and \(\lambda + 4, 2\lambda + 2\)

\[
\begin{align*}
&-A \lambda (\lambda + 1) \alpha \nu + \mu_1 A \lambda (\lambda + 1) \alpha^2 + 8\mu_2 A^2 \lambda^2 \alpha^2 \\
&- 4\mu_3 A \lambda (\lambda + 1) \alpha^2 (2\lambda + 1) \alpha \nu + \mu_4 A \lambda (\lambda + 1) \beta \alpha^2 = 0 \\
&- 2\mu_4 A^2 \lambda (2\lambda + 1) \alpha^2 + \mu_5 A \lambda (\lambda + 1) \alpha^2 = 0 \\
\end{align*}
\]

set \(\alpha = 2\)

\[
A = \frac{6\mu_3 (\alpha^2 \mu_1 + \beta^2 \mu_4)}{\mu_2 + 8\alpha^2 \mu_3}, \quad \nu = \frac{\alpha^2 \mu_1 + \beta^2 \mu_4}{\alpha + 8\alpha^3 \mu_3}
\]

thus

\[ u(x, y, t) = A \tanh^2 (ax + \beta y - vt) \]

(23)

3.2. (3+1)-D KP-BBM equation

To calculate the shock wave solitons for (1.2), suppose

\[ u(x, y, z, t) = A \tanh^3 \psi \quad \text{where} \quad \psi = ax + \beta y + \gamma z - vt \quad \text{and} \quad \lambda > 0 \]  \hspace{1cm} (24)
from (3.25)

\[ \begin{align*}
    u_{tx} &= -A\lambda\lambda - 1)\alpha\psi\tanh^{1-2}\psi + 2A\lambda^2\alpha^2\tanh^{1}\psi - A\lambda\lambda + 1)\alpha\psi\tanh^{1+2}\psi \\
    u_{xx} &= A\lambda\lambda - 1)\alpha\psi\tanh^{1-2}\psi - 2A\lambda^2\alpha^2\tanh^{1}\psi + A\lambda\lambda + 1)\alpha\psi\tanh^{1+2}\psi \\
    (u^2)_{xx} &= A\lambda\lambda - 1)(\lambda - 2)(\lambda - 3)\alpha^3\psi\tanh^{1-4}\psi + 4A\lambda\lambda - 1)\lambda - 2\lambda + 2)\alpha^3\psi\tanh^{1+2}\psi \\
    u_{xxt} &= -A\lambda\lambda - 1)(\lambda - 2)(\lambda - 3)\alpha^3\psi\tanh^{1-4}\psi - 2A\lambda^2(3\lambda^2 + 5)\alpha\psi\tanh^{1}\psi + 4A\lambda\lambda + 1)(\lambda^2 + 2\lambda + 2)\alpha^3\psi\tanh^{1+2}\psi \\
    u_{yy} &= A\lambda\lambda - 1)\beta^2\tanh^{1-2}\psi - 2A\lambda^2\beta^2\tanh^{1}\psi + A\lambda\lambda + 1)\beta^2\tanh^{1+2}\psi \\
    u_{zz} &= A\lambda\lambda - 1)\gamma^2\tanh^{1-2}\psi - 2A\lambda^2\gamma^2\tanh^{1}\psi + A\lambda\lambda + 1)\gamma^2\tanh^{1+2}\psi \\
\end{align*} \]

substituting (3.26)-(3.31) into (1.2)

by comparing the powers 2\lambda, \lambda + 2 and \lambda + 4, 2\lambda + 2

\[ \begin{align*}
    & -A\lambda\lambda - 1)\alpha\psi + A\lambda\lambda + 1)\alpha^2 + 8\mu_2A^2\lambda^2\alpha^2 \\
    & -4\mu_3A\lambda\lambda + 1)(\lambda^2 + 2\lambda + 2)\alpha^3\psi + \mu_4A\lambda\lambda + 1)\beta^2\psi + \mu_5A\lambda\lambda + 1)\gamma^2\psi = 0 \\
    & -2\mu_2A^2(2\lambda + 1)\alpha^2 + \mu_3A\lambda\lambda + 1)(\lambda + 2)\alpha^3\psi = 0
\end{align*} \]

set \( \alpha = 2 \)

\[ A = \frac{6\mu_3(\alpha^2\mu_1 + \beta^2\mu_4 + \gamma^2\mu_5)}{\mu_2 + 8\lambda^2\mu_3}, \quad \nu = \frac{\alpha^2\mu_1 + \beta^2\mu_4 + \gamma^2\mu_5}{\alpha + 8\lambda^3\mu_3} \]

thus

\[ u(x, y, z, t) = A\tanh^2(\alpha x + \beta y + \gamma z - vt) \]  

4. Singular wave solitons

4.1. Form-I (2+1)-D KP-BBM equation

To calculate the the singular wave Form I solution for (1.1), suppose

\[ u(x, y, t) = A\coth^4 \psi \quad \text{where} \quad \psi = ax + by - vt \quad \text{and} \quad \lambda > 0 \]
from (4.33)

\[ u_{tx} = -A\lambda(\lambda - 1)\alpha\nu \coth^{\lambda - 2} \psi + 2A\lambda^2 \alpha^2 \coth^{\lambda} \psi - A\lambda(\lambda + 1)\alpha\nu \coth^{\lambda + 2} \psi \]  
(33)

\[ u_{xx} = A\lambda(\lambda - 1)\alpha^2 \coth^{\lambda - 2} \psi - 2A\lambda^2 \alpha^2 \coth^{\lambda} \psi + A\lambda(\lambda + 1)\alpha^2 \coth^{\lambda + 2} \psi \]  
(34)

\[ (u^2)_{xx} = 2A^2 \lambda(2\lambda - 1)\alpha^2 \coth^{3\lambda - 2} \psi - 8A^2 \lambda^2 \alpha^2 \coth^{2\lambda} \psi + 2A^2 \lambda(2\lambda + 1)\alpha^2 \coth^{2\lambda + 2} \psi \]  
(35)

\[ u_{sxtx} = -A\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)\alpha^3 \nu \coth^{\lambda - 4} \psi + 4A\lambda(\lambda - 1)(\lambda^2 - 2\lambda + 1)\alpha^3 \nu \coth^{\lambda + 2} \psi \]  
\[ -2A^2(3\lambda^2 + 5)\alpha^3 \nu \coth^{\lambda} \psi + 4A\lambda(\lambda + 1)(\lambda^2 + 2\lambda + 2)\alpha^3 \nu \coth^{\lambda + 2} \psi \]  
\[ -A\lambda(\lambda + 1)(\lambda + 2)(\lambda + 3)\alpha^3 \nu \coth^{\lambda + 4} \psi \]  
(36)

\[ u_{yy} = A\lambda(\lambda - 1)\beta^2 \coth^{\lambda - 2} \psi - 2A\beta^2 \coth^{\lambda} \psi + A\lambda(\lambda + 1)\beta^2 \coth^{\lambda + 2} \psi \]  
(37)

substituting (4.34)-(4.38) into (1.1)

\[ -A\lambda(\lambda - 1)\alpha\nu \coth^{\lambda - 2} \psi + 2A\lambda^2 \alpha^2 \coth^{\lambda} \psi - A\lambda(\lambda + 1)\alpha\nu \coth^{\lambda + 2} \psi \]  
\[ +\mu_1A\lambda(\lambda - 1)\alpha^2 \coth^{\lambda - 2} \psi - 2\mu_1A\lambda^2 \alpha^2 \coth^{\lambda} \psi + \mu_1A\lambda(\lambda + 1)\alpha^2 \coth^{\lambda + 2} \psi \]  
\[ -2\mu_2A^2 \lambda(2\lambda - 1)\alpha^2 \coth^{3\lambda - 2} \psi + 8\mu_2A^2 \lambda^2 \alpha^2 \coth^{2\lambda} \psi - 2\mu_2A^2 \lambda(2\lambda + 1)\alpha^2 \coth^{2\lambda + 2} \psi \]  
\[ +\mu_3A\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)\alpha^3 \nu \coth^{\lambda - 4} \psi - 4\mu_3A\lambda(\lambda - 1)(\lambda^2 - 2\lambda + 1)\alpha^3 \nu \coth^{\lambda + 2} \psi \]  
\[ -4\mu_3A\lambda(\lambda + 1)(\lambda^2 + 2\lambda + 2)\alpha^3 \nu \coth^{\lambda + 2} \psi + 2\mu_3A\lambda(3\lambda^2 + 5)\alpha^3 \nu \coth^{\lambda} \psi \]  
\[ +\mu_3A\lambda(\lambda + 1)(\lambda + 2)(\lambda + 3)\alpha^3 \nu \coth^{\lambda + 4} \psi + \mu_4A\lambda(\lambda - 1)\beta^2 \coth^{\lambda - 2} \psi \]  
\[ -2\mu_4A\lambda^2 \beta^2 \coth^{\lambda} \psi + \mu_4A\lambda(\lambda + 1)\beta^2 \coth^{\lambda + 2} \psi = 0 \]

by comparing the powers \(2\lambda, \lambda + 2\) and \(\lambda + 4, 2\lambda + 2\)

\[-A\lambda(\lambda + 1)\alpha\nu + \mu_1A\lambda(\lambda + 1)\alpha^2 + 8\mu_2A^2 \lambda^2 \alpha^2 \]  
\[-4\mu_3A\lambda(\lambda + 1)(\lambda^2 + 2\lambda + 2)\alpha^3 \nu + \mu_4A\lambda(\lambda + 1)\beta^2 = 0 \]  
\[-2\mu_2A^2 \lambda(2\lambda + 1)\alpha^2 + \mu_3A\lambda(\lambda + 1)(\lambda + 2)(\lambda + 3)\alpha^3 \nu = 0 \]  

set \(\alpha = 2\)

\[ A = \frac{6\mu_3(\alpha^2 \mu_1 + \beta^2 \mu_4)}{\mu_2 + 8\alpha^2 \mu_2 \mu_3}, \quad \nu = \frac{\alpha^2 \mu_1 + \beta^2 \mu_4}{\alpha + 8\alpha^3 \mu_3} \]

thus

\[ u(x, y, t) = A \coth^2(\alpha x + \beta y - \nu t) \]  
(38)
Figure (1) represented solitary wave: $\mu_1=1, \mu_2=1, \mu_3=-1, \mu_4=1, \alpha=0.1, \beta=0.01, t=2$ and figure (2) showed shock wave: $\mu_1=1, \mu_2=1, \mu_3=1, \mu_4=1, \alpha=0.1, \beta=0.01, t=2$ of the (2+1)-D KP-BBM equation.

Figure (3) represented Singular Form-I: $\mu_1=1, \mu_2=1, \mu_3=1, \mu_4=1, \alpha=0.1, \beta=0.05, t=1$ and figure (4) showed Singular Form-II: $\mu_1=1, \mu_2=1, \mu_3=1, \mu_4=1, \alpha=0.1, \beta=0.05, t=1$ of the (2+1)-D KP-BBM equation.

4.2. Form-I (3+1)-D KP-BBM equation

To calculate the singular wave Form I solution for (1.2), suppose

$$u(x,y,z,t) = A \coth^\lambda \psi \quad \text{where} \quad \psi = ax + \beta y + \gamma z - vt \quad \text{and} \quad \lambda > 0$$

(39)
from (4.40)

\[
\begin{align*}
u_{tx} &= -AA\lambda(\lambda - 1)\alpha_1\psi + 2A\alpha_1^2\psi + 2A\alpha_1^2\alpha_1\psi \\
&
- AA(\lambda + 1)\alpha_2\psi \\
u_{xx} &= AA(\lambda - 1)\alpha_2\psi - 2A\alpha_2\alpha_2\psi \\
&+ AA(\lambda + 1)\alpha_2\psi \\
(u^2)_{xx} &= 2A^2\lambda(2\lambda - 1)\alpha_2\psi - 8A^2\lambda_2^2\psi \\
&+ 2A^2\lambda(2\lambda + 1)\alpha_2\psi \\
u_{xxt} &= AA(\lambda - 1)(\lambda - 1)\alpha_3\psi - 2A\alpha_3\psi \\
&+ AA(\lambda + 1)\alpha_3\psi \\
u_{xy} &= AA(\lambda - 1)\alpha_1\psi - 2A\alpha_1\alpha_1\psi \\
&+ AA(\lambda + 1)\alpha_1\psi \\
u_{xz} &= AA(\lambda - 1)\alpha_2\psi - 2A\alpha_2\alpha_2\psi \\
&+ AA(\lambda + 1)\alpha_2\psi
\end{align*}
\]

substituting (4.41)-(4.46) into (1.2)

\[
\begin{align*}
&-AA(\lambda - 1)\alpha_1\psi + 2A\alpha_1^2\psi - AA(\lambda + 1)\alpha_1\psi \\
&+ \mu_1AA(\lambda - 1)\alpha_1\psi - 2\mu_1A\alpha_2\alpha_2\psi \\
&+ AA(\lambda + 1)\alpha_1\psi \\
&-2\mu_2A^2\alpha_1(2\lambda - 1)\alpha_2\psi + 8\mu_2A^2\lambda_2\alpha_2\psi \\
&- 2\mu_2A^2\lambda(2\lambda + 1)\alpha_2\psi \\
&+ 2\mu_2A^2\alpha_2\psi + 2\mu_2A^2\lambda(3\lambda^2 + 5)\alpha_2\psi \\
&+ AA(\lambda - 1)(\lambda - 1)\alpha_3\psi - 4\mu_3AA(\lambda - 1)(\lambda - 2)\alpha_3\psi \\
&+ AA(\lambda + 1)(\lambda + 2)\alpha_3\psi \\
&-2\mu_3AA(\lambda + 1)(\lambda + 2)\alpha_3\psi \\
&+ 2\mu_3AA(\lambda + 1)(\lambda + 2)\alpha_3\psi \\
&+ AA(\lambda + 1)\alpha_3\psi = 0
\end{align*}
\]

by comparing the powers \(2\lambda, \lambda + 2\) and \(\lambda + 4, 2\lambda + 2\)

\[
\begin{align*}
&-AA(\lambda + 1)\alpha_1\psi + \mu_1AA(\lambda + 1)\alpha_1^2 + 8\mu_2A^2\lambda_2^2\alpha_2 \\
&- 4\mu_3AA(\lambda + 1)(\lambda^2 + 2\lambda + 1)^2\alpha_3\psi \\
&+ \mu_4AA(\lambda + 1)(\lambda + 2)\alpha_3\psi + 2\mu_5AA(\lambda + 1)(\lambda + 2)\alpha_3\psi \\
&-2\mu_5AA(\lambda + 1)(\lambda + 2)\alpha_3\psi \\
&+ \mu_5AA(\lambda + 1)(\lambda - 1)\gamma_2\alpha_3\psi = 0
\end{align*}
\]

set \(\alpha = 2\)

\[
A = \frac{6\mu_5(\alpha^2\mu_1 + \beta^2\mu_4 + \gamma^2\mu_5)}{\mu_2 + 8\alpha^2\mu_2\mu_3}, \quad \nu = \frac{\alpha^2\mu_1 + \beta^2\mu_4 + \gamma^2\mu_5}{\alpha + 8\alpha^2\mu_3}
\]

thus

\[
u(x, y, z, t) = A \coth^2(ax + \beta y + \gamma z - vt) \quad (46)
\]

4.3. Form-II (2+1)-D KP-BBM equation

To calculate the the singular wave Form II solution for (1.1), suppose

\[
u(x, y, t) = A \csch^4 \psi \quad \text{where} \quad \psi = ax + \beta y + \gamma z - vt \quad \text{and} \quad \lambda > 0
\]

(47)
from (4.48)

\[ u_{tx} = -AA(\lambda + 1)\alpha v \csc h^{4+2}\psi - AA^2\alpha v \csc h^3\psi \]  
\[ u_{xx} = AA(\lambda + 1)\alpha^2 \csc h^{4+2}\psi + AA^2\alpha^2 \csc h^3\psi \]  
\[ (u_t^2)_{xx} = 4A^2\lambda^2 \csc h^{21}\psi + 2A^2\lambda(2\lambda + 1)\alpha^2 \csc h^{3+2}\psi \]  
\[ u_{sxtx} = -AA^4\alpha^3 v \csc h^4\psi - 2AA(\lambda + 1)(\lambda^2 + 2\lambda + 2)\alpha^3 \csc h^{3+2}\psi \]  
\[ u_{xyy} = AA(\lambda + 1)\beta^2 \csc h^{4+2}\psi + AA^2\beta^2 \csc h^3\psi \]  
\[ u_{zz} = AA(\lambda + 1)\gamma^2 \csc h^{4+2}\psi + AA^2\gamma^2 \csc h^3\psi \]

substituting (4.49)-(4.53) into (1.1)

\[-AA(\lambda + 1)\alpha v \csc h^{1+2}\psi - AA^2\alpha v \csc h^1\psi \]  
\[ + \mu_1AA(\lambda + 1)\alpha^2 \csc h^{1+2}\psi + \mu_1AA^2\alpha^2 \csc h^1\psi \]  
\[ - 4\mu_2A^2\lambda^2 \csc h^{21}\psi - 2\mu_2A^2\lambda(2\lambda + 1)\alpha^2 \csc h^{3+2}\psi \]  
\[ + \mu_3AA^4\alpha^3 v \csc h^4\psi + 2\mu_3AA(\lambda + 1)(\lambda^2 + 2\lambda + 2)\alpha^3 \csc h^{4+2}\psi \]  
\[ + \mu_3AA(\lambda + 1)(\lambda + 2)(\lambda + 3)\alpha^3 v \csc h^{1+4}\psi \]  
\[ + \mu_4AA(\lambda + 1)\beta^2 \csc h^{4+2}\psi + \mu_4AA^2\beta^2 \csc h^3\psi = 0 \]

by comparing the powers \(2\lambda, \lambda + 2\) and \(\lambda + 4, 2\lambda + 2\)

\[-AA(\lambda + 1)\alpha v + \mu_1AA(\lambda + 1)\alpha^2 - 4\mu_2A^2\lambda^2 \alpha^2 \]  
\[ + 2\mu_3AA(\lambda + 1)(\lambda^2 + 2\lambda + 2)\alpha^3 v + \mu_4AA(\lambda + 1)\beta^2 + \mu_5AA(\lambda + 1)\gamma^2 = 0 \]  
\[-2\mu_2A^2\lambda(2\lambda + 1)\alpha^2 + \mu_3AA(\lambda + 1)(\lambda + 2)(\lambda + 3)\alpha^3 v = 0 \]

set \(\lambda = 2\) \(\lambda = 2\)

\[ A = \frac{6\mu_3(\alpha^2\mu_1 + \beta^2\mu_4)}{\mu_2 - 4\alpha^2\mu_3}, \quad v = \frac{\alpha^2\mu_1 + \beta^2\mu_4}{\alpha - 4\alpha^3\mu_3} \]

thus

\[ u(x, y, t) = A \csc h^2(ax + \beta y - vt) \]  

4.4. Form-II (3+1)-D KP-BBM equation

To calculate the the singular wave Form II solution for (1.2), suppose

\[ u(x, y, z, t) = A \csc h^4\psi \]  
\[ \psi = ax + \beta y + \gamma z - vt \]  
\[ \lambda > 0 \]  

from (4.55)

\[ u_{tx} = -AA(\lambda + 1)\alpha v \csc h^{4+2}\psi - AA^2\alpha v \csc h^3\psi \]  
\[ u_{xx} = AA(\lambda + 1)\alpha^2 \csc h^{4+2}\psi + AA^2\alpha^2 \csc h^3\psi \]  
\[ (u_t^2)_{xx} = 4A^2\lambda^2 \csc h^{21}\psi + 2A^2\lambda(2\lambda + 1)\alpha^2 \csc h^{3+2}\psi \]  
\[ u_{sxtx} = -AA^4\alpha^3 v \csc h^4\psi - 2AA(\lambda + 1)(\lambda^2 + 2\lambda + 2)\alpha^3 \csc h^{4+2}\psi \]  
\[ u_{xyy} = AA(\lambda + 1)\beta^2 \csc h^{4+2}\psi + AA^2\beta^2 \csc h^3\psi \]  
\[ u_{zz} = AA(\lambda + 1)\gamma^2 \csc h^{4+2}\psi + AA^2\gamma^2 \csc h^3\psi \]
Figure (5) represented Solitary wave: \( \mu_1=1, \mu_2=1, \mu_3=-1, \mu_4=1, \mu_5=1, \alpha=0.05, \beta=0.1, \gamma=0.1, t=2 \) and figure (6) showed Shock wave: \( \mu_1=1, \mu_2=1, \mu_3=1, \mu_4=1, \mu_5=-1, \alpha=0.05, \beta=0.1, \gamma=0.1, t=2 \) of the (3+1)-D KP-BBM equation.

Figure (7) represented Singular Form-I: \( \mu_1=1, \mu_2=1, \mu_3=1, \mu_4=1, \mu_5=1, \alpha=0.1, \beta=0.1, \gamma=0.05, t=2 \) and figure (8) showed Singular Form-II: \( \mu_1=1, \mu_2=1, \mu_3=1, \mu_4=1, \mu_5=1, \alpha=0.1, \beta=0.1, \gamma=0.05, t=2 \) of the (3+1)-D KP-BBM equation.
substituting (4.56)-(4.61) into (1.2)
\[-A_1(\lambda + 1)\alpha v \cosh^{1+2} \psi - A_2 \alpha v \cosh^4 \psi \]
\[+\mu_2 A_3 \lambda^2 \alpha^2 \cosh^2 \lambda \psi - 2\mu_2 A_3 \lambda (2\lambda + 1) \alpha^2 \cosh^{2+1+2} \psi \]
\[+\mu_3 A_4 \lambda^4 \alpha^3 \nu \cosh^4 \psi + 2\mu_3 A_4 \lambda (\lambda + 1) (\lambda^2 + 2\lambda + 2) \alpha^3 \nu \cosh^{4+2} \psi \]
\[+\mu_3 A_5 \lambda (\lambda + 1) (\lambda + 2) (\lambda + 3) \alpha^3 \nu \cosh^{4+4} \psi \]
\[+\mu_4 A_6 \lambda (\lambda + 1) \beta^2 \cosh^{4+2} \psi + \mu_4 A_6 \lambda^2 \beta^2 \cosh^4 \psi \]
\[+\mu_5 A_7 \lambda (\lambda + 1) \gamma^2 \cosh^{4+2} \psi + \mu_5 A_7 \lambda^2 \gamma^2 \cosh^4 \psi = 0 \]

by comparing the powers $2\lambda, \lambda + 2$ and $\lambda + 4, 2\lambda + 2$
\[-A_1(\lambda + 1)\alpha v + \mu_1 A_3 \lambda (\lambda + 1) \alpha^2 - 4\mu_2 A_3 \lambda^2 \alpha^2 \]
\[+2\mu_2 A_3 \lambda (\lambda + 1) (\lambda^2 + 2\lambda + 2) \alpha^3 \nu + \mu_4 A_3 (\lambda + 1) \beta^2 + \mu_5 A_3 (\lambda + 1) \gamma^2 = 0 \]
\[-2\mu_2 A_3 \lambda (2\lambda + 1) \alpha^2 + \mu_3 A_5 \lambda (\lambda + 1) (\lambda + 2) (\lambda + 3) \alpha^3 \nu = 0 \]

set $\lambda = 2$
\[A = \frac{4\mu_4 (\mu_1 + \mu_2 \mu_4 + \gamma^2 \mu_5)}{\mu_2 - 4\mu_2 \mu_3}, \quad \nu = \frac{\alpha^2 \mu_1 + \beta^2 \mu_4 + \gamma^2 \mu_5}{\alpha - 4\alpha^2 \mu_3} \]

thus
\[u(x, y, z, t) = A \cosh^2 (\alpha x + \beta y + \gamma z - vt) \quad (61) \]

5. Physical description

In Fig.1, we take $\mu_1 = 1, \mu_2 = 1, \mu_3 = 1, \mu_4 = 1, \alpha = 0.1, \beta = 0.01, t = 2$ for solitary wave and $\mu_1 = 1, \mu_2 = 1, \mu_3 = 1, \mu_4 = 1, \alpha = 0.1, \beta = 0.01, t = 2$ for shock waves; while $\mu_1 = 1, \mu_2 = 1, \mu_3 = 1, \mu_4 = 1, \alpha = 0.1, \beta = 0.05, t = 1$ for singular wave Form-I and $\mu_1 = 1, \mu_2 = 1, \mu_3 = 1, \mu_4 = 1, \alpha = 0.1, \beta = 0.05, t = 1$ for singular wave Form-II, to study the behaviour of (2+1) dimensional KP equation. In Fig.2, we choose $\mu_1 = 1, \mu_2 = 1, \mu_3 = 1, \mu_4 = 1, \alpha = 0.1, \beta = 0.05, t = 2$ for solitary wave and $\mu_1 = 1, \mu_2 = 1, \mu_3 = 1, \mu_4 = 1, \alpha = 0.1, \beta = 0.05, t = 2$ for shock waves, while $\mu_1 = 1, \mu_2 = 1, \mu_3 = 1, \mu_4 = 1, \alpha = 0.1, \beta = 0.05, t = 1$ for singular wave Form-I and $\mu_1 = 1, \mu_2 = 1, \mu_3 = 1, \mu_4 = 1, \alpha = 0.1, \beta = 0.05, t = 1$ for singular wave Form-II of (2+1) dimensional Boussinesq equation.

6. Conclusion

In this article, the solitary wave ansatz method are successfully employed to (2+1) and (3+1)-Dimensional Kadomtsev-Petviashvili-Benjamin-Bona-Mahony (KP-BBM) equations. The solitary wave ansatz method is used which is rather heuristic and processes significant features that make it practical for the determination of single soliton solutions for a wide class of nonlinear evolution equations. The constraint conditions for the existence of solutions are also listed.

References


[17] Bin Zheng, Traveling Wave Solutions For The (2+1) Dimensional Boussinesq Equation And The Two-Dimensional Burgers Equation By (G'/G)-expansion method Wseas Transactions on Computers (2010), ISSN: 1109-2750


