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THE INVERSE EIGENVALUE PROBLEM FOR REAL EVENTUALLY POSITIVE MATRICES[☆]

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Abstract

The eventually positive inverse eigenvalue problem (EPIEP) asks when a list $\sigma = (\rho, \lambda_2, \dots, \lambda_n)$ of n complex numbers is the spectrum of a real $n \times n$ eventually positive matrix. In this paper, a constructive method is presented to solve the EPIEP. Using this method, it is easy to find the realizing matrix, i.e., a real eventually positive matrix with the spectrum σ . And a complete answer to the EPIEP prescribed the left and the right Perron eigenvectors of the realizing matrix is also given.

1. Introduction

A real $n \times n$ matrix A is said to be eventually positive if there exists a positive integer K such that A^k is a positive matrix (a matrix whose entries are positive) for every integer $k \geq K$. It is well known that the nonnegative inverse eigenvalue problem (NIEP) asks when a list $\sigma = (\rho, \lambda_2, \dots, \lambda_n)$ of

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n complex numbers is the spectrum of an $n \times n$ nonnegative matrix. In NIEP, a list $(\lambda_1, \dots, \lambda_n)$ is said to be realizable if there exists an $n \times n$ nonnegative matrix A with the spectrum $\sigma = (\lambda_1, \lambda_2, \dots, \lambda_n)$. Analogously, the eventually positive inverse eigenvalue problem (abbreviated hereafter as EPIEP) asks when a list $\sigma = (\rho, \lambda_2, \dots, \lambda_n)$ of n complex numbers is the spectrum of a real $n \times n$ eventually positive matrix.

Initially our work of the EPIEP was motivated by a result proved by Wuwen[1] in the context of the NIEP. For convenience, we repeat that result as follows.

Theorem 1.1 ([1, Theorem 2.1]). *Let $(\lambda_2, \dots, \lambda_n)$ be a list of complex numbers closed under the conjugation. Then there exists a real number λ_0 , greater than or equal to $\max\{|\lambda_i|, i = 2, \dots, n\}$ such that the list $(\lambda_1, \dots, \lambda_n)$ is realizable if and only if $\lambda_1 \geq \lambda_0$. Furthermore, $\lambda_0 \leq 2n \max\{|\lambda_i|, i = 2, \dots, n\}$.*

Wuwen also presented a result[1, Corollary 2.4], which states that if there exists an $n \times n$ nonnegative matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, given any $\varepsilon > 0$, then there exists an $n \times n$ positive matrix with eigenvalues $\lambda_1 + \varepsilon, \lambda_2, \dots, \lambda_n$. From Theorem 1.1 and the result[1, Corollary 2.4] one immediately sees that for any list of complex numbers $(\lambda_2, \dots, \lambda_n)$ closed under the conjugation, there exists an $n \times n$ positive matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ for any real number λ_1 large enough compared with $\max\{|\lambda_i|, i = 2, \dots, n\}$. Thus, the following corollary can be easily seen.

Corollary 1.1. *Let $(\lambda_2, \dots, \lambda_n)$ be a list of complex numbers closed under the conjugation. Suppose $\rho \in \mathbb{R}$ and $\rho > |\lambda_i|, i = 2, \dots, n$. Then there exists an $n \times n$ positive matrix A_k with eigenvalues $\rho^k, \lambda_2^k, \dots, \lambda_n^k$ for any positive integer k large enough.*

This naturally brings us to the following question.

Let $(\lambda_2, \dots, \lambda_n)$ be a list of complex numbers closed under the conjugation. Suppose $\rho > |\lambda_i|, i = 2, \dots, n$. Does there exist an $n \times n$ real matrix A such that A^k possesses positive entries and eigenvalues $\rho^k, \lambda_2^k, \dots, \lambda_n^k$ for every positive integer k large enough, and if so, how do we construct one? we term this the EPIEP as mentioned earlier. This paper constructively settles the question.

A similar problem has been investigated by Friedland. We know only from [2] that Friedland has proved a sufficient condition for a list of complex numbers to be the spectrum of a real eventually non-negative matrix in [5]. Thus it is natural to seek a constructive solution of the EPIEP.

The rest of the paper is organized as follows. In Section 2, we introduce some preliminary definitions and notations. In Section 3, we study the structure of real eventually positive matrices. Section 4 contains our main results, these results give a constructive answer to the EPIEP. Finally, a computational example is given in Section 5 to illustrate Theorem 4.1.

2. Preliminary definitions and notations

Let \mathbb{C}^n , \mathbb{R}^n , $\mathbb{R}^{n \times n}$ be the set of column vectors of n complex components, the set of column vectors of n real components, and the set of all $n \times n$ real matrices respectively. Let \mathcal{P} denote the set of $n \times 1$ real vectors with positive components and I_n be the $n \times n$ identity matrix. The symbols A^T , $\text{rank } A$, and $\rho(A)$ stand for the transpose, the rank, and the spectral radius of a matrix A , respectively.

Definition 2.1. A matrix $A \in \mathbb{R}^{n \times n}$ is said to be eventually positive if there exists a positive integer K such that A^k is a positive matrix for every integer $k \geq K$.

Definition 2.2. A matrix $A \in \mathbb{R}^{n \times n}$ is said to possess the Perron-Frobenius property if its eigenvalue of maximum modulus λ_1 is positive and the corresponding eigenvector $x^{(1)}$ is nonnegative.

Definition 2.3. A matrix $A \in \mathbb{R}^{n \times n}$ is said to possess the strong Perron-Frobenius property if its eigenvalue of maximum modulus λ_1 is positive, the only one in the circle λ_1 ($\lambda_1 > |\lambda_i|$, $i = 2, 3, \dots, n$, where $\lambda_2, \lambda_3, \dots, \lambda_n$ are the other eigenvalues of A) and the corresponding eigenvector $x^{(1)}$ is positive. And the right and the left Perron eigenvectors are the right and the left eigenvectors respectively, corresponding to the eigenvalue $\lambda_1 = \rho(A)$ of A .

For a matrix A , its eigenvalue λ is said usually an algebraically simple eigenvalue if the algebraic multiplicity of λ equal to 1. Thus, the another statement of the strong Perron-Frobenius property is that the eigenvalue of maximum modulus is unique, positive, algebraically simple and the corresponding eigenvector is positive.

3. The structure of real eventually positive matrices

The following lemmas will be needed in the sequel.

Lemma 3.1. *Let $A, B_1,$ and $B_2 \in \mathbb{R}^{n \times n}$. Suppose $p_A(\lambda), p_{B_1}(\lambda),$ and $p_{B_2}(\lambda)$ are the characteristic polynomials of $A, B_1,$ and $B_2,$ respectively. If $A = B_1 + B_2$ and $B_1B_2 = 0$ or $B_2B_1 = 0,$ then*

$$\lambda^n \cdot p_A(\lambda) = p_{B_1}(\lambda) \cdot p_{B_2}(\lambda).$$

Proof. If $A = B_1 + B_2$ and $B_1B_2 = 0,$ then we have

$$\begin{aligned} \lambda^n \cdot p_A(\lambda) &= \lambda^n \cdot \det(\lambda I_n - A) \\ &= \lambda^n \cdot \det(\lambda I_n - B_1 - B_2) \\ &= \det(\lambda^2 I_n - \lambda B_1 - \lambda B_2) \\ &= \det(\lambda^2 I_n - \lambda B_1 - \lambda B_2 + B_1 B_2) \\ &= \det((\lambda I_n - B_1) \cdot (\lambda I_n - B_2)) \\ &= \det(\lambda I_n - B_1) \cdot \det(\lambda I_n - B_2) \\ &= p_{B_1}(\lambda) \cdot p_{B_2}(\lambda). \end{aligned}$$

The other case can be proved in the same way. \square

This leads at once to the following result.

Lemma 3.2. *Let $A, B_1,$ and $B_2 \in \mathbb{R}^{n \times n}$. If $A = B_1 + B_2$ and $B_1B_2 = 0$ or $B_2B_1 = 0,$ then the following assertions hold.*

- (i) $\rho(A) = \max\{\rho(B_1), \rho(B_2)\}.$
- (ii) *If $\rho(A) > 0,$ then a non-zero number λ is an algebraically simple eigenvalue of A if and only if λ is an algebraically simple eigenvalue of B_1 and not an eigenvalue of B_2 or λ is an algebraically simple eigenvalue of B_2 and not an eigenvalue of $B_1.$*
- (iii) *If $\rho(A) > 0$ and $\rho(B_1) \neq \rho(B_2),$ then $\rho(A)$ is the only one eigenvalue of maximum modulus of A if and only if one of the following holds: (1) $\rho(B_1)$ is the only one eigenvalue of maximum modulus of B_1 and $\rho(B_1) > \rho(B_2);$ (2) $\rho(B_2)$ is the only one eigenvalue of maximum modulus of B_2 and $\rho(B_2) > \rho(B_1).$*

We know that every eventually positive matrix possesses the strong Perron-Frobenius property (see [2, Remark 4.2] or [3, Theorem 2.3]). So, to study the structure of real eventually positive matrices we may first consider the structure of matrices possessing the strong Perron-Frobenius property, which is stated in the following theorem.

Theorem 3.1. *Let $A \in \mathbb{R}^{n \times n}$. Then, A possesses the strong Perron-Frobenius property if and only if there exist $\alpha \in \mathcal{P}$, $\beta \in \mathbb{R}^n$, and $Y \in \mathbb{R}^{n \times n}$, such that*

$$A = \frac{1}{(\beta^T \alpha)^2} \alpha \beta^T + \frac{1}{\beta^T \alpha} (I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T) Y (I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T), \quad (3.1)$$

$\beta^T \alpha > 0$, and $\rho((I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T) Y (I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T)) < 1$.

Proof. “If” part: Let $P = \frac{1}{\beta^T \alpha} \alpha \beta^T$, $\mu = \frac{1}{\beta^T \alpha}$. Then

$$A = \mu P + \mu (I_n - P) Y (I_n - P). \quad (3.2)$$

Note that $P^2 = P$. Hence

$$(\mu P) \cdot (\mu (I_n - P) Y (I_n - P)) = \mu^2 (P - P^2) Y (I_n - P) = 0. \quad (3.3)$$

Combine (3.2) and (3.3) with Lemma 3.2(i) to deduce

$$\rho(A) = \max \{ \rho(\mu P), \rho(\mu (I_n - P) Y (I_n - P)) \}. \quad (3.4)$$

By $\rho((I_n - P) Y (I_n - P)) < 1$,

$$\rho(\mu (I_n - P) Y (I_n - P)) < \mu = \rho(\mu P). \quad (3.5)$$

Combining (3.4) with (3.5), we have

$$\rho(A) = \rho(\mu P) = \mu = \frac{1}{\beta^T \alpha} > 0. \quad (3.6)$$

Since $\text{rank } P = 1$, we see that $\rho(\mu P) = \mu$ is an algebraically simple eigenvalue of μP . And $\rho(\mu P)$ is not the eigenvalue of $\mu (I_n - P) Y (I_n - P)$, from (3.5). Thus we know from Lemma 3.2(ii) that $\rho(A) = \mu$ is an algebraically simple eigenvalue of A .

Note that $\rho(\mu P) = \mu$ is the only one eigenvalue of maximum modulus of μP . Combine (3.6), (3.5), and Lemma 3.2(iii) to deduce that $\rho(A) = \mu$ is the only one eigenvalue of maximum modulus of A .

Finally, it is seen from $A\alpha = \rho(A)\alpha$ that the corresponding eigenvector α is positive. This completes the proof of the “if” part.

“Only if” part: Suppose A possesses the strong Perron-Frobenius property with eigenvalues $\rho(A) = \lambda_1, \lambda_2, \dots, \lambda_n$ ($\lambda_1 > |\lambda_i|, i = 2, \dots, n$). Let

α, β_0 be the real right and the real left eigenvectors, respectively, corresponding to eigenvalues $\rho(A)$ of A , then $\beta_0^T \alpha \neq 0$ (see [4, p371]). Thus, one can take $\beta = \frac{1}{\rho(A)\beta_0^T \alpha} \beta_0$ with

$$\beta^T \alpha = \frac{1}{\rho(A)} > 0 \quad (3.7)$$

and take $Y = \beta^T \alpha A - \frac{1}{\beta^T \alpha} \alpha \beta^T$. Note that $A \alpha \beta^T = \rho(A) \alpha \beta^T = \frac{1}{\beta^T \alpha} \alpha \beta^T$ and $\alpha \beta^T A = \rho(A) \alpha \beta^T = \frac{1}{\beta^T \alpha} \alpha \beta^T$. A straightforward computation shows that

$$(I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T) Y (I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T) = \beta^T \alpha A - \frac{1}{\beta^T \alpha} \alpha \beta^T,$$

which implies that

$$\frac{1}{(\beta^T \alpha)^2} \alpha \beta^T + \frac{1}{\beta^T \alpha} (I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T) Y (I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T) = A. \quad (3.8)$$

Let B_1, B_2 denote the first and second terms on the left-hand side of (3.8), respectively. Since $B_1 B_2 = 0$, $\rho(A) = \frac{1}{\beta^T \alpha}$ is an eigenvalue of B_1 and $\rho(A)$ is an algebraically simple eigenvalue of A , it follows from Lemma 3.2(ii) that $\rho(A)$ is not an eigenvalue of B_2 . So, it is immediate from Lemma 3.2(i) that

$$\rho(B_2) < \rho(A) = \frac{1}{\beta^T \alpha}$$

and hence that

$$\rho((I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T) Y (I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T)) = \rho(\beta^T \alpha B_2) = \beta^T \alpha \cdot \rho(B_2) < 1. \quad (3.9)$$

Combine (3.7), (3.8), and (3.9) to conclude the proof. \square

With the aid of the above theorem, we can now give the structure of real eventually positive matrices, which plays a fundamental role in this work.

Theorem 3.2. *Let $A \in \mathbb{R}^{n \times n}$. Then, A is a real eventually positive matrix if and only if there exist $\alpha, \beta \in \mathcal{P}$, and $Y \in \mathbb{R}^{n \times n}$, such that*

$$A = \frac{1}{(\beta^T \alpha)^2} \alpha \beta^T + \frac{1}{\beta^T \alpha} (I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T) Y (I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T) \quad (3.10)$$

and

$$\rho((I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T) Y (I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T)) < 1. \quad (3.11)$$

Proof. “Only if” part: Suppose that A is a real eventually positive matrix. Then, it follows from [2, Remark 4.2] or [3, Theorem 2.3] that A possesses the strong Perron-Frobenius property and there exists a left eigenvector $\beta_1 \in \mathcal{P}$ corresponding to the eigenvalue $\rho(A)$ of A . Thus, by Theorem 3.1, there exist $\alpha \in \mathcal{P}$, $\beta \in \mathbb{R}^n$, and $Y \in \mathbb{R}^{n \times n}$, such that (3.10) and (3.11) hold. So, to complete the proof of necessity it is enough to show $\beta \in \mathcal{P}$.

First notice that (3.10) and (3.11) hold, by induction on k , we obtain that

$$(\beta^T \alpha A)^k = \left(\frac{1}{\beta^T \alpha} \alpha \beta^T\right)^k + \left((I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T) Y (I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T)\right)^k. \quad (3.12)$$

Since (3.11) holds, then the second term of right-hand side of (3.12) tends to zero matrix as $k \rightarrow \infty$. Also note that $\left(\frac{1}{\beta^T \alpha} \alpha \beta^T\right)^k = \frac{1}{\beta^T \alpha} \alpha \beta^T$. Therefore, letting $k \rightarrow \infty$ and using (3.12), we have

$$\lim_{k \rightarrow \infty} (\beta^T \alpha A)^k = \frac{1}{\beta^T \alpha} \alpha \beta^T. \quad (3.13)$$

Using the same arguments in the proof of necessity of Theorem 3.1, one can see from (3.10) and (3.11) that $\rho(A) = \frac{1}{\beta^T \alpha}$. Thus (3.13) can be written as

$$\rho(A) \alpha \beta^T = \lim_{k \rightarrow \infty} (\rho(A))^{-k} A^k. \quad (3.14)$$

Since $\beta_1^T A = \rho(A) \beta_1^T$, so $\beta_1^T (\rho(A))^{-k} A^k = \beta_1^T$. By the above equality,

$$\beta_1^T \rho(A) \alpha \beta^T = \lim_{k \rightarrow \infty} \beta_1^T ((\rho(A))^{-k} A^k) = \beta_1^T, \quad (3.15)$$

which shows that $\beta = (\rho(A) \beta_1^T \alpha)^{-1} \beta_1 \in \mathcal{P}$, by noting that $\rho(A) \beta_1^T \alpha > 0$ and $\beta_1 \in \mathcal{P}$.

“If” part: If there exist $\alpha \in \mathcal{P}$, $\beta \in \mathcal{P}$, and $Y \in \mathbb{R}^{n \times n}$, such that (3.10) and (3.11) hold. By the same arguments as above, one can readily see that

$$\lim_{k \rightarrow \infty} (\rho(A))^{-k} A^k = \rho(A) \alpha \beta^T. \quad (3.16)$$

Note that the matrix on the right-hand side of (3.16) is positive under the assumption on α and β . Hence from (3.16), it is straightforward that $(\rho(A))^{-k} A^k$ and also A^k are positive matrices for all sufficient large positive integers k . This completes the proof. \square

4. Main results

We need an auxiliary lemma to prove our main results.

Lemma 4.1. *Let $\alpha, \beta \in \mathbb{R}^n$. Suppose $M \in \mathbb{R}^{(n-1) \times (n-1)}$ is a nonsingular matrix and satisfies*

$$(I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T) \begin{bmatrix} O \\ M \end{bmatrix} \begin{bmatrix} I_{n-2} \\ \tilde{O} \end{bmatrix} \begin{bmatrix} M \\ O \end{bmatrix} \begin{bmatrix} \tilde{O} \\ I_{n-2} \end{bmatrix} \quad (4.1)$$

and

$$e_n^T (I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T) \begin{bmatrix} O \\ M \end{bmatrix} \tilde{e}_{n-1} = 1. \quad (4.2)$$

Let

$$B_0 = (I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T) \begin{bmatrix} O & 0 \\ I_{n-1} & -Mu - v \end{bmatrix} (I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T), \quad (4.3)$$

where $u = (c_1, c_2, \dots, c_{n-1})^T \in \mathbb{C}^n$ and

$$v = \begin{bmatrix} I_{n-1} \\ O \end{bmatrix}^T (I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T) \begin{bmatrix} O \\ M \end{bmatrix} \tilde{e}_{n-1}. \quad (4.4)$$

Then the characteristic polynomial of B_0 is given by

$$p_{B_0}(\lambda) = \lambda^n + c_{n-1} \lambda^{n-1} + \dots + c_1 \lambda.$$

Here and in the sequel O and \tilde{O} are $1 \times (n-1)$ and $1 \times (n-2)$ zero matrices respectively, $e_i \in \mathbb{R}^n$ is the i -th column of the identity matrix I_n , and $\tilde{e}_i \in \mathbb{R}^{n-1}$ is the i -th column of the identity matrix I_{n-1} .

Proof. Let $Q = I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T$. Note that

$$\begin{bmatrix} O & 0 \\ I_{n-1} & -Mu - v \end{bmatrix} = \begin{bmatrix} 1 & O \\ O^T & M \end{bmatrix} \begin{bmatrix} O & 0 \\ I_{n-1} & -u \end{bmatrix} \cdot \begin{bmatrix} M^{-1} & -M^{-1}v \\ O & 1 \end{bmatrix}$$

and $Q^2 = Q$. By using $\det(\lambda I_n - AB) = \det(\lambda I_n - BA)$, for $A, B \in \mathbb{C}^{n \times n}$ (see [4,p53]), we first have

$$\begin{aligned} p_{B_0}(\lambda) &= \det \left(\lambda I_n - Q \begin{bmatrix} O & 0 \\ I_{n-1} & -Mu - v \end{bmatrix} Q \right) \\ &= \det \left(\lambda I_n - Q \begin{bmatrix} 1 & O \\ O^T & M \end{bmatrix} \begin{bmatrix} O & 0 \\ I_{n-1} & -u \end{bmatrix} \cdot \begin{bmatrix} M^{-1} & -M^{-1}v \\ O & 1 \end{bmatrix} Q \right) \\ &= \det \left(\lambda I_n - \begin{bmatrix} M^{-1} & -M^{-1}v \\ O & 1 \end{bmatrix} Q \begin{bmatrix} 1 & O \\ O^T & M \end{bmatrix} \begin{bmatrix} O & 0 \\ I_{n-1} & -u \end{bmatrix} \right). \quad (4.5) \end{aligned}$$

we next claim

$$\begin{bmatrix} M^{-1} & -M^{-1}v \\ O & 1 \end{bmatrix} Q \begin{bmatrix} 1 & O \\ O^T & M \end{bmatrix} \begin{bmatrix} O & 0 \\ I_{n-1} & -u \end{bmatrix} \begin{bmatrix} O & 0 \\ I_{n-1} & -u \end{bmatrix}. \quad (4.6)$$

To prove this, we note that

$$\begin{aligned} & Q \begin{bmatrix} 1 & O \\ O^T & M \end{bmatrix} \begin{bmatrix} O & 0 \\ I_{n-1} & -u \end{bmatrix} \\ = & Q \begin{bmatrix} 1 & O \\ O^T & M \end{bmatrix} \begin{bmatrix} O \\ I_{n-1} \end{bmatrix} [I_{n-1}, -u] \\ = & Q \begin{bmatrix} O \\ M \end{bmatrix} [I_{n-1}, -u] \\ = & Q \left[\begin{bmatrix} O \\ M \end{bmatrix} \begin{bmatrix} I_{n-2} \\ \tilde{O} \end{bmatrix}, \begin{bmatrix} O \\ M \end{bmatrix} \tilde{e}_{n-1} \right] [I_{n-1}, -u] \\ = & \left[Q \begin{bmatrix} O \\ M \end{bmatrix} \begin{bmatrix} I_{n-2} \\ \tilde{O} \end{bmatrix}, Q \begin{bmatrix} O \\ M \end{bmatrix} \tilde{e}_{n-1} \right] [I_{n-1}, -u] \\ = & \left[Q \begin{bmatrix} O \\ M \end{bmatrix} \begin{bmatrix} I_{n-2} \\ \tilde{O} \end{bmatrix}, \begin{bmatrix} [I_{n-1}]^T Q \begin{bmatrix} O \\ M \end{bmatrix} \tilde{e}_{n-1} \\ e_n^T Q \begin{bmatrix} O \\ M \end{bmatrix} \tilde{e}_{n-1} \end{bmatrix} \right] [I_{n-1}, -u]. \quad (4.7) \end{aligned}$$

A straightforward substitution of assumptions (4.1), (4.2), and (4.4) into (4.7) show that

$$Q \begin{bmatrix} 1 & O \\ O^T & M \end{bmatrix} \begin{bmatrix} O & 0 \\ I_{n-1} & -u \end{bmatrix} \left[\begin{bmatrix} M \\ O \end{bmatrix} \begin{bmatrix} \tilde{O} \\ I_{n-2} \end{bmatrix}, \begin{bmatrix} v \\ 1 \end{bmatrix} \right] [I_{n-1}, -u]. \quad (4.8)$$

Pre-multiplying both sides of (4.8) by

$$\begin{bmatrix} M^{-1} & -M^{-1}v \\ O & 1 \end{bmatrix},$$

a direct computation shows that (4.6) holds. Finally, combine (4.5) with (4.6) to deduce that

$$p_{B_0}(\lambda) = \det \left(\lambda I_n - \begin{bmatrix} O & 0 \\ I_{n-1} & -u \end{bmatrix} \right) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda. \quad \square$$

we now present our main results as following theorems.

Theorem 4.1. *Let $\sigma = (\rho, \lambda_2, \dots, \lambda_n)$ be a list of n complex numbers. Then there exists a real eventually positive matrix with the spectrum σ if and only if σ is closed under conjugation and ρ is a positive number which satisfies $\rho > |\lambda_i| (i = 2, \dots, n)$.*

In fact, under the assumptions on σ and ρ , given any $\alpha = (a_1, a_2, \dots, a_n)^T \in \mathcal{P}$. Let $\beta_0 = (1, 1, \dots, 1)^T$ and $\beta = \frac{1}{\rho \beta_0^T \alpha} \beta_0 \frac{1}{\rho \cdot (a_1 + a_2 + \dots + a_n)} \beta_0$ with $\rho = \frac{1}{\beta^T \alpha}$. Let

$$M = \begin{bmatrix} 1 + \frac{a_2}{a_1} & -1 & 0 & \cdots & \cdots & 0 \\ \frac{a_3}{a_1} & 1 & -1 & \ddots & \cdots & \vdots \\ \frac{a_4}{a_1} & 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \cdots & \ddots & \ddots & -1 \\ \frac{a_n}{a_1} & 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{(n-1) \times (n-1)}. \quad (4.9)$$

Suppose that

$$\left(\lambda - \frac{\lambda_2}{\rho}\right)\left(\lambda - \frac{\lambda_3}{\rho}\right)\cdots\left(\lambda - \frac{\lambda_n}{\rho}\right) = \lambda^{n-1} + c_{n-1}\lambda^{n-2} + \cdots + c_1. \quad (4.10)$$

Let

$$A = \frac{1}{(\beta^T \alpha)^2} \alpha \beta^T + \frac{1}{\beta^T \alpha} \left(I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T \right) \begin{bmatrix} O & 0 \\ I_{n-1} & -Mu - v \end{bmatrix} \left(I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T \right), \quad (4.11)$$

where $u = (c_1, c_2, \dots, c_{n-1})^T$ and

$$v = \begin{bmatrix} I_{n-1} \\ O \end{bmatrix}^T \left(I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T \right) \begin{bmatrix} O \\ M \end{bmatrix} \tilde{e}_{n-1}. \quad (4.12)$$

Then A is, a realizing matrix, a real eventually positive matrix with the spectrum $\sigma = (\rho, \lambda_2, \dots, \lambda_n)$.

As mentioned earlier, here O denotes $1 \times (n-1)$ zero matrix and $\tilde{e}_{n-1} = (0, \dots, 0, 1)^T \in \mathbb{R}^{n-1}$.

Proof. “If” part: Since ρ is a positive number and σ is closed under conjugation, it is seen that $u = (c_1, c_2, \dots, c_{n-1})^T \in \mathbb{R}^{n-1}$ by (4.10). So

$$\begin{bmatrix} O & 0 \\ I_{n-1} & -Mu - v \end{bmatrix} \in \mathbb{R}^{n \times n}$$

and thus $A \in \mathbb{R}^{n \times n}$.

To show that $\sigma = (\rho, \lambda_2, \dots, \lambda_n)$ is the spectrum of A , we will compute the characteristic polynomial of A . Let B_1, B_2 be the first and the second terms on the right-hand side of (4.11) respectively. Then $A = B_1 + B_2$ with $B_1 B_2 = 0$. Note that $p_{B_1}(\lambda) = \det(\lambda I_n - \rho^2 \cdot \alpha \beta^T) = \lambda^{n-1}(\lambda - \rho)$. By Lemma 3.1,

$$p_A(\lambda) = \lambda^{-n} p_{B_1}(\lambda) \cdot p_{B_2}(\lambda) = \lambda^{-1}(\lambda - \rho) \cdot p_{B_2}(\lambda). \quad (4.13)$$

Moreover, we claim

$$p_{B_2}(\lambda) = (\lambda - \lambda_2)(\lambda - \lambda_3) \cdots (\lambda - \lambda_n)\lambda. \quad (4.14)$$

The claim will follow from Lemma 4.1 provided that we can show that M is a nonsingular matrix and satisfies

$$\left(I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T\right) \begin{bmatrix} O \\ M \end{bmatrix} \cdot \begin{bmatrix} I_{n-2} \\ \tilde{O} \end{bmatrix} \begin{bmatrix} M \\ O \end{bmatrix} \begin{bmatrix} \tilde{O} \\ I_{n-2} \end{bmatrix} \quad (4.15)$$

and

$$e_n^T \cdot \left(I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T\right) \begin{bmatrix} O \\ M \end{bmatrix} \cdot \tilde{e}_{n-1} = 1. \quad (4.16)$$

In fact, a direct computation shows that

$$\det(M) = \frac{a_1 + a_2 + \cdots + a_n}{a_1} > 0 \quad (4.17)$$

and

$$\left(I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T\right) \begin{bmatrix} O \\ M \end{bmatrix} \begin{bmatrix} -1 & 0 & \cdots & \cdots & 0 \\ 1 & -1 & \ddots & \cdots & \vdots \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & 1 & \ddots & 0 \\ \vdots & \cdots & \ddots & \ddots & -1 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}_{n \times (n-1)}. \quad (4.18)$$

By (4.17) and (4.18), it is immediate that M is a nonsingular matrix and satisfies (4.15) and (4.16). Let

$$B_0 = \frac{1}{\rho} B_2 = \left(I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T\right) \begin{bmatrix} O & 0 \\ I_{n-1} & -Mu - v \end{bmatrix} \left(I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T\right). \quad (4.19)$$

Now, combine Lemma 4.1 with (4.19), (4.12), and (4.15)-(4.17) to deduce

$$\begin{aligned} p_{B_0}(\lambda) &= \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda \\ &= \left(\lambda - \frac{\lambda_2}{\rho}\right)\left(\lambda - \frac{\lambda_3}{\rho}\right)\cdots\left(\lambda - \frac{\lambda_n}{\rho}\right)\lambda \end{aligned} \quad (4.20)$$

by noting the assumption (4.10). We also have

$$p_{B_2}(\lambda) = \det(\lambda I_n - \rho B_0) = \rho^n \cdot \det\left(\frac{\lambda}{\rho} I_n - B_0\right) = \rho^n \cdot p_{B_0}\left(\frac{\lambda}{\rho}\right). \quad (4.21)$$

So, one sees that (4.14) holds, using (4.21) and (4.20). A direct substitution (4.14) into (4.13) shows

$$p_A(\lambda) = (\lambda - \rho)(\lambda - \lambda_2)(\lambda - \lambda_3)\cdots(\lambda - \lambda_n). \quad (4.22)$$

Finally, by (4.19), (4.20), and the assumption on ρ , we have

$$\rho\left((I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T) \begin{bmatrix} O & 0 \\ I_{n-1} & -Mu - v \end{bmatrix} (I_n - \frac{1}{\beta^T \alpha} \alpha \beta^T)\right) = \rho(B_0) < 1. \quad (4.23)$$

Therefore it can be seen readily that A is a real eventually positive matrix from Theorem 3.2 by noting (4.23) and (4.11). Combine this with (4.22) to conclude the proof of “if” part.

“Only if” part: Assume that there exists a real eventually positive matrix A with the spectrum $\sigma = (\rho, \lambda_2, \dots, \lambda_n)$. Then it readily follows from [2, Corollary 3.4] and the Perron-Frobenius theorem that σ is closed under conjugation and ρ is a positive number which satisfies $\rho > |\lambda_i| (i = 2, \dots, n)$, as desired. \square

We are now ready to prove the following result, which gives a complete answer to the EPIEP prescribed the left and the right Perron eigenvectors of the realizing matrix.

Theorem 4.2. *Let $\sigma = (\rho, \lambda_2, \dots, \lambda_n)$ be a list of n complex numbers. Given any $\tilde{\alpha} = (\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n)^T \in \mathcal{P}$ and any $\tilde{\beta} = (\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n)^T \in \mathcal{P}$. Then there exists a real eventually positive matrix \tilde{A} with the spectrum σ and with the right and the left eigenvectors $\tilde{\alpha}$ and $\tilde{\beta}$ respectively, corresponding to the eigenvalue ρ of \tilde{A} if and only if σ is closed under conjugation and ρ is a positive number which satisfies $\rho > |\lambda_i| (i = 2, \dots, n)$.*

In fact, under the assumptions on σ and ρ . Define A as in the Theorem 4.1 where we take $\alpha = (a_1, a_2, \dots, a_n)^T = (\tilde{a}_1 \tilde{b}_1, \tilde{a}_2 \tilde{b}_2, \dots, \tilde{a}_n \tilde{b}_n)^T \in \mathcal{P}$. Let $\tilde{A} = D^{-1}AD$, where D is a diagonal matrix defined as

$$D = \begin{bmatrix} \tilde{b}_1 & & & \\ & \tilde{b}_2 & & \\ & & \ddots & \\ & & & \tilde{b}_n \end{bmatrix}.$$

Then \tilde{A} is a real eventually positive matrix with the spectrum $\sigma = (\rho, \lambda_2, \dots, \lambda_n)$ and with the right and the left eigenvectors $\tilde{\alpha}$ and $\tilde{\beta}$ respectively, corresponding to the eigenvalue ρ of \tilde{A} .

Proof. “Only if” part: This is immediate from Theorem 4.1.

“If” part: Since by Theorem 4.1 we know that A is a real eventually positive matrix with the spectrum $\sigma = (\rho, \lambda_2, \dots, \lambda_n)$, thus it is easy to see that \tilde{A} is also a real eventually positive matrix with the spectrum $\sigma = (\rho, \lambda_2, \dots, \lambda_n)$, by noting that $\tilde{A} = D^{-1}AD$ and D is a diagonal matrix with positive diagonal entries. Hence, to complete the proof, it is sufficient to prove that $\tilde{\alpha}$ and $\tilde{\beta}$ are the right and the left eigenvectors respectively, corresponding to the eigenvalue ρ of \tilde{A} . Let $\beta_0 = (1, 1, \dots, 1)^T$ as in Theorem 4.1. By the definition of A , it follows that α and $\beta = \frac{1}{\rho \beta_0^T \alpha} \beta_0$ (also β_0) are the right and the left eigenvectors respectively, corresponding to the eigenvalue ρ of A , and in other words $A\alpha = \rho\alpha$ and $\beta_0^T A = \rho\beta_0^T$. Equivalently,

$$\tilde{A}D^{-1}\alpha = \rho D^{-1}\alpha \quad \text{and} \quad \beta_0^T D\tilde{A} = \rho\beta_0^T D, \quad (4.24)$$

by noting that $A = D\tilde{A}D^{-1}$. Now direct computations show that

$$D^{-1}\alpha = \tilde{\alpha} \quad \text{and} \quad \beta_0^T D = \tilde{\beta}^T. \quad (4.25)$$

Combine (4.24) and (4.25) to deduce that $\tilde{\alpha}$ and $\tilde{\beta}$ are the right and the left eigenvectors respectively, corresponding to the eigenvalue ρ of \tilde{A} . This completes the proof \square

5. An example

In this section we illustrate Theorem 4.1 with a computational example.

Example 5.1. Consider the list $\sigma = (\rho, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (19, -1 + 18i, -1 - 18i, -3 + 8i, -3 - 8i)$, partly borrowed from [6, Example 6]. From [6, Theorem 1], we know that there exists no nonnegative matrix with the spectrum σ . However, in the eventually positive matrix case, Theorem 4.1 says that one can construct an eventually positive matrix A (the realizing matrix) with the spectrum σ .

First, We have

$$\begin{aligned} & (x - \frac{\lambda_2}{\rho})(x - \frac{\lambda_3}{\rho})(x - \frac{\lambda_4}{\rho})(x - \frac{\lambda_5}{\rho}) \\ &= (x - \frac{-1 + 18i}{19})(x - \frac{-1 - 18i}{19})(x - \frac{-3 + 8i}{19})(x - \frac{-3 - 8i}{19}) \\ &= \frac{23725}{130321} + \frac{2096}{6859}x + \frac{410}{361}x^2 + \frac{8}{19}x^3 + x^4, \end{aligned}$$

hence

$$u = (c_1, c_2, c_3, c_4)^T = \left(\frac{23725}{130321}, \frac{2096}{6859}, \frac{410}{361}, \frac{8}{19} \right)^T. \quad (5.1)$$

Second, if we take

$$\alpha = (a_1, a_2, a_3, a_4, a_5)^T = (1, 1, 3, 8, 8)^T, \quad (5.2)$$

then

$$M = \begin{bmatrix} 1 + \frac{a_2}{a_1} & -1 & 0 & 0 \\ \frac{a_3}{a_1} & 1 & -1 & 0 \\ \frac{a_4}{a_1} & 0 & 1 & -1 \\ \frac{a_5}{a_1} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 3 & 1 & -1 & 0 \\ 8 & 0 & 1 & -1 \\ 8 & 0 & 0 & 1 \end{bmatrix}, \quad (5.3)$$

and thus

$$v = \begin{bmatrix} I_4 \\ O \end{bmatrix}^T \left(I_5 - \frac{1}{\beta^T \alpha} \alpha \beta^T \right) \begin{bmatrix} O \\ M \end{bmatrix} \tilde{e}_4 = (0, 0, 0, -1)^T \quad (5.4)$$

by noting that

$$\beta = \frac{1}{\rho \cdot (a_1 + a_2 + a_3 + a_4 + a_5)} \beta_0 = \frac{1}{399} (1, 1, 1, 1, 1)^T. \quad (5.5)$$

Combine (5.1), (5.3), and (5.4) to deduce that

$$-Mu - v = \left(-\frac{7626}{130321}, \frac{37011}{130321}, -\frac{282938}{130321}, -\frac{114351}{130321} \right)^T. \quad (5.6)$$

Finally, from Theorem 4.1, we know that

$$A = \frac{1}{(\beta^T \alpha)^2} \alpha \beta^T + \frac{1}{\beta^T \alpha} (I_5 - \frac{1}{\beta^T \alpha} \alpha \beta^T) \begin{bmatrix} O & 0 \\ I_4 & -Mu - v \end{bmatrix} (I_5 - \frac{1}{\beta^T \alpha} \alpha \beta^T),$$

hence that

$$A = \begin{bmatrix} -\frac{8497}{20577} & -\frac{8497}{20577} & -\frac{8497}{20577} & -\frac{8497}{20577} & \frac{62678}{20577} \\ \frac{2607949}{144039} & -\frac{128792}{144039} & -\frac{128792}{144039} & -\frac{128792}{144039} & \frac{209287}{144039} \\ -\frac{31834}{7581} & \frac{112205}{7581} & -\frac{31834}{7581} & -\frac{31834}{7581} & \frac{87740}{7581} \\ \frac{3869}{399} & \frac{3869}{399} & \frac{11450}{399} & \frac{3869}{399} & -\frac{1549}{399} \\ -\frac{88}{21} & -\frac{88}{21} & -\frac{88}{21} & \frac{311}{21} & \frac{143}{21} \end{bmatrix}, \quad (5.7)$$

using (5.2), (5.5), and (5.6). With the aid of Mathematica one can verify that A defined as (5.7) has the spectrum $\sigma = (19, -1 + 18i, -1 - 18i, -3 + 8i, -3 - 8i)$ and A^k are positive matrices for $34 \leq k \leq 67$, so A^k are positive matrices for $k \geq 34$. Thus A defined as (5.7) is a real eventually positive matrix with the spectrum $\sigma = (19, -1 + 18i, -1 - 18i, -3 + 8i, -3 - 8i)$, as desired.

Remark. One easily verifies that the spectrum of $(\frac{1}{19}A)^{34}$ is

$$\begin{aligned} \sigma &= (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \\ &= \left(1, \frac{(-1 + 18i)^{34}}{19^{34}}, \frac{(-1 - 18i)^{34}}{19^{34}}, \frac{(-3 + 8i)^{34}}{19^{34}}, \frac{(-3 - 8i)^{34}}{19^{34}} \right), \end{aligned} \quad (5.8)$$

with $\max\{|\lambda_i|, i = 2, \dots, 5\}$ to be approximately 0.167647. Hence, the list σ defined as (5.8) can be realized by a nonnegative matrix (more precisely, a positive matrix). But this can not be deduced from Theorem 1.1 ([1, Theorem 2.1]), taking into account that $2n \cdot \max\{|\lambda_i|, i = 2, \dots, 5\} > |\lambda_1|$, here $n = 5$. Also this can not be deduced from [6, Theorem 1], since λ_2 and λ_3 have positive real parts. Thus, using our constructive skill, it is helpful to construct various nonnegative matrices in the NIEP.

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