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SEMILATTICE DECOMPOSITIONS OF SEMIGROUPS

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Preface

General decomposition problems hold a central place in the general structure theory of semigroups, as they look for different ways to break a semigroup into parts, with as simple a structure as possible, in order to examine these parts in detail, as well as the relationships between the parts within the whole semigroup. The main problem is to determine whether the greatest decomposition of a given type exists, the decomposition having the finest components, and to give a characterization and construction of this greatest decomposition. Another important issue is whether a given type of decomposition is atomic, in the sense that the components of the greatest decomposition of the given type cannot further be broken down by decomposition of the same type. In semigroup theory only five types of atomic decompositions are known so far. The atomicity of semilattice decompositions was proved by Tamura [Osaka Math. J. 8 (1956) 243–261], of ordinal decompositions by Lyapin [Semigroups, Fizmatgiz, Moscow, 1960], of the so-called U-decompositions by Shevrin [Dokl. Akad. Nauk SSSR 138 (1961) 796–798], of orthogonal decompositions by Bogdanović and Ćirić [Israel J. Math 90 (1995) 423–428], whereas the atomicity of subdirect decompositions follows from a more general result of universal algebra proved by Birkhoff [Bull. AMS 50 (1944) 764–768]. Semilattice decompositions of semigroups were first defined and studied by A. H. Clifford [Annals of Math. 42 (1941) 1037–1049]. Later T. Tamura and N. Kimura [Kodai Math. Sem. Rep. 4 (1954) 109–112] proved the existence of the greatest semilattice decomposition of an arbitrary semigroup, and as we have already noted, while T. Tamura [Osaka Math. J. 8 (1956) 243–261] proved the atomicity of semilattice decompositions. The theory of the greatest semilattice decompositions of semigroups has been developed from the middle of the 1950s to the middle of the 1970s by T. Tamura, M. S. Putcha, M. Petrich, and others. For a long time after that there were no new results in this area. In the mid of 1990s, the authors of this book initiated the further development of this theory by introducing completely new ideas and methodology. The purpose of this book is to give an overview of the main results on semilattice decompositions of semigroups which appeared in the last 15 years, as well as to connect them with earlier results.

The structure of the book is as follows. The first three chapters of the book provide an introduction to the basic concepts of semigroup theory, various types of regularity and the concepts of simple, 0-simple, Archimedean and 0-Archimedean semigroups. Chapter 4 develops the general theory of the greatest semilattice decompositions of semigroups, using the methodology that was built by the authors. This methodology is based on the computation of the principal radicals of a semigroup, which is an iterative process that, in general, may consist of infinitely many iterations. For this reason, later this chapter discusses the various cases where the greatest semilattice decompositions can be achieved by methods that involve only finitely many iterations.

The first effective construction of the smallest semilattice congruence on a semigroup, provided by T. Tamura [Semigroup Forum 4 (1972) 255–261], was based on the arrow relation $\to$, which was defined as a natural generalization of the division relation. Namely, two elements a and b of a semigroup are said to be in the relation $\to$, written as $a \to b$, if the element b divides some power of the element a. If each pair of elements of a semigroup is in that relation, then this semigroup is said to satisfy the famous Archimedean property, which Archimedes proved for natural numbers, and such a semigroup is called an Archimedean semigroup. In the above mentioned paper, T. Tamura proved that the smallest semilattice congruence on a semigroup can be constructed as the symmetric opening of the transitive closure of the arrow relation, whereas M. S. Putcha [Trans. Amer. Math. Soc. 189 (1974), 93–106] showed that these two operations can be permuted, i.e., the smallest semilattice congruence can be computed as the transitive closure of the symmetric opening of the arrow relation.

In Chapter 4 the authors discuss various situations where the transitive closure of the arrow relation can be computed in a finite number of steps, and in Chapter 5 they consider the situation when the arrow relation is transitive.

Semigroups with the latter property are actually semigroups that can be represented as a semilattice of Archimedean semigroups. Chapter 5 also deals with various special types of semilattices of Archimedean semigroups. A particular case of Archimedean semigroups are semigroups in which each element divides a fixed power of any other element, and such semigroups are called k-Archimedean. The semilattices of k-Archimedean semigroups and many of their special cases are studied in Chapter 6.

A very important special case of semilattices of Archimedean semigroups are semilattices of completely Archimedean semigroups, or equivalently, semilattices of nil-extensions of completely simple semigroups. At a scientific conference held back in 1977, L. N. Shevrin announced that a semigroup can be decomposed into a semilattice of completely Archimedean semigroups if and only if each of its elements has a regular power, and each of its regular elements is completely regular (i.e., belongs to a subgroup of this semigroup). However, this result along with other related results was published with proof 17 years later [Mat. Sbornik 185 (8) (1994) 129–160, 185 (9) (1994) 153–176]. In the meantime, other authors have studied these decompositions building their own methodology, for example J. L. Galbiati and M. L. Veronesi [Rend. Ist. Lomb. Cl. Sc. (A) 116 (1982) 180–189; Riv. Mat. Univ. Parma (4) 10 (1984) 319–329], and others. The first author of this book began his research in this area in 1985, and later the other two authors joined him. In a series of papers, the authors of this book built their own methodology, which not only led to the same results announced by L. N. Shevrin, but also provided some significant improvements. A complete theory of the decompositions of a semigroup into a semilattice of completely Archimedean semigroups was presented for the first time in the book by the first two authors [Semigroups, Prosveta, Niš, 1993]. Chapter 7 of this book outlines not only these results, but also many results obtained later.

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Niš, On Saint Petka, 2011

Authors

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Chapter 1

Introduction

In this chapter we will outline the basic notions and results of the theory of semigroups which will be used in the main part of this book. Also, we will present some basic notions of general lattice theory and the theory of Boolean algebra. For more details, we refer to special monographs from these areas.

1.1 The Definition of a Semigroup

Let S be a non-empty set. The mapping $\circ$ from a Cartesian product $S \times S$ into a set S, which to every ordered pair (a, b) of elements of S associates an element of S, denoted by $a \circ b$, we call a *binary operation* on the set S, or a (binary) *operation* of S. An ordered pair $(S, \circ)$ is called a *groupoid*.

A binary operation $\circ$ of a groupoid $(S, \circ)$ is *associative* if $(a \circ b) \circ c = a \circ (b \circ c)$, for all $a, b, c \in S$. Then, the pair $(S, \circ)$ is a *semigroup*.

For the sake of simplicity, we introduce the following agreement: the operation of a groupoid we will denote by "$\cdot$", and refer to it as the *multiplication* or the *product*, and the element $a \cdot b$ we will call the *multiplication* of elements a and b. Without any loss of generality, the pair $(S, \cdot)$ we will, for short, denote as S, so instead of "*the goupoid* $(S, \cdot)$" we will simply say "*the goupoid* S". As a substitution for the term "$a \cdot b$" we use the term "ab". In the case when we use some different symbols for the notation of operations, we will stress this additionally.

Often, it is not easy to determine that some binary operation on a groupoid S is associative. A. H. Clifford and G. B. Preston in their book "*The algebraic theory of semigroups* I" give *Light's associativity test* for finite groupoids. The procedure is consists of: Let $(S, \cdot)$ be a groupoid. We define for S two new binary operations $*$ and $\circ$ with

$$x * y = x \cdot (a \cdot y), \qquad x \circ y = (x \cdot a) \cdot y, \qquad x, y \in S,$$

where $a \in S$ is a fixed element. It is evident that associativity hold in S if and only if both operations $*$ and $\circ$ are equal on S, for every $a \in S$.

This procedure we will shown on an example. Let the groupoid $(S, \cdot)$ be given by Cayley's table

$$\begin{array}{c|cc} \cdot & \alpha & \beta \\ \hline \alpha & \alpha & \alpha \\ \beta & \beta & \alpha \end{array} \ .$$

Then for $a = \alpha$ the product $a \cdot y$ is in the first row $(\alpha\alpha)$, and for $\alpha = \beta$ the product $a \cdot y$ is in the second row $(\beta\alpha)$.

Now, the given table extends to the right side first by the first row, then by the second row, and does all the multiplications with the elements from S. In this way we obtain the operation $*$ for both elements of the groupoid S. Similarly, the given table extends down throught columns from S. Then we obtain the operation $\circ$ for all the elements of S.

$$\begin{array}{c|cc|cc|cc|} \cdot & \alpha & \beta & \alpha & \alpha & \beta & \alpha \\ \hline \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\ \beta & \beta & \alpha & \beta & \boxed{\beta} & \alpha & \beta \\ \hline \alpha & \alpha & \alpha \\ \beta & \beta & \boxed{\alpha} \\ \hline \alpha & \alpha & \alpha \\ \alpha & \alpha & \alpha \\ \hline \end{array}$$

Now, it is easy to see that for $a = \alpha$ the tables for $*$ and $\circ$ do not coincide, because

$$\beta * \beta = \beta \cdot (\alpha \cdot \beta) = \beta \cdot \alpha = \beta, \quad \beta \circ \beta = (\beta \cdot \alpha) \cdot \beta = \beta \cdot \beta = \alpha,$$

as we can see in the extended table. Thus, the given table does not define a semigroup.

By $\mathbf{Z}^+$ we denote the set of all positive integers.

Theorem 1.1 *Every semigroup S satisfies the general associative law, i.e. for every $n \in \mathbf{Z}^+$, a product of n elements from S does not depend on the positioning of the parentheses.*

Proof. Let $a_1, a_2, \dots, a_n \in S$ and let

$$a_1 a_2 \cdots a_n = a_1(a_2(a_3 \cdots (a_{n-1}a_n)\dots)).$$

The statement of the theorem immediately follows for $n = 1$ and $n = 2$. Also, it is true for $n = 3$, by supposition, because S is a semigroup.

Assume $n > 3$ and that the statement of the theorem holds for some $r < n$. Assume that $u \in S$ is equal to the product of elements $a_1, a_2, \dots, a_n$ with an arbitrary disposition of parentheses. Then the element u we can write as $u = vw$, where v is the product of elements $a_1, a_2, \dots, a_r$ and w is the product of elements $a_{r+1}, a_{r+2}, \dots, a_n$, (with some disposition of parentheses), where $1 \le r < n$. Using induction we obtain that $v = a_1 a_2 \cdots a_r$ and $w = a_{r+1}a_{r+2} \cdots a_n$ and

$$\begin{aligned} u &= (a_1 a_2 \cdots a_r)(a_{r+1}a_{r+2} \cdots a_n) = (a_1(a_2 \cdots a_r))(a_{r+1}a_{r+2} \cdots a_n) \\ &= a_1((a_2 \cdots a_r)(a_{r+1}a_{r+2} \cdots a_n)) = a_1(a_2 \cdots a_r a_{r+1}a_{r+2} \cdots a_n) \\ &= a_1 a_2 \cdots a_n. \end{aligned}$$

for $r > 1$, and $u = vw = a_1(a_2 \cdots a_n) = a_1 a_2 \cdots a_n$, for $r = 1$. This proves the theorem. □

Namely, the general associative law says that the product of n elements of a semigroup is not dependent on the order in which we calculate this product, while it is dependent on the order in which we write the elements in this product, from left to right. By Theorem 1.1, in a semigroup S we can omit all the parentheses in products of elements from S, so the product of elements $a_1, a_2, \dots, a_n \in S$, in this order, we will simply denote with $a_1 a_2 \cdots a_n$, $n \in \mathbf{Z}^+$. If $a_i = a$, for every $i \in \{1, 2, \dots, n\}$, then the product $a_1 a_2 \cdots a_n$ we denote as a^n, and it is called the n-th *power of the element* $a \in S$. If A is a non-empty subset of a semigroup S, then the set

$$\sqrt{A} = \{x \in S \mid (\exists n \in \mathbf{Z}^+)\, x^n \in A\}$$

we call the *radical* of set A.

Let S be a semigroup. Elements $a, b \in S$ *commute* if $ab = ba$. If A is a non-empty subset of a semigroup S, then with $C(A)$ we denote the set of all the elements of S which commute with every element of A. The set $C(S)$ we call the *center* of a semigroup S, and its elements are the *central elements* of S. A semigroup S is *commutative* if all of its elements commute with each other. A semigroup S is *anti-commutative* if for all $a, b \in S$, from $ab = ba$ it follows that $a = b$.

If S is an arbitrary semigroup, then we define a binary operation $*$ on S, with: $a * b = ba$. The set S with such a defined operation is a semigroup, which we call a *dual semigroup* of a semigroup S, and we denote it by $\overleftarrow{S}$. A semigroup need not be commutative, i.e. the value of a product depends on the order of elements which are in the product, and as a consequence of this in terms corresponding to the semigroup, for its subsets or for its elements, very often we use terms "left" or "right". The *dual* of a term which corresponding to a semigroup, or its subsets or its elements, is the term which we obtain when the word "left" is replaced with the word "right" and conversely, every product ab we replace with ba.

An element a of a semigroup S is *idempotent* if $a^2 = a$. The set of all idempotents of a semigroup S we denote by $E(S)$. A semigroup in which all the elements are idempotents is a *band*. A commutative band is a *semilattice*. A semilattice S is a *chain* if $ab = a$ or $ab = b$, for all $a, b \in S$.

Let S be a semigroup and let $a \in S$. An element $e \in S$ is a *left* (*right*) *identity of element* a if $ea = a$ ($ae = a$), and e is an *identity of element* a if $ae = ea = a$. If $e \in S$ is an identity (left identity, right identity) for all the elements of S, then e is an *identity* (*left identity, right identity*) *of a semigroup* S. By definition, every (left, right) identity of a semigroup is an idempotent of S. It is easy to prove that a semigroup has exactly one identity. A semigroup which has an identity is a *semigroup with an identity* or *monoid*.

Let S be a semigroup and let e be an element which is not contained in S. On the set $S \cup \{e\}$ we define multiplication with: $ae = ea = a$, $a \in S$, $ee = e$, and the product of the elements from S stays the same. Then, the set $S \cup \{e\}$ with such a defined multiplication is a semigroup with the identity e, which we call the *identity extension* of a semigroup S by e. If S is a semigroup, then with S^1 we denote a semigroup obtained from S in the following way: if S has an identity, then $S = S^1$, if S has no identity, then S^1 is an identity extension of S by 1. The identity element of a semigroup S we usually denote with e or 1. Using the identity extension of a semigroup, we extend the definition of the power in the semigroup: if S is a semigroup and if a is an element of S, then a^0 is the identity of the monoid S^1.

Let S be a semigroup and let $z \in S$. An element z is a *left* (*right*) *zero* of S if $za = z$ ($az = z$), for every $a \in S$, and z is a *zero* of S if z is both the left and right zero of S. Every (left, right) zero of a semigroup is an idempotent. Thus, a semigroup whose every element is left (right) zero is a band, which we call *left* (*right*) *zero band*. Hence, a semigroup S is a left (right) zero band if $ab = a$ ($ab = b$), for all $a, b \in S$. It is evident that a semigroup has exactly one zero. A semigroup which has a zero we call a *semigroup with a zero*.

Let S be a semigroup and let z be an element which is not contained in S, on the set $S \cup \{z\}$ we define multiplication with: $az = za = z$, $a \in S$, $zz = z$, and the product of elements from S stays the same, then, the set $S \cup \{z\}$ with such a defined multiplication is a semigroup with zero z, which we call the *zero extension* of a semigroup S by z. If S is a semigroup, then S^0 denotes a semigroup obtained from S in the following way: if S has a zero, then $S = S^0$, if S has no zero, then S^0 is a zero extension of S by 0. The zero of a semigroup we often denote with 0, and very often the term "$\{0\}$" we replace with the term "0". According to the previous notations, with $S = S^0$ we denote a semigroup S with zero 0. If $S = S^0$ and if $A \subseteq S$, then we use the notations $A^0 = A \cup 0$, $A^\bullet = A - 0$. If $S = S^0$, then the element $a \in S^\bullet$ is a *divisor of zero* if there is an element $b \in S^\bullet$ such that $ab = 0$ or $ba = 0$. A semigroup $S = S^0$ which has no divisors of zero, i.e. if $S^\bullet$ is a subsemigroup of S, is called a *semigroup without a zero divisor*.

A *partial* (*binary*) *operation* on a non-empty set S is a mapping of a non-empty subset of $S \times S$ into S. A non-empty set with a partial binary operation is a *partial groupoid*. If S is a partial groupoid with a partial operation "$\cdot$", and for arbitrary $x, y, z \in S$, the product $x \cdot (y \cdot z)$ is defined if and only if the product $(x \cdot y) \cdot z$ is defined, and where these products are equal, then S is a *partial semigroup*. It is evident that every subset of a semigroup is a partial semigroup. On the other hand, if Q is a partial semigroup and if 0 is an element which is not contained in Q, then the set $Q \cup \{0\}$ with a operation "$\cdot$" defined with:

$$x \cdot y = \begin{cases} xy, & \text{if } x, y, xy \in Q \\ 0, & \text{otherwise} \end{cases} ,$$

where xy is a product in Q, is a semigroup which we denote as Q^0, and we refer to it as a *zero extension of a partial semigroup* Q.

If X is a non-empty set, then with $\mathcal{P}(X)$ we denote the *partitive set* of the set X, i.e. the set of all the subsets of X. Let S be a semigroup. On the partitive set of a semigroup S we define a multiplication with:

$$AB = \{ \ x \in S \,|\, (\exists a \in A)(\exists b \in B) \ \ x = ab \ \}, \qquad A, B \in \mathcal{P}(S).$$

Then, under this operation the set $\mathcal{P}(S)$ is a semigroup which we call a *partitive semigroup* of a semigroup S. It is evident that $\mathcal{P}(S)$ is a semigroup with zero $\emptyset$ (the empty set), without a divisor of zero. Definitions and notations which we use for the multiplication of elements of a semigroup S, we will also use for the multiplication of elements of a semigroup $\mathcal{P}(S)$. For an element a of a semigroup S, in terms of the products of subsets of S, often the term "$\{a\}$" will be replaced with the term "a".

A non-empty subset T of a semigroup S is a *subsemigroup* of S if T is *closed under an operation* of S, i.e. if $ab \in T$, for all $a, b \in T$. If T is a subsemigroup of a semigroup S, then we say that S is an *over semigroup* of T. It is evident that the intersection of an arbitrary family of subsemigroups of a semigroup S, if it is non-empty, is also a subsemigroup of S. Thus, if A is a non-empty subset of S, then the intersection of all the subsemigroups of S containing A is a subsemigroup of S, which we denote by $\langle A \rangle$, and which we call a *subsemigroup of S generated by A*. A semigroup $\langle A \rangle$, under the set inclusion, is the smallest subsemigroup of S containing A. If $A = \{a_1, a_2, \dots, a_n\}$, then instead $\langle \{a_1, a_2, \dots, a_n\} \rangle$ we write $\langle a_1, a_2, \dots, a_n \rangle$, and we say that $\langle A \rangle$ is generated by elements $a_1, a_2, \dots, a_n$. A subsemiogroup $\langle a \rangle$ of a semigroup S generated by the one element subset $\{a\}$ of S we call a *monogenic* or a *cyclic subsemigroup* of S. If A is a subset of a semigroup S such that $\langle A \rangle = S$, then we say that A generates a semigroup S and A is a *generating set* of a semigroup S. The elements from A we call *generator elements* or *generators* of S. A semigroup generated by its one element subset we call a *monogenic* or a *cyclic semigroup*. The proof of the following statement is elementary, so we will omit it.

Lemma 1.1 *Let A be a non-empty subset of a semigroup S. Then*

$$\langle A \rangle = \cup_{n \in \mathbf{Z}^+} A^n.$$

Let A be a non-empty subset of a semigroup S. An element $a \in S$ has a *decomposition into a product of elements from A* if there are $a_1, a_2, \dots, a_n \in A$ such that $a = a_1 a_2 \cdots a_n$. According to Lemma 1.1, A is a set of generators of a semigroup S if and only if every element of S has a decomposition into a product of elements from A. An element $a \in S$ has a unique decomposition into a product of elements from A, if from $a = a_1a_2\cdots a_n$ and $a = b_1b_2\cdots b_m$, $a_i, b_j \in A$, it follows that $n = m$ and $a_i = b_i$, for every $i \in \{1, 2, \ldots, n\}$.

Exercises

1. If e is a left identity (left zero) and f is a right identity (right zero) of a semigroup S, then $e = f$ and e is a unit (zero) of S.

2. Prove that a subsemigroup of a monogenic semigroup need not be monogenic.

3. A semigroup S is a left zero band if and only if its dual semigroup is a right zero band.

4. Give an example of (finite) semigroup in which the set of all idempotents is not a subsemigroup.

5. Give examples of semigroups with zero, and with or without a zero divisor.

1.2 Semigroups of Relations and Mappings

Let A be a non-empty set. Every subset of a Cartesian product $A \times A$ (including the empty set) is a (*binary*) *relation* on A. The set $\Delta_A = \{(a,a) \mid a \in A\}$ is an *identical relation* (*diagonal* or *equality relation*) on A. The set $\omega_A = A \times A$ is a *universal* (*full*) *relation* on A. If there is no danger of confusion (if we know the set), then the identical and universal relation we denote by Δ and ω for short, respectively. The empty subset of $A \times A$ we call the *empty relation* on A. If ξ is a binary relation on A, and if $(a,b) \in \xi$, then we say that a and b are in the relation ξ, and often the term "$(a,b) \in \xi$" we replace with the term "$a\xi b$".

Let A be a non-empty set and let $\mathcal{B}(A)$ be *the set of all binary relations* in A. For $\alpha, \beta \in \mathcal{B}(A)$, a *product of relations* α and β is the relation $\alpha\beta$ in A defined by:

$$\alpha\beta = \{(a,b) \in A \times A \mid (\exists x \in A)\, (a,x) \in \alpha \wedge (x,b) \in \beta\}.$$

The set $\mathcal{B}(A)$ with such a defined multiplication is a semigroup which we call a *semigroup of* (*binary*) *relations* in the set A. For $n \in \mathbf{Z}^+$, by ξ^n we denote the n-th *power of the relation* ξ in A in a semigroup $\mathcal{B}(A)$.

Let A be a non-empty set and let $\xi \in \mathcal{B}(A)$. The set $\mathrm{dom}\xi = \{a \in A \mid (\exists b \in A)\, a\xi b\}$ we call a *domain of relation* ξ. The set $\mathrm{ran}\xi = \{b \in A \mid (\exists a \in A)\, a\xi b\}$ we call a *range of relation* ξ. For $a \in S$ is $a\xi = \{x \in A \mid a\xi x\}$, $\xi a = \{x \in A \mid x\xi a\}$, and for $X \subseteq A$ is $X\xi = \cup\{a\xi \mid a \in X\}$,

$\xi X = \cup\{\xi a \,|\, a \in X\}$. The relation $\xi^{-1} = \{(a,b) \in A \times A \,|\, b\xi a\}$ is an *inverse relation* of a relation ξ. It is evident that $\mathrm{dom}(\xi^{-1}) = \mathrm{ran}\xi$, and $\mathrm{ran}(\xi^{-1}) = \mathrm{dom}\xi$. The relation $\{(a,b) \in A \times A \,|\, (a,b) \notin \xi\}$ is a *converse relation* of ξ.

Let A be a non-empty set. An element $\phi \in \mathcal{B}(A)$ is a *partial mapping* (*partial transformation*) of a set A if $|a\phi| = 1$, for every $a \in \mathrm{dom}\phi$ (by $|X|$ we denote the cardinality of the set X), i.e. if for every $a \in \mathrm{dom}\phi$ there exists a unique $b \in A$ such that $(a,b) \in \phi$. Using this definition, the empty relation on A is a partial mapping in the set A. A set $\mathcal{PT}(A)$ of all the partial mappings in the set A is a subsemigroup of a semigroup $\mathcal{B}(A)$, which we call a *semigroup of partial mappings* (*transformations*) of the set A. For $\varphi, \psi \in \mathcal{PT}(A)$, $\mathrm{dom}(\varphi\psi) = [\mathrm{ran}\varphi \cap \mathrm{dom}\psi]\varphi^{-1}$, $\mathrm{ran}(\varphi\psi) = [\mathrm{ran}\varphi \cap \mathrm{dom}\psi]\psi$, the following condition holds

$$a(\varphi\psi) = (a\varphi)\psi, \qquad \text{for every } a \in \mathrm{dom}(\varphi\psi),$$

which we use as a definition of a multiplication of partial mappings.

Let φ and ψ be a partial mappings of a set A such that $\varphi \subseteq \psi$. Then $\mathrm{dom}\varphi \subseteq \mathrm{dom}\psi$ and $\mathrm{ran}\varphi \subseteq \mathrm{ran}\psi$. If we introduce notions $X = \mathrm{ran}\varphi$, $Y = \mathrm{dom}\psi$, then we say that φ is a *restriction* of ψ on X, in notation, $\varphi = \psi/X$, and that ψ is an *extension* of φ on Y.

Let X and Y be non-empty sets. If ϕ is a partial mapping of some set such that $\mathrm{dom}\phi = X$ and $\mathrm{ran}\phi \subseteq Y$, then we say that ϕ is a *mapping of the set X into the set Y* (or *ϕ maps X into Y*), and we write $\phi : X \mapsto Y$. Based on the definition of partial mapping, for every $x \in X$ there exists a unique $y \in Y$ such that $(x,y) \in \phi$, and then we write $y = x\phi$ and $\phi : x \mapsto y$, and we say that *ϕ maps x into y*. If $\phi : X \mapsto Y$, and if $X = Y$, then we say that ϕ is a mapping of the set X (into itself). If $\phi : X \mapsto Y$, $U \subseteq X$ and $V \subseteq Y$, then the set $U\phi = \{y \in Y \,|\, (\exists u \in U)\, u\phi = y\}$ is an *image* of the subset U (under a mapping ϕ), and the set $V\phi^{-1} = \{x \in X \,|\, x\phi \in V\}$ is an *inverse image* of the subset V (under a mapping ϕ).

Let X and Y be non-empty sets and $\phi : X \mapsto Y$. A mapping ϕ is an *injection* (*injective, one-to-one*) if for $a, b \in X$ from $a\phi = b\phi$ it follows $a = b$. A mapping ϕ is a *surjection* (*surjective, onto*) if $X\phi = Y$, i.e. if for every $y \in Y$ there exists $x \in X$ such that $x\phi = y$. If ϕ is a surjection, then we say that ϕ is a *mapping of X onto Y*, or that maps X onto Y. A mapping ϕ is a *bijection* (*bijective*) if ϕ is both one-to-one and onto.

A mapping $i_X : X \mapsto X$ of a non-empty set X defined by $xi_X = x$, $x \in X$ is an *identical mapping* of a set X. Let X and Y be non-empty sets and let

$\varphi : X \mapsto Y$. If there exists $\psi : Y \mapsto X$ such that $\varphi\psi = i_X$ and $\psi\varphi = i_Y$, then ψ is an *inverse mapping* of φ. Let a mapping φ be a partial mapping of some set A. If ψ is an inverse mapping of φ, then $\psi = \varphi^{-1}$, where φ^{-1} is an inverse relation of φ. Conversely, if φ^{-1} is a partial mapping of a set A, then $\varphi^{-1} : Y \mapsto X$ and φ^{-1} is an inverse mapping of φ. The proof of the following lemma is elementary.

Lemma 1.2 *Let X and Y be non-empty sets. A mapping $\varphi : X \mapsto Y$ has an inverse mapping if and only if φ is a bijective mapping.*

Let X be a non-empty set. For a mapping φ on a set X, we use two types of notations. First one, a *right notation* of mapping: $\varphi : x \mapsto x\varphi$, $x \in X$. In this case we say that φ is a mapping of X right writing. A product of mappings α and β of a set X right writing is a mapping $\alpha\beta$ of a set X which is defined by

$$x(\alpha\beta) = (x\alpha)\beta, \qquad x \in X.$$

A set $\mathcal{T}_r(X)$ of all the mappings of a set X right writing with a previous multiplication is a semigroup which we call a *full semigroup of transformation* (*mapping*) *of a set X right writing.* A semigroup $\mathcal{T}_r(X)$ is a subsemigroup of a semigroup $\mathcal{PT}(X)$. The second way, a *left notation* of mapping: $\varphi : x \mapsto \varphi x$, $x \in X$. In this case we say that φ is a mapping of X left writing. A product of mappings α and β of a set X left writing is a mapping $\alpha\beta$ of a set X which is defined by

$$(\alpha\beta)x = \alpha(\beta x), \qquad x \in X.$$

A set $\mathcal{T}_l(X)$ of all the mappings of a set X left writing with a previous multiplication is a semigroup which we call a *full semigroup transformation* (*mapping*) *of a set X left writing.* It is clear that semigroups $\mathcal{T}_r(X)$ and $\mathcal{T}_l(X)$ are dual. Thus, we usually discuss only one of these semigroups, most often a semigroup $\mathcal{T}_r(X)$, so this semigroup is called a *full semigroup transformation* (*mapping*) *of a set X*, for short.

Let a be an element of a semigroup S. A mapping $\lambda_a \in \mathcal{T}_r(X)$ defined by $x\lambda_a = ax$, $x \in S$, is an *inner left translation* of a semigroup S. A mapping $\rho_a \in \mathcal{T}_r(X)$ defined by $x\rho_a = xa$, $x \in S$, is an *inner right translation* of a semigroup S.

Except (partial) mappings, some other types of relations are very interesting, especially partial ordering and equivalence relations. Let A be a non-empty set. A relation ξ in a set A is:

- *reflexive*, if $a\xi a$, for every $a \in A$, i.e. if $\Delta \subseteq \xi$;
- *symmetric*, if for $a, b \in A$, from $a\xi b$ it follows $b\xi a$, i.e. if $\xi \subseteq \xi^{-1}$;
- *anti-symmetric*, if for $a, b \in A$, from $a\xi b$ and $b\xi a$ it follows $a = b$, i.e. if $\xi \cap \xi^{-1} \subseteq \Delta$;
- *transitive*, if for $a, b, c \in A$, from $a\xi b$ and $b\xi c$ it follows $a\xi c$, i.e. if $\xi^2 \subseteq \xi$.

A reflexive and transitive relation is a *quasi-order.* A reflexive, anti-symmetric and transitive relation is a *partial ordering.* A reflexive, symmetric and transitive relation is an *equivalence relation* or *equivalence*, for short. There will be more talk of partial ordering in Section 1.5. Here we will discuss equivalence relations.

Let ξ be a binary relation on a set A. The relations ξ_l and ξ_r on A defined by:

$$a\xi_l b \;\Leftrightarrow\; a\xi = b\xi, \qquad a\xi_r b \;\Leftrightarrow\; \xi a = \xi b, \qquad a, b \in A,$$

are equivalences on A.

Let ξ be an equivalence relation on a set A. Elements $a, b \in A$ are ξ-equivalent if $a\xi b$. A set $a\xi$ we call the *equivalence class* of an element a, or ξ-*class* of an element a. It is evident that $a \in a\xi$. The set of all ξ-classes we denote by A/ξ and call it the *factor set* of a set A. A mapping $\xi^\natural : a \mapsto a\xi$ of a set A onto a factor set A/ξ is a *natural mapping* of A determined with an equivalence ξ. Let A and B be non-empty sets and $\phi : A \mapsto B$. A relation $\ker\phi = \{(x, y) \in A \times A \,|\, x\phi = y\phi\}$ in A we call the *kernel of mapping* ϕ. A connection between equivalences and mappings gives the following lemma, whose proof is elementary, so it is omitted.

Lemma 1.3 *Let A be a non-empty set. If ϕ is a mapping on a set A into a set B, then* $\ker\phi$ *is an equivalence relation on A.*

Also, if ξ is an equivalence on A, then $\ker(\xi^\natural) = \xi$.

The family $\{A_i \,|\, i \in I\}$ of subsets on a set A is a *partition* of A if $A_i \neq \emptyset$, for every $i \in I$, $A = \cup_{i \in I} A_i$, and for all $i, j \in I$, $A_i = A_j$ or $A_i \cap A_j = \emptyset$. The following lemma, whose proof is elementary, gives us a connection between partitions of A and equivalences on that set.

Lemma 1.4 *Let $\omega = \{A_i \mid i \in I\}$ be a partition of a set A. Then the relation ξ_ω on A defined by*

$$a\xi_\omega b \;\Leftrightarrow\; (\exists i \in I)\; a, b \in A_i, \qquad a, b \in A,$$

is an equivalence relation on a set A.

Conversely, let ξ be an equivalence on a set A. Then a family $\omega_\xi = \{a\xi \mid a \in A\}$ is a partition of A.

Also, mappings $\omega \mapsto \xi_\omega$ and $\xi \mapsto \omega_\xi$ are mutually inverse bijections from the set of all partitions of A onto the set of all equivalences on A, and conversely.

Let A be a non-empty set. An intersection of an arbitrary family of transitive relations on A, if it is not empty, is also a transitive relation on A. If ξ is a binary relation on the set A, an intersection of all transitive relations on A containing ξ is a transitive relation, denoted by ξ^∞. It is easy to prove that $\xi^\infty = \cup_{n\in\mathbf{Z}^+}\xi^n$. The relation ξ^∞ we call the *transitive closure* of ξ. An intersection of an arbitrary family of equivalences on A is not empty, because it contains the identical relation on A, and this intersection is an equivalence on A. If ξ is a relation on A, then the intersection of all equivalences containing ξ we call the *equivalence relation generated by* ξ, and we denote it by ξ^e. It is evident that $\xi^e = (\xi \cap \xi^{-1} \cup \Delta)^\infty$.

A mapping ν which every semigroup S joins with some relation on S, we call the *type of relation* and denote by ν_S. Then we say that ν_S is a relation of type ν on a semigroup S. If a semigroup is fixed, then the term "ν_S" we replace with "ν". If ν is some type of relation and if ν_S is an equivalence, for every semigroup S, then we say that ν is a type of equivalence relation. Let ν be a type of equivalence relation. A semigroup S is *ν-simple* if ν_S is a universal relation on S, i.e. if S has only one ν_S-class.

Exercises

1. The empty relation on a set A is a zero of a semigroup $\mathcal{B}(A)$.

2. Let $\phi \in \mathcal{PT}(A)$. Then $\ker\phi = \phi\phi^{-1}$.

3. For $\phi \in \mathcal{PT}(A)$, the element $a \in \mathrm{dom}\phi$ is a *fix point* of the partial mapping ϕ if $a\phi = a$. The set of all fix points of the partial mapping ϕ we denote by $\mathrm{fix}\phi$. Prove that ϕ is an idempotent of $\mathcal{PT}(A)$ if and only if $\mathrm{fix}\phi = \mathrm{ran}\phi$.

4. For an infinite countable set A, $S = \{\alpha \in \mathcal{T}_r(A) \mid A - A\alpha \text{ is the infinite set}\}$ is a subsemigroup of $\mathcal{T}_r(A)$ which we call *Baer-Levi's semigroup*. Prove that Baer-Levi's semigroup has no idempotents.

References

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1.3 Congruences and Homomorphisms

Let ξ be an equivalence relation on a semigroup S. A relation ξ is a *left* (*right*) *congruence* if for all $a, b, c \in S$, $a\xi b$ implies $ca\xi cb$ ($ac\xi bc$). A relation ξ is a *congruence relation* if it is both a left and right congruence relation. The following lemma follows immediately:

Lemma 1.5 *An equivalence relation ξ on a semigroup S is a congruence if and only if for all $a, b, c, d \in S$, $a\xi b$ and $c\xi d$ imply $ac\xi bd$.*

It is evident that the intersection of an arbitrary family of congruences on a semigroup S is also a congruence on S. Here we determine that for an arbitrary relation ξ on S, the intersection of all congruences on S containing ξ is a congruence relation on S, which we call the *congruence relation generated by* ξ, and denote by $\xi^{\#}$.

Let ξ be an equivalence on a semigroup S. Then we define $\xi^{\flat}$ by

$$\xi^{\flat} = \{(a, b) \in S \times S \mid (\forall x, y \in S^1)\ (xay, xby) \in \xi\}.$$

The important characteristic of a relation $\xi^{\flat}$ is outlined in the following theorem:

Theorem 1.2 *Let ξ be an equivalence relation on a semigroup S. Then the relation $\xi^{\flat}$ is a congruence on S contained in ξ.*

Also, for an arbitrary congruence η on S contained in ξ is $\eta \subseteq \xi^{\flat}$.

Proof. It is clear that $\xi^{\flat}$ is an equivalence on S. Also, if $(a, b) \in \xi^{\flat}$ and $c \in S$, then $(xcay, xcby) \in \xi$, for all $x, y \in S^1$. Hence, $(ca, cb) \in \xi^{\flat}$. Similarly, we have that $(ac, bc) \in \xi^{\flat}$. Thus, $\xi^{\flat}$ is a congruence. It is clear that $\xi^{\flat} \subseteq \xi$.

Let η be an arbitrary congruence on S contained in ξ. Assume $(a, b) \in \eta$. Since η is a congruence, then $(xay, xby) \in \eta$, for all $x, y \in S^1$, whence $(xay, xby) \in \xi$, for all $x, y \in S^1$, so $(a, b) \in \xi^{\flat}$. Therefore, $\eta \subseteq \xi^{\flat}$. $\square$

Let S and T be semigroups. A mapping $\phi : S \mapsto T$ is a *homomorphism* if $(a\phi)(b\phi) = (ab)\phi$, for all $a, b \in S$. Let ϕ be a homomorphism of a semigroup S into a semigroup T. If ϕ is one-to-one, then ϕ is a *monomorphism* or *embedding*, and then we say that S can be embeddable into T. If ϕ is onto, then ϕ is an *epimorphism*. If ϕ is bijective, then ϕ is an *isomorphism* and then semigroups S and T are *isomorphic*, in notation $S \cong T$. It is easy to prove that an inverse mapping of isomorphism is also an isomorphism. Namely, two semigroups are isomorphic if and only if we can obtain one of them from another by different notations of the elements. So, if semigroups are isomorphic then we mean that they are the same. A homomorphism of a semigroup S into itself is an *endomorphism*, and an isomorphism of S into itself is an *automorphism*. If ϕ is a homomorphism of a semigroup S into a semigroup T, then $S\phi$ is a subsemigroup of T. A semigroup T is a *homomorphic image* of a semigroup S, if there exists an epimorphism of S onto T. A semigroup T *divides* a semigroup S, and T is a *divisor* of S if T is a homomorphic image of some subsemigroup of S.

Let A be a subsemigroup of semigroups S and T. A homomorphism $\phi : S \mapsto T$ is an *A-homomorphism* if $a\phi = a$, for every $a \in A$.

Let S and T be semigroups. A mapping $\phi : S \mapsto T$ is an *anti-homomorphism* if $(ab)\phi = (b\phi)(a\phi)$, for all $a, b \in S$. A bijective anti-homomorphism we call *anti-isomorphism*. Semigroups S and T are *anti-isomorphic* if there is an anti-isomorphism of S onto T. It is evident that semigroups S and T are anti-isomorphic if and only if S is isomorphic onto a semigroup $\overleftarrow{T}$.

A mapping $\phi : S \mapsto T$ is a *partial homomorphism* of partial semigroup S into a partial semigroup T if for all $a, b \in S$ the following holds: if a product ab is defined in S, then a product $(a\phi)(b\phi)$ is defined in T and holds $(ab)\phi = (a\phi)(b\phi)$. A bijective partial homomorphism is a *partial isomorphism*.

Let ξ be a congruence on a semigroup S. Then the factor set S/ξ by the multiplication defined with: $(a\xi)(b\xi) = (ab)\xi$, is a semigroup which we call a *factor semigroup*, or *factor* for short, of a semigroup S under a congruence ξ. A theorem immediately follows which gives a connection between congruences and homomorphisms, and it is known as *Homomorphism's theorem.*

Theorem 1.3 *If ξ is a conguence on a semigroup S, then $\xi^\natural$ is a homomorphism of S onto S/ξ.*

Conversely, if ϕ is a homomorphism of a semigroup S into a semigroup T, then $\ker\phi$ *is a congruence on S and a mapping $\Phi : S/\ker\phi \mapsto T$ defined by:* $(a\ker\phi)\Phi = a\phi$, $a \in S$, *is an isomorphism.*

For congruence ξ, a homomorphism $\xi^\natural$ is called the *natural homomorphism* induced by congruence ξ, while for homomorphism ϕ, a congruence $\ker\phi$ is called the *kernel of homomorphism* ϕ. According to Homomorphism's theorem, we will make no difference between terms "factor" and "homomorphic image".

Theorem 1.4 *Let ξ and η be congruences on a semigroup S and let $\xi \subseteq \eta$. Then*

$$\eta/\xi = \{(a\xi, b\xi) \in S/\xi \times S/\xi \,|\, (a,b) \in \eta\}$$

is a congruence on S/ξ and $(S/\xi)/(\eta/\xi) \cong S/\eta$.

Proof. Let $\phi : S/\xi \mapsto S/\eta$ be a mapping defined by: $(a\xi)\phi = a\eta$. For $a\xi, b\xi \in S/\xi$, we have that $[(a\xi)(b\xi)]\phi = [(ab)\xi]\phi = (ab)\eta = (a\eta)(b\eta) = [(a\xi)\phi][(b\xi)\phi]$. Hence, ϕ is a homomorphism. Also, $(a\xi)\phi = (b\xi)\phi$ if and only if $a\eta = b\eta$, i.e. $(a,b) \in \eta$. Thus, $\ker\phi = \eta/\xi$, so η/ξ is a congruence and by means of Theorem 1.3 we obtain that $(S/\xi)/(\eta/\xi) \cong S/\eta$. $\square$

Let $\{A_i \,|\, i \in I\}$ be a family of sets and let $A = \prod_{i\in I} A_i$ be a Cartesian product of family $\{A_i \,|\, i \in I\}$. The elements from A we denote by $(a_i)_{i\in I}$ ($a_i \in A_i$, for every $i \in I$), or (a_i) for short if the index set is well known. For $i \in I$, the mapping $\pi_i : A \mapsto A$ defined with: $a\pi_i = a_i$, if $a = (a_j)_{j\in I}$, we call the *i-th projection*, and the element a_i we call the *i-th coordinate* of an element a.

Let $\{S_i \,|\, i \in I\}$ be a family of semigroups and let S be a Cartesian product of family $\{S_i \,|\, i \in I\}$. We define the multiplication on S a *componentwise*, i.e. $(a_i)_{i\in I}(b_i)_{i\in I} = (a_ib_i)_{i\in I}$, for $(a_i)_{i\in I}, (b_i)_{i\in I} \in S$. Then S along with this multiplication is a semigroup, and for every $i \in I$, a projection π_i is an epimorphism. Every semigroup isomorphic to a semigroup S we call a *direct product of the family of semigroups* $\{S_i \,|\, i \in I\}$.

A semigroup S is a *subdirect product of the family of semigroups* $\{S_i \,|\, i \in I\}$, if S is isomorphic to some subsemigroup T of a direct product $\prod_{i\in I} S_i$ such that the following holds: $T\pi_i = S_i$, for every $i \in I$.

A congruence ξ on a semigroup S *divides elements* a and b from S if a and b are in different ξ-classes, i.e. if $(a,b) \notin \xi$. A family $\{\xi_i \,|\, i \in I\}$ of non-identical congruences on a semigroup S divides elements from S if for every pair of different elements a and b from S there is a congruence from this family which divide it. It is easy to prove:

Lemma 1.6 *A family $\{\xi_i \,|\, i \in I\}$ of non-identical congruences on a semigroup S divides elements from S if and only if $\cap_{i\in I}\xi_i = \Delta$.*

Theorem 1.5 *Let a semigroup S be a subdirect product of a family of semigroups $\{S_i \,|\, i \in I\}$. Then, the family $\{\xi_i \,|\, i \in I\}$ of congruences on S which corresponds to congruences* $\ker\pi_i$*, $i \in I$, is the family of congruences on S which divide elements from S.*

Conversely, if $\{\xi_i \,|\, i \in I\}$ is a family of non-identical congruences on a semigroup S which divides elements from S, then S is a subdirect product of the family of semigroups $\{S/\xi_i \,|\, i \in I\}$.

Proof. Let $\{\xi_i \,|\, i \in I\}$ be a family of non-identical congruences on a semigroup S. We define a mapping $\phi : S \mapsto \prod_{i\in I} S_i$, with $a\phi = (a\xi_i)_{i\in I}$, $a \in S$. It is easy to prove that ϕ is a homomorphism and $(S\phi)\pi_i = S/\xi_i$, for every $i \in I$. If $a, b \in S$ are some different elements, then there is $i \in I$ such that $(a, b) \notin \xi_i$, i.e. $a\xi_i \neq b\xi_i$, so $a\phi \neq b\phi$. Thus, ϕ is a monomorphism. Hence, S is a subdirect product of the family $\{S/\xi_i \,|\, i \in I\}$.

The converse follows immediately. □

According to the Homomorphism theorem, we can present Theorem 1.5 in a different way.

Corollary 1.1 *Let S be a semigroup and let $\{S_i \,|\, i \in I\}$ be a family of semigroups. Then S is a subdirect product of the family $\{S_i \,|\, i \in I\}$ if and only if the following conditions hold*

(i) *for every $i \in I$ there exists an epimorphism φ_i of S onto S_i;*
(ii) *for $a, b \in S$, $a \neq b$, there is $i \in I$ such that $a\phi_i \neq b\varphi_i$.*

According to Corollary 1.1 we determine

Corollary 1.2 *Let a semigroup S be a subdirect product of a family of semigroups $\{S_\alpha \,|\, \alpha \in Y\}$, and for every $\alpha \in Y$, let S_α be a subdirect product of a family of semigroups $\{T_i^\alpha \,|\, i \in I_\alpha\}$. Then S is a subdirect product of the family of semigroups $\{T_i^\alpha \,|\, i \in I_\alpha, \alpha \in Y\}$.*

On a Cartesian product $I \times \Lambda$ of the non-empty sets I and Λ we define a multiplication by

$$(i, \lambda)(j, \mu) = (i, \mu), \qquad i, j \in I, \lambda, \mu \in \Lambda.$$

Then $I \times \Lambda$ with this multiplication is a band, $I \times \Lambda$ is isomorphic to a direct product of a left zero and a right zero band. Every semigroup isomorphic to a direct product of a left zero and a right zero band we call a *rectangular band*.

Let $\mathfrak{C}$ be a class of semigroups. A congruence ξ on a semigroup S is a $\mathfrak{C}$-*congruence* on S if the factor S/ξ is from class $\mathfrak{C}$. Decomposition of a semigroup S which corresponds to a $\mathfrak{C}$-congruence we call the $\mathfrak{C}$-*decomposition* of a semigroup S, and a corresponding factor semigroup we call the $\mathfrak{C}$-*homomorphic image* of S.

If $\mathfrak{C}$ is a class of bands, then we have *band congruence, band decomposition* and *a band homomorphic image.* If $\mathfrak{C}$ is a class of semilattices, then we have *semilattice congruence, semilattice decomposition* and *a semilattice homomorphic image.* If $\mathfrak{C}$ is a class of rectangular bands, then we have *matrix congruence* and *matrix decomposition*, and if $\mathfrak{C}$ is a class of left (right) zero bands, then we have *left* (*right*) *zero band congruence* and *left* (*right*) *zero band decomposition.*

A congruence ξ on a semigroup S is a band congruence if and only if $a\xi a^2$, for every $a \in S$, i.e. if and only if every ξ-class of S is a subsemigroup of S. Let ξ be a band congruence on a semigroup S and let $B = S/\xi$. For $i \in B$, let $S_i = i(\xi^\natural)^{-1}$. Then S_i is a subsemigroup of S, for every $i \in B$, $S = \cup_{i \in B} S_i$, and for all $i, j \in B$ is $S_i S_j \subseteq S_{ij}$, and then we say that S is a *band B of semigroups S_i, $i \in B$.* The semigroups S_i, $i \in B$ are *components* of this band decomposition. If $\mathfrak{C}$ is a class of semigroups and if for every $i \in B$, S_i belongs to $\mathfrak{C}$, then we say that S is a *band B of semigroups S_i, $i \in B$, from* $\mathfrak{C}$. If B is a semilattice (chain, rectangular band, left zero band, right zero band), then S is a *semilattice* (*chain, rectangular band* or *matrix, left zero band, right zero band*) *B of semigroups S_i, $i \in B$.* When ξ is the smallest band (semilattice) congruence on S, S/ξ will be called a *greatest band* (*semilattice*) *homomorphic image* of S. By analogy, we introduce definitions for some other types of bands and semilattices.

Exercises

1. Every semigroup S can be embeddable into a semigroup $\mathcal{T}_r(S^1)$.

2. Let φ and ψ be homomorphisms of a semigroup S onto semigroups T and U, respectively, such that $\ker\varphi \subseteq \ker\psi$. Then, there is a unique homomorphism θ of T onto U such that $\varphi\theta = \psi$.

3. If ξ is a relation on a semigroup S, then $\xi^\# = (\xi^c)^c = [\xi^c \cup (\xi^c)^{-1} \cup \Delta_S]^\infty$, where $\xi^c = \{(e,f) \mid (\exists x, y \in S^1)(\exists a, b \in S)\, (a,b) \in \xi, e = xay, f = xby\}$.

4. A semigroup S is *subdirectly irreducible* if whenever S is a subdirect product of the family of semigroups $\{S_i \mid i \in I\}$, then π_i is an isomorphism, for some $i \in I$.

The following conditions on a semigroup S are equivalent:

(a) S is subdirectly irreducible;
(b) the intersection of an arbitrary family of non-identical congruences on S is a non-identical congruence on S;
(c) S has the smallest non-identical congruence.

5. Every semigroup is a subdirect product of subdirectly irreducible semigroups.

References

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1.4 Maximal Subgroups and Monogenic Semigroups

A semigroup S is a *group* if S has an identity e and for every $a \in S$ there exists $b \in S$ such that $ab = ba = e$. The element b is unique in a group G with such properties, we denote it by a^{-1} and call the *group inverse* of a, or the *inverse* of *a in a group* G. A subsemigroup G of a semigroup S is a *subgroup* of S, if G is a group. It is easy to prove that a non-empty subset G of a semigroup S is a subgroup of S if and only if $aG = Ga = G$, for every $a \in G$.

A subgroup G of a semigroup S is a *maximal subgroup* of S if there is no subgroup H of S such that $G \subset H$. The following theorem describes a maximal subgroup of a semigroup.

Theorem 1.6 *Let e be an idempotent of a semigroup S. Then there exists a maximal subgroup of S with an identity e, which we denote by G_e, and*

$$\begin{aligned} G_e &= \{a \in S \mid a = ea = ae, (\exists a' \in S)\, e = aa' = a'a\} \\ &= \{a \in S \mid a \in eS \cap Se, e \in aS \cap Sa\}. \end{aligned}$$

Proof. It is evident that every subgroup of S with an identity e is contained in the first set and one is contained in the second. The first set is a subgroup of S with an identity e. Let a be an element of the second set. Then $a = ex = ye$, $e = az = wa$, for some $x, y, z, w \in S$. From this it follows $ea = eex = ex = a$, and similarly $ae = a$. Furthermore, $eze = eeze = ewaze = ewee = ewe$, whence $e = ee = aze = a(eze)$ and $e = ee = ewa = (ewe)a$. Thus, $e = aa' = a'a$, where $a' = eze = ewe$, so the element a belongs to the first set. □

Theorem 1.7 *If e and f are two different idempotents from a semigroup S, then $G_e \cap G_f = \emptyset$.*

Proof. Assume $a \in G_e \cap G_f$. Then $a = ea = ae = fa = af$, $e = aa' = a'a$ and $f = aa'' = a''a$, for some $a', a'' \in S$. Hence $e = aa' = faa' = fe = a''ae = a''a = f$. Thus, from $e \neq f$ it follows $G_e \cap G_f = \emptyset$. □

If S is a semigroup with an identity e, an element $a \in S$ is *invertible* if there is $b \in S$ such that $ab = ba = e$. Then a maximal subgroup G_e is called the *group of identity*, and all of its elements are invertible elements of a semigroup S.

Lemma 1.7 *An element a of a semigroup S with an identity is invertible if and only if $aS = Sa = S$.*

The following result is very useful for further work and it is known as *Munn's lemma.*

Lemma 1.8 *Let S be a semigroup and let x be an element of S such that x^n belongs to a subgroup G of S for some $n \in \mathbf{Z}^+$. If e is the identity of G, then*

(1) *$ex = xe \in G_e$;*
(2) *$x^m \in G_e$, for any $m \in \mathbf{Z}^+$, $m \geq n$.*

Proof. (1) Let y be an inverse element of the element x^n in G. Then

$$ex = yx^{n+1} = yxx^n = yxx^n e = yxx^n x^n y = yx^{2n+1}y,$$

and similarly we prove that $xe = yx^{2n+1}y$. Thus, $ex = xe$. Since $ey = ye = y$, then

$$xy = xey = exy = yx^n xy = yxx^n y = yxe = yex = yx,$$

whence by induction we obtain that $x^k y = yx^k$, for every $k \in \mathbf{Z}^+$. Assume $z = x^{n-1}y = yx^{n-1}$. Then $zxe = yx^{n-1}xe = yx^n e = e$, and similarly $exz = e$. Furthermore, $e(ex) = (ex)e = ex$, so $ex = xe \in G_e$.

(2) Let $m \in \mathbf{Z}^+$, $m > n$. Assume $r \in \mathbf{Z}^+$ such that $nr > m$, and assume that y is an inverse of the element x^n in G_e. Then $x^{nr-m}y^r = y^r x^{nr-m}$, and if assume that $w = x^{nr-m}y^r$, then we have

$$wx^m = y^r x^{nr-m}x^m = y^r x^{nr} = (yx^n)^r = e.$$

In a similar way we prove that $x^m w = e$. On the other hand, $ex^m = ex^n x^{m-n} = x^n x^{m-n} = x^m$, and similarly $x^m e = x^m$. Thus, by Theorem 1.6, $x^m \in G_e$. □

Let S be a semigroup. The cardinality $|S|$ of a semigroup S we call the *order of a semigroup* S. If $|S|$ is a finite number, then we say that S is a *finite order* or a *finite* semigroup. Otherwise, we say that S is an *infinite order* or an *infinite semigroup.* A semigroup S is *trivial* if $|S| = 1$. For an element $a \in S$, the *order of element* a is the order of a monogenic subsemigroup $\langle a\rangle$ of S. The order of an element a we denote by $r(a)$. If $\langle a\rangle$ is a finite semigroup, then the order of a is finite, otherwise, the order of a is infinite.

An element a of a semigroup S is *periodic* if there are $m, n \in \mathbf{Z}^+$, such that $a^m = a^{m+n}$. Let a be a periodic element of a semigroup S. The set $\{m \in \mathbf{Z}^+ \mid (\exists n \in \mathbf{Z}^+)\ a^m = a^{m+n}\}$ is a subset of integers, so it has the smallest element which we call the *index of the element* a (*index of a semigroup* $\langle a\rangle$) and denote by $i(a)$. The smallest element of the set $\{n \in \mathbf{Z}^+ \mid a^{i(a)} = a^{i(a)+n}\}$ we call the *period of the element* a (*period of a semigroup* $\langle a\rangle$) and denote it by $p(a)$.

Theorem 1.8 *Let a be an element of a semigroup S.*

If a is not a periodic element, then the order of a is infinite and the monogenic subsemigroup $\langle a\rangle$ of S is isomorphic to the additive semigroup $(\mathbf{Z}^+, +)$ of integers.

If a is a periodic element, then the order $r(a) = i(a) + p(a) - 1$ of a is finite, $K_a = \{a^{i(a)}, a^{i(a)+1}, \dots, a^{i(a)+p(a)-1}\}$ is a maximal subgroup of $\langle a\rangle$, and K_a is a monogenic group whose order is $p(a)$.

Proof. If a is non-periodic, then it is evident that the order of a is infinite and the mapping $\phi : \mathbf{Z}^+ \mapsto \langle a\rangle$ defined by $n\phi = a^n$, $n \in \mathbf{Z}^+$ is an isomorphism.

Let a be a periodic element. According to the definition of an index and the period of an element, it is clear that $a, a^2, a^3, \dots, a^{i(a)+p(a)-1}$ are different. Assume an arbitrary $n \in \mathbf{Z}^+$. Then $n = kp(a) + m$, $0 \leq k$, $0 \leq m \leq p(a) - 1$, so $a^{i(a)+n} = a^{i(a)+kp(a)+m} = a^{i(a)+m} \in K_a$. Hence, $\langle a \rangle = \{a, a^2, \dots, a^{i(a)+p(a)-1}\}$, and the order of $\langle a \rangle$ is $r(a) = i(a) + p(a) - 1$. It is evident that K_a is isomorphic to the additive group of the rest of integers modulo $p(a)$, that the order of K_a is $p(a)$ and that K_a is a maximal subgroup of $\langle a \rangle$. $\square$

Based on the previous theorems, monogenic semigroups are isomorphic if and only if they are the same index and the same period. A monogenic semigroup with an index i and period p we denote by $M(i, p)$.

A semigroup S is *periodic* if each of its elements is periodic.

Exercises

1. Denote as $\mathcal{S}(X)$ the set of all bijective mappings of the set X. Then $\mathcal{S}(X)$ is a group of identity of monoid $\mathcal{T}_r(X)$.

The group $\mathcal{S}(X)$ we call the *symmetric group* or the *group of permutations* of X.

2. Every group can be embeddable into the group of permutations of some set.

3. An element a of a semigroup S is periodic if and only if there exists $n \in \mathbf{Z}^+$ such that $a^n \in E(S)$.

4. Every finite semigroup is periodic.

5. An infinite monogenic semigroup is a subdirect product of finite monogenic semigroups.

References

J. Bosák [1]; A. H. Clifford and D. D. Miller [1]; A. H. Clifford and G. B. Preston [1]; H. Hashimoto [1]; J. M. Howie [1]; N. Kimura [1]; W. D. Munn [3]; Š. Schwarz [3]; G. Thierrin [1], [4].

1.5 Ordered Sets and Lattices

Let us once again be reminded that a reflexive, antisymmetric and transitive relation on a set A is a *partial ordering* on A. Usually, we denote it by $\leq$. A set A supplied with partial ordering is a *partially ordered set.* The notion *poset* will be used as a synonym for the notion "partially ordered set".

If partial ordering $\leq$ on a set A is *linear*, i.e. if for all $a, b \in A$ is $a \leq b$ or $b \leq a$, then A is a *linear partially ordered set* or a *chain*. If $\leq$ is a partial ordering on a set A, then by $<$ we denote a relation on A defined by:

$$a < b \;\Leftrightarrow\; a \leq b \;\wedge\; a \neq b, \qquad a, b \in A,$$

and by $\geq$ and $>$ we denote the inverse relations of $\leq$ and $<$, respectively.

Let A and B be ordered sets and $\varphi : A \mapsto B$. A mapping φ is an *isotone* (*save order*) if for $a, b \in A$, from $a \leq b$ it follows that $a\varphi \leq b\varphi$. A mapping φ is *antitone* if for $a, b \in A$, from $a, b \in A$ it follows that $a\varphi \geq b\varphi$. The ordered sets A and B are *isomorphic* if there is a bijection $\varphi : A \mapsto B$ such that for every $x, y \in A$ holds

$$x \leq y \quad \Leftrightarrow \quad \varphi(x) \leq \varphi(y).$$

Let A be an ordered set. An element $a \in A$ is a *minimal* (*maximal*) *element* of the set A if there is no $x \in A$ such that $x < a$ ($x > a$), i.e. if for $x \in A$, from $x \leq a$ ($x \geq a$) it follows that $x = a$. An element $a \in A$ is the *smallest* (the *biggest*) *element* of a set A if $a \leq x$ ($a \geq x$), for every $x \in A$. The smallest (the biggest) element of a set A, if it exists there, is a *minimal* (*maximal*) element of a set A, while the opposite does not hold. A set A can have a lot of minimal (maximal) elements, while it can have only one smallest (biggest) element.

Let X be a non-empty subset of an ordered set A. An element $a \in A$ is an *upper bound* (a *lower bound*) of a set X if $x \leq a$ ($x \geq a$), for every $x \in X$. An element $a \in A$ is a *least upper bound* or *join* (a *greatest lower bound* or *meet*) of the set X, in notation $a = \vee X$ ($a = \wedge X$), if the following holds:

(i) a is an upper (lower) bound of a set X;
(ii) if $b \in A$ is an upper (lower) bound of a set X, then $a \leq b$ ($a \geq b$).

If $X = \{x_i \,|\, i \in I\}$, then we write $\vee_{i \in I} x_i$ ($\wedge_{i \in I} x_i$) instead of $\vee X$ ($\wedge X$), and if $I = \{1, 2, \ldots, n\}$, $n \in \mathbf{Z}^+$, $n \geq 2$, then we write

$$x_1 \vee x_2 \vee \cdots \vee x_n \qquad (x_1 \wedge x_2 \wedge \cdots \wedge x_n),$$

instead of $\vee_{i \in I} x_i$ ($\wedge_{i \in I} x_i$).

An ordered set A is an *upper* (*lower*) *semilattice* if every two-element subsets of A have a join (a meet). Using induction in that case we prove that every finite subset of A has a join (a meet). For infinite subsets of A

it does not hold. An ordered set A is a *lattice* if A is both an upper and a lower semilattice.

If A is an upper (lower) semilattice, then the mapping $\vee : A \times A \mapsto A$ ($\wedge : A \times A \mapsto A$) defined by

$$(1) \quad \vee : (a, b) \mapsto a \vee b, \quad a, b \in A, \qquad (\wedge : (a, b) \mapsto a \wedge b, \quad a, b \in A),$$

is an associative and commutative operation on the set A. Using this lower semilattice (upper semilattice, lattice) we can define it in some other way.

We would like to remind the reader that we use the term *semilattice* in the theory of semigroups for a commutative band. Here we give an explanation of the connection between this term and the term for lower semilattice. If S is a semigroup, then the relation $\leq$ of the set $E(S)$ of all the idempotents of S, defined by

$$e \leq f \;\Leftrightarrow\; ef = fe = e, \qquad e, f \in E(S),$$

is a partial order which we call a *natural partial order* on $E(S)$. If S is a band, then we have an order on S. If S is a commutative band, then under its natural order S is a lower semilattice. Conversely, if A is a lower semilattice, then under the operation $\wedge$, A is a commutative band. The operations $\vee$ and $\wedge$ we call a *union* and an *intersection*, respectively.

Now, we give an another definition of a lattice: If L is a non-empty set and if $\wedge$ and $\vee$ are binary operations on the set L which satisfies the following conditions:

(L1) *idempotent*: $x \wedge x = x, \quad x \vee x = x$;
(L2) *commutative*: $x \wedge y = y \wedge x, \quad x \vee y = y \vee x$;
(L3) *associative*: $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \quad x \vee (y \vee z) = (x \vee y) \vee z$;
(L4) *absorption*: $x \wedge (x \vee y) = x, \quad x \vee (x \wedge y) = x$;

for all $x, y, z \in L$, then L is a *lattice*. If L is a lattice in the sense of the first definition, then under the operations $\wedge$ and $\vee$ defined by (1) L is a lattice in the sense of the second definition. Conversely, if L is a lattice in the sense of the second definition, then on L we define an order by

$$a \leq b \;\Leftrightarrow\; a \wedge b = a, \qquad a, b \in L,$$

or, equivalently, by

$$a \leq b \;\Leftrightarrow\; a \vee b = b, \qquad a, b \in L,$$

and under this order the set L is a lattice in the sense of the first definition. So, for a lattice we can use both definitions.

Also, we immediately prove that the definitions of a chain, as a linear order set and as a semilattice for which is $xy = x$ or $xy = y$, for all x, y, are equivalent.

A subset K of a lattice L is a *sublattice* of L if $x \wedge y, x \vee y \in K$, for all $x, y \in K$. If L is a lattice and $a, b \in L$ such that $a \leq b$, then the *interval* $[a, b]$ of a lattice L is a sublattice of L defined by: $[a, b] = \{x \in L \mid a \leq x \leq b\}$.

Let L and K be lattices and $\phi : L \mapsto K$. A mapping ϕ is a *homomorphism of lattice* L into a lattice K if $(a \vee b)\phi = a\phi \vee b\phi$ and $(a \wedge b)\phi = a\phi \wedge b\phi$, for all $a, b \in L$. A mapping ϕ is a *monomorphism* or *embedding* of a lattice L into K if ϕ is homomorphism and one-to-one, and then we say that a lattice L can be embedded into K. A mapping ϕ is an *isomorphism* of lattices L and K if ϕ is a homomorphism and bijection.

Theorem 1.9 *Let $L_1 = (L_1, \leq_1)$ and $L_2 = (L_2, \leq_2)$ be lattices and let $\varphi : L_1 \mapsto L_2$ be a bijection. Then the following conditions are equivalent*

(i) *φ is an isomorphism of lattice order sets L_1 and L_2;*
(ii) *for all $x, y \in L_1$ the following holds*

$$\varphi(x \wedge_1 y) = \varphi(x) \wedge_2 \varphi(y), \qquad \varphi(x \vee_1 y) = \varphi(x) \vee_2 \varphi(y).$$

Proof. (i)⇒(ii) Let $x, y \in L_1$. If we want to prove the equation $\varphi(x \wedge_1 y) = \varphi(x) \wedge_2 \varphi(y)$, we should prove that $\varphi(x \wedge_1 y)$ is a meet of the set $\{\varphi(x), \varphi(y)\}$. Since $x \wedge_1 y \leq_1 x$ and $x \wedge_1 y \leq_1 y$ and since φ is isotone, we have that $\varphi(x \wedge_1 y) \leq_2 \varphi(x)$ and $\varphi(x \wedge_1 y) \leq_2 \varphi(y)$, i.e. $\varphi(x \wedge_1 y) \leq_2 \varphi(x) \wedge_2 \varphi(y)$.

Suppose that for any $a \in L_2$, $a \leq_2 \varphi(x)$ and $a \leq_2 \varphi(y)$. Since φ is isotone, then it follows that $\varphi^{-1}(a) \leq_1 x$ and $\varphi^{-1}(a) \leq_1 y$, whence $\varphi^{-1}(a) \leq_1 x \wedge_1 y$. From this we obtain that $a \leq_2 \varphi(x \wedge_1 y)$. Therefore, $\varphi(x \wedge_1 y)$ is the greatest lower bound of the set $\{\varphi(x), \varphi(y)\}$.

Similarly, we prove that $\varphi(x \vee_1 y) = \varphi(x) \vee_2 \varphi(y)$.

(ii)⇒(i) Let $x \leq_1 y$, for some $x, y \in L_1$. Then $x \wedge_1 y = x$, so we have

$$\varphi(x) = \varphi(x \wedge_1 y) = \varphi(x) \wedge_2 \varphi(y),$$

whence $\varphi(x) \leq_2 \varphi(y)$, i.e. φ is an isotone mapping.

Now, let $a \leq_2 b$, for some $a, b \in L_2$, where $x = \varphi^{-1}(a)$ and $y = \varphi^{-1}(b)$. Since

$$\varphi(x \wedge_1 y) = \varphi(x) \wedge_2 \varphi(y) = a \wedge_2 b = a,$$

then it follows that

$$\varphi^{-1}(a) \wedge_1 \varphi^{-1}(b) = x \wedge_1 y = \varphi^{-1}(\varphi(x \wedge_1 y)) = \varphi^{-1}(a).$$

Hence, $\varphi^{-1}(a) \leq_1 \varphi^{-1}(b)$, so φ^{-1} is an isotone mapping. □

Lemma 1.9 *Any isotone bijection with an isotone inverse is a lattice isomorphism.*

Proof. Let L_1 and L_2 be lattices and let $\varphi : L_1 \to L_2$ be an isotone bijection with the isotone inverse $\varphi^{-1} : L_2 \to L_1$. Let $x, y \in L_1$. If we want to prove the equation $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$, we should prove that $\varphi(x \wedge y)$ is a meet of the set $\{\varphi(x), \varphi(y)\}$. Since $x \wedge y \leq x$ and $x \wedge y \leq y$ and since φ is isotone, we have that $\varphi(x \wedge y) \leq \varphi(x)$ and $\varphi(x \wedge y) \leq \varphi(y)$, whence $\varphi(x \wedge y) \leq \varphi(x) \wedge \varphi(y)$.

Suppose that $a \in L_2$ and let $a \leq \varphi(x)$ and $a \leq \varphi(y)$. Since φ^{-1} is isotone, then $\varphi^{-1}(a) \leq x$ and $\varphi^{-1}(a) \leq y$, whence $\varphi^{-1}(a) \leq x \wedge y$. Hence $a \leq \varphi(x \wedge y)$. Therefore, $\varphi(x \wedge y)$ is the greatest lower bound of the set $\{\varphi(x), \varphi(y)\}$, i.e. $\varphi(x \wedge y) \leq \varphi(x) \wedge \varphi(y)$. Thus, $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$.

Similarly, throught duality we can prove $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$.

According to Theorem 1.9, φ is a lattice isomorphism. □

Let $\{L_i \,|\, i \in I\}$ be a family of lattices. On a Cartesian product $L = \prod_{i \in I} L_i$ we define the binary operations $\vee$ and $\wedge$ by means of coordinates, i.e. by

$$(x_i)_{i \in I} \vee (y_i)_{i \in I} = (x_i \vee y_i)_{i \in I}, \qquad (x_i)_{i \in I} \wedge (y_i)_{i \in I} = (x_i \wedge y_i)_{i \in I},$$

for $(x_i)_{i \in I}, (y_i)_{i \in I} \in L$. Then L with such a defined operation is a lattice and every lattice isomorphic to L we call a *direct product of lattices* L_i, $i \in I$. Just like in the theory of semigroups, a projection π_i is a homomorphism of a lattice L onto a lattice L_i. Every lattice L is isomorphic to a direct product $\prod_{i \in I} L_i$, where for some $i \in I$ a lattice L_i is isomorphic to L and $|L_j| = 1$, for every $j \in I$, $j \neq i$. This decomposition we call a *trivial decomposition into a direct product* of lattices. A lattice L is *directly indecomposable* if L only has a trivial decomposition into a direct product of lattices.

A lattice L is *distributive for a meet* (*for a join*) if

$$(2) \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \qquad (x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)),$$

for all $x, y, z \in L$. It is easy to prove that a lattice L is distributive for a meet if and only if it is distributive for a join, so a lattice for which one of the conditions from (2) holds we call a *distributive* lattice.

An element $0 \in L$ is a *zero* of a lattice L if $x \wedge 0 = 0$, $x \vee 0 = x$, for every $x \in L$. If a lattice L has a zero, then it is unique and it is the smallest element in L, and conversely, if a lattice L has the smallest element, then it is the zero in L. An element $1 \in L$ is an *identity* of a lattice L if $x \wedge 1 = x$, $x \vee 1 = 1$, for every $x \in L$. If a lattice L has an identity, then it is unique and it is the greatest element in L, and conversely, if a lattice L has the greatest element, then it is an identity in L. If a lattice L has a zero (an identity), then we denote it by 0 (1). A lattice with a zero and an identity we call a *bounded lattice*.

A lattice L is *complete for a join* (*complete for a meet*) if for every $A \subseteq L$ there exists $\vee A$ ($\wedge A$), and a lattice is *complete* if it is both complete for a join and for a meet. If a lattice L is complete for a join (complete for a meet), then $\vee L$ ($\wedge L$) is an identity (a zero) of a lattice L. If a lattice L is complete for a join (for a meet) and has a zero (identity), then we can prove that L is also complete for a meet (for a join).

By means of the inductive method, we prove that in a distributive lattice L, for every $a \in L$ and every finite subset $\{x_i \,|\, i \in I\}$ of L the following holds:

$$a \wedge (\vee_{i\in I} x_i) = \vee_{i\in I}(a \wedge x_i), \qquad a \vee (\wedge_{i\in I} x_i) = \wedge_{i\in I}(a \vee x_i).$$

If $\{x_i \,|\, i \in I\}$ is an infinite subset, previous equations in distributive lattices do not hold. For this reason we introduce the following definitions: a lattice L is complete for a join (for a meet), i.e. it is *infinitely distributive for a meet* (*for a join*) if for every $a \in L$ and every subset $\{x_i \,|\, i \in I\}$ of L the following holds:

$$a \wedge (\vee_{i\in I} x_i) = \vee_{i\in I}(a \wedge x_i), \qquad (a \vee (\wedge_{i\in I} x_i) = \wedge_{i\in I}(a \vee x_i)).$$

A lattice L is *infinitely distributive* if it is both infinitely distributive for a join and for a meet.

Let L be a lattice with a zero 0 and an identity 1. An element $y \in L$ is a *complement* of an element $x \in L$ if $x \wedge y = 0$ and $x \vee y = 1$. In that case, the element x is a complement of y, i.e. the relation "to be a complement" is symmetric. If L is a distributive lattice with a zero and an identity, then every element from L has only one complement, and a complement of $x \in L$ we denote by x'. *Boolean algebra* is a bounded distributive lattice in which every element has a complement. An example of Boolean algebra is a partitive set $\mathcal{P}(A)$ of all the subsets of the set A, under the operations of sets union and sets intersection. The Boolean algebra $\mathcal{P}(A)$ we call the *Boolean algebra of all the subsets of the set* A.

We immediately prove the following lemma:

Lemma 1.10 *Let L be a distributive lattice with a zero 0 and an identity 1, and $\mathfrak{B}(L)$ be the set of all the elements from L which have a complement. Then $\mathfrak{B}(L)$ is a Boolean algebra.*

If B is an arbitrary sublattice of L which is a Boolean algebra with a zero 0 and an identity 1, then $B \subset \mathfrak{B}(L)$.

The Boolean algebra $\mathfrak{B}(L)$ we call the *greatest Boolean subalgebra* of a distributive lattice L.

Theorem 1.10 *Every complete Boolean algebra is infinitely distributive.*

Proof. Let B be a complete Boolean algebra, let $a \in B$ and let $\{x_i \mid i \in I\}$ be a subset of B. Assume $u = \vee_{i\in I}(a \wedge x_i)$. For every $i \in I$ is $a \wedge x_i \leq a \wedge (\vee_{i\in I} x_i)$, whence

$$u = \vee_{i\in I}(a \wedge x_i) \leq a \wedge (\vee_{i\in I} x_i).$$

On the other hand, $a \wedge x_i \leq u$, for every $i \in I$, so

$$x_i = 1 \wedge x_i = (a \wedge x_i) \vee (a' \wedge x_i) \leq u \vee a',$$

for every $i \in I$. Now, we determine that $\vee_{i\in I} x_i \leq u \vee a'$, whence

$$a \wedge (\vee_{i\in I}) \leq a \wedge (u \vee a') = (a \wedge u) \vee (a \wedge a') = a \wedge u \leq u.$$

Thus, B is infinitely distributive for a meet. Similarly, we prove that B is infinitely distributive for a join. □

Let L be a lattice with a zero 0. An element $a \in L$, $a \neq 0$, is an *atom* of a lattice L if there is no $x \in L$ such that $0 < x < a$, i.e. if a is a minimal element in the ordered set $L - \{0\}$. A lattice L with a zero is *atomic* if for every $x \in L$, $x \neq 0$, there exists an atom $a \in L$ such that $a \leq x$.

Theorem 1.11 *Let B be a complete Boolean algebra with the set of atoms A. Then B is atomic if and only if for every $x \in B$ there is $A_x \subseteq A$ such that $x = \vee A_x$.*

Also, the set A_x is uniquely determined.

Proof. Let B be an atomic Boolean algebra and let $x \in B$. Let A_x be the set of all the atoms contained in the interval $[0, x]$, and let $y = \vee A_x$. Let $z = y' \wedge x$. If $z \neq 0$, then there exists $b \in A$ such that $b \leq z$. Since $z \leq x$, then $b \leq x$, so $b \in A_x$, thus it follows that $b \leq \vee A_x = y$, i.e. $b \wedge y = b$. On the other hand,

$$b = b \wedge z = b \wedge y \wedge z = b \wedge y \wedge y' \wedge x = 0,$$

that contradicts the definition of atoms. Thus, $z = 0$, whence

$$x = x \wedge 1 = x \wedge (y \vee y') = (x \wedge y) \vee (x \wedge y') = (x \wedge y) \vee 0 = x \wedge y,$$

so $x \leq y$. Since $y \leq x$, then $x = y$, i.e. $x = \vee A_x$.

The converse follows immediately.

Now, we will prove the second part of the theorem. Assume that $\vee P = \vee Q$, for some $P, Q \subseteq A$. Assume $a \in P$. Then $a \leq \vee P = \vee Q$, i.e. $a \wedge (\vee Q) = a$. If $a \notin Q$, then $a \wedge b = 0$, for every $b \in Q$, because a and b are atoms. According to Theorem 1.10 we have that B is infinitely distributive, so $a = a \wedge (\vee Q) = \vee_{b \in Q}(a \wedge b) = 0$, which is a contradiction based on the definition of atoms. Thus, $a \in Q$, so $P \subseteq Q$. Similarly we prove the converse inclusion. Therefore, $P = Q$. □

Corollary 1.3 *Let B be a complete Boolean algebra. Then B is atomic if and only if B is isomorphic to a Boolean algebra of subsets of some set.*

Proof. If B is a complete Boolean algebra with a set of atoms A, then B is isomorphic to a Boolean algebra $\mathcal{P}(A)$.

Conversely, the Boolean algebra $\mathcal{P}(A)$ of all the subsets of a non-empty set A is atomic and atoms in $\mathcal{P}(A)$ are singleton sets $\{a\}$, $a \in A$. □

At the end of this section we give the Axiom of choice and without the proof of its most famous equivalent - Zorn's lemma.

Axiom of choice [1]

If A is a non-empty set, then there exists a mapping $\psi : \mathcal{P}(A) \mapsto A$ such that $X\psi \in X$, for every non-empty subset X of A.

Lemma 1.11 (**Zorn's lemma**) *Let A be an ordered set with the property that every chain in A has an upper bound. Then for every element $x \in A$ there exists at least one maximal element $a \in A$ such that $x \leq a$.*

More about the Axiom of choice and its equivalents, about the ordered sets, the reader can find in the books by M. R. Tasković [1], [2]. For more on the lattice theory, we suggest books by G. Birkhof [1], G. Grätzer [1] and G. Szász [2].

The *radicals* $R(\varrho)$ and $T(\varrho)$ of a binary relation ϱ on a semigroup S are defined as follows:

$$(a,b) \in R(\varrho) \Leftrightarrow (\exists m,n \in \mathbf{Z}^+)\, a^m \varrho b^n, \qquad (a,b) \in T(\varrho) \Leftrightarrow (\exists n \in \mathbf{Z}^+)\, a^n \varrho b^n.$$

Consider the mappings $R : \varrho \mapsto R(\varrho)$ and $T : \varrho \mapsto T(\varrho)$ on the lattice $\mathcal{B}(S)$ of all binary relations on S. For an arbitrary $\varrho \in \mathcal{B}(S)$ we have that $\varrho \subseteq T(\varrho) \subseteq R(\varrho)$, which means that T and R are *extensive* mappings. Furthermore, for $\varrho_1, \varrho_2 \in \mathcal{B}(S)$, $\varrho_1 \subseteq \varrho_2$ implies $T(\varrho_1) \subseteq T(\varrho_2)$ and $R(\varrho_1) \subseteq R(\varrho_2)$. The mappings satisfying such a condition are called *isotone*. Also, $T(T(\varrho)) = T(\varrho)$ and $R(R(\varrho)) = R(\varrho)$, for each $\varrho \in \mathcal{B}(S)$, so T and R are *idempotent* mappings. Finally, we have that $R(T(\varrho)) = T(R(\varrho)) = R(\varrho)$, for each $\varrho \in \mathcal{B}(S)$, i.e. $RT = TR = R$ in the semigroup of mappings on $\mathcal{B}(S)$. Recall that extensive, isotone and idempotent mappings on lattices are known as *closure mappings*. Thus, the previous observations can be summarized by the following lemma:

Lemma 1.12 *Let S be a semigroup. Then the mappings $R : \varrho \mapsto R(\varrho)$ and $T : \varrho \mapsto T(\varrho)$ are closure mappings on the lattice $\mathcal{B}(S)$ of all the binary relations on S and $RT = TR = R$.*

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[1] One example of the axiom of choice can be found in The Mountain Wreath in 1847 written by the great Serbian poet Petar Petrović Njegoš and published in serbian in Vienna.
The verse (2310) in Vasa D. Mihailović's translation is cited here:

"Various tree - barks, wings, and speed of feet, **and the array of seeming disorder, always follow some definite order**".

Exercises

1. The set $\mathcal{E}(A)$ of all the equivalence relations on the set A, ordered by inclusion, is a lattice, where $\xi \wedge \eta = \xi \cap \eta$ and $\xi \vee \eta = (\xi \cup \eta)^e$, for all $\xi, \eta \in \mathcal{E}(A)$. The lattice $\mathcal{E}(A)$ is complete and it has the identity ω_A and the zero Δ_A.

The lattice $\mathcal{E}(A)$ we call the *lattice of equivalences* on A.

2. Let $\xi, \eta \in \mathcal{E}(A)$. Then $\xi \vee \eta = (\xi\eta)^\infty$. If $\xi\eta = \eta\xi$, then $\xi\eta \in \mathcal{E}(A)$ and $\xi \vee \eta = \xi\eta$.

3. The set $\mathcal{C}on(S)$ of all congruences on a semigroup S, ordered by inclusion, is a lattice, where $\xi \wedge \eta = \xi \cap \eta$ and $\xi \vee \eta = (\xi \cup \eta)^\#$, for all $\xi, \eta \in \mathcal{C}on(S)$. The lattice $\mathcal{C}on(S)$ is complete and it has the identity ω_S and the zero Δ_S.

The lattice $\mathcal{C}on(S)$ we call the *lattice of congruences* on S.

4. Let L be a lattice. Then for all $a, b, c \in L$, from $a \leq c$ it follows that $a \vee (b \wedge c) \leq (a \vee b) \wedge c$.

5. A lattice L is *modular* if for all $a, b, c \in L$, from $a \leq c$ it follows that $a \vee (b \wedge c) = (a \vee b) \wedge c$. Prove that the lattice L is modular if and only if $a \vee (b \wedge (a \vee c)) = (a \vee b) \wedge (a \vee c)$, for all $a, b, c \in L$.

6. Let $\mathfrak{S}(S)$ be the set of all subsemigroups of a semigroup S, and let $\mathfrak{S}^0(S) = \mathfrak{S}(S) \cup \emptyset$. Then, the set $\mathfrak{S}^0(S)$, ordered by inclusion, is a lattice, where $A \wedge B = A \cap B$, $A \vee B = \langle A \cup B \rangle$, for all $A, B \in \mathfrak{S}(S)$. The empty set is the zero of this lattice.

The lattice $\mathfrak{S}^0(S)$ we call the *lattice of subsemigroups* of S.

7. The set $\mathfrak{L}(G)$ of all the subgroups of a group G, ordered by inclusion, is a lattice, where for all $A, B \in \mathfrak{L}(G)$, $A \wedge B = A \cap B$ and $A \vee B$ is the intersection of all the subgroups of G which contain the set $A \cup B$.

The lattice $\mathfrak{L}(G)$ we call the *lattice of the subgroups* of G.

8. The relation $\leq$ defined by: $a \leq b \Leftrightarrow (\exists x, y \in S^1)\, a = xb = by, xa = a = ay$, $a, b \in S$, is the order on an arbitrary semigroup S. This order we call the *natural order* on S. The restriction of this order on $E(S)$ (if $E(S) \neq \emptyset$) is the natural order on $E(S)$.

References

M. I. Arbib [1]; G. Birkhof [1]; S. Burris and H. P. Sankappanavar [1]; A. H. Clifford and G. B. Preston [1]; M. P. Drazin [2]; P. Edwards [2]; G. Grätzer [1]; J. M. Howie [1]; J. Kovačević [1]; H. Mitsch [1]; M. Petrich [10]; L. N. Shevrin [1]; L. N. Shevrin and A. Ya. Ovsyanikov [1], [2]; B. Stamenković and P. Protić [1], [2]; G. Szász [2]; M. R. Tasković [1], [2].

1.6 Ideals

Let S be a semigroup. A subsemigroup A of a semigroup S is a

- *left ideal* of S, if $SA \subseteq A$;
- *right ideal* of S, if $AS \subseteq A$;
- (*two-sided*) *ideal* of S, if A is both a left and a right ideal of S, i.e. if $SA \cup AS \subseteq A$;
- *quasi-ideal* of S, if $SA \cap AS \subseteq A$;
- *bi-ideal* of S, if $ASA \subseteq A$.

Every quasi-ideal of a semigroup is its bi-ideal, every left (right) ideal of a semigroup is its quasi-ideal, and every ideal of a semigroup is its left (right) ideal. Every semigroup S is its own ideal, while an (left, right, quasi-, bi-) ideal of S different than S we call a *proper* (*left, right, quasi-, bi-*) *ideal* of S. If L is a left ideal of S, R a right ideal of S and A subset of S, then LA is a left ideal, AR is a right ideal and LR is an ideal of S. Also, $RL \subseteq L \cap R$ holds, so the intersection of a left ideal and a right ideal of a semigroup is always non-empty. Moreover, the intersection of a left ideal and a right ideal of a semigroup is its quasi-ideal. Conversely, if A is a quasi-ideal of S, then $A \cup SA$ is a left and $A \cup AS$ is a right ideal of S, where $(A \cup AS) \cap (A \cup SA) = A$. Thus, a subsemigroup A of a semigroup S is its quasi-ideal if and only if A is equal to the intersection of a left ideal and a right ideal of S.

Based on the aforementioned, we can determine that the intersection of two ideals A and B of a semigroup S is non-empty, and AB and BA are ideals of S contained in $A \cap B$. Also, the intersection of an arbitrary finite family of ideals of a semigroup is non-empty. For an infinite family of ideals it does not hold. However, if so far the intersection of some family of (left, right) ideals of a semigroup S is non-empty, then it is an (left, right) ideal of S. Thus, if A is a non-empty subset of a semigroup S, the intersection of all (left, right) ideals of S which contain A is an (left, right) ideal of S which we call the (*left, right*) *ideal of S generated by A*. The set A in that case is the *generate set* of that (left, right) ideal, and the elements of A are its *generate elements* or the *generators*. For an element a of a semigroup S, the left ideal, the right ideal, the ideal and the bi-ideal of S generated by a we denote with $L(a)$, $R(a)$, $J(a)$ and $B(a)$, respectively, and we call *the principal left ideal*, *the principal right ideal*, *the principal ideal* and *the* *principal bi-ideal of* S *generated by* a. It is easy to prove that

$$L(a) = S^1 a, \qquad R(a) = aS^1, \qquad J(a) = S^1 a S^1, \qquad B(a) = \{a, a^2\} \cup aSa.$$

Let a and b be elements of a semigroup S. Then:

$$a \mid b \Leftrightarrow b \in J(a), \quad a \mid_l b \Leftrightarrow b \in L(a), \quad a \mid_r b \Leftrightarrow b \in R(a).$$

If $a \mid b$ $(a \mid_l b, a \mid_r b)$, then we say that $a \in S$ is a *factor* (a *right factor*, a *left factor*) *of the element* b. The relations $\mid$, $\mid_l$ and $\mid_r$ are quasi-orders on S. Using the previous relations we will define the following relations:

$$a \longrightarrow b \Leftrightarrow (\exists n \in \mathbf{Z}^+) a \mid b^n, \; a \xrightarrow{l} b \Leftrightarrow (\exists n \in \mathbf{Z}^+) a|_l\, b^n, \; a \xrightarrow{r} b \Leftrightarrow (\exists n \in \mathbf{Z}^+) a|_r\, b^n,$$

$$\xrightarrow{t} = \xrightarrow{l} \cap \xrightarrow{r}, \qquad \text{——} = \longrightarrow \cap (\longrightarrow)^{-1}, \qquad \overset{l}{\text{——}} = \xrightarrow{l} \cap (\xrightarrow{l})^{-1},$$

$$\overset{r}{\text{——}} = \xrightarrow{r} \cap (\xrightarrow{r})^{-1}, \qquad \overset{t}{\text{——}} = \overset{r}{\text{——}} \cap \overset{l}{\text{——}}, \qquad a \overset{p}{\text{——}} b \Leftrightarrow (\exists m, n \in \mathbf{Z}^+)\; a^m = b^n.$$

If T is a subsemigroup of S and $a, b \in T$, then we say that *a divides b into* T, in notation $a|b$ in T or $a|_T b$, if $b = xay$, for some $x, y \in T^1$.

A set $\mathcal{I}d(S)$ of all the ideals of a semigroup S, ordered by the set inclusion, is a lattice in which the operations of union and intersection are equal to the set union and the set intersection of the ideals, and it we call a *lattice of ideals of a semigroup* S. For the left ideals this does not hold, because the intersection of two left ideals of a semigroup can be an empty set. So we can make a distinction between two cases: if S is a semigroup with zero, then the intersection of every two ideals of S is non-empty, because it contains the zero. In that case, a set $\mathcal{LI}d(S)$, ordered by the set inclusion, is a lattice with a union and intersection which are equal to the set union and the set intersection. If S is a semigroup without zero, then we assume that the set $\mathcal{LI}d(S)$ consists of the empty set and of all the left ideals of S, then the lattice $\mathcal{LI}d(S)$ is isomorphic to the lattice $\mathcal{LI}d(S^0)$. In both cases, a lattice $\mathcal{LI}d(S)$ we call the *lattice of left ideals of a semigroup* S. Similarly we define the *lattice of right ideals of a semigroup*, in notation $\mathcal{RI}d(S)$.

Let S be a semigroup. Because that intersection of every two ideals of a semigroup S is non-empty, and it is an ideal of S, a lattice $\mathcal{I}d(S)$ can have only one minimal element, and it is the smallest element in $\mathcal{I}d(S)$. The smallest element of a lattice $\mathcal{I}d(S)$, if it exists there, we call the *kernel of a semigroup* S. It is easy to prove that a semigroup S has a kernel if and only if the intersection of all the ideals of S is non-empty, and in that case the kernel is equal to this intersection. An infinite monogenic semigroup is an example of a semigroup which has no kernel. A minimal element of the ordered set of all the left (right) ideals of S we call the *minimal left* (*right*) *ideals* of S.

If $S = S^0$, then $\{0\}$ is an ideal of S, which we call a *null ideal* and a null ideal is a kernel of S. So, if a semigroup has a zero, then we investigate some other important ideal: minimal elements in the ordered set of all the ideals of S different than the null ideal we call the 0-*minimal ideal* of S, while the smallest element of this set, if it exists there, we call the 0-*kernel* of S. If the minimal elements of the ordered set of all the left (right) ideals of S are different, then the null ideals we call the 0-*minimal left* (*right*) *ideals* of S.

A semigroup S is *simple* (*left simple, right simple*) if S has no proper ideals (left ideals, right ideals). Since a semigroup S with zero has a null ideal, then the case when the null ideal is a unique proper two-sided (left, right) ideal of S is very interesting. We introduce the following definitions: a semigroup $S = S^0$ is a *null semigroup*, if $S^2 = 0$, i.e. if $ab = 0$, for all $a, b \in S$. A semigroup $S = S^0$ is 0-*simple* (*left* 0-*simple, right* 0-*simple*) if the following conditions hold:

(i) S is not a null semigroup;
(ii) the null ideal is the unique proper two-sided (left, right) ideal of S.

The important property of a 0-minimal left ideal of a semigroup with zero gives

Theorem 1.12 *Let L be a left 0-minimal ideal of a semigroup $S = S^0$. Then one of the following conditions holds:*

(i) *$Sa = L$, for every $a \in L^\bullet$;*
(ii) *$L = \{0, a\}$ and $Sa = 0$.*

Proof. For $a \in L^\bullet$, Sa is a left ideal of S contained in L, so $Sa = L$ or $Sa = 0$. If $Sa = L$, for every $a \in L^\bullet$, then (i) holds. Let $Sa = 0$, for some $a \in L^\bullet$. Then $\{0, a\}$ is a left ideal of S contained in L, whence $L = \{0, a\}$, so (ii) holds. □

Based on Theorem 1.12, what immediately follows is

Corollary 1.4 *A semigroup $S = S^0$ is a left 0-simple if and only if $Sa = S$, for every $a \in S^\bullet$.*

If S is a semigroup without zero, by using Corollary 1.4 on a semigroup S^0, we obtain

Corollary 1.5 *A semigroup S is a left simple if and only if $Sa = S$, for every $a \in S$.*

The following result gives one very important characteristic of 0-minimal ideals.

Theorem 1.13 *Let M be a 0-minimal ideal of a semigroup S. Then $M^2 = 0$ or $MaM = M$, for every $a \in M^\bullet$.*

Proof. Let $M^2 \neq 0$. Since M^2 is an ideal of S contained in M, then $M^2 = M$, whence $M^3 = M$. Let $a \in M^\bullet$. Then $J(a) = S^1 a S^1$ is a non null ideal of S contained in M, so $M = S^1 a S^1$. Thus, $M = M^3 = MS^1 a S^1 M \subseteq MaM \subseteq M$, so $M = MaM$. □

As a consequence of Theorem 1.13 we determine the following

Corollary 1.6 *A semigroup $S = S^0$ is a 0-simple if and only if $SaS = S$, for every $a \in S^\bullet$.*

Theorem 1.14 *A minimal two-sided (left, right) ideal of a semigroup S is a simple (left simple, right simple) subsemigroup of S.*

Proof. Let K be a minimal two-sided ideal of S and let A be an ideal of K, $A \neq K$. Then KAK is an ideal of S. Since K is minimal, then we have that $K = KAK \subseteq A$, which is not possible.

The remaining cases can be proved in a similar way. □

Corollary 1.7 *Let M be a 0-minimal ideal of a semigroup S. Then $M^2 = 0$ or M is a 0-simple subsemigroup of S.*

If S is a semigroup without zero, using Corollary 1.7 on a semigroup S^0, we find

Corollary 1.8 *A semigroup S is simple if and only if $SaS = S$, for every $a \in S$.*

Corollary 1.9 *Let K be an ideal of a semigroup S. Then K is the kernel of S if and only if K is a simple semigroup.*

Proof. Let K be the kernel of S. For an arbitrary $a \in S$, KaK is an ideal of S contained in K, so since K is the kernel, then $K = KaK$. Thus, according to Corollary 1.8, K is a simple semigroup.

Conversely, let K be a simple semigroup. For an arbitrary ideal A of S, $A \cap K$ is an ideal of K, so since K is simple, then $A \cap K = K$, i.e. $K \subseteq A$. Therefore, K is the kernel. □

A maximal element of the ordered set of all the proper left (right) ideals of S we call the *maximal left* (*right*) *ideal* of S. Based on the following theorem we describe a maximal left ideal of a semigroup.

Theorem 1.15 *Let L be a proper left ideal of a semigroup S. Then L is maximal if and only if one of the following conditions holds:*

(i) $S - L = \{a\}$ *and* $a^2 \in L$;
(ii) $S - L \subseteq Sa$, *for every* $a \in S - L$.

Proof. Let L be a maximal left ideal of S. Then we have two cases:

(i) there exists $a \in S - L$ such that $Sa \subseteq L$, then $L \cup \{a\} = S$, whence $S - L = \{a\}$, $a^2 \in L$;

(ii) for every $a \in S - L$, $Sa \nsubseteq L$, then $L \cup Sa = S$, whence $S - L \subseteq Sa$, for every $a \in S - L$.

The converse follows immediately. □

Let $L(S)$ be the *union of all the proper left ideals* of a semigroup S.

Theorem 1.16 *Let $L(S)$ be as same as (ii) in Theorem 1.15. Then $S - L(S) = \{a \in S \mid Sa = S\}$ and $S - L(S)$ is a subsemigroup of S.*

Proof. For $a \in S - L(S)$ we have that $S = L(S) \cup (S - L(S)) = a \cup Sa$, so $L(S) \subseteq Sa$. From this and from $S - L(S) \subseteq Sa$ we have that $S = Sa$, for every $a \in S - L(S)$.

Conversely, let $S = Sa$, for every $a \in S - L(S)$. Then $S - L(S) \subseteq Sa$, $a \in S - L(S)$. Thus, $S - L(S) = \{a \in S \mid Sa = S\}$, and it is evident that $S - L(S)$ is a subsemigroup of S. □

Corollary 1.10 *Let A be a proper ideal of a semigroup S which is not a proper subset of any one left ideal of S. Then one of the following conditions holds:*

(i) *$S - A$ is a left simple semigroup;*
(ii) *$S - A = \{a\}$ and $a^2 \in A$.*

Proof. Let (i) $S - A = T$ have at least two elements. Then by Theorem 1.16, T is a subsemigroup of S. Since $A \cup Sa = A \cup (A \cup T)a = A \cup T = S$, for every $a \in T$, and $A \cap T = \emptyset$, then $T \subseteq Ta \subseteq T$, i.e. $Ta = T$, for every $a \in T$, so T is a left simple semigroup. Hence, in this case (i) holds.

Let $S - A = \{a\}$. Then $a^2 = a$ and $S - A$ is a group, so (i) holds, or $a^2 \neq a$, i.e. $a^2 \in A$, so (ii) holds. □

If A is a minimal element of the set of all the bi-ideals of a semigroup S, then we it call the *minimal bi-ideal* of S.

We prove the following lemma immediately.

Lemma 1.13 *Let A be a bi-ideal of a semigroup S and let $x, y \in S$. Then xAy is also a bi-ideal of S.*

Lemma 1.14 *Let M be a minimal bi-ideal of a semigroup S, let $x, y \in M$ and let A be a bi-ideal of S. Then $M = xAy$.*

Proof. According to Lemma 1.13, xAy is a bi-ideal of S. Since $xAy \subseteq MAM \subseteq MSM \subseteq M$ and since M is a minimal bi-ideal, then $xAy = M$. □

Lemma 1.15 *Let M be a minimal bi-ideal of a semigroup S, let $x, y \in S$. Then xMy is also a minimal bi-ideal of S.*

Proof. According to Lemma 1.13, xMy is a bi-ideal of S. Assume that A is a bi-ideal of S contained in xMy. Then $A = \{xay \,|\, a \in H\}$, where $H \subseteq M$. Assume $a, b \in H$, $u \in S$. Then $xayuxby \in A$, so $ayuxb \in H$. Hence, $aySxb \subseteq H$. Since $a, b \in M$ and ySx is a bi-ideal of S, then by Lemma 1.14, $M = aySxb \subseteq H$. Thus, $M = H$, whence $A = xMy$, so xMy is a minimal bi-ideal of S. □

By Lemmas 1.14 and 1.15 we determine

Lemma 1.16 *Let M be a minimal bi-ideal of S. Then every minimal bi-ideal of S is of the form xMy, for $x, y \in S$.*

A minimal bi-ideal we characterize by means of the following lemma.

Lemma 1.17 *A bi-ideal M of a semigroup S is minimal if and only if M is a group.*

Proof. Let M be a minimal bi-ideal of S. For $x, y \in M$, by Lemma 1.14, $M = xMy$, whence $M = aM = Ma$, for $a \in M$, so M is a subgroup of S.

Conversely, let M be a group. Let A be a bi-ideal of S contained in M. Assume $a \in M$, $x, y \in A$. Let x^{-1} and y^{-1} be the inverse of x and y in a group M, respectively. Then $a = x(x^{-1}ay^{-1})y \in ASA \subseteq A$. Thus, $M = A$, so M is a minimal bi-ideal of S. □

Theorem 1.17 *Let K be the union of all the minimal bi-ideals of a semigroup S. If $K \neq \emptyset$, then K is the kernel of S.*

Proof. Let M be a minimal bi-ideal of S. According to Lemma 1.16, $K = \cup\{xMy \mid x,\, y \in S\} = SMS$, so K is an ideal of S. Assume $a, b \in K$. Then $a \in M$, $b \in N$, for some minimal bi-ideals M and N of S, and by Lemma 1.16, $N = xMy$, for some $x, y \in S$, whence $b = xcy$, for some $c \in M$. Since M is a group, then $c = caa^{-1}$, so $b = xcy = (xc)a(a^{-1}y) \in KaK$. Thus, $KaK = K$, for every $a \in K$, so by Corollaries 1.8 and 1.9, K is the kernel of S. □

Let A and B be the subsets of a semigroup S, and let $A \subseteq B$. Then A is a *consistent* (*right consistent, left consistent*) *subset* of B, in notation $A \leq_C B$ ($A \leq_{RC} B, A \leq_{LC} B$), if for $x, y \in B$

$$xy \in A \Rightarrow x \in A \wedge y \in A \qquad (xy \in A \Rightarrow y \in A, \;\; xy \in A \Rightarrow x \in A).$$

The empty set is also a consistent subset of B. If $A \leq_C S$ ($A \leq_{RC} S, A \leq_{LC} S$), then we say, in short, that A is a *consistent* (*right consistent, left consistent*) *subset*.

The proofs of the following lemmas are elementary.

Lemma 1.18 *The relation $\leq_C$ is a partial order on a partitive set $\mathcal{P}(S)$ of a semigroup S, $\leq_C = \leq_{LC} \cap \leq_{RC}$, $\leq_{RC} \cdot \leq_C = \leq_{RC}$ and $\leq_{LC} \cdot \leq_C = \leq_{LC}$, where "$\cdot$" is a multiplication of binary relations.*

Lemma 1.19 *The intersection and union of an arbitrary family of consistent (right consistent, left consistent) subsets of a subset A of a semigroup S are consistent (right consistent, left consistent) subsets of A.*

Lemma 1.20 *Let A be a subset of a semigroup S different from S. Then*

(i) *$A \leq_{RC} S$ $(A \leq_{LC} S)$ if and only if $S - A$ is a left (right) ideal of S;*
(ii) *$A \leq_C S$ if and only if $S - A$ is an ideal of S.*

A subset A of a semigroup S is a *completely prime subset* of S if for $x, y \in S$

$$xy \in A \ \Rightarrow\ (x \in A \ \vee\ y \in A).$$

A subset A of a semigroup S is a *completely semiprime subset* of S if for $x \in S$, from $x^2 \in A$ it follows that $x \in A$. It is evident that every completely prime subset of S is completely semiprime. The empty set is also a completely prime subset of S.

A subsemigroup A of a semigroup S is a *filter* (*left filter, right filter*) of S if A is a consistent (right consistent, left consistent) subset of S. For an element a of a semigroup S, the intersection of all the filters of S which contain a we call the *principal filter* of S *generated* by a, and denote by $N(a)$. It is the smallest filter containing an element a of a semigroup S.

We immediately prove

Lemma 1.21 *Let A be a non-empty subset of a semigroup S different from S. Then*

(i) *A is a completely prime subset of S if and only if $S - A$ is a subsemigroup of S;*
(ii) *A is a completely prime left (right) ideal of S if and only if $S - A$ is a left (right) filter of S;*
(iii) *A is a completely prime ideal of S if and only if $S - A$ is a filter of S.*

Lemma 1.22 *The intersection of an arbitrary family of completely semiprime subsets of a semigroup S is a completely semiprime subset of S.*

Corollary 1.11 *The intersection of an arbitrary family of completely prime (completely semiprime) ideals of a semigroup S, if it is non-empty, is a completely semiprime ideal of S.*

Let A be an ideal of a semigroup S. The ideal A is a *semiprime ideal* of S if for $a \in S$, from $aSa \subseteq A$ it follows that $a \in A$. The ideal A is a *prime ideal* of S if for $a, b \in S$, from $aSb \subseteq A$ it follows that $a \in A$ or $b \in A$. The ideal A is a *completely semiprime ideal* of S if for $a \in S$, from $a^2 \in A$ it follows that $a \in A$. The ideal A is a *completely prime ideal* of S if for $a, b \in S$, from $ab \in A$ it follows that $a \in A$ or $b \in A$. By $\mathcal{I}d^{cs}(S)$ will denote the lattice of all the completely semiprime ideals of S.

The following lemma gives another definition of prime ideals.

Lemma 1.23 *Let A be an ideal of a semigroup S. Then A is a prime ideal of S if and only if for ideals M, N of S, from $MN \subseteq A$ it follows that $M \subseteq A$ or $N \subseteq A$.*

Proof. Let A be a prime ideal of S, and let M and N be the ideals of S such that $MN \subseteq A$. Assume that there exists $x \in M - A$ and $y \in N - A$. Then $xSy \subseteq MSN \subseteq MN \subseteq A$, so $x \in A$ or $y \in A$, because A is a prime ideal. So, it is a contradiction. Hence, $M - A = \emptyset$ or $N - A = \emptyset$, i.e. $M \subseteq A$ or $N \subseteq A$.

Conversely, for ideals M and N of S, from $MN \subseteq A$, let it follow that $M \subseteq A$ or $N \subseteq A$. Assume $x, y \in S$ such that $xSy \subseteq A$. Then $J(x)J(y) \subseteq A$, whence $J(x) \subseteq A$ or $J(y) \subseteq A$, i.e. $x \in A$ or $y \in A$. Therefore, A is a prime ideal of S. $\square$

Exercises

1. Let ϕ be a homomorphism of a semigroup S into a semigroup T. If A is a left (right) ideal of S, then $A\phi$ is a left (right) ideal of T. If B is a left (right) ideal of T, then $B\phi^{-1}$ is a left (right) ideal of S.

2. If X is a finite set, then every ideal of a semigroup $\mathcal{T}_r(X)$ is principal. If X is an infinite countable set, then the unique non-principal ideal of $\mathcal{T}_r(X)$ is the set of all mapping from $\mathcal{T}_r(X)$, such that its image is the finite subset of X.

3. A semigroup S is left (right) 0-simple if and only if $S^\bullet$ is a left (right) simple subsemigroup of S.

4. Let M be a 0-minimal ideal of a semigroup $S = S^0$ which contains at least one 0-minimal left ideal of S. Then M is the union of all 0-minimal left ideals of S contained in M.

If, also, $M^2 \neq 0$, then every left ideal of M is a left ideal of S.

5. A semigroup S has no proper quasi-ideals (bi-ideals) if and only if S is a group.

6. If L is a left and R is a right ideal of a semigroup S, and if B is a subset of S such that $RL \subseteq B \subseteq R \cap L$, then B is a bi-ideal of S.

7. A semigroup S is a group if and only if S is left simple and right simple.

8. Prove that in a monogenic semigroup $S = \langle a\rangle = M(i,p)$, the group $K_a = \{a^i, a^{i+1}, \dots, a^{i+p+1}\}$ is the kernel of S.

9. A semigroup cannot have the proper left consistent left ideals and cannot have the proper consistent ideals.

10. If B is a bi-ideal of a semigroup S, then $\mathcal{P}(B)$ is a bi-ideal of $\mathcal{P}(S)$.

References

S. Bogdanović [11]; S. Bogdanović and M. Ćirić [18]; K. S. Carman [1]; A. H. Clifford [2]; A. H. Clifford and G. B. Preston [1], [2]; P. Dubreil [1]; R. A. Good and D. R. Hughes [1]; H. B. Grimble [1]; J. M. Howie [1]; A. Iampan [1]; J. Kist [1]; D. Krgović [1], [2], [3]; S. Lajos [1]; G. Lallement [2], [3], [4]; T. Malinović [1], [2]; S. Milić and V. Pavlović [1]; T. Saito and S. Hori [1]; Š. Schwarz [1], [3], [4], [5]; O. Steinfeld [3]; E. G. Šutov [1]; A. D. Wallace [2], [3].

1.7 Ideal and Retractive Extensions of Semigroups

Let T be an ideal of a semigroup S. We define a relation θ on S with:

$$a\theta b \;\Leftrightarrow\; a = b \;\vee\; a, b \in T, \qquad a, b \in S,$$

i.e. $\theta = \Delta_S \cup T \times T$. It is evident that θ is a congruence on S, and we call it *Rees's congruence* determined by the ideal T. A factor semigroup S/θ we call *Rees's factor semigroup* under the ideal T, and denote it by S/T. Assume that $S/T = Q$. According to the definition of Rees's congruence, T is one of θ-classes of S, which is a zero in Q. Hence, a Rees's factor semigroup is a semigroup with zero. For $a \in S - T$, a θ-class of the element a is a singleton. Thus, we can informally discuss, a semigroup Q as a semigroup obtained from S contracting the ideal T into one element (zero), while a partial semigroup $S - T$ stays the same. Formally, a semigroup Q is isomorphic to the zero extension of a partial semigroup $S - T$. So, we usually identify partial semigroups $Q^\bullet$ and $S - T$.

A semigroup S is an *ideal extension* of a semigroup T by a semigroup Q with a zero if T is isomorphic to an ideal T' of S and a factor semigroup S/T is isomorphic to Q. In that case we identify semigroups T and T', semigroups S/T' and Q, and semigroups $S - T$ and $Q^\bullet$. One of the main problems with an ideal extension is: If there is a given semigroup T and a semigroup Q with zero, how do we construct an ideal extension S of T by a semigroup Q? Namely, if we assume that $S = T \cup Q^\bullet$, the question is: How do we define a multiplication $*$ on S such that S is a semigroup, T an ideal of S and a factor semigroup S/T is isomorphic to Q, i.e. such that the following conditions hold:

(M1) $x * y = xy$, if $xy \neq 0$; (M2) $x * y \in T$, if $xy = 0$;
(M3) $a * b = ab$; (M4) $a * x \in T$; (M5) $x * a \in T$;

for all $x, y \in Q^\bullet$, $a, b \in T$? One very useful method for the construction of some ideal extension gives us partial homomorphisms. We defined partial homomorphisms in Section 1.3. The following lemma gives its role in the construction of some ideal extensions.

Lemma 1.24 *Let T and $Q = Q^0$ be the semigroups, and let $\varphi : Q^\bullet \mapsto T$ be a partial homomorphism. We define a multiplication $*$ on $S = T \cup Q^\bullet$ with:*

$$a * y = \begin{cases} xy, & \text{if } xy \neq 0 \text{ in } Q \\ (x\varphi)(y\varphi), & \text{if } xy = 0 \text{ in } Q \end{cases},$$

$$a * x = a(x\varphi), \qquad x * a = (x\varphi)a, \qquad a * b = ab,$$

for $x, y \in Q^\bullet$, $a, b \in T$. Then S with this operation $*$ is a semigroup and S is an ideal extension of T by Q.

Proof. Follows immediately. □

An ideal extension constructed in Lemma 1.24 we call an *extension of T by Q determined with partial homomorphism.*

Retractive extensions are in very close relation with ideal extensions determined by partial homomorphisms, which we are about to discuss.

An endomorphism φ of a semigroup S is a *retraction* if $\varphi^2 = \varphi$, i.e. if $(x\varphi)\varphi = x\varphi$, for every $x \in S$. If φ is a retraction of a semigroup S, then a subsemigroup $T = S\varphi$ of S we call a *retract* of S and say that φ is a *retraction* of S onto T. Namely, a subsemigroup T of a semigroup S is a retract of S if there exists a retraction of S onto T, i.e. if there exists a homomorphism φ of S onto T such that $x\varphi = x$, for every $x \in T$.

Here we are especially interested in the retracts of the given semigroup which are equal to its ideals. If T is both, a retract of a semigroup S and an ideal of S, then T is a *retractive ideal* of S and the corresponding retraction of S onto T is an *ideal retraction*. Namely, a retraction φ of a semigroup S is an *ideal retraction* of S if $S\varphi$ is an ideal of S. Based on the following lemma we give one characterization of ideal retractions:

Lemma 1.25 *A retraction φ of a semigroup S is an ideal retraction of S if and only if $(xy)\varphi = x(y\varphi) = (x\varphi)y$, for all $x, y \in S$.*

Proof. Let φ be an ideal retraction of S, i.e. let $T = S\varphi$ be an ideal of S. Assume $x, y \in S$. Since $y\varphi \in T$, then $x(y\varphi) \in T$, whence

$$x(y\varphi) = [x(y\varphi)]\varphi = (x\varphi)(y\varphi^2) = (x\varphi)(y\varphi) = (xy)\varphi.$$

Similarly, we prove that $(x\varphi)y = (xy)\varphi$.

Conversely, let $(xy)\varphi = x(y\varphi) = (x\varphi)y$, for all $x, y \in S$, and let $T = S\varphi$. Assume $a \in T$, $x \in S$. Then $ax = (a\varphi)x = (ax)\varphi \in T$, and similar, $xa \in T$. Hence, T is an ideal of S. □

Lemma 1.26 *Let T be a semigroup. To every element $a \in T$ we associated the set Y_a such that*

$$a \in Y_a, \quad Y_a \cap Y_b = \emptyset \quad \text{if } a \neq b, \quad a, b \in T.$$

For $a, b \in T$, let $\varphi^{(a,b)} : Y_a \times Y_b \mapsto Y_{ab}$ be a mapping for which

$$(x, b)\varphi^{(a,b)} = (a, y)\varphi^{(a,b)} = ab, \tag{1}$$

for all $x \in Y_a$, $y \in Y_b$, $a, b \in T$, and

$$((x, y)\varphi^{(a,b)}, z)\varphi^{(ab,c)} = (x, (y, z)\varphi^{(b,c)})\varphi^{(a,bc)}, \tag{2}$$

for all $x \in Y_a - \{a\}$, $y \in Y_b - \{b\}$, $z \in Y_c - \{c\}$, $a, b, c \in T$. We define a multiplication $*$ on $S = \cup_{a \in T} Y_a$ with:

$$x * y = (x, y)\varphi^{(a,b)}, \quad \text{if } x \in Y_a, y \in Y_b, a, b \in T.$$

Then S with this multiplication is a semigroup, in notation $(T; Y_a, \varphi^{(a,b)})$.

Proof. Assume $x, y, z \in S$, $x \in Y_a$, $y \in Y_b$, $z \in Y_c$, $a, b, c \in T$. According to (2) we obtain that

$$\begin{aligned}(x * y) * z &= (x, y)\varphi^{(a,b)} * z = ((x, y)\varphi^{(a,b)}, z)\varphi^{(ab,c)} \\ &= (x, (y, z)\varphi^{(b,c)})\varphi^{(a,bc)} = x * (y, z)\varphi^{(b,c)} = x * (y * z).\end{aligned}$$

Thus, S is a semigroup. □

A subset A of a semigroup S is a *transversal* of S if a congruence ξ on S exists such that every ξ-classe contains only one element from A.

By the following theorem we give a characterization of a retractive extension, i.e. of an ideal extension determined by partial homomorphisms.

Theorem 1.18 *Let T be an ideal of a semigroup S. Then the following conditions are equivalent:*

(i) *S is an ideal extension of T determined by partial homomorphism;*
(ii) *S is a retractive extension of T;*
(iii) *T is a transversal of S;*
(iv) *S is isomorphic to some semigroup $(T; Y_a, \varphi^{(a,b)})$.*

Proof. (i)⇒(ii) Let φ be a partial homomorphism which determined a multiplication on S. We define a mapping $\psi : S \mapsto T$ with

$$x\psi = \begin{cases} x\varphi, & \text{if } x \in S/T \\ x, & \text{if } x \in T \end{cases} .$$

It is easy to prove that ψ is a retraction of S onto T.

(ii)⇒(i) Let φ be a retraction of S onto T. Then, by the usual identification of partial semigroups $S - T$ and $Q^\bullet$, where $Q = S/T$, a retraction ψ of a retraction φ on $Q^\bullet$ is a partial homomorphism of $Q^\bullet$ into T and multiplication on S is determined by this partial homomorphism, in the way which we saw in Lemma 1.24.

(ii)⇒(iv) Let φ be a retraction of S onto T. For $a \in T$, let $Y_a = a\varphi^{-1} = \{x \in S \mid x\varphi = a\}$. Then $S = \cup_{a \in T} Y_a$, and for sets Y_a, $a \in T$ the conditions of Lemma 1.26 hold.

For an arbitrary $x, y \in S$ there are $a, b \in T$ such that $x \in Y_a$, $y \in Y_b$, i.e. $x\varphi = a$, $y\varphi = b$, whence $(xy)\varphi = (x\varphi)(y\varphi) = ab \in Y_{ab}$. It is easy to prove that for $a, b \in T$, a mapping $\varphi^{(a,b)} : Y_a \times Y_b \mapsto Y_{ab}$ defined by

$$(x, y)\varphi^{(a,b)} = (xy)\varphi,$$

satisfied the condition (2) and a multiplication on S is defined the same as in Lemma 1.26. Since T is an ideal of S, then (1) holds.

(iv)⇒(ii) Let $S = (T; Y_a, \varphi^{(a,b)})$. We define a mapping $\varphi : S \mapsto T$ with $x\varphi = a$ if $x \in Y_a$, $a \in T$. It is easy to prove that φ is a retraction of S onto T.

(iii)⇒(ii) Let ξ be a congruence on S such that in every ξ-classe there is only one element from T. For $a \in T$, let $C_a = \{x \in S \mid a\xi x\}$, and we define a mapping $\varphi : S \mapsto T$ with $x\varphi = a$ if $x \in C_a$, $a \in T$. It is evident that φ is a retraction of S onto T.

(ii)$\Rightarrow$(iii) Let $\varphi : S \mapsto T$ be a retraction. Then $\xi = \ker\varphi$ is a congruence on S. Let C be an arbitrary ξ-class of S, and let $a, b \in C \cap T$. Then $a = a\varphi = b\varphi = b$. Therefore, T is a transversal of S. □

Theorem 1.19 *A semigroup T is a retract of every one of its ideal extensions if and only if T has a unit.*

Proof. Let T be a retract of every one of its ideal extensions. Then T is also a retract of a semigroup $S = T^1$. Let φ be a retraction of S onto T. Then for an arbitrary $x \in T$ we have

$$x(1\varphi) = (x\varphi)(1\varphi) = (x1)\varphi = x\varphi = x = (1x)\varphi = (1\varphi)(x\varphi) = (1\varphi)x,$$

so 1φ is an identity in T.

Conversely, let T be a semigroup with an identity e. Let S be an arbitrary ideal extension of T. Then it is easy to prove that the mapping $\varphi : S \mapsto T$ defined by

$$x\varphi = xe, \qquad x \in S,$$

is a retraction of S onto T. □

Lemma 1.27 *Let ξ be a congruence on a semigroup S. For every congruence η on S which contains ξ we define a relation η' on S/ξ with*

$$(x\xi)\eta'(y\xi) \quad \Leftrightarrow \quad x\eta y, \quad x, y \in S.$$

Then η' is a congruence on S/ξ and a mapping $\eta \mapsto \eta'$ of the set of all congruences on S which contains ξ into the set of all congruences of a semigroup S/ξ is a bijection which preserves an order.

Proof. The proof follows immediately. □

Let T be an ideal of a semigroup S. A congruence ξ on S is a *T-congruence* if its restriction on T is Δ_T. An ideal extension S of a semigroup T is a *dense extension* of T if the equality relation is the unique T-congruence on S.

Lemma 1.28 *Let S be an ideal extension of a semigroup T, let ξ be a T-congruence on S and let S/ξ be an ideal extension of T. Then S/ξ is a dense extension of T if and only if ξ is a maximal T-congruence on S.*

Proof. Follows from Lemma 1.27. □

Theorem 1.20 *Let D be an ideal extension of a semigroup T, and let $Q = Q^0$ be a semigroup such that $T \cap Q = \emptyset$. Let $\varphi : Q^\bullet \mapsto D$ be a partial homomorphism such that $(a\varphi)(b\varphi) \in T$, whenever $ab = 0$ in Q, $a, b \in Q$. We define a multiplication $*$ on $S = T \cup Q^\bullet$ with*

$$a * b = \begin{cases} (a\varphi)b, & \text{if } a \in Q^\bullet, b \in T, \\ a(b\varphi), & \text{if } a \in T, b \in Q^\bullet, \\ (a\varphi)(b\varphi), & \text{if } a, b \in Q^\bullet, ab = 0 \text{ in } Q, \\ ab, & \text{otherwise.} \end{cases}$$

Then S is an ideal extension of T by Q.

Conversely, every ideal extension of a semigroup T by a semigroup Q can be constructed in the previous way, for any extension D of T and any partial homomorphism φ from $Q^\bullet$ into D, where we can choose that D is a dense extension of T and that is $D = T \cup Q^\bullet\varphi$.

Proof. Let S be an ideal extension of T by Q. In a partially ordered set of all T-congruences on S, by Lemma 1.11, there exists a maximal element, i.e. there exists a maximal T-congruence ξ on S. Let $D = S/\xi$ and let φ be a restriction of a natural homomorphism $\xi^\natural$ on $Q^\bullet = S - T$.

If $a, b \in Q^\bullet$ and $ab \neq 0$ in Q, then $(a\varphi)(b\varphi) = (a\xi^\natural)(b\xi^\natural) = (ab)\xi^\natural = (ab)\varphi$, so φ is a partial homomorphism. If $a, b \in Q^\bullet$ and $ab = 0$ in Q, i.e. $ab \in T$ in S, then $(a\varphi)(b\varphi) = (a\xi^\natural)(b\xi^\natural) = (ab)\xi^\natural = ab \in S$. Furthermore, $D = S\xi^\natural = T \cup Q^\bullet\varphi$. Based on Lemma 1.28, D is a dense extension of T.

For $a \in S$, $b \in Q^\bullet$, $ab \in S$, so $ab = (ab)\xi^\natural = (a\xi^\natural)(b\xi^\natural) = a(b\varphi)$. Similarly we prove the other cases from the multiplication $*$. Thus, a semigroup S can be constructed in this the way from the formulation of a theorem.

The converse follows immediately. □

Let $S = S^0$. An element $a \in S$ is *nilpotent* if there is $n \in \mathbf{Z}^+$ such that $a^n = 0$. The set of all nilpotent elements from a semigroup S we denote by $\mathrm{Nil}(S)$. A semigroup S is a *nil-semigroup* if $S = \mathrm{Nil}(S)$. An ideal extension S of a semigroup T is a *nil-extension* of T if S/T is a nil-semigroup, i.e. if $\sqrt{T} = S$. A semigroup $S = S^0$ is *nilpotent* if there is $n \in \mathbf{Z}^+$ such that $S^{n+1} = 0$. If $S^{n+1} = 0$, then we say that S is $(n+1)$-*nilpotent.* A semigroup S is *nilpotent*, the class of nilpotency $n+1$, if S is $(n+1)$-nilpotent and it is not n-nilpotent. Let $n \in \mathbf{Z}^+$. An ideal extension S of a semigroup T by nilpotent ($(n+1)$-nilpotent) semigroup we call a *nilpotent* (*$(n+1)$-nilpotent*) *extension* of T. A retractive $(n+1)$-nilpotent extension of a semigroup T we call *n-inflation* of a semigroup T, 1-inflation is an *inflation*, and 2-inflation is a *strong inflation.*

Exercises

1. Let I and J be the ideals of a semigroup S. Then $I \cap J$ and $I \cup J$ are ideals of S and $(I \cup J)/J \cong I/(I \cap J)$.

2. A semigroup S is a *semigroup with unique decomposition* if every non-zero element from S has a unique decomposition into a product of the elements from $S - S^2$.

Let $T = T^0$ and S be semigroups. Then

(a) there exists a semigroup U with a unique decomposition and a homomorphism ϕ of U onto T such that $|0\phi^{-1}| = 1$;
(b) if α is a partial homomorphism of $U^\bullet$ into S such that $\ker\phi \subseteq \ker\alpha$ on $U^\bullet$, then the mapping $\alpha' : T^\bullet \mapsto S$ defined by $y\alpha' = x\alpha$, where $x \in y\phi^{-1}$, $y \in T^\bullet$, is a partial homomorphism of $T^\bullet$ into S.

Conversely, every partial homomorphism of $T^\bullet$ into S is determined in this way. Also, the mapping $\alpha \mapsto \alpha'$ is injective.

3. Let $IR(S)$ be the set of all ideal retractions of a semigroup S and let $RI(S)$ be the set of all retractive ideals of S. Then

(a) If $IR(S)$ is a semilattice under the product of mappings, then $RI(S)$ is a semilattice under the intersection and $RI(S)$ is the homomorphic image of $IR(S)$;
(b) If $S^2 = S$ or for all $a, b \in S$, from $a^2 = b^2 = ab = ba$ it follows that $a = b$, then $IR(S)$ is a semilattice and $RI(S) \cong IR(S)$.

4. Let S be a semigroup such that $S^2 = S$ or for all $a, b \in S$, from $a^2 = b^2 = ab = ba$ it follows that $a = b$, and if I is an ideal of S, then there exists at most one retraction of S onto I.

5. Let T be a semigroup, let Q be the non-empty set and let φ be an arbitrary mapping from Q into T. Then $S = Q \cup T$ with the multiplication defined by: $x * y = (x\varphi)(y\varphi)$, $x * a = (a\varphi)a$, $a * x = a(x\varphi)$, $a * b = ab$, for $x, y \in Q$, $a, b \in T$, is a semigroup and S is an *inflation* of T. Conversely, every inflation of a semigroup T can by constructed in this way.

References

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1.8 Green's Relations

On a semigroup S we define the relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, $\mathcal{H}$ and $\mathcal{D}$ in the following way

$$\begin{aligned} a\,\mathcal{L}\,b &\Leftrightarrow L(a) = L(b),\ a, b \in S; \\ a\,\mathcal{R}\,b &\Leftrightarrow R(a) = R(b),\ a, b \in S; \\ a\,\mathcal{J}\,b &\Leftrightarrow J(a) = J(b),\ a, b \in S; \end{aligned}$$

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}, \qquad \mathcal{D} = \mathcal{L}\mathcal{R}.$$

These relations are equivalence relations and we call them *Green's relations* or *Green's equivalences.* By L_a (R_a, J_a, H_a, D_a) we denote a $\mathcal{L}$- ($\mathcal{R}$-, $\mathcal{J}$-, $\mathcal{H}$-, $\mathcal{D}$-) class containing a fixed element $a \in S$.

Lemma 1.29 *Let a and b be the elements of a semigroup S, then*

$$\begin{aligned} a\,\mathcal{L}\,b &\Leftrightarrow (\exists\, x, y \in S^1)\ xa = b,\ yb = a; \\ a\,\mathcal{R}\,b &\Leftrightarrow (\exists\, u, v \in S^1)\ au = b,\ bv = a; \\ a\,\mathcal{J}\,b &\Leftrightarrow (\exists\, x, y, u, v \in S^1)\ xay = b,\ ubv = a. \end{aligned}$$

According to Lemma 1.29 it is evident that the following corollary holds.

Corollary 1.12 *Every idempotent e of a semigroup S is a left identity element of R_e and a right identity element of L_e.*

Lemma 1.30 *On a semigroup S, $\mathcal{L}$ is a right and $\mathcal{R}$ is a left congruence relation.*

Lemma 1.31 *On a semigroup S the relations $\mathcal{L}$ and $\mathcal{R}$ commute.*

Proof. Assume $a\mathcal{LR}b$, $a, b \in S$. Then there exists $c \in S$ such that $a\mathcal{L}c$ and $c\mathcal{R}b$. According to Lemma 1.29 we have that $a = xc$, $b = cy$, $c = ua = bv$, for some $x, y, u, v \in S^1$. Let $d = ay$. Then

$$d = xcy = xb, \quad a = xc = xbv = dv, \quad b = cy = uay = ud.$$

Hence, $a\mathcal{R}d$ and $d\mathcal{L}b$, so $\mathcal{LR} \subseteq \mathcal{RL}$. Similarly, we can prove that $\mathcal{RL} \subseteq \mathcal{LR}$. Therefore, $\mathcal{LR} = \mathcal{RL}$. $\square$

It is evident that $\mathcal{L} \cup \mathcal{R} \subseteq \mathcal{J}$ and $\mathcal{D} \subseteq \mathcal{J}$. There are semigroups on which some Green's relations are equal. For instance, if S is a commutative semigroup then all of Green's relations are equal to each other. There are semigroups on which the relation $\mathcal{D}$ is the proper subset of the relation $\mathcal{J}$. Here, we will prove that the relations $\mathcal{D}$ and $\mathcal{J}$ are equal to each other on an important class of semigroups, on the class of completely π-regular semigroups.

An element a of a semigroup S is *regular* if there exists $x \in S$ such that $a = axa$. A semigroup S is *regular* if all its elements are regular.

An element $a \in S$ is *π-regular* if there exists $n \in \mathbf{Z}^+$ and $x \in S$ such that $a^n = a^n x a^n$. A semigroup S is *π-regular* if all its elements are π-regular.

An element $a \in S$ is *completely π-regular* if there exists $n \in \mathbf{Z}^+$ and $x \in S$ such that $a^n = a^n x a^n$ and $a^n x = x a^n$. A semigroup S is *completely π-regular* if all its elements are completely π-regular.

Lemma 1.32 *If S is a completely π-regular semigroup, then $\mathcal{D} = \mathcal{J}$.*

Proof. Let S be a completely π-regular semigroup. Assume $a, b \in S$ such that $a\mathcal{J}b$. Then $a = xby$ and $b = uav$, for some $x, y, u, v \in S^1$. So, $a = x(uav)y = (xu)a(vy)$, whence $a = (xu)^m a(vy)^m$, for all $m \in \mathbf{Z}^+$. Assume $n \in \mathbf{Z}^+$ and $z \in S$ such that $(xu)^n = (xu)^n z(xu)^n$ and $(xu)^n z = z(xu)^n$. Then $a = (xu)^n a(vy)^n = (xu)^n z(xu)^n a(vy)^n = (xu)^n za = z(xu)^n a \in S^1 ua$. Thus $a\mathcal{L}ua$. Similarly, we prove that $a\mathcal{R}av$. Since $\mathcal{L}$ is a right congruence if follows that $b = uav\mathcal{L}av$. Therefore, $a\mathcal{D}b$, i.e. $\mathcal{J} \subseteq \mathcal{D}$. Since the opposite inclusion always holds we have that $\mathcal{J} = \mathcal{D}$. $\square$

More will be said about completely π-regular semigroups in Section 2.1.

Let ρ be an equivalence relation on a semigroup S, let A and B be a subset of S and let $\varphi : A \mapsto B$ be a mapping. We say that the mapping φ *preserves the ρ-classes* if $x \,\rho\, (x\varphi)$ for all $x \in A$.

The next two results are well known as Green's lemmas.

Lemma 1.33 *Let a and b be $\mathcal{R}$-equivalent elements of a semigroup S and let $u, v \in S^1$ such that $au = b$ and $bv = a$. Then the mappings*

$$x \mapsto xu, \quad x \in L_a, \qquad y \mapsto yv, \quad y \in L_b, \tag{1}$$

are mutually inverse bijections, $\mathcal{R}$-class preserving, of L_a onto L_b and of L_b ontp L_a, respectively.

Proof. First, we note that the given mappings (1) are right translation ρ_u and ρ_v restricted to L_a and L_b, respectively. For $x \in L_a$, from $x\mathcal{L}a$ we get $xu\mathcal{L}au = b$, because $\mathcal{L}$ is a right congruence. Thus ρ_u maps L_a into L_b. Similarly, ρ_v maps L_b into L_a. Also, for $x \in L_a$ from $x\mathcal{L}a$ it follows that $x = wa$ for some $w \in S^1$, whence $x\rho_u\rho_v = xuv = wauv = wbv = wa = x$. Similarly, we prove that $y\rho_v\rho_u = y$ for every $y \in L_b$. Therefore, the mappings (1) are mutually inverse bijections of L_a onto L_b and of L_b onto L_a, respectively.

For $x \in L_a$ we have that $x = x\rho_u\rho_v = (xu)v$, whence $x\mathcal{R}xu$. Similarly, we prove that $y\mathcal{R}yv$, for every $y \in L_b$. Thus, the mapping (1) preserves $\mathcal{R}$-classes. □

Lemma 1.34 *Let a and b be $\mathcal{L}$-equivalent elements of a semigroup S and let $s, t \in S^1$ such that $sa = b$ and $tb = a$. Then the mappings*

$$x \mapsto sx, \quad x \in R_a, \qquad y \mapsto ty, \quad y \in R_b, \tag{2}$$

are mutually inverse bijections, $\mathcal{L}$-class preserving, of R_a onto R_b and of R_b onto R_a, respectively.

Lemma 1.35 *Let a and b be the elements of a semigroup S, then:*

(i) *If $ab \in H_a$, then the mapping $x \mapsto xb$, $x \in H_a$ is a bijection from H_a onto H_a;*
(ii) *If $ab \in H_b$, then the mapping $x \mapsto ax$, $x \in H_b$ is a bijection from H_b onto H_b.*

Proof. (i) From $ab \in H_a$ it follows that $ab\mathcal{R}a$, whence $a = (ab)u$ for some $u \in S^1$, so by Lemma 1.33 the mappings $\xi : x \mapsto xb$, $x \in L_a$, and $\xi' : y \mapsto yu$, $y \in L_{ab} = L_a$ are mutually inverse bijections from L_a onto itself which preserve $\mathcal{R}$-classes. Let η and η' be the restrictions of ξ and ξ' on H_a, respectively. For $x \in H_a$ we have that $x\eta = x\xi \in L_a$. On the other hand, since ξ preserves $\mathcal{R}$- classes, then $x\mathcal{R}x\xi = x\eta$, i.e. $x\eta \in R_x = R_a$. Thus $x\eta \in L_a \cap R_a = H_a$, so η maps H_a into itself. Similarly, we prove that η' maps H_a into itself. It is evident that η and η' are mutually inverse bijections from H_a onto H_a.

(ii) This is proved in a similar way as (i). □

The following result is as famous as *Green's theorem.*

Theorem 1.21 *Let H be an $\mathcal{H}$-class of a semigroup S, then $H^2 \cap H = \emptyset$ or $H^2 = H$.*

If $H^2 = H$ holds, then H is a (maximal) subgroup of S.

Proof. Assume that $H^2 \cap H \neq \emptyset$, then there exist $a, b \in H$ such that $ab \in H$. According to Lemma 1.35 the mappings

$$x \mapsto xb,\ x \in H, \qquad y \mapsto ay,\ y \in H,$$

are bijections from H onto itself. Thus $ah, hb \in H$ for every $h \in H$ and again by Lemma 1.35, for every $h \in H$, the mappings

$$x \mapsto xh,\ x \in H, \qquad y \mapsto hy,\ y \in H,$$

are bijections from H onto itself. Hence, $hH = Hh = H$ for every $h \in H$, so, we have that $H^2 = H$ and H is a subgroup of S. It is easy to prove that H is a maximal subgroup of S. □

Corollary 1.13 *If e is an idempotent of a semigroup S, then H_e is a subgroup of S. Also, the $\mathcal{H}$-class cannot contain more than one idempotent element.*

Lemma 1.36 *If a $\mathcal{D}$-class D of a semigroup S contains a regular element, then every element of D is regular.*

Proof. Let a be a regular element of a class D and let $b \in D$. Then $a\mathcal{D}b$, i.e. $ua = c$, $vc = a$, $cs = b$ and $bt = c$ for some $c \in S$ and $u, v, s, t \in S^1$. If $x \in S$ such that $a = axa$, then we have that

$$b = cs = uas = uaxas = cxas = cxvcs = cxvb = btxvb.$$

Therefore, b is a regular element too. □

According to Lemma 1.36 a $\mathcal{D}$-class of a semigroup S which contains a regular element (i.e. whose elements are all regular) we call a *regular* $\mathcal{D}$*-class*.

Lemma 1.37 *If D is a regular $\mathcal{D}$-class, then every $\mathcal{L}$-class and every $\mathcal{R}$-class contained in D contains an idempotent.*

Proof. If $a \in D$ and $a = axa$, for some $x \in S$, then $ax, xa \in E(S)$, and $ax \in R_a$ and $xa \in L_a$. □

Let A and B be the ideals of a semigroup S such that $A \subseteq B$. It is easy to prove that the factor set B/A can be embedded into a factor set S/A, and usually we assume that B/A is a subsemigroup of S/A.

According to Theorem 1.4 and Lemma 1.27 the next result immediately follows:

Lemma 1.38 *Let A be an ideal of a semigroup S:*

(i) *If B is an ideal of S such that $A \subseteq B$, then B/A is an ideal of S/A and $(S/A)/(B/A) \cong S/B$.*
(ii) *The mapping $\theta : B \mapsto B/A$ is a bijection from $Id(S)$ onto $Id(S/A)$ which preserves the partial order.*

Let a be an element of a semigroup S. Based on $I(a)$ we denote the set $I(a) = J(a) - J_a = \{x \in S \mid J(x) \subset J(a)\}$.

Lemma 1.39 *Let a be an element of a semigroup S such that $I(a) \neq \emptyset$. Then $I(a)$ is an ideal of S. Moreover, $I(a)$ is the greatest element in the partial ordered set of all the ideals of S which are strictly contained in $J(a)$.*

Proof. Assume $b \in I(a)$ and $x \in S$. Then $J(bx) \subseteq J(b) \subset J(a)$ and $bx \in J(a)$, so $bx \in I(a)$. Similarly, we prove that $xb \in I(a)$. Thus, $I(a)$ is an ideal of S.

Let A be an arbitrary ideal of S strictly contained in $J(a)$. For $x \in A$ we have that $J(x) \subseteq A \subset J(a)$ and $x \in J(a)$, so, $x \in I(a)$. Thus, $A \subseteq I(a)$. Therefore, $I(a)$ is the greatest ideal of S strictly contained in $J(a)$. □

For reasons of simplicity we use the following notation: the factor set $S/\emptyset$ is S.

For an element a of a semigroup S, the factor semigroup $J(a)/I(a)$ we call the *principal factor* of a semigroup S which contains the element a.

The important characteristics of the principal factors give the following result.

Theorem 1.22 *Let a be an element of a semigroup S. Then one of the following statements holds:*

(i) *$J(a)$ is the kernel of a semigroup S;*
(ii) *$I(a) \neq \emptyset$ and the principal factor $J(a)/I(a)$ is a 0-simple semigroup or a zero-semigroup.*

Proof. Let $J(a)$ be the kernel of a semigroup S. Then there exists an ideal A of S such that $A \subset J(a)$. For $x \in A$ we have that $J(x) \subseteq A \subset J(a)$, so $x \in I(a)$. Therefore, $I(a) \neq \emptyset$.

Let A be a non zero ideal of a semigroup $S/I(a)$. Using the bijection from Lemma 1.38, the ideal B corresponds to the ideal A such that $I(a) \subset B \subseteq J(a)$. According to Lemma 1.39, it follows that $B = J(a)$, whence $A = J(a)/I(a)$. Thus $J(a)/I(a)$ is a 0-minimal ideal of $S/I(a)$ and by Corollary 1.7 $J(a)/I(a)$ is a 0-simple semigroup or a zero-semigroup. □

Exercises

1. Let T be a monoid and let H be a group of its identity. Let θ be a homomorphism of T into H, and let N be the set of all non-negative integers. Then, $S = N \times T \times N$ with the multiplication defined by:

$$(m; a; n)(p; b; q) = (m - n + t; (a\theta^{t-n})(b\theta^{t-p}); q - p + t),$$

for $(m; a; n), (p; b; q) \in S$ and $t = \max\{n, p\}$, is a semigroup, in notation $S = BR(T, \theta)$, which we call the *Bruck-Reilly's extension* of T by θ.

Prove the following conditions:

(a) S is a simple semigroup;
(b) $(m; a; n)\mathcal{D}_S(p; b; q) \Leftrightarrow a\mathcal{D}_T b, \quad (m; a; n), (p; b; q) \in S$;
(c) every semigroup T can be embedded into $BR(T^1, \theta)$, where $\theta : T^1 \mapsto \{1\}$;
(d) if T is a semigroup without an identity, $\theta : T^1 \mapsto \{1\}$ and $S = BR(T^1, \theta)$, then $\mathcal{D} \neq \mathcal{J}$ on S.

2. If $\alpha, \beta \in \mathcal{T}_r(X)$, then

(a) $\alpha\mathcal{L}\beta \Leftrightarrow X\alpha = X\beta$;
(b) $\alpha\mathcal{R}\beta \Leftrightarrow \ker\alpha = \ker\beta$;

(c) $\alpha\mathcal{D}\beta \Leftrightarrow |X\alpha| = |X\beta|$;
(d) $\mathcal{D} = \mathcal{J}$.

3. Let a and b be the elements of a semigroup S. Then $(a, b) \in \mathcal{L}^\dagger$ if and only if a and b are $\mathcal{L}$-equivalents in any semigroup of S. The relation $\mathcal{L}^\dagger$ is the generalization of Green's relation $\mathcal{L}$. Dually, we define the relation $\mathcal{R}^\dagger$. By $\mathcal{H}^\dagger$ we denote the intersection of relations $\mathcal{L}^\dagger$ and $\mathcal{R}^\dagger$. Prove the following conditions:

(a) $a\mathcal{L}^\dagger b \Leftrightarrow ((\forall x, y \in S^1)\, ax = ay \Leftrightarrow bx = by)$;
(b) $a\mathcal{R}^\dagger b \Leftrightarrow ((\forall x, y \in S^1)\, xa = ya \Leftrightarrow xb = yb)$;
(c) $\mathcal{L}^\dagger$ ($\mathcal{R}^\dagger$) is a right (left) congruence on S;
(d) $\mathcal{H}^\dagger$-class which contains an idempotent is a cancellative monoid.

4. If e and f are the idempotents of a semigroup S, then

$$e\mathcal{L}f \Leftrightarrow e = ef, f = fe \quad \text{and} \quad e\mathcal{R}f \Leftrightarrow e = fe, f = ef.$$

References

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Chapter 2

Regularity on Semigroups

The notion of the regularity in semigroups and rings was introduced by J. von Neumann, in 1936, who defined an element a of a semigroup (ring) S a being regular if the equation $a = axa$, with a variable x, has a solution in S. His work initiated an investigation of many other types of regularity.

R. Croisot, in 1953, stated a very interesting problem of the classification of all types of the regularity of semigroups defined by equations of the type $a = a^m x a^n$, with $m, n \geq 0$, $m+n \geq 2$. He proved that any of these equations determines either ordinary regularity, left, right or complete regularity (see also the book by A. H. Clifford and G. B. Preston, Section 4.1). A similar problem, concerning all types of the regularity of semigroups and their elements defined by equations of the type $a = a^p x a^q y a^r$, with $p, q, r \geq 0$, was treated by S. Lajos and G. Szász, 1975. S. Bogdanović, M. Ćirić, P. Stanimirović and T. Petković, 2004, determined all types of the regularity of elements defined by linear equations, and proved that there are exactly 14 types of the regularity of semigroups defined by such equations.

R. Arens and I. Kaplansky, in 1948, introduced the notion of π-regularity which is a generalization of regularity. π-regularity is in very close connection with the nil-extensions of semigroups, about which we will talk throughout this book. In particular, we will investigate completely π-regular semigroups which M. P. Drazin, in 1958, called pseudo-inverse semigroups, while L. N. Shevrin and his students, for a short time, called them epigroups. These semigroups we meet as eventually regular or quasi-periodic semigroups.

2.1 π-regular Semigroups

In this section we outline the general characterizations of π-regular semigroups. The set of all the regular elements of a semigroup S we denote by $\mathrm{Reg}(S)$ and we call it *the regular part* of S. A semigroup S is regular if $S = \mathrm{Reg}(S)$. We remind the reader that a semigroup S is called *π-regular* if for every $a \in S$ there exists $n \in \mathbf{Z}^+$ such that a^n is a regular element.

Lemma 2.1 *The following conditions for an element a of a semigroup S are equivalent:*

(i) *a is π-regular;*
(ii) *there exists $n \in \mathbf{Z}^+$ such that $R(a^n)$ $(L(a^n))$ has an idempotent as a generator;*
(iii) *there exists $n \in \mathbf{Z}^+$ such that $R(a^n)$ $(L(a^n))$ has a left (right) identity.*

Proof. (i)$\Rightarrow$(ii) Let a be a π-regular element, i.e. let there exists $n \in \mathbf{Z}^+$ and $x \in S$ such that $a^n = a^n x a^n$. Assume $e = a^n x$. Then $R(a^n) = R(e)$ and $e \in E(S)$, so (ii) holds.

(ii)$\Rightarrow$(i) If (ii) holds, then $R(a^n) = R(e)$ for some $n \in \mathbf{Z}^+$ and $e \in E(S)$, so there are $x, y \in S$ such that $a^n = ex$, $e = a^n y$ whence we have that

$$a^n = ex = e^2 x = ea^n = a^n y a^n.$$

Thus, a is π-regular.

(i)$\Rightarrow$(iii) Let $a^n = a^n x a^n$, for some $n \in \mathbf{Z}^+$ and $x \in S$ and let $e = a^n x$. Assume an arbitrary $b \in R(a^n)$. Then $b = a^n y$ for some $y \in S$, so

$$eb = a^n x b = a^n x a^n y = a^n y = b.$$

Therefore, e is a left identity of $R(a^n)$.

(iii)$\Rightarrow$(i) Let $n \in \mathbf{Z}^+$ such that $R(a^n)$ has a left identity e. Then $e = a^n x$, for some $x \in S^1$, so $a^n = ea^n = a^n x a^n$. Thus, a is π-regular. $\square$

Corollary 2.1 *The following conditions on a semigroup S are equivalent:*

(i) *S is a π-regular semigroup;*
(ii) *for every $a \in S$ there exists $n \in \mathbf{Z}^+$ and $e \in E(S)$ such that $R(a^n) = eS$ $(L(a^n) = Se)$;*

(iii) *for every $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $R(a^n)$ ($L(a^n)$) has a left (right) identity.*

Corollary 2.2 *An element a of a semigroup S is regular if and only if there is an idempotent $e \in E(S)$ such that $aS^1 = eS$.*

Theorem 2.1 *The following conditions on a semigroup S are equivalent:*

(i) *S is simple and π-regular;*
(ii) *S is simple and regular;*
(iii) $(\forall a, b \in S)$ *$a \in aSbSa$;*
(iv) *every bi-ideal of S is a simple semigroup.*

Proof. (i)⇒(ii) Suppose that S is π-regular and simple. Let $a \in S$. Then there exist $x, y \in S$ such that $a = xay = x^n ay^n$, for every $n \in \mathbf{Z}^+$. For some $n \in \mathbf{Z}^+$ and $v \in S$ we have $y^n = y^n vy^n$, and then $a = x^n ay^n vy^n = avy^n$, so based on the simplicity of S we obtain that $a \in aSa^2S$. From this it follows that $a = apa^2q$, for some $p, q \in S$, whence $a = (apa)^n aq^n$, for every $n \in \mathbf{Z}^+$. Since S is π-regular, then we have that $a = (apa)^n aq^n = (apa)^n u(apa)^n aq^n = (apa)^n ua$, for some $n \in \mathbf{Z}^+$ and $u \in S$. Therefore, $a \in aSa$ and we have proved that S is a regular semigroup.

(ii)⇒(i) This is obvious.

(ii)⇒(iii) Let $a, b \in S$. Then $a \in SbS$, and also, there exists $x \in S$ such that $a = axa$. But, then we have that $a = axaxa \in axSbSxa \subseteq aSbSa$.

(iii)⇒(ii) This is obvious.

(iii)⇒(iv) Let B be a bi-ideal of S and let $a, b \in B$. According to (iii) we have that $a \in aSb^3Sa$, which yields

$$a \in aSb^3Sa = (aSb)b(bSa) \subseteq (BSB)b(BSB) \subseteq BbB.$$

Thus, we have proved that B is a simple semigroup.

(iv)⇒(iii) Consider the arbitrary elements $a, b \in S$ and the principal bi-ideal $B = B(a) = \{a\} \cup \{a^2\} \cup aSa$. Based on the hypothesis, B is a simple semigroup, and $a, aba \in B$, so we have that

$$a \in BabaB \subseteq aS^1abaS^1a \subseteq aSbSa.$$

Therefore, (iii) holds. □

The element x of a semigroup S is the *inverse* of an element $a \in S$ if $a = axa$ and $x = xax$. The set of all the inverse elements of the element a we denote by $V(a)$. We mention that it must make a difference between the notion of the "inverse of an element a" and the "inverse of an element a in a subgroup - group inverse". A semigroup S is *inverse* if every one of its elements has an unique inverse element.

Lemma 2.2 *An element a of a semigroup S has an inverse element if and only if a is a regular element.*

Proof. Assume that a is a regular element. Then $a = axa$ for some $x \in S$, so the element $y = xax$ is an inverse of the element a.

The converse follows immediately. $\square$

Lemma 2.3 *Let ξ be a congruence relation on a π-regular semigroup S and let $A, B \in S/\xi$ such that $A = ABA$ and $B = BAB$ in S/ξ. Then there exists $a, b \in S$ such that $a \in A$, $b \in B$, and $a = aba$ and $b = bab$ in S.*

Proof. Let $x \in A$, $y \in B$. Also, let $n \in \mathbf{Z}^+$ such that $(xy)^{2n} \in Reg(S)$ and let z be the inverse element of $(xy)^{2n}$. If we assume that $a = xyz(xy)^{2n-1}x$, $b = yz(xy)^{2n-1}$ then we have $a = aba$ and $b = bab$. On the other hand, from $A = ABA$, $B = BAB$ in S/ξ we have that $x\xi xyx$, $y\xi yxy$, so

$$xy\xi(xy)^k, \quad \text{for every } k \in \mathbf{Z}^+.$$

Hence, it follows that

$$xyz\xi(xy)^{2n}z, \quad (xy)^{2n-1}x\xi(xy)^{2n}x,$$

and by Lemma 1.5 we have that

$$a = xyz(xy)^{2n-1}x\xi(xy)^{2n}z(xy)^{2n}x = (xy)^{2n}x\xi xyx\xi x.$$

Thus, $a \in A$. Similarly we prove that $b \in B$. $\square$

The following corollary is famous in the literature as the Lallement lemma. In the Section 6.6 we give some new generalizations on Lallement's lemma.

Corollary 2.3 *Let ξ be a congruence relation on a π-regular semigroup semigroup S. Then every ξ-class which is an idempotent in S/ξ, contains an idempotent from S.*

Proof. Let E be an arbitrary idempotent from S/ξ. Since $E = EEE$ in S/ξ then there exists $a, b \in E$ such that $a = aba$ and $b = bab$ (by Lemma 2.3). Now we have that $ab \in EE = E$ and ab is an idempotent in S. □

Theorem 2.2 *Let ξ be a congruence on a π-regular semigroup S, let $n \in \mathbf{Z}^+$ and let $A, B_1, B_2, \dots, B_n \in S/\xi$ such that $A = AB_iA$ and $B_i = B_iAB_i$, for all $i \in \{1, 2, \dots, n\}$. Then there exist $a, b_1, b_2, \dots, b_n \in S$ such that $a \in A$, $b_i \in B_i$ and $a = ab_ia$, $b_i = b_iab_i$, for all $i \in \{1, 2, \dots, n\}$.*

Proof. The theorem we will prove by induction. According to Lemma 2.3 the statement of the theorem is true for $n = 1$. Assume that the statement of theorem is true for some positive integer $k < n$. Then there are elements $x, y_1, y_2, \dots, y_k \in S$ such that $x \in A$, $y_i \in B_i$, $x = xy_ix$ and $y_i = y_ixy_i$ for $i \in \{1, 2, \dots, k\}$. Assume that the element $y_{k+1} \in B_{k+1}$. Since S is a π-regular then there exists $m \in \mathbf{Z}^+$ such that $(xy_{k+1})^{2m} \in \mathrm{Reg}(S)$. Let $z \in V((xy_{k+1})^{2m})$ and let

$$\begin{aligned}
&u = xy_{k+1}z(xy_{k+1})^{2m-1}x,\\
&v_{k+1} = y_{k+1}z(xy_{k+1})^{2m-1},\\
&v_i = y_ixy_{k+1}z(xy_{k+1})^{2m-1}xy_i, \quad \text{for } i \in \{1, 2, \dots, k\}.
\end{aligned}$$

It is easy to prove that $u \in A$, $v_i \in B_i$, $u = uv_iu$ and $v_i = v_iuv_i$, for all $i \in \{1, 2, \dots, k+1\}$. □

Exercises

1. A semigroup S is regular if and only if $L \cap R = RL$, for every left ideal L and every right ideal R of S.

2. Let S be a regular subsemigroup of a semigroup T. Then Green's relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ on S are restrictions of the corresponding relations on T.

3. The statement that a full semigroup of transformations $\mathcal{T}_r(X)$ is regular, for every set X, is equivalent to the axiom of choice.

4. A semigroup satisfies the conditions TC (term conditions) if

(C1) $xy = xz \;\Rightarrow\; uy = uz$;
(C2) $yx = zx \;\Rightarrow\; yu = zu$;
(C3) $y_1xy_2 = z_1xz_2 \;\Rightarrow\; y_1uy_2 = z_1uz_2$.

A semigroup S which satisfies the TC conditions we call a TC-*semigroup.*

Let G be a commutative group, I, Λ and Q be non-empty sets and ϕ, λ and β be mappings from G into I, Λ and Q, respectively. Then the set $S = Q \cup (G \times I \times \Lambda)$ with a multiplication defined by

$$\begin{aligned}
p * q &= ((p\phi)(q\phi); p\alpha, q\beta), & (a; i, \lambda) * (b; j, \mu) &= (ab; i, \mu),\\
p * (a; i, \lambda) &= ((p\phi)a; p\alpha, \lambda) & (a; i, \lambda) * p &= (a(p\phi); i, p\beta),
\end{aligned}$$

for $p, q \in Q$, $(a; i, \lambda), (b; j, \mu) \in G \times I \times \Lambda$, is a π-regular TC-semigroup.

Conversely, every π-regular TC-semigroup can be constructed in this way.

5. A semigroup S is a periodic TC-semigroup if and only if S is isomorphic to some semigroup constructed in Exercise 4., where G is a periodic group.

6. Let $\mathcal{I}(X)$ be the set of all injective partial mappings of a set X, including the empty relation. Prove that $\mathcal{I}(X)$ is an inverse subsemigroup of $\mathcal{B}(X)$.

A semigroup $\mathcal{I}(X)$ we call a *symmetric inverse semigroup* of the set X.

7. Every inverse semigroup can be embedded into some symmetric inverse semigroup.

8. A congruence ξ on a semigroup S *divides idempotents* if for all $e, f \in E(S)$, from $e\xi f$ it follows that $e = f$. On an arbitrary semigroup S we define a relation μ with

$$\begin{aligned}\mu = \{(a,b) \in S \times S \,|\, (\forall x \in \mathrm{Reg}(S))((x\mathcal{R}xa \vee x\mathcal{R}xb) &\Rightarrow xa\mathcal{H}xb \;\wedge \\ (x\mathcal{L}ax \vee x\mathcal{L}bx) &\Rightarrow ax\mathcal{H}bx)\}.\end{aligned}$$

Prove that μ is a congruence which divides idempotents. If S is a π-regular semigroup, then μ is the greatest congruence which divides idempotents.

9. The following conditions for the congruence μ, from Exercise 8., on a semigroup S are equivalent:

(a) $\xi \subseteq \mu$;
(b) $(\forall e \in E(S))(\forall b \in S)\ e\xi b \Rightarrow L(e) \subseteq L(b) \wedge R(e) \subseteq R(b)$;
(c) $(\forall a \in \mathrm{Reg}(S))(\forall b \in S)\ a\xi b \Rightarrow L(a) \subseteq L(b) \wedge R(a) \subseteq R(b)$.

If S is a π-regular semigroup, then every one of given conditions are equivalent with

(d) ξ divides idempotents.

References

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2.2 Completely π-regular Semigroups

As we know, an element a of a semigroup S is *completely regular* if there is $x \in S$ such that $a = axa$ and $ax = xa$. A semigroup S is *completely regular* if all its elements are completely regular.

The set of all the completely regular elements of a semigroup S we denote by $\mathrm{Gr}(S)$ and we call it the *group part* of S. This name is justified from the following lemma.

Lemma 2.4 *The following conditions for an element a of a semigroup S are equivalent:*

(i) *a is completely regular;*
(ii) *a has inverse which commutes with a;*
(iii) *$a \in a^2Sa^2$;*
(iv) *a is both right and left regular;*
(v) *a is contained in some subgroup of S.*

Proof. (i)$\Rightarrow$(ii) Assume $x \in S$ such that $a = axa$ and $ax = xa$. Then for $y = xax$ we have that $y \in V(a)$ and $ay = ya$.

(ii)$\Rightarrow$(iii)$\Rightarrow$(iv) This follows immediately.

(iv)$\Rightarrow$(v) Let $a \in a^2S \cap Sa^2$. Then we have $a = a^2x = ya^2$, for some $x, y \in S$, whence $ax = ya^2x = ya$. Let $e = ax = ya$. Since $e^2 = yaax = ya^2x = ya = e$, $e \in aS \cap Sa$, $ae = a(ax) = a^2x = a$, $ea = (ya)a = ya^2 = a$, then $a \in eS \cap Se$, so by Theorem 1.6 we have that $a \in G_e$.

(v)$\Rightarrow$(i) This follows immediately. $\square$

An element a of a semigroup S is *completely π-regular* if there exists $n \in \mathbf{Z}^+$ and $x \in S$ such that $a^n = a^nxa^n$ and $a^nx = xa^n$, i.e. if some power of the element a is completely regular. A semigroup S is *completely π-regular* if all its elements are completely π-regular.

An element a of a semigroup S is *pseudo inverse* if there exists $x \in S$ and $n \in \mathbf{Z}^+$ such that $a^n = a^{n+1}x$, $ax = xa$ and $x = x^2a$. In that case x is the *pseudo inverse* of a. A semigroup S is *pseudo inverse* if all its elements are pseudo inverse.

An element a of a semigroup S is *left* (*right*) *regular* if $a \in Sa^2$ ($a \in a^2S$). A semigroup S is *left* (*right*) *regular* if all its elements are left (right) regular.

The set of all left (right) regular elements of a semigroup S we denote by $\mathrm{LReg}(S)$ $(\mathrm{RReg}(S))$.

An element a of a semigroup S is *left* (*right*) *π-regular* if there is $n \in \mathbf{Z}^+$ such that $a^n \in Sa^{n+1}$ $(a^n \in a^{n+1}S)$. A semigroup S is *left* (*right*) *π-regular* if all its elements are left (right) π-regular.

Theorem 2.3 *The following conditions on a semigroup S are equivalent:*

(i) *S is completely π-regular;*

(ii) *for every element from S some of its power is in some subgroup of S;*

(iii) *for every $a \in S$ there exist $n \in \mathbf{Z}^+$ such that $a^n \in a^n S a^{n+1}$;*

(iii') *for every $a \in S$ there exist $n \in \mathbf{Z}^+$ such that $a^n \in a^{n+1} S a^n$;*

(iv) *S is π-regular and left π-regular;*

(v) *S is pseudo inverse.*

Proof. (i)$\Rightarrow$(ii)$\Rightarrow$(iii) This follows by Lemma 2.4.

(iii)$\Rightarrow$(iv) This is evident.

(iv)$\Rightarrow$(i) Let (iv) hold. Assume $a \in S$. Since a is left π-regular, then there exists $m \in \mathbf{Z}^+$ and $x \in S$ such that $a^m = xa^{m+1}$, whence

$$a^m = x^k a^{m+k}, \tag{1}$$

for every $k \in \mathbf{Z}^+$. Since a^m is π-regular, then there exists $p \in \mathbf{Z}^+$ and $y \in S$ such that $a^{mp} = a^{mp} y a^{mp}$. Then from (1) we have that $a^{mp} = a^{mp} y (x^{2mp} a^{m+2mp})^p \in a^{mp} S a^{mp}$, i.e.

$$a^n = a^n z a^{2n}, \tag{2}$$

for $n = mp$ and some $z \in S$. By (2) it is easy to prove that

$$a^n = a^n (za^n)^k a^{nk}, \tag{3}$$

for every $k \in \mathbf{Z}^+$. Since za^n is left π-regular, then there exists $q \in \mathbf{Z}^+$ and $u \in S$ such that $(za^n)^q = u(za^n)^{q+1}$. Then $(za^n)^q = u^2 (za^n)^{q+2}$, so by (3) we have

$$\begin{aligned} a^n &= a^n (za^n)^q a^{nq} = a^n u^2 (za^n)^{q+2} a^{nq} = a^n u^2 z a^n z [a^n (za^n)^q a^{nq}] \\ &= a^n u^2 z a^n z a^n = a^n u^2 (za^n)^2, \end{aligned}$$

whence it follows that

$$\begin{aligned} a^{2n} z a^n &= a^n (a^n z a^n) = a^n u^2 (za^n)^2 (a^n z a^n) = a^n u^2 z (a^n z a^{2n}) z a^n \\ &= a^n u^2 z a^n z a^n = a^n. \end{aligned}$$

From this and (2) using Lemma 2.4 we get that a is a completely π-regular element. Thus (i) holds.

(ii)$\Rightarrow$(v) Let a be an arbitrary element from S. Then $a^n \in G_e$ for some $e \in E(S)$ and $n \in \mathbf{Z}^+$. According to Lemma 1.8, $ae = ea \in G_e$ so there is $x \in G_e$ such that $xea = aex = e$. Since $x = xe = ex$ then $xa = ax = e$ and $x = xe = x^2a$. Finally, $a^n = a^n e = a^{n+1}x$. Thus a is pseudo inverse.

(v)$\Rightarrow$(iii) Let a be a pseudo inverse element of S. Then there are $x \in S$ and $n \in \mathbf{Z}^+$ such that

$$a^n = a^{n+1}x = a^{n+2}x^2 = \cdots = a^{3n}x^{2n} = a^n x^{2n} a^{2n} \in a^n S a^{n+1}.$$

□

Lemma 2.5 *Let S be a completely π-regular semigroup. If K is a subsemigroup of S and completely π-regular, then*

$$\mathrm{Gr}(K) = K \cap \mathrm{Gr}(S).$$

Proof. If g is a group element of a completely π-regular semigroup, then its group inverse belongs to the same maximal subgroup as g. □

Thus, that the pseudo inverse is unique proves the following lemma.

Lemma 2.6 *The element a of a semigroup S has at most one pseudo inverse. If x is a pseudo inverse of a then x commutes with every element from S which commutes with a.*

Proof. Let x and y be two pseudo inverses of the element a and let k and m be corresponding integers from the definition of pseudo inverse. Assume that $n = \max\{k, m\}$. Then

$$xa^{n+1} = a^n = a^{n+1}y, \quad x = x^2a, \quad y = ay^2.$$

Hence

$$\begin{aligned} x &= x^2a = x^3a^2 = \cdots = x^{n+1}a^n = x^{n+1}a^{n+1}y = xay = xaay^2 \\ &= xa^2y^2 = \cdots = xa^{n+1}y^{n+1} = a^n y^{n+1} = \cdots = y. \end{aligned}$$

Thus, a has at most one pseudo inverse x.

Now, assume $u \in S$ such that $au = ua$. Then $xa^n u = xua^n = xua^{n+1}x = xa^{n+1}ux = a^n ux$ whence we have $x^{n+1}a^n u = a^n u x^{n+1}$. Namely, since $x = x^{n+1}a^n$ then $xu = x^{n+1}a^n u = a^n u x^{n+1} = ux^{n+1}a^n = ux$. □

A pseudo inverse is a generalization of a group inverse. Using Lemma 1.8 and Theorem 2.3, pseudo inverses can be represented in another way. Namely, if x is pseudo invertible, or equivalently, a completely π-regular element of a semigroup S, then $x^n \in G_e$, for some $n \in \mathbf{Z}^+$ and $xe \in G_e$, and the pseudo inverse $\overline{x}$ of x is given by $\overline{x} = (xe)^{-1}$, i.e. $\overline{x}$ is the group inverse of the element xe in the group G_e. If x is an element of a completely π-regular semigroup S and $x^n \in G_e$, for some $n \in \mathbf{Z}^+$ and $e \in E(S)$, then x^0 denotes the identity of G_e, $x^0 = e$. A pseudo inverse is in fact Drazin's inverse.

An element a of a semigroup S is *intra regular* if $a \in Sa^2S$. The set of all intra regular elements of a semigroup S we denote by $\mathrm{Intra}(S)$ and we call it the *intra regular part* of S. A semigroup S is *intra regular* if all its elements are intra regular.

An element a of a semigroup S is *intra π-regular* if there is $n \in \mathbf{Z}^+$ such that $a^n \in Sa^{2n}S$, i.e. if some its power is intra regular. A semigroup S is *intra π-regular* if all its elements are intra π-regular.

Theorem 2.4 *A semigroup S is left π-regular if and only if it is intra π-regular and* $\mathrm{Intra}(S) = \mathrm{LReg}(S)$.

Proof. Let S be left π-regular. Clearly, S is intra π-regular and $\mathrm{LReg}(S) \subseteq \mathrm{Intra}(S)$. Assume $a \in \mathrm{Intra}(S)$. Then $a = xa^2y$, for some $x, y \in S$, whence $a = (xa)^n ay^n$, for each $n \in \mathbf{Z}^+$. Since S is left π-regular, then $(xa)^n = z(xa)^{2n}$, for some $n \in \mathbf{Z}^+$ and $z \in S$, whence

$$a = (xa)^n ay^n = z(xa)^{2n} ay^n = z(xa)^n a \in Sa^2.$$

Therefore, $a \in \mathrm{LReg}(S)$, so $\mathrm{Intra}(S) = \mathrm{LReg}(S)$.

The converse follows immediately. □

Lemma 2.7 *Let $\mathfrak{C}$ be one of the following classes of semigroups: regular, π-regular, intra regular, intra π-regular, completely regular, completely π-regular, left π-regular, right π-regular, and let ξ be a semilattice congruence on a semigroup S. Then S is from a class $\mathfrak{C}$ if and only if every ξ-class of S is from $\mathfrak{C}$.*

Proof. We will prove only for a class of π-regular semigroups, in the other cases the proofs are similar.

Let S be a π-regular semigroup, let A be an arbitrary ξ-class of S and let $a \in A$. Then there are $n \in \mathbf{Z}^+$ and $x \in S$ such that $a^n = a^n xa^n$

and $x = xa^nx$. Since $x\xi = (xa^nx)\xi = (x\xi)((a^n)\xi)(x\xi) = (x\xi)(a\xi) = ((a^n)\xi)(x\xi)((a^n)\xi) = (a^n)\xi = a\xi$, so $a \in A$. Thus, A is a π-regular semigroup.

The converse follows immediately. $\square$

Similarly we prove the following result.

Lemma 2.8 *Let $\mathfrak{C}$ be a class of completely regular semigroups or a class of a completely π-regular semigroups, and let ξ be a band congruence on a semigroup S. Then S is from a class $\mathfrak{C}$ if and only if every ξ-class of S is from $\mathfrak{C}$.*

Exercises

1. Let N be the set of all non-negative integers. Then $S = N \times N$ with a multiplication defined by

$$(m,n)(p,q) = (m - n + max\{n,p\}, q - p + max\{n,p\}), \quad (m,n),(p,q) \in S,$$

is a semigroup which we call a *bi-cyclic semigroup.* Prove that a bi-cyclic semigroup is simple and inverse, and it is not completely simple, i.e. it is not completely π-regular.

2. The following conditions on a semigroup S are equivalent:

(a) S is completely π-regular;
(b) S is left and right π-regular;
(c) every proper bi-ideal of S is π-regular.

3. Let S be a π-regular semigroup and $m \in \mathbf{Z}^+$. If every $\mathcal{D}$-class of S contains at most m $\mathcal{L}$-classes, then S is completely π-regular and for every $a \in S$, a^{mn} belongs to some subgroup of S, where $n \in \mathbf{Z}^+$ is the smallest number for which $a^n \in \mathrm{Reg}(S)$.

4. Every ideal of a π-regular (completely π-regular, regular, completely regular) semigroup is π-regular (completely π-regular, regular, completely regular).

5. Let S be a completely π-regular semigroup, and for $e \in E(S)$ let $T_e = \sqrt{G_e}$. Then G_e is an ideal of $\langle T_e \rangle$, $xe = ex$ for every $x \in \langle T_e \rangle$, and $M_e = \{u \in S \mid (\exists x \in \langle T_e \rangle) xu \in \langle T_e \rangle\} = \{u \in S \mid (\exists x \in G_e) xu \in G_e\}$ is a subsemigroups of S with the ideal G_e.

6. Let $e, f \in E(S)$ and $(ef)^n, (fe)^n \in G_g$, for some $n \in \mathbf{Z}^+$. Then $(ef)^n = (fe)^n = g$.

7. The following conditions on a semigroup S are equivalent:

(a) S is completely π-regular and $E(S) = \mathrm{Gr}(S)$;
(b) S is a union of nil-semigroups;
(c) $(\forall a \in S)(\exists n \in \mathbf{Z}^+)\ a^n = a^{n+1}$.

8. A semigroup S is inverse if and only if S is regular and its idempotents commute.

References

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2.3 The Union of Groups

An idempotent e of a semigroup S without zero is *primitive* if it is the minimal element with respect to the natural partial order $\leq$ on $E(S)$, i.e. if

$$f^2 = f = ef = fe \quad \Rightarrow \quad f = e.$$

A semigroup S is *completely simple* if S is simple and if contains a primitive idempotent.

The next result, is a known as *Munn's theorem* in the relevant literature.

Theorem 2.5 *Let S be a simple semigroup. Then S is completely simple if and only if S is a completely π-regular semigroup.*

Proof. Let S be a completely simple semigroup, let $a \in S$ be an arbitrary element and let $e \in E(S)$ be an primitive idempotent. Then $S = SeS = Sea^3eS$, because S is simple, so there are $u, v, x, y \in S$ such that $a = uev$ and $e = x(ea^3e)y$. Assume $f = evaeyexeaue$. Then

$$\begin{aligned} f^2 &= evaeyexeaueevaeyexeaue = evaeyexea(uev)aeyexeaue \\ &= evaeye(xea^3ey)exeaue = evaeyeeexeaue = f, \end{aligned}$$

and since $f \leq e$ then we have $f = e$. Thus

$$a = uev = ufev = (uev)(aeyexea)(uev) = a^2(eyexe)a^2 \in a^2Sa^2,$$

and by Lemma 2.4 a is a completely regular element.

Conversely, let S be completely π-regular and let $a \in S$. Since S is simple then $a = xay$ for some $x, y \in S$. It is clear that $a = x^r a y^r$, for every $r \in \mathbf{Z}^+$. Since S is completely π-regular then $x^s \in G_e$ for some $s \in \mathbf{Z}^+$ and $e \in E(S)$. We will prove that e is primitive. Assume that $ef = fe = f$. Since S is simple then $e = pfq$ for some $p, q \in S$. Let $h = epf$ and $k = fqe$. Then we have that $eh = h = hf = hfe = he$ and $ke = k = fk = efk = ek$. Also, $hk = epf^2qe = e^3 = e$, so

$$e = hk = hek = h(hk)k = h^2k^2 = h^3k^3 = \cdots = h^r k^r,$$

for every $r \in \mathbf{Z}^+$. Since S is completely π-regular then $h^n \in G_g$ for some $n \in \mathbf{Z}^+$ and $g \in E(S)$. Assume that $u = h^n$, $v = k^n$ and let w be the group inverse of u in G_g. Then

$$eu = u = ue,\ ev = v = ve,\ e = uv = u^2v^2,\ gu = u = ug,\ wu = g = uw,$$

whence we have that $gv^2u^2 = w^2u^2v^2u^2 = w^2eu^2 = w^2u^2 = g$ so

$$e = uv = ugv = ugv^2u^2v = (ugv)(vu)(uv) = e(vu)e = vu.$$

On the other hand, $fv = fk^n = k^n = v$ because $fk = k$. Thus, $f = fe = fvu = vu = e$. □

Corollary 2.4 *A semigroup S is completely simple if and only if S is simple and a completely regular semigroup.*

The following theorem offers the structural characterization of intra regular semigroups.

Theorem 2.6 *A semigroup S is intra regular if and only if S is a semilattice of a simple semigroup.*

Proof. Let S be an intra regular semigroup. Assume $a \in S$. Then $a = xa^2y$ for some $x, y \in S$, so $J(a) \subseteq J(a^2)$. Since the opposite inclusion always holds we have that $J(a) = J(a^2)$ for every $a \in S$.

Assume $a, b \in S$. Then, based on the previous it follows that $J(ab) = J(abab) \subseteq J(ba)$ and $J(ba) \subseteq J(ab)$. Thus $J(ab) = J(ba)$ for every $a, b \in S$.

Assume $a, b \in S$ such that $J(a) = J(b)$ and assume $x \in S$. Then $a = ubv$ for some $u, v \in S$ so

$$J(ax) = J(ubvx) \subseteq J(bvx) = J(bvxbvx) \subseteq J(xb) = J(bx).$$

Similarly we prove that $J(bx) \subseteq J(ax)$ whence $J(ax) = J(bx)$ and $J(xa) = J(xb)$. Thus $\mathcal{J}$ is a semilattice congruence on S.

It is evident that J_a is a subsemigroup of S, for all $a \in S$. Assume $a \in S$ and $x, y \in J_a$. Then $J(y) = J(x) = J(x^3)$ so we have $y = ux^3v = (ux)x(xv)$ for some $u, v \in S^1$. Since

$$J_a = J_y = J_{ux}J_xJ_{xv} = J_{ux}J_aJ_{xv}$$

is in $S/\mathcal{J}$, then we have that

$$J_a = J_{ux}J_a = J_uJ_xJ_a = J_uJ_x = J_{ux},$$

and similarly $J_{xv} = J_a$. Thus, $y \in J_axJ_a$, so J_a is a simple semigroup. Therefore, S is a semilattice of simple semigroups.

The converse follows based on the fact that every simple semigroup is intra regular and by Lemma 2.7. □

A semigroup S is a *union of groups* if S can be represented as a union of its maximal subgroups. According to Theorem 1.7 this union is disjoint.

Theorem 2.7 *The following conditions on a semigroup S are equivalent:*

(i) *S is completely regular;*

(ii) *S is a union of groups;*

(iii) *S is a semilattice of completely simple semigroups;*

(iv) *$(\forall a \in S)$ $a \in aSa^2$;*

(iv') *$(\forall a \in S)$ $a \in a^2Sa$.*

Proof. (i)⇒(ii) and (ii)⇒(iv) This follows from Lemma 2.4.

(iv)⇒(iii) Let $a = axa^2$ for some $x \in S$. Then

$$a = axa^2 = (ax)aa = (ax)(axa^2)a \in Sa^2S,$$

so S is intra regular. According to Theorem 2.6 S is a semilatice of simple semigroups, and now by Theorem 2.3, Lemma 2.7 and Theorem 2.5, S is a semilattice of completely simple semigroups.

(iii)⇒(i) This follows from Corollary 2.4 and Lemma 2.7. □

The condition (iv) from the previous theorem can be replaced with: S is a regular and a left (right) regular semigroup.

Exercises

1. A semigroup S is intra regular if and only if $R(\mathcal{J}) = \mathcal{J}$.

2. The following conditions on a semigroup S are equivalent:

(a) S is a union of groups;
(b) $R(\mathcal{L}) = \mathcal{L}, \quad R(\mathcal{R}) = \mathcal{R}$;
(c) $R(\mathcal{H}) = \mathcal{H}$.

3. A semigroup S is a semilattice of groups if and only if $R(\mathcal{L}) = \mathcal{R}$.

4. The following conditions on a semigroup S are equivalent:

(a) S is a union of groups;
(b) S is left and right regular;
(c) S is regular and left (right) regular;
(d) every $\mathcal{H}$-class of S is a group.

References

O. Anderson [1]; S. Bogdanović, M. Ćirić and A. Stamenković [1]; A. H. Clifford [1]; A. H. Clifford and G. B. Preston [1]; R. Croisot [1]; P. Kržovski [1]; W. D. Munn [3]; M. Petrich [6], [8].

2.4 π-inverse Semigroups

A semigroup S is *right* (*left*) *π-inverse* if S is π-regular and if for all $a, x, y \in S$ the following implication holds

$$a = axa = aya \quad \Rightarrow \quad xa = ya \quad (ax = ay).$$

Theorem 2.8 *The following conditions on a semigroup S are equivalent:*

(i) *S is right π-inverse;*
(ii) *S is π-regular and for all $e, f \in E(S)$ there exists $n \in \mathbf{Z}^+$ such that $(ef)^n = (fef)^n$;*
(iii) *for every $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $L(a^n)$ has a unique idempotent as a generator;*

(iv) *for every* $a \in S$ *there exists* $n \in \mathbf{Z}^+$ *such that* $L(a^n)$ *has a unique right identity;*

(v) S *is* π*-regular and for every* $e, f \in E(S)$ *there exists* $n \in \mathbf{Z}^+$ *such that* $(ef)^n \mathcal{R} (fe)^n$.

Proof. (i)⇒(ii) Let $e, f \in E(S)$ and let a be an inverse element of the element $(ef)^n$, for some $n \in \mathbf{Z}^+$. Then

$$(ef)^n = (ef)^n a (ef)^n = (ef)^n fa(ef)^n,$$

and by supposition we have that $a(ef)^n = fa(ef)^n$, so $a(ef)^n a = fa(ef)^n a$. Thus

$$a = fa. \tag{4}$$

Now we have

$$a = a(ef)^n a = a(efe)^{n-1} fa = a(efe)^{n-1} a. \tag{5}$$

Hence, based on (4)

$$\begin{aligned} a(efe)^{n-1} &= a(efe)^{n-1} a(efe)^{n-1} a(efe)^{n-1} \\ &= a(efe)^{n-1} efa(efe)^{n-1} a(efe)^{n-1} \end{aligned}$$

so by supposition we have that

$$a(efe)^{n-1} a(efe)^{n-1} = efa(efe)^{n-1} a(efe)^{n-1},$$

i.e.

$$a(efe)^{n-1} = efa(efe)^{n-1}.$$

Hence and according to (5) it follows that $a = efa$ and from (4) we get

$$a = ea. \tag{6}$$

Using (4) and (6) we have that

$$\begin{aligned} (ef)^n &= (ef)^n a (ef)^n = (ef)^n ea (ef)^n = (ef)^n efa (ef)^n \\ &= ef(ef)^n a (ef)^n = ef(ef)^n = (ef)^{n+1}. \end{aligned}$$

Now, we have $(ef)^n = (ef)^n ef (ef)^n = (ef)^n f (ef)^n$, so $ef(ef)^n = f(ef)^n$. Thus, $(ef)^n = (fef)^n$, i.e. (ii) holds.

(ii)⇒(i) If $a = axa = aya$ then $(xaya)^n = (yaxaya)^n$ for some $n \in \mathbf{Z}^+$, so $xa = ya$. Thus, S is a right π-inverse.

(i)$\Rightarrow$(iii) Let $a^n = a^n x a^n$ for some $n \in \mathbf{Z}^+$ and $x \in S$. Then by Lemma 2.1, $L(a^n)$ has an idempotent e as a generator. Assume $f \in E(S)$ such that $L(a^n) = Sf$. Then $Se = Sf$, so $e = yf, f = xe$ for some $x, y \in S$. Now we have $ef = (yf)f = yf = e$, $fe = f$ whence $e = efe = e(efe)e$. From this, by supposition, we have that $fe = efee = efe$. Thus, $f = fe = efe = e$. Therefore, $L(a^n)$ has a unique idempotent as a generator.

(iii)$\Rightarrow$(iv) Let $L(a^n)$ have a unique idempotent e as a generator. Then by Lemma 2.1 $L(a^n)$ has a unique right identity.

(iv)$\Rightarrow$(i) Let $L(a^n)$ have a unique right identity. According to Lemma 2.1 a is π-regular. Assume $a = axa = aya$. Then since the identity is unique we have $xa = ya$. Thus, S is right π-regular.

(ii)$\Rightarrow$(v) For an arbitrary $e, f \in E(S)$ there exists $m, n \in \mathbf{Z}^+$ such that $(efe)^m = (fe)^m$ and $(fef)^n = (ef)^n$. Hence

$$(ef)^{mn}e = (fe)^{mn} \quad \text{and} \quad (fe)^{mn}f = (ef)^{mn}.$$

Thus $(ef)^k \mathcal{R} (fe)^k$ for $k = mn$.

(v)$\Rightarrow$(ii) For $e, f \in E(S)$ let $(fe)^n \mathcal{R} (ef)^n$ for some $n \in \mathbf{Z}^+$. Then $(fe)^n u = (ef)^n$ for some $u \in S$, so $f(ef)^n = f(fe)^n u = (fe)^n u = (ef)^n$, i.e. $(fef)^n = (ef)^n$. $\square$

A semigroup S is *right* (*left*) *completely* π-*inverse* if S is completely π-regular and for all $a, x, y \in S$, $a = axa = aya$ implies $xa = ya$ ($ax = ay$), i.e. if S is completely π-regular and right (left) π-inverse.

Theorem 2.9 *The following conditions on a semigroup S are equivalent:*

(i) *S is right completely π-inverse;*
(ii) *S is π-regular and for all $a \in S$, $f \in E(S)$ there exists $n \in \mathbf{Z}^+$ such that $(af)^n = (faf)^n$;*
(iii) *S is π-regular and for all $a \in \mathrm{Reg}(S)$, $f \in E(S)$ there exists $n \in \mathbf{Z}^+$ such that $(af)^n = (faf)^n$.*

Proof. (i)$\Rightarrow$(ii) Assume $a \in S$ and $f \in E(S)$. According to Theorem 2.3 there exist $k, m \in \mathbf{Z}^+$ such that $(af)^k \in G_g$ and $(faf)^m \in G_h$ for some $g, h \in E(S)$. According to Lemma 1.8 there is $n \in \mathbf{Z}^+$ such that $(af)^n \in G_g$ and $(faf)^n \in G_h$. Now we have

$$g = ((af)^n)^{-1}(af)^n = ((af)^n)^{-1}(af)^n f = gf.$$

Similarly, we prove that $h = hf = fh$. Since $f(af)^r = (af)^r f = (faf)^r$, for all $r \in \mathbf{Z}^+$, then $f(af)^n = (faf)^n = h(faf)^n = hf(af)^n = h(af)^n$. Thus

$$f(af)^n((af)^n)^{-1} = h(af)^n((af)^n)^{-1},$$

i.e. $fg = hg$, whence $g(fg) = g(hg)$. Since $gf = g$ then $g = ghg = g^2$ and since S is right π-inverse then $hg = g$. Thus, the following holds

$$fg = hg = g. \tag{7}$$

Also, we have

$$\begin{aligned} h = hf &= ((faf)^n)^{-1}(faf)^n f = ((faf)^n)^{-1} f(af)^n f \\ &= ((faf)^n)^{-1} f(af)^n gf = ((faf)^n)^{-1}(faf)^n gf = hgf = hg. \end{aligned}$$

Hence, using (7) it follows that $g = h$. Thus, the elements $(af)^n$ and $(faf)^n$ belong to the same subgroup G_g of S and since $gf = g$ then

$$(faf)^n = g(faf)^n = gf(af)^n = g(af)^n = (af)^n.$$

(ii)$\Rightarrow$(iii) This follows immediately.

(iii)$\Rightarrow$(i) We will prove that S is completely π-regular. Let $a = axa$ for some $x \in S$. Then based on the hypothesis of the theorem there is $r \in \mathbf{Z}^+$ such that

$$a^r = (a(xa))^r = ((xa)a)^r = (xa^2)^r = xa^{r+1}.$$

Thus, every regular element from S is left π-regular. Since S is π-regular then for every $a \in S$ there exists $m \in \mathbf{Z}^+$ such that $a^m \in \mathrm{Reg}(S)$. From this it follows that there are $r \in \mathbf{Z}^+$ and $x \in S$ such that $(a^m)^r = x(a^m)^{r+1}$, i.e. $a^{mr} \in Sa^{mr+1}$. Thus, S is π-regular and left π-regular, so by Theorem 2.3, S is a completely π-regular semigroup. Based on Theorem 2.8, S is a right π-inverse. $\square$

A semigroup S is a *π-inverse* if S is π-regular and for every $a \in \mathrm{Reg}(S)$ there is a unique $x \in S$ such that $a = axa$ and $x = xax$, i.e. if every regular element has a unique inverse.

Theorem 2.10 *The following conditions on a semigroup S are equivalent:*

(i) *S is π-inverse;*

(ii) *S is π-regular and for all $e, f \in E(S)$ there exists $n \in \mathbf{Z}^+$ such that $(ef)^n = (fe)^n$;*

(iii) *S is left and right π-inverse.*

Proof. (i)$\Rightarrow$(ii) For an arbitrary $e, f \in E(S)$ there exists $z \in S$ and $k \in \mathbf{Z}^+$ such that $(ef)^k = (ef)^k z (ef)^k$ and $z = z(ef)^k z$. Hence $(ef)^k = (ef)^k ze(ef)^k$ and $ze = ze(ef)^k ze$. Now, since z is unique, we have that $z = ze$, and similarly $z = fz$. There are two cases.

Assume that $k > 1$. Then

$$z = z(ef)^k z = ze(fe)^{k-1} fz = z(fe)^{k-1} z,$$

and if $t = (fe)^{k-1} z (fe)^{k-1}$ then we have $ztz = z$ and $tzt = t$. From this, based on uniqueness we have that $(ef)^k = t = (fe)^{k-1} z (fe)^{k-1}$, so

$$(ef)^k e = (fe)^{k-1} z (fe)^{k-1} e = (ef)^k.$$

Now, $(ef)^k ef = (ef)^k f$, i.e. $(ef)^{k+1} = (ef)^k \in E(S)$, and based on uniqueness we have that $z = (ef)^k$.

If $k = 1$ then

$$z^2 = zz = (ze)(fz) = z(ef)z = z,$$

i.e. $z \in E(S)$. Hence, based on uniqueness $z = ef$.

Thus in both cases we have that

$$z = (ef)^k = (ef)^{k+1}.$$

Since $z = ze = fz$ we have that $(ef)^k = z = fze = f(ef)^k e = (fe)^{k+1}$. Therefore, for $n \geq k+1$ is $(ef)^n = (fe)^n$.

(ii)$\Rightarrow$(iii) This follows from Theorem 2.8 and its dual.

(iii)$\Rightarrow$(i) Assume that $a \in \mathrm{Reg}(S)$ has two inverse elements b and c. Then

$$abS = aS = acS \quad \text{and} \quad Sbc = Sa = Sca.$$

According to Theorem 2.8 and its dual, $L(a)$ and $R(a)$ have a unique idempotent as a generator, so $ab = ac$ and $ba = ca$, whence $b = bab = bac = cac = c$ $\square$

A semigroup S is *completely π-inverse* if S is completely π-regular and a π-inverse semigroup. Based on Theorems 2.9 and 2.10 we immediately have the following result.

Theorem 2.11 *The following conditions on a semigroup S are equivalent:*

(i) *S is completely π-inverse;*
(ii) *S is π-regular and for all $a \in S$, $f \in E(S)$ there exists $n \in \mathbf{Z}^+$ such that $(af)^n = (fa)^n$;*
(iii) *S is π-regular and for all $a \in \mathrm{Reg}(S)$, $f \in E(S)$ there exists $n \in \mathbf{Z}^+$ such that $(af)^n = (fa)^n$.*

A semigroup S is *strongly π-inverse* if S is π-regular and if its idempotents commute each other.

Theorem 2.12 *The following conditions on a semigroup S are equivalent:*

(i) *S is strongly π-inverse;*
(ii) *S is π-regular and $\mathrm{Reg}(S)$ is an inverse subsemigroup of S;*
(iii) *S is π-inverse and the product of every two idempotents from S is also an idempotent.*

Proof. (i)$\Rightarrow$(ii) Let $a, b \in \mathrm{Reg}(S)$. Then there are $x, y \in S$ such that $a = axa$ and $b = byb$. Now we have

$$ab = (axa)(byb) = a(xa)(by)b = a(by)(xa)b = (ab)(yx)(ab).$$

Thus $\mathrm{Reg}^2(S) = \mathrm{Reg}(S)$. Let $a \in \mathrm{Reg}(S)$ and $x, y \in V(a)$. Since idempotents from $\mathrm{Reg}(S)$ commute then we have

$$x = xax = x(aya)x = x(ay)(ax) = x(ax)(ay) = xay.$$

Similarly, we have $x = yax$. So, it follows that

$$x = xax = (yax)a(xay) = y(axaxa)y = yay = y.$$

Thus, $\mathrm{Reg}(S)$ is an inverse semigroup.

(ii)$\Rightarrow$(i) Assume $e, f \in E(S) \subseteq \mathrm{Reg}(S)$ then $ef \in \mathrm{Reg}(S)$. Let $x \in V(ef)$ then

$$x = x(ef)x \quad \text{and} \quad (ef) = (ef)x(ef). \tag{8}$$

From this we have

$$fxe = f(xefx)e = (fxe)(ef)(fxe) \quad \text{and} \quad (ef) = (ef)(fxe)(ef),$$

i.e. $fxe \in V(ef)$. Since in $\mathrm{Reg}(S)$ the inverse element is unique then $x = fxe$. Now, we have $x^2 = (fxe)(fxe) = f(xefx)e = fxe = x$, whence $x \in E(S) \subseteq \mathrm{Reg}(S)$. So, for $x \in E(S)$ and based on (8) it follows that $x \in V(x)$ and $ef \in V(x)$, and since an inverse is unique we have that $x = ef \in E(S)$ and $ef \in V(ef)$. Similarly, we prove that $fe \in E(S)$. For this element the following also holds

$$\begin{aligned} ef &= (ef)^2 = (ef)(ef) = (ef)(fe)(ef), \\ fe &= (fe)^2 = (fe)(fe) = (fe)(ef)(fe), \end{aligned}$$

i.e. $fe \in V(ef)$. Thus, $ef \in V(ef)$ and $fe \in V(ef)$ and since the inverse is unique then we have $ef = fe$. Therefore, S is π-regular and its idempotents commute, so S is strongly π-regular.

(i)$\Rightarrow$(iii) This follows from Theorem 2.10.

(iii)$\Rightarrow$(i) Let (iii) hold, then S is π-regular and for all $e, f \in E(S)$ there exists $n \in \mathbf{Z}^+$ such that $(ef)^n = (fe)^n$. From this, for $n = 1$ we have $ef = fe$, for all $e, f \in E(S)$. Thus, S is strongly π-inverse. $\square$

A semigroup S is *Clifford's semigroup* if S is regular and $E(S) \subseteq C(S)$. It is evident that every Clifford's semigroup is inverse and completely regular. The following concept is more general: element b of a semigroup S is the *σ-inverse* of an element $a \in S$ if $a = aba$ and $b = bab$ and there is $n \in \mathbf{Z}^+$ such that $a^n b = ba^n$. A semigroup S is a *σ-inverse* if all its elements have a unique σ-inverse.

Theorem 2.13 *The following conditions on a semigroup S are equivalent:*

(i) *S is σ-inverse;*
(ii) *S is inverse and completely π-regular;*
(iii) *S is regular and for all $a \in S$, $e \in E(S)$ there exists $n \in \mathbf{Z}^+$ such that $(ae)^n = (ea)^n$.*

Proof. (i)$\Rightarrow$(ii) Let (i) hold, then S is inverse and regular. Assume $a \in S$ then there exists a unique $b \in S$ and $n \in \mathbf{Z}^+$ such that $a = aba$, $b = bab$ and $a^n b = ba^n$. From this we have $a^n = aba^n = a^{n+1}b = ba^{n+1}$ whence S is left π-regular. Since S is regular and left π-regular then by Theorem 2.3, S is completely π-regular.

(ii)$\Rightarrow$(i) Assume $a \in S$. Then there exists $x \in S$ such that $a = axa$ and $x = xax$. Also $ax, xa \in E(S)$. According to Theorem 2.11 there exist

$n, m \in \mathbf{Z}^+$ such that

$$(aax)^n = (axa)^n = a^n \quad \text{and} \quad (xaa)^m = (axa)^m = a^m.$$

Then $(a^2x)^t = a^t = (xa^2)^t$ for $t = nm$, so we have that

$$(a^2x)^t = a(axa)^{t-1}ax = a(axa)^{t-1}axax = a(axa)^t x = aa^t x = a^{t+1}x.$$

Similarly, we prove $(xa^2)^t = xa^{t+1}$. Thus $a^{t+1}x = xa^{t+1}$. Therefore, S is a σ-inverse semigroup.

(ii)⇔(iii) This follows from Theorem 2.11. □

Recall that a subsemigroup B of a semigroup S is a *bi-ideal* of S if $BSB \subseteq B$. For $a \in S$, $B(a) = \{a\} \cup \{a^2\} \cup aSa$ is the smallest bi-ideal containing a, and it is called the *principal bi-ideal* of S generated by a.

Recall also that a semigroup S is called *globally idempotent* if $S^2 = S$ (i.e. every element of S is decomposable).

Exercises

1. If for every $a \in S$ there exists $m \in \mathbf{Z}^+$ such that $L(a^m)$ has an identity, then S is a completely π-regular and right π-inverse semigroup.

2. A semigroup S is π-inverse if and only if S is π-regular and from $a = axa = aya$ it follows that $xax = yay$.

3. If S is a π-inverse semigroup, then for all $e, f \in E(S)$ there exists $n \in \mathbf{Z}^+$ such that $(ef)^n \in E(S)$.

References

S. A. Duply [1]; G. Thierrin [3], [4]; Y. Wang and Y. F. Luo [1]; J. P. Yu, Y. Sun and S. H. Li [1].

2.5 Quasi-regular Semigroups

The following lemma establishes an interesting connection between intra quasi-regular and intra regular, left quasi-regular and left regular, and right quasi-regular and right regular elements.

An element a of a semigroup S is *intra quasi-regular* if $a = xayaz$, for some $x, y, z \in S$. A semigroup S is *intra quasi-regular* if all its elements are intra quasi-regular.

An element a of a semigroup S is *left* (*right*) *quasi-regular* if $a = xaya$ ($a = axay$), for some $x, y \in S$. A semigroup S is *left* (*right*) *quasi-regular* if all its elements are left (right) quasi-regular.

Lemma 2.9 *The following conditions on a semigroup S are true:*

(a) *S has an intra quasi-regular element if and only if it has an intra regular element;*

(b) *S has a left quasi-regular element if and only if it has a left regular element;*

(c) *S has a right quasi-regular element if and only if it has a right regular element.*

Proof. (a) Let a be an intra quasi-regular element of S, i.e. $a = xayaz$, for some $x, y, z \in S$. Then

$$\begin{aligned} yaz &= y(xayaz)z = (yx)a(yaz^2) = (yx)(xayaz)(yaz^2) \\ &= (yx^2a)(yaz)^2z \in S(yaz)^2S, \end{aligned}$$

so we have that yaz is an intra regular element of S. The converse is clear.

Further, let a be a left quasi-regular element of S, i.e. $a = xaya$, for some $x, y \in S$. Then

$$\begin{aligned} ya &= y(xaya) = (yx)a(ya) = (yx)(xaya)ya) \\ &= (yx^2a)(ya)^2 \in S(ya)^2, \end{aligned}$$

so ya is a left regular element of S. The converse is evident.

The assertions (b) and (c) can be proved similarly. □

It is well-known that an element a of a semigroup S is regular if and only if the principal left ideal $L(a)$ (or the principal right ideal $R(a)$) has an idempotent generator. In a similar way we characterize the left, right and intra quasi-regular elements.

Theorem 2.14 *Let a be any element of a semigroup S. Then the following assertions are true:*

(a) *a is intra quasi-regular if and only if the principal ideal $J(a)$ of S has an intra regular generator;*

(b) *a is left quasi-regular if and only if the principal left ideal $L(a)$ of S has a left regular generator;*

(c) *a is right quasi-regular if and only if the principal right ideal $R(a)$ of S has a right regular generator.*

Proof. (a) Let a be an intra quasi-regular element. Then $a = xayaz$, for some $x, y, z \in S$, so $J(a) = J(yaz)$. According to Lemma 2.9 it follows that yaz is an intra regular element, so we have proved that $J(a)$ is generated by an intra regular element.

Conversely, let $J(a)$ be generated by an intra regular element b. Then $J(a) = J(b)$ and $b = pb^2q$, for some $p, q \in S$, from which it follows that $a \in J(b) = J(pb^2q) =\subseteq Sb^2S$. On the other hand, from $b \in J(a)$ it follows that $b^2 \in SaSaS$. Therefore, $a \in SaSaS$, which has to be proved.

The assertions (b) and (c) can be proved similarly. □

By LQReg(S), IQReg(S) and IReg(S) we denote respectively the sets of all the left quasi-regular, intra quasi-regular and intra regular elements of a semigroup S.

Theorem 2.15 *A semigroup S is left quasi-π-regular if and only if it is intra quasi-π-regular and* $\mathrm{IQReg}(S) = \mathrm{LQReg}(S)$.

Proof. Let S be left quasi-π-regular. Then it is also intra quasi-π-regular and $\mathrm{LQReg}(S) \subseteq \mathrm{IQReg}(S)$. To prove the opposite inclusion, consider an arbitrary $a \in \mathrm{IQReg}(S)$. Then $a = xayaz$ for some $x, y, z \in S$, so $a = (xay)^n az^n$, for every $n \in \mathbf{Z}^+$. On the other hand, since S is left quasi-π-regular, then there exists $n \in \mathbf{Z}^+$ and $p, q \in \mathrm{LQReg}(S)$ such that $(xay)^n = p(xay)^n q(xay)^n$. Now

$$a = (xay)^n az^n = p(xay)^n q(xay)^n az^n = p(xay)^n qa \in SaSa,$$

so $a \in \mathrm{LQReg}(S)$. Thus, $\mathrm{LQReg}(S) = \mathrm{IQReg}(S)$, which has to be proved.

The converse is obvious. □

References

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2.6 Idempotent-Generated Semigroups

In this section we give some properties of semigroups and subsemigroups generated by idempotent elements. These results will be useful in the further discussion. We remind the reader that by $\langle E(S)\rangle$ we denote the idempotent-generated subsemigroup of a semigroup S. This subsemigroup is the *core* of S. Also, based on $V(E^n)$, $n \in \mathbf{Z}^+$, we denote the set $\{V(a) \mid a \in E^n\}$, where $E = E(S)$ and $V(a)$ is the set of all the inverse elements of the element $a \in S$.

Theorem 2.16 *Let $E(S) \neq \emptyset$, then the following conditions are equivalent on a semigroup S:*

(i) $\mathrm{Reg}(S)$ *is a subsemigroup of S;*

(ii) $\langle E(S)\rangle$ *is a regular subsemigroup of S;*

(iii) $V(E) = E^2$*;*

(iv) $V(E^n) = E^{n+1}$*, for every $n \in \mathbf{Z}^+$.*

Proof. (i)⇒(ii) Let $a = e_1e_2\dots e_n \in \langle E(S)\rangle$, $e_i \in E(S)$, $i = 1, 2, \dots, n$ and let b be an inverse of a in $\mathrm{Reg}(S)$. If $n = 1$, then $b = bab = ba^2b = (ba)(ab) \in \langle E(S)\rangle$. Let $n > 1$. For every $i = 1, 2, \dots, n$ assume that

$$t_i = e_1e_2\cdots e_i, \quad u_i = e_ie_{i+1}\cdots e_n, \quad f_i = u_ibt_{i-1},\ i > 1.$$

Then $t_iu_i = a = t_n = u_1$ and $t_{i-1}u_i = a$, $f_i^2 = u_ibabt_{i-1} = f_i$. Thus

$$\begin{aligned} b &= b(ab)^n = b(t_nu_nb)(t_{n-1}u_{n-1}b)\cdots(t_2u_2b)(t_1u_1b)\\ &= (bt_n)(u_nbt_{n-1})\cdots(u_2bt_1)(u_1b)\\ &= (ba)f_n\cdots f_2(ab) \in E^{n+1}(S) \subseteq \langle E(S)\rangle. \end{aligned}$$

Hence, $\langle E(S)\rangle$ is regular.

(ii)⇒(i) Assume $a, b \in \mathrm{Reg}(S)$. Then $a = axa$ and $b = byb$, for some $x, y \in S$. Based on the hypothesis there is a $z \in \langle E(S)\rangle$ such that $(xa)(by) = (xa)(by)z(xa)(by)$. Thus

$$\begin{aligned} ab &= axabyb = a(xabyzxaby)b = (axa)(byzxa)(byb)\\ &= abyzxab \in abSab. \end{aligned}$$

Hence, $ab \in \mathrm{Reg}(S)$, i.e. $\mathrm{Reg}(S)$ is a subsemigroup of S.

(i)⇒(iv) This follows from the proof of (i)⇒(ii).

(iv)⇒(iii) This is evident for $n = 1$.

(iii)⇒(i) This is like (ii)⇒(i). □

Lemma 2.10 *If S is a completely simple semigroup, then $\langle E(S)\rangle$ is completely simple.*

Proof. According to Theorem 2.16, $\langle E(S)\rangle$ is a regular semigroup and since its idempotents are primitive, because it is primitive in S, then $\langle E(S)\rangle$ is a completely simple semigroup. $\square$

Lemma 2.11 *If a semigroup S is (completely) π-regular, then $\langle E(S)\rangle$ is (completely) π-regular.*

Proof. If $x \in \langle E(S)\rangle$ and x^n, $n \in \mathbf{Z}^+$, is regular in S, then by Theorem 2.3 x^n, $n \in \mathbf{Z}^+$, is regular in $\langle E(S)\rangle$. If $x^n \mathcal{H}^S e$, where $\mathcal{H}^S$ is Green's relation with respect to S, then the inverse of x^n contained in the $\mathcal{H}^S$-class of e is contained in $\langle E(S)\rangle$, so that also $x^n \mathcal{H}^{\langle E(S)\rangle} e$. Thus the lemma follows. $\square$

Theorem 2.17 *For a π-inverse semigroup S, $\langle E(S)\rangle$ is a periodic semigroup.*

Proof. Let $e_1, e_2, \dots e_n \in E(S)$, $i = 1, 2, \dots, n$ with $n \in \mathbf{Z}^+$. Since S is a π-inverse semigroup, there exists $m \in \mathbf{Z}^+$ and a unique $x \in S$ such that

$$x = x(e_1e_2\dots e_n)^m x, \quad (e_1e_2\dots e_n)^m = (e_1e_2\dots e_n)^m x (e_1e_2\dots e_n)^m.$$

Clearly,

$$xe_1(e_1e_2\dots e_n)^m xe_1 = xe_1,$$

$$(e_1e_2\dots e_n)^m xe_1(e_1e_2\dots e_n)^m = (e_1e_2\dots e_n)^m.$$

Based on the definition of a π-inverse semigroup, we have $xe_1 = x$. Symmetrically, we have $e_n x = x$.

If $m = 1$, then $x = xe_1e_2\dots e_n x = xe_2\dots e_n x$. Let $y = e_2\dots e_n xe_2\dots e_n$. Then $xyx = x$, $yxy = y$. Based on the uniqueness of inverses of x, we have

$$y = e_2e_3\dots e_n xe_2e_3\dots e_n = e_1e_2\dots e_n.$$

Hence, $e_2 y = y = e_2(e_1e_2\dots e_n)$. It follows that

$$xe_2(e_1e_2\dots e_n)xe_2 = xe_2,$$

$$e_1e_2\dots e_n = (e_1e_2\dots e_n)xe_2(e_1e_2\dots e_n).$$

Based on the uniqueness of inverses $e_1e_2\dots e_n$, we have $xe_2 = x$.

Repeating this process, we have that

$$xe_1 = xe_2 = \cdots = xe_n = x.$$

Symmetrically

$$e_n x = e_{n-1}x = \cdots = e_2 x = e_1 x = x.$$

Hence

$$e_1e_2 \dots e_n = e_1e_2 \dots e_n x e_1 e_2 \dots e_n = x = x(e_1e_2 \dots e_n)x = x^2,$$

and so $e_1e_2 \dots e_n$ is an idempotent of S.

If $m \geq 2$, let $y = e_2 \dots e_n(e_1e_2 \dots e_n)^{m-2}e_1e_2 \dots e_{n-1}$, then $xyx = x$. Hence yxy is an inverse of x. Since S is a π-inverse semigroup, then

$$\begin{aligned} yxy &= (e_1e_2 \dots e_n)^m \\ &= e_2 \dots e_n(e_1 \dots e_n)^{m-2}e_1 \dots e_{n-1}xe_2 \dots e_n(e_1 \dots e_n)^{m-2}e_1 \dots e_{n-1}. \end{aligned}$$

Thus $e_2(e_1e_2 \dots e_n)^m = (e_1e_2 \dots e_n)^m = (e_1e_2 \dots e_n)^m e_{n-1}$ and so

$$xe_2(e_1e_2 \dots e_n)^m xe_2 = xe_2,$$

$$(e_1e_2 \dots e_n)^m = (e_1e_2 \dots e_n)^m xe_2(e_1e_2 \dots e_n)^m.$$

Based on the uniqueness of the inverses of $(e_1e_2 \dots e_n)^m$, we have $xe_2 = x$. Symmetrically, $e_{n-1}x = x$. Hence

$$x = x(e_1e_2 \dots e_n)^m x = x(e_3e_4 \dots e_n)(e_1e_2 \dots e_n)^{m-2}e_1e_2 \dots e_{n-2}x.$$

Repeating the abovementioned process, we have

$$xe_1 = xe_2 = \cdots = xe_n = x = e_n x = e_{n-1}x = \cdots = e_1 x.$$

Hence

$$(e_1e_2 \dots e_n)^m = (e_1e_2 \dots e_n)^m x(e_1e_2 \dots e_n)^m = x = x(e_1e_2 \dots e_n)^m x = x^2$$

and so $(e_1e_2 \dots e_n)^m$ is an idempotent of S.

Thus we have proved that $e_1e_2 \dots e_n$ is a periodic element of S and so $\langle E(S) \rangle$ is a periodic semigroup. □

In view of Theorem 2.17 we have the following corollary.

Corollary 2.5 *For a π-inverse semigroup S,* $\mathrm{Reg}(S) \cap \langle E(S)\rangle = E(S)$ *and* $\langle E(S)\rangle$ *is a π-inverse subsemigroup of S.*

Exercises

1. A semigroup S is a *semiband* if it is idempotent-generated.

An ideal of a regular semiband is itself a regular semiband.

References

S. Bogdanović [1], [16], [17]; S. Bogdanović and M. Ćirić [4], [6]; A. H. Clifford and G. B. Preston [1]; D. Easdown [1], [2]; D. Easdown and T. E. Hall [1]; C. Eberhart, W. Williams and L. Kinch [1]; P. Edwards [1]; D. G. Fitzgerald [1]; R. Jones, Z. Tian and Z. B. Xu [1]; Z. Tian [3].

2.7 Left Regular Semigroups

In this section, we will give various structural characterizations of left regular semigroups.

A semigroup S is a *band* Y of *left* (*right*) *ideals* $L_\alpha, \alpha \in Y$ if

$$S = \bigcup_{\alpha\in Y} L_\alpha, \quad L_\alpha \cap L_\beta = \emptyset, \quad \alpha \neq \beta.$$

Lemma 2.12 *A semigroup S is a left (right) zero band of a semigroup from the class $\mathcal{K}$ if and only if S is a band of right (left) ideals from $\mathcal{K}$.*

Proof. Let S be a left zero band Y of semigroups $S_\alpha, \alpha \in Y$ and $S_\alpha \in \mathcal{K}$. Then for each $\alpha \in Y$ we have that

$$S_\alpha S = S_\alpha\left(\bigcup_{\beta\in Y} S_\beta\right) = \bigcup_{\beta\in Y} S_\alpha S_\beta \subseteq \bigcup_{\beta\in Y} S_{\alpha\beta} \subseteq S_\alpha.$$

Hence, S is a band of right ideals from $\mathcal{K}$.

Conversely, let S be a band of right ideals $S_\alpha \in \mathcal{K}$, $\alpha \in Y$. Let τ be the congruence relation on S induced by the decomposition of S. For $a \in S_\alpha$, $b \in S_\beta$ we have $ab \in S_\alpha S_\beta \subseteq S_{\alpha\beta}$ and $ab \in S_\alpha S_\beta \subseteq S_\alpha$. So $S_{\alpha\beta} = S_\alpha$ and therefore τ is a left zero band congruence. □

A semigroup S will be called *left* (*right*) *completely simple* if it is simple and left (right) regular. It is well-known that a semigroup S is completely simple if and only if it is simple and completely regular, whence we have that S is completely simple if and only if it is both left and right completely simple.

Now we will characterize left completely simple semigroups.

Theorem 2.18 *The following conditions on a semigroup S are equivalent:*

(i) *S is left completely simple;*
(ii) *S is simple and left π-regular;*
(iii) *every principal left ideal of S is a left simple subsemigroup of S;*
(iv) *S is a right zero band of left simple semigroups;*
(v) $(\forall a, b \in S)\ a \in Sba$*;*
(vi) *S is a matrix of left simple semigroups;*
(vii) $|_l$ *is a symmetric relation on S;*
(viii) $S/\mathcal{L}$ *is a discrete partially ordered set.*

Proof. (i)⇒(ii) This is obvious.

(ii)⇒(i) Since S is simple, then $S = \mathrm{Intra}(S)$. Now by Theorem 2.4 we obtain that $S = \mathrm{Intra}(S) = \mathrm{LReg}(S)$, so S is left regular.

(iii)⇒(iv) If all the principal left ideals of S are left simple, then the principal left ideals are minimal, so the principal left ideals are disjoint. From this and Lemma 2.12 it follows that S is a right zero band of left simple semigroups.

(iv)⇒(v) If S is a right zero band Y of left simple semigroups $S_\alpha, \alpha \in Y$, then for $a \in S_\alpha$, $b \in S_\beta$ we have that

$$ba \in S_\beta S_\alpha \subseteq S_{\beta\alpha} \subseteq S_\alpha, \quad \text{so } a \in S_\alpha ba \subseteq Sba.$$

(v)⇒(iii) Let condition (iii) hold. Assume $a \in S$ and $x, y \in L(a)$. Then we have

(a) $x = a$, $y = a$. Then $x = a \in Saa \subseteq L(a)y$. Hence,

$$L(a) = L(a)y \quad \text{for every } y \in L(a). \tag{$\star$}$$

(b) $x = za$, $y = a$. Then $x = za \in zSaa \subseteq L(a)y$, i.e. condition $(\star)$ holds.

(c) $x = a$, $y = ua$. Then $x = a \in S(au)a \subseteq L(a)ua \subseteq L(a)y$, i.e. $(\star)$ holds.

(d) $x = za$, $y = ua$. Then $x = za \in zS(au)a \subseteq L(a)ua = L(a)y$ i.e. $(\star)$ holds. By (a), (b), (c) and (d) we have that $L(a)$ is a left simple.

(vi)$\Rightarrow$(iv) If S is a matrix of left simple semigroups, then it is a right zero band of semigroups that are left zero bands of left simple semigroups. Since a left zero band of left simple semigroups is also a left simple semigroup, then we obtain (iv).

(iv)$\Rightarrow$(vi). This is clear.

(i)$\Rightarrow$(v) For $a, b \in S$ we have that $a = xby$, for some $x, y \in S^1$, and $xb = z(xb)^2$, for some $z \in S$, whence

$$a = xby = z(xb)^2 y = zxb(xby) = zxba \in Sba.$$

(v)$\Rightarrow$(i) This is immediate.

(iv)$\Rightarrow$(vii) Let S be a right zero band I of left simple semigroups S_i, $i \in I$. Assume $a, b \in S$ such that $a \,|_l\, b$, i.e. $b = xa$, for some $x \in S^1$. Then $a, b \in S_i$, for some $i \in I$, and S_i is left simple, whence $b \,|_l\, a$.

(vii)$\Rightarrow$(v) For all $a, b \in S$, $a \,|_l\, ba$, and based on the hypothesis, $ba \,|_l\, a$, i.e. $a \in S^1 ba$, which yields $a \in Sba$.

(vii)$\Rightarrow$(viii) Assume $L_a, L_b \in S/\mathcal{L}$ such that $L_a \leq L_b$, i.e. such that $a \in S^1 b$. Then $b \,|_l\, a$, so by (vi) we obtain that $a \,|_l\, b$, i.e. $b \in S^1 a$, whence $L_b \leq L_a$. Thus, $L_a = L_b$. This proves (vii).

(viii)$\Rightarrow$(vii) Assume $a, b \in S$ such that $a \,|_l\, b$. Then $L_b \leq L_a$, and from (vii) it follws that $L_b = L_a$, whence $b \,|_l\, a$. Hence, $|_l$ is symmetric. □

An element a of a semigroup S is *left* (*right*) *reproduced* if $a = xa$ ($a = ax$), for some $x \in S$. A semigroup S is *left* (*right*) *reproduced* if all its elements are left (right) reproduced.

Note that several known characterizations of completely simple semigroups can be obtained immediately from the previous theorem and its dual.

Here we give some new characterizations of left regular semigroups.

Theorem 2.19 *The following conditions on a semigroup S are equivalent:*

(i) *S is left regular;*

(ii) *S is intra regular and left π-regular;*

(iii) *S is a semilattice of left completely simple semigroups;*

(iv) *S is a union of left completely simple semigroups;*

(v) *S is a semilattice of right zero bands of left simple semigroups;*

(vi) $(\forall a, b \in S)\ a \mid b \ \Rightarrow\ ab \mid_l b$*;*

(vii) *every left ideal of S is a left quasi-regular semigroup;*

(viii) *every left ideal of S is a left reproduced semigroup.*

Proof. (i)⇒(ii) This is clear.

(ii)⇒(iii) According to Theorem 2.6, S is a semilattice of simple semigroups S_α, $\alpha \in Y$. For any $\alpha \in Y$, S_α is also left π-regular, so by Theorem 2.18, it is left completely simple.

(iii)⇒(vi) Assume $a, b \in S$ such that $a \mid b$. Based on the hypothesis, there exists a left completely simple subsemigroup A of S such that $b, ba \in A$, and by Theorem 2.18, $b \in Abab \subseteq Sab$.

(vi)⇒(i) This is obvious.

(iii)⇔(iv)⇔(v) This follows immediately from Theorem 2.18.

(i)⇒(vii) Let L be a left ideal of S and let $a \in L$. Based on the left regularity of S we have that $a = xa^2$ for some $x \in S$, so

$$a = xa^2 = x^3a^4 = (x^3a)aaa \in LaLa.$$

Hence, L is a left quasi-regular semigroup.

(vii)⇒(viii) This implication is evident.

(viii)⇒(i) Consider an arbitrary $a \in S$ and the principal left ideal $L = L(a) = S^1a$. Based on the hypothesis, L is a left reproduced semigroup, so $a \in La \subseteq S^1aa$. Accordingly, we easily conclude that $a \in Sa^2$. Thus, S is a left regular semigroup. □

Similarly, we prove the following theorem.

Theorem 2.20 *The following conditions on a semigroup S are equivalent:*

(i) *S is completely regular;*

(ii) *S is left (resp. right) regular and right (resp. left) quasi-regular;*

(iii) *S is left (resp. right) regular and right (resp. left) quasi-π-regular;*

(iv) *every left (resp. right) ideal of S is a right (resp. left) regular semigroup;*

(v) *every left (right) ideal of S is a completely quasi-regular semigroup;*

(vi) *every bi-ideal of S is a left* (*right*) *quasi-regular semigroup.*

References

S. Bogdanović [17]; S. Bogdanović and M. Ćirić [17]; S. Bogdanović, M. Ćirić and M. Mitrović [3]; S. Bogdanović, M. Ćirić, P. Stanimirović and T. Petković [1]; J. Calais [1]; A. H. Clifford [1]; A. H. Clifford and G. B. Preston [1]; R. Croisot [1]; J. M. Howie [3]; N. Kuroki [1]; S. Lajos and G. Szász [1]; W. D. Munn [2]; J. von Neumann [1]; M. Satyanarayana [1]; G. Szász [1]; [3], [4], [5].

Chapter 3

(0)-Archimedean Semigroups

In 1928, A. K. Suškevič gave the construction of a semigroup kernel, i.e. the construction of the smallest ideal of a finite semigroup. What we are dealing with a simple semigroup, i.e. a simple semigroup with a primitive idempotent. In 1941, D. Rees proved the structural theorem for completely 0-simple semigroups. This theorem, which we call *the theorem of Suškevič-Rees*, was later used as one of the most explored models for "making" new classes of semigroups. Studying the decompositions of commutative semigroups T. Tamura and N. Kimura, and independently G. Thierrin in 1954, gave the definition of the notion of an Archimedean semigroup. What we are dealing with is a semigroup in which for every two elements, any one of them divides some power of the others. Simple semigroups, i.e. semigroups with no proper ideals are Archimedean semigroups. The converse does not hold, an Archimedean semigroup with a primitive idempotent is a completely Archimedean semigroup. These semigroups will play an important role in a semilattice decomposition of completely π-regular semigroups (Chapter 4). By analogy to Rees's construction of a completely 0-simple semigroup using a completely simple semigroup, S. Bogdanović and M. Ćirić in 1993 introduced the notion of a (weakly) 0-Archimedean semigroup. It is a structural reach class of semigroups. Archimedean and (weakly) 0-Archimedean semigroups will be discussed later in this chapter. At the end of the chapter we will give the results regarding the semigroups, whose proper (left) ideals are Archimedean semigroups.

3.1 Completely 0-simple Semigroups

An idempotent e of a semigroup $S = S^0$ is called 0-*primitive* if it is minimal in the set of all the non-zero idempotents of a semigroup S with respect to the natural partial order on $E(S)$. A semigroup $S = S^0$ is *completely* 0-*simple* if S is 0-simple and if it contains an 0-primitive idempotent.

As in the case of Theorem 2.5, we prove another form of Munn's theorem.

Theorem 3.1 *Let S be an 0-simple semigroup. Then S is completely 0-simple if and only if S is completely π-regular.*

Lemma 3.1 *Let e be an 0-primitive idempotent of an 0-simple semigroup S. Then $L_e^0 = Se$.*

Proof. It is evident that $L_e^0 \subseteq Se$. Assume $b \in Se, b \neq 0$. Then $b = be$ and since S is 0-simple we have that $e = xby$ for some $x, y \in S$. For $f = eyexb$ it follows that $ef = fe = f$ and $f^2 = eyexbeyexb = eyexbyexb = eyexb = f$, and since e is an 0-primitive idempotent we have that $f = e$ or $f = 0$. If $f = 0$, then $0 = xbfy = xbeyexby = e$ which is impossible. Hence, $f = e$, i.e. $e = eyexb \in Sb$. Thus, $e\mathcal{L}b$, i.e. $b \in L_e^0$. So, we have $Se \subseteq L_e^0$. Therefore, $Se = L_e^0$. □

Lemma 3.2 *Let S be a completely 0-simple semigroup and let L be an arbitrary $\mathcal{L}$-class of S. Then L^0 is a 0-minimal left ideal of S.*

Proof. Assume that $L = L_x, x \neq 0$. According to Lemma 3.1 we have that $S = SeS = L_e^0 S$ so $x = ua$ for some $u \in L_e^0$ and $a \in S$.

Assume $y \in L$. Then $y = sx$ for some $s \in S^1$ whence $y = sua \in L_e^0 a$ because $su \in L_e^0$ and since by Lemma 3.1 L_e^0 is a left ideal of S. Thus $L^0 \subseteq L_e^0 a$. Assume $y \in L_e^0 a$. Then $y = va$ for some $v \in L_e^0$. If $v = 0$, then $y = 0 \in L^0$. If $v \neq 0$, then $v\mathcal{L}u$ whence $va\mathcal{L}ua$, because $\mathcal{L}$ is a right congruence, i.e. $y\mathcal{L}x$. Thus, $y \in L$ i.e. $L_e^0 a \subseteq L^0$. Therefore, by Lemma 3.1 $L^0 = L_e^0 a \subseteq Sea$. So L^0 is a left ideal of S.

Assume that $A \subseteq L^0, A \neq 0$ is a left ideal of S. Let $a \in A, a \neq 0$ and assume $x \in L$. Then $x\mathcal{L}a$, whence $x = ua \in A$ for some $u \in S$. Thus, $A = L^0$. Therefore, L^0 is a 0-minimal left ideal of S. □

From Lemma 3.2 we have the following

Corollary 3.1 *Let S be a completely 0-simple semigroup and let $a \in S$. Then $L_a^0 = Sa$.*

Lemma 3.3 *Let S be a completely 0-simple semigroup. For all $a, b \in S$ from $aSb = 0$ it follows that $a = 0$ or $b = 0$.*

Proof. Let $aSb = 0$ and let $a \neq 0$ and $b \neq 0$. According to Corollary 1.6 we have $SaS = SbS = S$, whence $S = S^2 = SaSSbS = SaSbS = 0$, which is impossible. Thus, $a = 0$ or $b = 0$. □

A semigroup S is 0-*bi-simple* if S has only one non-zero $\mathcal{D}$-class.

Lemma 3.4 *Every completely 0-simple semigroup is 0-bi-simple.*

Proof. Assume $a, b \in S^{\bullet}$. According to Lemma 3.3 we have that $aSb \neq 0$. Let $x \in aSb$ and $x \neq 0$. Based on Corollary 3.1 we have that $x \in aSb \subseteq Sb = L_b^0$, so $x\mathcal{L}b$. Similarly we can prove that $x\mathcal{R}a$. Thus, $a\mathcal{D}b$, i.e. S is 0-bi-simple. □

From Lemmas 3.4 and 1.36 immediately follows

Corollary 3.2 *Every completely 0-simple semigroup is a regular semigroup.*

Lemma 3.5 *Let H be an $\mathcal{H}$-class of a completely 0-simple semigroup S. Then, $H^2 = 0$ or H is a group.*

Proof. Assume $H \neq H_0 = 0$ and $a \in H$. There are two cases:

(i) Let $a^2 = 0$. Assume $x, y \in H$. Then $x\mathcal{L}a$ and $y\mathcal{R}a$, whence $x = ua$ and $y = av$ for some $u, v \in S^1$, so $xy = ua^2v = 0$. Thus $H^2 = 0$.

(ii) Let $a^2 \neq 0$. According to Lemma 3.2, L_a^0 is a left ideal of S whence $a^2 \in L_a^0$ and by assertion we have that $a^2 \in L_a$. Thus $a\mathcal{L}a^2$. In the same way we prove that $a\mathcal{R}a^2$. Therefore, from $a\mathcal{H}a^2$ and Green's theorem it follows that $H = H_a$ is a group. □

A semigroup $S = S^0$ is a 0-*group* if $S^{\bullet}$ is a group.

Let G be a group, let I, Λ be non-empty sets and let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix over a 0-group G^0. On $S = (G \times I \times \Lambda) \cup \{0\}$ we define the multiplication by

$$(a; i, \lambda) \cdot (b; j, \mu) = \begin{cases} (ap_{\lambda j}b; i, \mu), & \text{if } p_{\lambda j} \neq 0 \\ 0, & \text{if } p_{\lambda j} = 0 \end{cases}.$$

$$(a; i, \lambda) \cdot 0 = 0 \cdot (a; i, \lambda) = 0 \cdot 0 = 0.$$

It is easy to see that $(S, \cdot)$ is a semigroup which we denote by $S = \mathcal{M}^0(G; I, \Lambda; P)$ and which we call *Rees's matrix semigroup of the type* $\Lambda \times I$ *over a* 0*-group* G^0 *by a sandwich matrix* P.

A matrix P of the type $\Lambda \times I$ over a 0-group G^0 is regular if

$$(\forall i \in I)(\exists \lambda \in \Lambda)\ p_{\lambda i} \neq 0, \qquad (\forall \lambda \in \Lambda)(\exists i \in I)\ p_{\lambda i} \neq 0,$$

i.e. if every row and every column of a matrix P contains a non-zero element.

Lemma 3.6 *A Rees's matrix semigroup* $S = \mathcal{M}^0(G; I, \Lambda; P)$ *is regular if and only if the matrix* P *is regular.*

Proof. Let S be a regular semigroup, let $i \in I$, $\lambda \in \Lambda$ and let $a \in G$. Let $(b; j, \mu) \in S$ be the inverse element of the element $(a; i, \lambda)$. Then $p_{\lambda j} b p_{\mu i} = a^{-1}$ where $p_{\lambda j} \neq 0$ and $p_{\mu i} \neq 0$. Thus, P is a regular matrix.

Conversely, let P be a regular matrix. Assume $(a; i, \lambda) \in S^\bullet$. Then there exist $j \in I$ and $\mu \in \Lambda$ such that $p_{\lambda j}, p_{\mu i} \in G$ and the element $(p_{\lambda j}^{-1} a^{-1} p_{\mu i}^{-1}; j, \mu)$ is an inverse element of the element $(a; i, \lambda)$, so $(a; i, \lambda)$ is a regular element. It is evident that 0 is a regular element. Therefore, S is a regular semigroup. □

Lemma 3.7 *Let* $S = \mathcal{M}^0(G; I, \Lambda; P)$ *be a regular Rees's matrix semigroup and let* $(a; i, \lambda), (b; j, \mu) \in S$*. Then*

$$\begin{aligned}(a; i, \lambda)\, \mathcal{L}\, (b; j, \mu) &\quad\Leftrightarrow\quad \lambda = \mu,\\ (a; i, \lambda)\, \mathcal{R}\, (b; j, \mu) &\quad\Leftrightarrow\quad i = j.\end{aligned}$$

Proof. Assume $(a; i, \lambda)\mathcal{L}(b; j, \mu)$. Then $(a; i, \lambda) = (b; j, \mu)$ or there exists $(x; k, \nu) \in S$ such that $(a; i, \lambda) = (x; k, \nu)(b; j, \mu) = (x p_{\nu j} b; k, \mu)$, where $p_{\nu j} \neq 0$ because $(a; i, \lambda) \neq 0$. Therefore, $\lambda = \mu$.

Conversely, let $\lambda = \mu$ and let $\nu, \eta \in \Lambda$ such that $p_{\nu i} \neq 0$ and $p_{\eta j} \neq 0$ (these elements exist because P is a regular matrix). Then we have that

$$\begin{aligned}(ba^{-1}p_{\nu i}^{-1}; j, \nu) \cdot (a; i, \lambda) &= (b; j, \lambda),\\ (ab^{-1}p_{\eta j}^{-1}; i, \eta) \cdot (b; j, \lambda) &= (a; i, \lambda).\end{aligned}$$

Thus, $(a; i, \lambda)\mathcal{L}(b; j, \lambda)$. The similar proof exists for the $\mathcal{R}$ relation. □

Corollary 3.3 *Let $S = \mathcal{M}^0(G; I, \Lambda; P)$ be a regular Rees's matrix semigroup. Then $\{L_\lambda \mid \lambda \in \Lambda\}$ is the set of all non-zero $\mathcal{L}$-classes of S and $\{R_i \mid i \in I\}$ is the set of all non-zero $\mathcal{R}$-classes of S, where*

$$L_\lambda = \{(a; i, \lambda) \mid a \in G, i \in I\}, \quad R_i = \{(a; i, \lambda) \mid a \in G, \lambda \in \Lambda\},$$

and $\{H_{i\lambda} \mid i \in I, \lambda \in \Lambda\}$ is the set of all non-zero $\mathcal{H}$-classes of S, where $H_{i\lambda} = R_i \cap L_\lambda = \{(a; i, \lambda) \mid a \in G\}$.

Theorem 3.2 *A Rees's matrix semigroup $S = \mathcal{M}^0(G; I, \Lambda; P)$ is 0-simple if and only if S is regular, and in that case S is completely 0-simple.*

Proof. Let S be a 0-simple semigroup. Suppose that S is not regular. Then by Lemma 3.6 there exists some row or some column of a matrix P all of whose elements are equal to zero. Generally speaking we can assume that there exists $\lambda \in \Lambda$ such that $p_{\lambda j} = 0$ for all $j \in I$. Let $A = \{(a; i, \lambda) \mid a \in G, i \in I\} \cup \{0\}$. Then for $(a; i, \lambda) \in A$ and $(b; j, \mu) \in S^\bullet$ we have that $(a; i, \lambda)(b; j, \mu) = 0$, because $p_{\lambda j} = 0$, and

$$(b; j, \mu) \cdot (a; i, \lambda) = \begin{cases} (bp_{\mu i}a; j, \lambda) \in A & \text{if } p_{\mu i} \neq 0 \\ 0 \in A & \text{if } p_{\mu i} = 0 \end{cases}.$$

Thus, A is an ideal of S and $A \neq \{0\}$, $A \neq S$, which is a contradiction according to the hypothesis that S is a 0-simple semigroup. Therefore, S is a regular semigroup.

Conversely, let S be a regular semigroup. Assume $(a; i, \lambda), (b; j, \mu) \in G \times I \times \Lambda$. According to Lemma 3.6 there exist $k \in I$ and $\nu \in \Lambda$ such that $p_{\nu i} \neq 0$ and $p_{\lambda k} \neq 0$. Then

$$(b(p_{\nu i}ap_{\lambda k})^{-1}; j, \nu)(a; i, \lambda)(e; k, \mu) = (b; j, \mu),$$

where e is the identity of a group G, so from Corollary 1.6 it follows that S is a 0-simple semigroup.

Since $E(S) = \{(p_{\lambda i}^{-1}; i, \lambda) \mid i \in I, \lambda \in \Lambda\} \cup \{0\}$, it is easy to prove that every non-zero idempotent of a semigroup S is 0-primitive. Thus, S is 0-simple, i.e. S is completely 0-simple. □

The basic structural characterization of a completely 0-simple semigroup was given by means of the following theorem, which we call the *Suškevič-Rees theorem.*

Theorem 3.3 *A semigroup S is completely 0-simple if and only if S is isomorphic to some regular Rees's matrix semigroup over a 0-group.*

Proof. Let S be a completely 0-simple semigroup. According to Lemma 3.4, S is 0-bi-simple, i.e. $D = S - 0$ is a $\mathcal{D}$-class of S. Let $\{R_i \,|\, i \in I\}$ and $\{L_\lambda \,|\, \lambda \in \Lambda\}$ be the sets of all $\mathcal{R}$-classes and all $\mathcal{L}$-classes of S contained in D. Based on this notation the set of all the $\mathcal{H}$-classes of S contained in D is the set $\{H_{i\lambda} \,|\, i \in I, \lambda \in \Lambda\}$, where $H_{i\lambda} = R_i \cap L_\lambda$.

Let e be an arbitrary idempotent from D. According to Corollary 1.13 we have that H_e is a group. Denote R_e by R_1, L_e by L_1 and H_e by $R_1 \cap L_1$. Thus, here we take that sets I and Λ have element 1 in common, what no make mistake and without loss of generality.

For every $i \in I$ and $\lambda \in \Lambda$ fix the element $r_i \in H_{i1}$ and the element $q_\lambda \in H_{\lambda 1}$. Since $r_i \mathcal{L} e$, by Corollary 1.12 we have that $r_i e = r_i$ and by Lemma 1.34 the mapping $x \mapsto r_i x$ is a bijection from H_{11} onto H_{i1}. Similarly, we have that $eq_\lambda = q_\lambda$ and based on Lemma 1.33 the mapping $y \mapsto yq_\lambda$ is a bijection from H_{i1} onto $H_{i\lambda}$. Thus, the mapping $a \mapsto r_i a q_\lambda$ is a bijection from H_{11} onto $H_{i\lambda}$, so, every element of H_{11} has the unique representation of the form $r_i a q_\lambda$, where $a \in H_{11}$. Since $D = \cup\{H_{i\lambda} \,|\, i \in I, \lambda \in \Lambda\}$ and since this union is disjointed, the mapping $\phi : (H_{11} \times I \times \Lambda) \cup 0 \mapsto S$ defined by

$$(a; i, \lambda)\phi = r_i a q_\lambda, \qquad 0\phi = 0,$$

is a bijection. Let $M = \mathcal{M}^0(H_{11}; I, \Lambda; P)$, where the matrix P is defined by

$$p_{\lambda i} = q_\lambda r_i \quad (i \in I, \lambda \in \Lambda).$$

Assume $i \in I$ and $\lambda \in \Lambda$ and prove that $p_{\lambda i} \in H_{11}^0$. According to Lemma 3.5 we have that $H_{i\lambda}^2 = 0$ or $H_{i\lambda}$ is a group. First assume that $H_{i\lambda}^2 = 0$. Then for $c \in H_{i\lambda}$ there exist $u, v \in S^1$ such that $q_\lambda = uc$ and $r_i = cv$ so $p_{\lambda i} = uc^2 v = 0 \in H_{11}^0$. Let $H_{i\lambda}$ be a group and let f be its identity. Then from Corollary 1.12 we have $fr_i = r_i$ and by Lemma 1.33 it follows that the mapping $x \mapsto xr_i$ is a bijection from L_λ onto L_1 which is $\mathcal{R}$-class preserving. Hence $p_{\lambda i} = q_\lambda r_i \in H_{11}$. Thus P is a matrix over H_{11}^0. Also, we proved that $p_{\lambda i} = 0$ if and only if $H_{i\lambda}^2 = 0$. Since by Lemma 1.37 we have that every $\mathcal{L}$-class L_λ and every $\mathcal{R}$-class R_i of S contained in D has an idempotent, then for every $i \in I$ there exists $\lambda \in \Lambda$ such that $H_{i\lambda}$ is a group, i.e. $p_{\lambda i} \neq 0$. We prove the second condition for regularity of the matrix P in a similar way.

It is easy to prove that ϕ is an isomorphism. Therefore, a semigroup S is isomorphic to a Rees's matrix semigroup M.

The converse follows immediately from Theorem 3.2. $\square$

As we can see from the proof of Theorem 3.3, the representation of a completely 0-simple semigroup by a semigroup $\mathcal{M}^0(H_{11}; I, \Lambda; P)$ we get by means of the arbitrary election of a subgroup H_{11} and the sets $\{r_i \mid i \in I\}$ and $\{q_\lambda \mid \lambda \in \Lambda\}$. The natural question is: How do we make the selection which does not influence (up to the isomorphism) the structure of a semigroup $\mathcal{M}^0(H_{11}; I, \Lambda; P)$? The answer to this question is provided by the following theorem, which we give without proof.

Theorem 3.4 *Two regular Rees's matrix semigroups $S = \mathcal{M}^0(G; I, \Lambda; P)$ and $T = \mathcal{M}^0(H; J, M; Q)$ are isomorphic if and only if there is an isomorphism $\theta : G \mapsto H$, bijections $\varphi : I \mapsto J$, $\psi : \Lambda \mapsto M$ and sets $\{u_i \mid i \in I\}$, $\{v_\lambda \mid \lambda \in \Lambda\} \subseteq H$ such that $p_{\lambda i}\theta = v_\lambda q_{\lambda\psi, i\varphi} u_i$, for all $\lambda \in \Lambda$, $i \in I$.*

Let G be a group and I a non-empty set. If P is an $I \times I$-matrix over a 0-group G^0 such that $p_{ii} = 1$ for every $i \in I$, where 1 is the identity of a group G, then P is called the *identity $I \times I$-matrix*. A semigroup S is a *Brandt semigroup* if it is isomorphic to some semigroup $\mathcal{M}^0(G; I, I; P)$, where P is an identity $I \times I$-matrix. From Theorems 3.3 and 3.4 we have

Corollary 3.4 *A semigroup S is a Brandt semigroup if and only if S is completely 0-simple and an inverse semigroup.*

Proof. Let $S = \mathcal{M}^0(G; I, I; P)$ be a Brandt semigroup. For an arbitrary element $(a; i, j) \in S^\bullet$, from $(b; k, l) \in V((a; i, j))$ it follows that $k = j$, $l = i$ and $b = a^{-1}$, whence S is an inverse semigroup. According to Theorem 3.3, S is a completely 0-simple semigroup.

Conversely, let S be a completely 0-simple and an inverse semigroup. From Theorem 3.3 $S \cong \mathcal{M}^0(G; I, \Lambda; P)$, where P is a regular matrix. Now $((p_{\lambda i}^{-1})^2; i, \lambda) \in V((1; i, \lambda))$. If $\mu \in \Lambda$ such that $p_{\mu i} \neq 0$, then $(p_{\lambda i}^{-1} p_{\mu i}^{-1}; i, \mu) \in V((1; i, \lambda))$ which is contradicted by the hypothesis that S is an inverse semigroup. Thus for every $i \in I$ there exists only one $\lambda \in \Lambda$ such that $p_{\lambda i} \neq 0$. Similarly, we prove that for every $\lambda \in \Lambda$ there exists only one $i \in I$ such that $p_{\lambda i} \neq 0$. Thus, the mapping $\psi : \Lambda \mapsto I$, defined as $\lambda\psi = i$ if and only if is $p_{\lambda i} \neq 0$, is a bijection. If we now assume that Q is an identity $I \times I$-matrix over a group G^0 then by Theorem 3.4 we have that

$\mathcal{M}^0(G; I, \Lambda; P) \cong \mathcal{M}^0(G; I, I; Q)$ (where, for example, we assume $v_\lambda = 1$, for all $\lambda \in \Lambda$, $u_i = p_{i\psi^{-1}i}$, for all $i \in I$ and θ is an identical automorphism of a group G). □

Let G be a group, I, Λ be the non-empty sets and $P = (p_{\lambda i})$ be a $\Lambda \times I$-matrix over a group G. On the set $S = G \times I \times \Lambda$ we define the multiplication by

$$(a; i, \lambda) \cdot (b; j, \mu) = (a p_{\lambda i} b; i, \mu).$$

Then S is a semigroup which we denote by $S = \mathcal{M}(G; I, \Lambda; P)$ and which we call the *Rees's matrix semigroup of the type* $\Lambda \times I$ *over a group* G *with a sandwich matrix* P.

It is evident that such a constructed semigroup can be obtained from Rees's matrix semigroup $S = \mathcal{M}^0(G; I, \Lambda; P)$. Namely, since all the elements of a matrix P are different from zero, then $S - 0$ is a subsemigroup of S isomorphic to $\mathcal{M}(G; I, \Lambda; P)$. So, the proof of the following theorem immediately follows by Theorem 3.3.

Theorem 3.5 *A semigroup S is completely simple if and only if S is isomorphic to a Rees's matrix semigroup over a group.*

A semigroup which is isomorphic to a direct product of a rectangular band and a group is a *rectangular group*. The next lemma immediately follows:

Lemma 3.8 *If a rectangular group S is a direct product of a group G and a rectangular band E, then $E(S)$ is a rectangular band isomorphic to E.*

Theorem 3.6 *A semigroup S is a rectangular group if and only if S is a completely simple semigroup in which $E(S)$ is a subsemigroup.*

Proof. Let S be a completely simple semigroup in which $E(S)$ is a subsemigroup and denotes $E(S)$ with E. Then $S = \mathcal{M}(G; I, \Lambda; P)$. Since $E = \{(p_{\lambda i}^{-1}; i, \lambda) \mid i \in I, \lambda \in \Lambda\}$ from the hypothesis we have that

$$(p_{\lambda i}^{-1}; i, \lambda) \cdot (p_{\mu j}^{-1}; j, \mu) = (p_{\mu i}^{-1}; i, \mu),$$

so

$$p_{\lambda i}^{-1} p_{\lambda j} p_{\mu j}^{-1} = p_{\mu i}^{-1}, \quad \text{i.e. } p_{\lambda i}^{-1} p_{\lambda j} = p_{\mu i}^{-1} p_{\mu j}.$$

Choose and fix an arbitrary element $1 \in I$. Then we have

$$p_{\lambda 1}^{-1} p_{\lambda i} = p_{\mu 1}^{-1} p_{\mu i},$$

for all $i \in I$, $\lambda, \mu \in \Lambda$. Define the mapping $\phi : S \mapsto E \times G$ with

$$(a; i, \lambda)\phi = ((p_{\lambda i}^{-1}; i, \lambda), p_{\lambda 1}^{-1} p_{\lambda i} a p_{\lambda 1}).$$

It is easy to prove that ϕ is an isomorphism from a semigroup S onto a rectangular group $E \times G$.

The converse follows immediately. $\square$

From Theorem 3.6 we have the following

Corollary 3.5 *A band S is completely simple if and only if S is a rectangular band.*

Based on Theorem 2.7 and Corollary 3.5 we have:

Corollary 3.6 *Every band is a semilattice of rectangular bands.*

Corollary 3.7 *Let S be a band B of semigroups S_i, $i \in B$ and let B be a semilattice Y of rectangular bands B_α, $\alpha \in Y$. Then S is a semilattice Y of semigroups S_α, $\alpha \in Y$ and for all $\alpha \in Y$, S_α is a matrix B_α of semigroups S_i, $i \in B_\alpha$.*

A semigroup S is *right* (*left*) *cancellative* if for all $a, b \in S$ from $ac = bc$ ($ca = cb$) it follows $a = b$. A semigroup S is *cancellative* if it is both left and right cancellative. A semigroup S is a *left* (*right*) *group* if S is isomorphic to a direct product of a group and a left (right) zero band.

Theorem 3.7 *The following conditions on a semigroup S are equivalent:*

(i) *S is a left group;*
(ii) *S is a left zero band of groups;*
(iii) $(\forall a, x \in S)$ *$x \in xSa$;*
(iv) *S is regular and $E(S)$ is a left zero band;*
(v) *S is left simple and right cancellative;*
(vi) *for all $a, b \in S$ there exists only one $x \in S$ such that $xa = b$;*

(vii) *S is left simple and contains an idempotent;*

(viii) *S has a right identity e and $e \in Sa$, for all $a \in S$.*

Proof. (i)⇒(ii) If $S = G \times E$ is a direct product of a group G and a band E, then S is a left zero band E of a group $G_e = G \times \{e\}$, $e \in E$.

(ii)⇒(iii) Let S be a left zero band E of groups G_e, $e \in E$. Assume $x, a \in S$. Then $x \in G_e$, $a \in G_f$ for some $e, f \in E$, whence $x, xa \in G_e$ and since G_e is a group we have that $x \in xG_exa \subseteq xSa$.

(iii)⇒(iv) If (iii) holds it is clear that S is a regular semigroup. Assume $e, f \in E(S)$. Then $e \in Sf$ whence $ef = e$. Thus, $E(S)$ is a left zero band.

(iv)⇒(v) Let S be a regular semigroup and let $E(S)$ be a left zero band. Assume $a, b \in S$. Then for $x \in V(a)$, $y \in V(b)$ we have that $b = byb = bybxa \in Sa$. Thus, by Corollary 1.5, S is left simple.

Assume $a, b, c \in S$ such that $ac = bc$. Then for $x \in V(a)$, $y \in V(b)$ and $z \in V(c)$ we have that

$$a = axa = axacz = acz = bcz = bybcz = byb = b.$$

Thus, S is right cancellative.

(v)⇒(vi) This follows immediately.

(vi)⇒(vii) Let (vi) hold. Then by Corollary 1.5, S is a left simple semigroup. Assume an arbitrary $a \in S$. By (vi) there exists only one $x \in S$ such that $xa = a$. Hence we get $x^2a = xa = a$ and since x is unique, then $x^2 = x$. Thus, S contains an idempotent.

(vii)⇒(viii) Let S be a left simple and let S contains an idempotent. Assume an arbitrary $e \in E(S)$. Then by Corollary 1.5 for an arbitrary $a \in S$, we have that $e \in Sa$ and $a \in Se$. From $a \in Se$ we have that $ae = a$, so e is a right identity.

(viii)⇒(vii) Let (viii) hold. Then for arbitrary $a, b \in S$ we have that $b = be \in bSa \subseteq Sa$, so S is left simple. Since $e \in E(S)$ then (vii) holds.

(vii)⇒(i) Let S be a left simple semigroup and let S contain an idempotent. Then it is evident that S is simple. Also, for arbitrary $e, f \in E(S)$ from $e \in Sf$ we have that $ef = e$, so $E(S)$ is a subsemigroup of S and since $E(S)$ is a left zero band, then it immediately follows that all idempotents from S are primitive. Thus, S is completely simple, and by Theorem 3.6, S is a rectangular group, i.e. S is a direct product of a group G and a rectangular band E. Since $E(S)$ is a left zero band, based on Lemma 3.8 E is a left zero band. Therefore, S is a left group. □

Theorem 3.8 *Let $S = \mathcal{M}(G; I, \Lambda; P)$. Then:*

(i) *S is a disjoint union of minimal left ideals*

$$L_\lambda = \{(a; i, \lambda) \mid a \in G, i \in I\}, \quad (\lambda \in \Lambda),$$

which are left groups;

(ii) *S is a disjoint union of minimal right ideals*

$$R_i = \{(a; i, \lambda) \mid a \in G, \lambda \in \Lambda\}, \quad (i \in I),$$

which are right groups;

(iii) *S is a disjoint union of bi-ideals*

$$H_{i\lambda} = \{(a; i, \lambda) \mid a \in G\}, \quad (i \in I, \lambda \in \Lambda),$$

which are groups with an identity $(p_{\lambda i}^{-1}; i, \lambda)$; moreover, S is a matrix (rectangular band) $I \times \Lambda$ of groups $H_{i\lambda}$.

Corollary 3.8 *On a semigroup S the following conditions are equivalent:*

(i) *S is completely simple;*
(ii) *S is a left zero band of right groups;*
(iii) *S is a right zero band of left groups;*
(iv) *S is a matrix of groups.*

Exercises

1. A semigroup $S = S^0$ is a 0-group if and only if S is a left 0-simple and right 0-simple.

2. The following conditions on a semigroup S are equivalent:

(a) S is completely simple;
(b) S is regular and for any $a, x \in S$, $a = axa$ implies $x = xax$;
(c) $(\forall a, b \in S)\ a \in aSba$.

3. A semigroup S is a left group if and only if $(\forall a \in S)(\exists_1 x \in S)\ a = xa^2$.

References

D. Allen [1]; S. Bogdanović [7]; S. Bogdanović and S. Gilezan [1]; S. Bogdanović and B. Stamenković [1]; A. H. Clifford [1]; A. H. Clifford and G. B. Preston [1]; G. Čupona [1], [2], [3]; J. M. Howie [1]; J. Ivan [1], [2]; K. Kapp and H. Schneider [1]; G. Lallement [4]; G. Lallement and M. Petrich [1], [2]; H. Mitsch [2]; W. D. Munn [3]; R. P. Rich [1]; T. Saito and S. Hori [1]; Š. Schwarz [2]; O. Steinfeld [1], [2]; A. K. Suškevič [1], [2]; T. Tamura, R. B. Merkel and J. F. Latimer [1]; P. S. Vankatesan [2]; J. R. Warne [1], [2].

3.2 0-Archimedean Semigroups

In this section we consider (*completely*) 0-*Archimedean* semigroups as a generalization of (*completely*) 0-*simple* and (*completely*) *Archimedean* semigroups. We describe nil-extensions of (*completely*) 0-*simple* semigroups.

Recall that, an element a of a semigroup $S = S^0$ is a *nilpotent* if there exists $n \in \mathbf{Z}^+$ such that $a^n = 0$. The set of all nilpotent elements of S is denoted by $\mathrm{Nil}(S)$. S is a *nil-semigroup* if $S = \mathrm{Nil}(S)$, otherwise it is *non-nil*. An ideal I of S is a *nil-ideal* of S if I is a nil-semigroup. Based on $\mathfrak{R}(S)$ we denote *Clifford's radical* of a semigroup $S = S^0$, i.e. the union of all nil-ideals of S (it is the greatest nil-ideal of S). An ideal extension S of a semigroup K is a *nil-extension* of K if S/K is a nil-semigroup. Some characterizations of a Clifford's radical give the following lemmas.

Lemma 3.9 *For an arbitrary semigroup $S = S^0$, $\mathfrak{R}(S/\mathfrak{R}(S)) = 0$.*

Proof. Let $S/\mathfrak{R}(S) = Q$, let $\varphi : S \mapsto Q$ be a natural homomorphism and let I be a nil-ideal of Q. Let $J = \{x \in S \,|\, \varphi(x) \in I\}$. Then it is evident that J is a nil-ideal of S, whence $J \subseteq \mathfrak{R}(S)$, so I is a zero ideal of Q. $\square$

Let S be a semigroup. For $a, b \in S$, $a \,|\, b$ if $b \in J(a)$ and $a \longrightarrow b$ if $a \,|\, b^n$, for some $n \in \mathbf{Z}^+$. For $a \in S$, $\Sigma_1(a) = \{x \in S \mid a \longrightarrow x\}$ and an equivalence σ_1 on S is defined by: $a \ \sigma_1 \ b$ if and only if $\Sigma_1(a) = \Sigma_1(b)$, $a, b \in S$. More will be said about sets $\Sigma_n(a)$ and relations $\sigma_n, n \in \mathbf{Z}^+$ in Chapter 4.

An ideal I of a semigroup S is *prime* if for all $a, b \in S$, $aSb \subseteq I$ implies that either $a \in I$ or $b \in I$, or, equivalently, if for all ideals A, B of S, $AB \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$.

The purpose of this section is to give some generalizations of (completely) 0-simple semigroups and of (completely) Archimedean semigroups and to describe some of their characteristics.

First we will give a connection between Clifford's radical of a semigroup with zero and the relation σ_1.

Lemma 3.10 *The Clifford radical $\mathfrak{R}(S)$ of a semigroup $S = S^0$ is equal to the σ_1-class containing the zero 0.*

Proof. Let C be the σ_1-class of S containing the zero 0, and let $a \in C$, $x \in S$. Then $\Sigma_1(a) = \Sigma_1(0) = \mathrm{Nil}(S)$. Since $ab \longrightarrow x$ implies that $a \longrightarrow x$ and $b \longrightarrow x$, then we have that

$$\Sigma_1(ax) \subseteq \Sigma_1(a) = \mathrm{Nil}(S), \quad \Sigma_1(xa) \subseteq \Sigma_1(a) = \mathrm{Nil}(S).$$

Since $\mathrm{Nil}(S) \subseteq \Sigma_1(u)$ for all $u \in S$, then $\Sigma_1(ax) = \Sigma_1(xa) = \mathrm{Nil}(S) = \Sigma_1(0)$, so $ax, xa \in C$. Hence, C is an ideal of S. It is clear that $C \subseteq \mathrm{Nil}(S)$, so C is a nil-ideal, whence $C \subseteq \mathfrak{R}(S)$.

Let $a \in \mathfrak{R}(S)$ and $x \in \Sigma_1(a)$, i.e. $x^n \in SaS$ for some $n \in \mathbf{Z}^+$. Since $SaS \subseteq S\mathfrak{R}(S)S \subseteq \mathfrak{R}(S) \subseteq \mathrm{Nil}(S)$, then $x \in \mathrm{Nil}(S) = \Sigma_1(0)$. Thus, $\Sigma_1(a) \subseteq \Sigma_1(0)$. It is clear that $\Sigma_1(0) \subseteq \Sigma_1(a)$. Therefore, $a \in C$ so $\mathfrak{R}(S) = C$. □

Let A be a subsemigroup of a semigroup S. By $\mathcal{I}(A)$ we denote the set of all elements $x \in S$ which satisfied the condition $xA \cup Ax \subseteq A$. The set $\mathcal{I}(A)$ we call an *idealizer* of a subsemigroup A into a semigroup S. It is evident that $\mathcal{I}(A)$ is the greatest subemigroup of S containig A as an ideal.

Lemma 3.11 *Let A be a proper subsemigroup of a semigroup S. If A^n is an ideal of S, for some $n \in \mathbf{Z}^+$, then $A \neq \mathcal{I}(A)$.*

Proof. Assume $x \in S - A$. If $x \in \mathcal{I}(A)$, then the lemma holds. If $x \notin \mathcal{I}(A)$, then there is an element $a_1 \in A$ such that $x_1 = xa_1 \notin A$ (or $a_1 x \notin A$). The same holds for element x_1 as for element x. Hence, if we continue this procedure for elements x_i, then in no more than $2(n-1)$ steps, multiplying (left or right) by elements a_i from A we obtain that $x_k \in \mathcal{I}(A) - A$. □

Corollary 3.9 *If A is a proper nilpotent subsemigroup of a semigroup S and the zero of A is the zero of S, then $A \neq \mathcal{I}(A)$.*

Theorem 3.9 *A nil-semigroup is nilpotent if and only if the class of nilpotency of all its nilpotent subsemigroups is bounded.*

Proof. Let n be an upper bound of classes of nilpotency of all the nilpotent subsemigroups of a nil-semigroup S. Since the union of the increasing family of nilpotent semigroups of the class $\leq n$ is also a nilpotent semigroup of the class $\leq n$, then in S there is a maximal nilpotent subsemigroup A. If $A = S$, then the statement of the theorem holds. Let $A \neq S$. Then, by Lemma 3.11 $A \neq \mathcal{I}(A)$. Let x be an arbitrary element from $\mathcal{I}(A) - A$ and $k \in \mathbf{Z}^+$ such that $x^k \notin A$, then $x^{k+1} \in A$. Let F be a subsemigroup of S generated by A and x^k, $F = \langle A, x^k \rangle$. It is evident that F is nilpotent, F is not a proper subsemigroup of S, because A is a maximal. Hence, $F = S$.

The converse follows immediately. □

In the following lemma we describe the identities which should satisfy the nil-semigroup to be nilpotent.

Lemma 3.12 *A nil-semigroup with the identity $u = x_1 x_2 \cdots x_m$, where $|u| \geq m + 1$, is nilpotent.*

Proof. Let a nil-semigroup S satisfies the identity $u = x_1 x_2 \cdots x_m$, where $|u| \geq m + 1$. Then every nilpotent subsemigroup T of S has the power of nilpotency not more than m. Suppose that the equation $x_1 x_2 \cdots x_k = 0$, $k \geq m + 1$ is satisfied in T and let $y_1, y_2, \ldots, y_m \in T$. Then $y_1 \cdots y_m = u(y_1, \ldots, y_m)$. If on the letter u we apply the equation $x_1 \cdots x_m = u(x_1, \ldots, x_m)$, then we obtain $y_1 \cdots y_m = u_1(y_1, \ldots, y_m)$, where $|u_1| \geq m + 2$. If this procedure we apply again we obtain the equation $y_1 \cdots y_m = u_i(y_1, \ldots, y_m)$, where $|u_i| \geq k$. Hence, $y_1 y_2 \cdots y_m = 0$, for all $y_1, y_2, \ldots, y_m \in T$. According to Theorem 3.9 the rest of the proof follows immediately. □

Note that a semigroup $S = S^0$ is 0-*simple* if and only if $a \,|\, b$, for all $a, b \in S^\bullet$. Using the relation $\longrightarrow$, we can introduce a generalization of 0-simple semigroups. A semigroup $S = S^0$ is *0-Archimedean* if $a \longrightarrow b$, for all $a, b \in S^\bullet$. Also, we can introduce a more general notion: A semigroup $S = S^0$ is *weakly 0-Archimedean* if $a \longrightarrow b$, for all $a, b \in S - \mathfrak{R}(S)$.

A relationship between weakly 0-Archimedean and 0-Archimedean semigroups is given in the next theorem. Since every nil-semigroup is (weakly) 0-Archimedean, then a consideration of nil-semigroups will be omitted.

Theorem 3.10 *The following conditions on a non-nil semigroup $S = S^0$ are equivalent:*

(i) *S is weakly 0-Archimedean;*

(ii) *S is an ideal extension of a nil-semigroup by a 0-Archimedean semigroup;*

(iii) *S contains at most two σ_1-classes.*

Proof. (i)⇒(ii) Let S be weakly 0-Archimedean. Then S is an ideal extension of a nil-semigroup $R = \mathfrak{R}(S)$ by a semigroup Q. Assume $a, b \in Q^\bullet$. Then $a, b \in S - R$, so there exists $x, y \in S$ and $n \in \mathbf{Z}^+$ such that $b^n = xay$, since S is weakly 0-Archimedean. If $x \in R$ or $y \in R$, then $b^n \in R$, whence $b^n = 0 \in QaQ$ in Q, so $a \longrightarrow b$ in Q. Assume that $x, y \in S - R = Q^\bullet$. Then $b^n = xay \in QaQ$ in Q, so $a \longrightarrow b$ in Q. Thus, Q is 0-Archimedean.

(ii)⇒(i) Let S be an ideal extension of a nil-semigroup R by a 0-Archimedean semigroup Q. Assume $a, b \in S - \mathfrak{R}(S)$. Since $R \subseteq \mathfrak{R}(S)$, then $a, b \in S - R = Q^\bullet$. Thus, there exist $x, y \in Q$ and $n \in \mathbf{Z}^+$ such that $b^n = xay$. If $x = 0$ or $y = 0$, then $b^n = 0$ in Q, whence $b^n \in R \subseteq \mathrm{Nil}(S)$ in S, so $b^{nk} = (b^n)^k = 0 \in SaS$ in S, for some $k \in \mathbf{Z}^+$, i.e. $a \longrightarrow b$ in S. Assume that $x, y \neq 0$ in Q. Then $x, y \in Q^\bullet = S - R$, so $b^n = xay \in SaS$ in S, whence $a \longrightarrow b$ in S. Thus, S is weakly 0-Archimedean.

(i)⇒(iii) Let S be weakly 0-Archimedean. According to Lemma 3.10 we obtain that $\mathfrak{R}(S)$ is equal to the σ_1-class containing 0. Assume $a, b \in S - \mathfrak{R}(S)$. Let us prove that $a\,\sigma_1\,b$. Let $x \in \Sigma_1(a)$, i.e. let $x^n = uav$ for some $n \in \mathbf{Z}^+$, $u, v \in S$. If $uav \in \mathfrak{R}(S)$, then $x \in \mathrm{Nil}(S)$, so $b \longrightarrow x$, i.e. $x \in \Sigma_1(b)$. Let $uav \in S - \mathfrak{R}(S)$. Then $(uav)^k \in SbS$ for some $k \in \mathbf{Z}^+$, whence $x^{nk} \in SbS$, i.e. $x \in \Sigma_1(b)$. Thus, $\Sigma_1(a) \subseteq \Sigma_1(b)$. Similarly we prove the opposite inclusion. Therefore, (iii) holds.

(iii)⇒(i) This follows from Lemma 3.10. □

Lemma 3.13 *Let S=S^0 be a nil-extension of a 0-simple semigroup K. Then*

$$\mathfrak{R}(S) = \{x \in S \mid SxS \cap K = 0\}.$$

Proof. Let $A = \{x \in S \mid SxS \cap K = 0\}$. Assume $a \in A$, $x \in S$. Then $SaS \cap K = 0$ so

$$SaxS \cap K \subseteq SaS \cap K = 0, \quad SxaS \cap K \subseteq SaS \cap K = 0,$$

whence $ax, xa \in A$. Thus, A is an ideal of S. It is clear that A is a nil-semigroup. Assume a nil-ideal I of S. Then $I \cap K$ is an ideal of K, whence $I \cap K = 0$ or $I \cap K = K$. Since K contains a non-nilpotent element, then $I \cap K = 0$, so $SaS \cap K \subseteq SIS \cap K \subseteq I \cap K = 0$, for every $a \in I$. Therefore, $I \subseteq A$, whence $\mathfrak{R}(S) = A$. □

Note that the smallest ideal, if it exists, of a semigroup S is called a *kernel* of S. But, in a semigroup with zero, this notion degenerates, since the zero ideal is the kernel, so we introduce the following notion: the smallest element of a set of all nonzero ideals of a semigroup $S = S^0$, if it exists, is called the *0-kernel* of S.

Let $S = S^0$ and K be the 0-kernel of S. According to Corollary 1.7, $K^2 = 0$, and then we say that K is a *nilpotent 0-kernel*, or K is 0-simple, and we call it a *0-simple 0-kernel*.

Recall that, if a semigroup S is an ideal extension of a semigroup T by a semigroup Q, then we usually identify the partial semigroups $S - T$ and $Q^\bullet$. This fact will be used in the following:

Theorem 3.11 *A semigroup $S = S^0$ is a nil-extension of a 0-simple semigroup if and only if S is an ideal extension of a nil-semigroup R by a 0-Archimedean semigroup Q with a 0-simple 0-kernel K and the following conditions hold:*

(a) *for all $a \in K^\bullet,\ b \in S - R$*

$$\begin{array}{llll} ab = 0 & in\ \ Q & \Rightarrow ab = 0 & in\ \ S; \\ ba = 0 & in\ \ Q & \Rightarrow ba = 0 & in\ \ S; \end{array}$$

(b) $ab = ba = 0$*, for all $a \in K^\bullet,\ b \in R$.*

Proof. Let S be a nil-extension of a 0-simple semigroup T and let $R = \mathfrak{R}(S)$. Then R is a nil-semigroup and S is an ideal extension of R by a semigroup Q. Since T is 0-simple, then $R \cap T = 0$.

Assume $a \in T^\bullet,\ b \in S - R$. Then $ab \in T$, since T is an ideal of S. If $ab = 0$ in Q, then $ab \in R$ in S, so $ab = 0$ in S, since $R \cap T = 0$. Thus,

$$ab = 0 \in Q \Rightarrow ab = 0 \in S.$$

Similarly we prove the second implication from (a).

Assume $a \in T^\bullet,\ b \in R$. Then $ab = ba = 0$, since $ab, ba \in R \cap T = 0$.

Let $K = T^\bullet \cup 0 \subseteq Q$. Then K is a subsemigroup of Q isomorphic to T, whence K is 0-simple. Therefore, from the aforementioned we obtain (a) and (b).

Let I be an ideal of Q, $I \neq 0$. It is easy to verify that $I^\bullet \cup R$ is an ideal of S and $I^\bullet \cup R \neq 0$, whence $T \subseteq I^\bullet \cup R$, so $K^\bullet = T^\bullet \subseteq I^\bullet$, i.e. $K \subseteq I$. Thus K is a 0-simple 0-kernel of Q.

Assume $a, b \in S - R$. Based on Lemma 3.13 we obtain that $SaS \cap T \neq 0$, whence $T \subseteq SaS$. Thus, there exists $n \in \mathbf{Z}^+$ such that $b^n \in T \subseteq SaS$, i.e.

$a \longrightarrow b$. Hence, S is a weakly 0-Archimedean, so by the proof of Theorem 3.10 we obtain that Q is 0-Archimedean.

Conversely, let S be an ideal extension of a nil-semigroup R by a 0-Archimedean semigroup Q with a 0-simple 0-kernel K and let (a) and (b) hold. From (a) it follows that $T = K^\bullet \cup 0 \subseteq S$ is a subsemigroup of S isomorphic to K, so T is 0-simple. From (a) and (b) it follows that T is an ideal of S. According to Theorem 3.10, S is a weakly 0-Archimedean. Assume $x \in S$. If $x \in \mathfrak{R}(S)$, then $x \in \mathrm{Nil}(S)$, so $x^n = 0 \in T$ for some $n \in \mathbf{Z}^+$. Let $x \in S - \mathfrak{R}(S)$ and assume $a \in T - \mathrm{Nil}(S)$. Then $a \longrightarrow x$, whence $x^n \in SaS \subseteq T$, for some $n \in \mathbf{Z}^+$. Therefore, S is a nil-extension of T. $\square$

As we have seen, a 0-Archimedean semigroup is a generalization of a 0-simple semigroup. Similarly we generalize the notion of completely 0-simple semigroups. An idempotent e of a semigroup $S = S^0$ is a *0-primitive* idempotent of S if it is a minimal element in the partially ordered set of all nonzero idempotents of S. A 0-Archimedean semigroup containing a 0-primitive idempotent is called a *completely 0-Archimedean* semigroup.

Lemma 3.14 *Every completely 0-Archimedean semigroup contains a* (*completely*) *0-simple 0-kernel.*

Proof. Let S be a completely 0-Archimedean semigroup and let $e \in E(S)$ be a 0-primitive idempotent. Let K be an intersection of all non zero ideals of a semigroup S. It is clear that $0 \in K$, so K is a non-empty set, and also it is evident that K is an ideal of S. Assume an arbitrary non-zero ideal I of S and assume an arbitrary element $a \in I^\bullet$. Since S is a 0-Archimedean and $a, e \in S^\bullet$, then $a \longrightarrow e$, i.e. $e \in SaS \subseteq I$. Thus, e is an element of all non-zero ideals of S, so $e \in K$. Hence, K is a 0-minimal ideal of S and $K^2 \neq 0$ and by Corollary 1.7 we have that K is a 0-simple semigroup. Since e is a 0-primitive, then K is a completely 0-simple semigroup, i.e. K is a completely 0-simple 0-kernel of S. $\square$

Based on Lemma 3.14 and Theorem 3.11 we obtain the following:

Corollary 3.10 *A semigroup $S = S^0$ is a nil-extension of a completely 0-simple semigroup if and only if S is an ideal extension of a nil-semigroup R by a completely 0-Archimedean semigroup Q, and the conditions* (*a*) *and* (*b*) *hold, where K is the 0-kernel of Q.*

Theorem 3.12 *The following conditions on a non-nil semigroup* $S = S^0$ *are equivalent:*

(i) *S is a 0-Archimedean semigroup with a 0-simple 0-kernel;*
(ii) *S is a 0-Archimedean semigroup with a 0-minimal ideal;*
(iii) *S is a weakly 0-Archimedean semigroup with a 0-simple 0-kernel;*
(iv) *S is a 0-Archimedean intra-π-regular semigroup;*
(v) *S is a nil-extension of a 0-simple semigroup and* $\mathfrak{R}(S) = 0$*;*
(vi) *S is a nil-extension of a 0-simple semigroup and* 0 *is a prime ideal of* S.

Proof. (i)$\Rightarrow$(ii) This follows immediately.

(ii)$\Rightarrow$(i) Let S be a 0-Archimedean semigroup with a non-nil 0-minimal ideal M. Let $I \neq 0$ be an ideal of S, let $x \in I^\bullet$ and let $a \in M - \mathrm{Nil}(S)$. Then $x \longrightarrow a$, i.e. $a^n \in SxS \subseteq I$ for some $n \in \mathbf{Z}^+$, whence $a^n \in I \cap M$, $a^n \neq 0$. Thus $I \cap M \neq 0$ is an ideal of S contained in M, and since M is 0-minimal, we obtain that $I \cap M = M$, i.e. $M \subseteq I$. Hence, M is a 0-simple 0-kernel of S.

(i)$\Rightarrow$(v) Let S be a 0-Archimedean semigroup with a 0-simple 0-kernel K. Then K is 0-simple semigroup. Let $a \in K^\bullet$ and assume $x \in S^\bullet$. Then $a \longrightarrow x$, i.e. $x^n \in SaS \subseteq SKS \subseteq K$, for some $n \in \mathbf{Z}^+$. Thus S is a nil-extension of K. If $\mathfrak{R}(S) \neq 0$, then $K \subseteq \mathfrak{R}(S)$, which is not possible, since $K \neq \mathrm{Nil}(K)$. Thus $\mathfrak{R}(S) = 0$, so (v) holds.

(v)$\Rightarrow$(iv) Let S be a nil-extension of a 0-simple semigroup K and let $\mathfrak{R}(S) = 0$. Then it is clear that S is intra-π-regular and from the proof of Theorem 3.11 we obtain that S is a 0-Archimedean.

(iv)$\Rightarrow$(i) Let S be a non-nil 0-Archimedean intra π-regular semigroup. Assume $a \in S - \mathrm{Nil}(S)$. Then there exists $m \in \mathbf{Z}^+$ and $z, w \in S$ such that $a^m = za^{2m}w \in Sa^mS$. Let $K = Sa^mS$ and let $c, d \in K^\bullet$. Then $c = xa^my$ for some $x, y \in S$. On the other hand, by $a^m = za^{2m}w = za^m(a^mw)$ it follows that

$$a^m = z^n a^m (a^m w)^n, \tag{1}$$

for all $n \in \mathbf{Z}^+$. Since $d, a^mw \in S^\bullet$ and S is 0-Archimedean, then there exists $k \in \mathbf{Z}^+$ and $u, v \in S$ such that $(a^mw)^k = udv$. Now, from (1) we obtain that

$$\begin{aligned} c &= xa^my = (xz^{k+1}a^m)(a^mw)^k(a^mwy) = (xz^{k+1}a^m)udv(a^mwy) \\ &= (xz^{k+1}a^mu)d(va^mwy) \in KdK. \end{aligned}$$

Thus, by Corollary 1.6 we obtain that K is a 0-simple semigroup.

Let $I \neq 0$ be an ideal of S. Let $I \subseteq \mathrm{Nil}(S)$. Assume $x \in I^{\bullet}$. Then $x \longrightarrow a$, i.e. $a^n \in SxS \subseteq I$, for some $n \in \mathbf{Z}^+$, and since $I \subseteq \mathrm{Nil}(S)$, then $a \in \mathrm{Nil}(S)$, which leads to a contradiction. Thus, there exists $b \in I - \mathrm{Nil}(S) \subseteq S^{\bullet}$, so there exists $n \in \mathbf{Z}^+$ such that $b^n \in Sa^mS = K$, whence $b^n \in I \cap K$, $b^n \neq 0$, so $I \cap K \neq 0$. Now, since K is 0-simple, then $I \cap K = K$, so $K \subseteq I$. Thus, K is a 0-simple 0-kernel of S.

(iii)⇒(i) Let S be a weakly 0-Archimedean semigroup with a 0-simple 0-kernel K. Since K is 0-simple, then $K \nsubseteq \mathrm{Nil}(S)$ so $K \nsubseteq \mathfrak{R}(S)$, whence $\mathfrak{R}(S) = 0$, so by the proof of Theorem 3.10 we obtain that S is 0-Archimedean.

(i)⇒(iii) This follows immediately.

(v)⇒(vi) Let S be a nil-extension of a 0-simple semigroup K and let $\mathfrak{R}(S) = 0$. Let A and B be nonzero ideals of S and let $a \in A^{\bullet}$, $b \in B^{\bullet}$. According to Lemma 3.13 we obtain that $K \subseteq SaS \subseteq A$ and $K \subseteq SbS \subseteq B$, whence $K = K^2 \subseteq AB$. Thus $AB \neq 0$. Therefore, 0 is a prime ideal of S.

(vi)⇒(v) Let S be a nil-extension of a 0-simple semigroup K and let 0 be a prime ideal of S. Let $R = \mathfrak{R}(S)$. From the proof of Theorem 3.11 we obtain that $RK = 0$, whence $R = 0$ or $K = 0$. Since K is 0-simple, then $R = 0$, so (v) holds. □

In the following theorem a consideration of nil-semigroups will be omitted once again.

Theorem 3.13 *The following conditions on a non-nil semigroup $S = S^0$ are equivalent:*

(i) *S is a completely 0-Archimedean semigroup;*
(ii) *S is 0-Archimedean and completely π-regular;*
(iii) *S is a nil-extension of a completely 0-simple semigroup and $\mathfrak{R}(S) = 0$;*
(iv) *S is a nil-extension of a completely 0-simple semigroup and 0 is a prime ideal of S.*

Proof. (i)⇒(iii) Let S be a completely 0-Archimedean semigroup. According to Lemma 3.14, S has a completely 0-simple 0-kernel K, and it is clear that S is a nil-extension of K. Now, by Theorem 3.12 we have that $\mathfrak{R}(S) = 0$. Thus, (iii) holds.

(ii)⇒(iii) This follows from Theorem 3.12 and Theorem 2.5.

(iii)⇒(i), (iii)⇒(ii) and (iii)⇔(iv) This follows from Theorem 3.12. □

Exercises

1. A semigroup $S = S^0$ is a weakly 0-Archimedean and has a 0-primitive idempotent if and only if S is an ideal extension of a nil-semigroup by a completely 0-Archimedean semigroup.

2. Every periodic (finite) 0-Archimedean semigroup is completely 0-Archimedean.

3. Let $S = S^0$ be a 0-Archimedean semigroup. Then S has no divisors of zero if and only if S has no non-zero nilpotents.

References

S. Bogdanović [17]; S. Bogdanović and M. Ćirić [9]; A. H. Clifford and G. B. Preston [1], [2]; M. Ćirić and S. Bogdanović [5], [7]; B. M. Schein [1]; M. Yamada and T. Tamura [1].

3.3 Archimedean Semigroups

A semigroup S is *Archimedean* if $a \longrightarrow b$ for all $a, b \in S$. It is clear that a semigroup S is Archimedean if and only if S^0 is a 0-Archimedean semigroup. The Archimedean semigroups with kernels were described by the following theorem:

Theorem 3.14 *On a semigroup S the following conditions are equivalent:*

(i) *S is a nil-extension of a simple semigroup;*
(ii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in Sb^{2n}S$;
(iii) *S is an Archimedean intra π-regular semigroup.*

Proof. (i)⇒(ii) Let S be a nil-extension of a simple semigroup K. Assume $a, b \in S$. Then there exists $n \in \mathbf{Z}^+$ such that $a^n, b^{2n} \in K$ and since K is a simple semigroup then $a^n \in Kb^{2n}K \subseteq Sb^{2n}S$. Thus, (ii) holds.

(ii)⇒(iii) This follows immediately.

(iii)⇒(i) This implication we prove using Theorem 3.12 on a semigroup S^0. □

Corollary 3.11 *On a semigroup S the following conditions are equivalent:*

(i) *S is a nil-extension of a left simple semigroup;*

(ii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in Sb^{2n}$;
(iii) *S is a left Archimedean and left π-regular semigroup.*

Theorem 3.15 *The following conditions on a semigroup S are equivalent:*

(i) *S is π-regular and an Archimedean semigroup;*
(ii) *S is a nil-extension of a simple regular semigroup;*
(iii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in a^nSbSa^n$.

Proof. (i)⇒(ii) If S is a π-regular Archimedean semigroup, then $E(S) \neq 0$. Assume $e \in E(S)$ and let I be an ideal of S and let $b \in I$. Then $e \in SbS \subseteq I$. Hence the intersection K of all the ideals of S is the non-empty set and by Corollary 1.7, K is a simple kernel of S. Since S is Archimedean, we have that for every $a \in S$ there exists $m \in \mathbf{Z}^+$ such that $a^m \in K$. Thus, S is a nil-extension of a simple and clearly π-regular semigroup K. Hence, by Theorem 2.1 we have that S is a nil-extension of a simple regular semigroup K.

(ii)⇒(i) Let S be a nil-extension of a simple regular semigroup K. According to Theorem 3.14, S is an Archimedean semigroup. For $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in K$. But K is a regular semigroup, so we have $a^n \in a^nKa^n \subseteq a^nSa^n$, and S is a π-regular semigroup.

(ii)⇒(iii) Let S be a nil-extension of a simple regular semigroup K and let $a, b \in S$. Then there exists $n \in \mathbf{Z}^+$ such that $a^n, a^nb \in K$, so $a^n \in Ka^nbK$, and there exists $x \in K$ such that

$$a^n = a^nxa^n = a^nxa^nxa^n \in a^nxKa^nbKxa^n \subseteq a^nKbKa^n \subseteq a^nSbSa^n,$$

which has to be proved.

(iii)⇒(i) It is obvious that S is a π-regular semigroup. Assume $a, b \in S$. Then there exists $n \in \mathbf{Z}^+$ such that $a^n \in a^nSbSa^n \subseteq SbS$, so S is an Archimedean semigroup. □

Lemma 3.15 *Let S be a π-regular semigroup in which all the idempotents are primitive. Then S is completely π-regular, and maximal subgroups of S are of the form*

$$G_e = eSe, \qquad e \in E(S).$$

Proof. For $a \in S$ there exist $x \in S$ and $m \in \mathbf{Z}^+$ such that $a^m = a^mxa^m$. For a^k, where $k > m$, there exist $y \in S$ and $n \in \mathbf{Z}^+$ such that $a^{kn} = a^{kn}ya^{kn}$.

Assume that $e = xa^m$ and $f = xa^m ya^{kn}$. Then $e^2 = e$ and

$$\begin{aligned} f^2 &= xa^m ya^{kn} xa^m ya^{kn} = xa^m ya^{kn-m}(a^m xa^m) ya^{kn} = xa^m ya^{kn-m} a^m ya^{kn} \\ &= xa^m ya^{kn} ya^{kn} = xa^m ya^{kn} = f, \\ fe &= xa^m ya^{kn} xa^m = xa^m ya^{kn-m} a^m xa^m = xa^m ya^{kn-m} a^m = f = ef. \end{aligned}$$

Thus $ef = fe = f$ and since idempotents in S are primitive we have that $e = f$. Whence

$$a^m = a^m xa^m = a^m e = a^m f = a^m xa^m ya^{kn} \in a^m S a^{m+1},$$

and by Theorem 2.3, S is a completely π-regular semigroup.

Let $e \in E(S)$ and $u \in G_e$. Then $u = eue \in eSe$, so $G_e \subseteq eSe$. On the other hand, assume $u \in eSe$, i.e. let $u = ebe$ for some $b \in S$. Since S is a completely regular semigroup then $u^p \in G_f$ for some $p \in \mathbf{Z}^+$ and $f \in E(S)$. Now, we have that

$$ef = eu^p(u^p)^{-1} = e(ebe)^p(u^p)^{-1} = (ebe)^p(u^p)^{-1} = f,$$

where $(u^p)^{-1}$ is a group inverse of u^p in G_f, and dually we get $fe = f$, so based on the primitivity of idempotents from S we have that $e = f$. Thus $u^p \in G_e$ and based on Lemma 1.8 $u = ebe = e(ebe) = eu \in G_e$. Therefore, $eSe \subseteq G_e$. □

As completely 0-Archimedean semigroups are one generalization of completely 0-simple semigroups, in a similar way we can introduce one new generalization of completely simple semigroups.

A semigroup S is *completely Archimedean* if S is Archimedean and if it has a primitive idempotent.

Theorem 3.16 *The following conditions on a semigroup S are equivalent:*

(i) *S is a completely Archimedean semigroup;*
(ii) *S is a nil-extension of a completely simple semigroup;*
(iii) *S is Archimedean and completely π-regular;*
(iv) *S is π-regular and all idempotents from S are primitive;*
(v) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in a^n Sba^n$;
(v') $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in a^n bSa^n$;
(vi) *S is completely π-regular and $\langle E(S)\rangle$ is a (completely) simple semigroup.*

Proof. (i)⇒(ii)⇒(iii)⇒(i) These implications hold by Theorem 3.13 if a semigroup S adds zero.

(ii)⇒(v) Let S be a nil-extension of a completely simple semigroup K. Assume arbitrary $a, b \in S$. Then there exists $n \in \mathbf{Z}^+$ such that $a^n \in K$, so by Corollary 3.8 K is a matrix of groups, whence there exists $e \in E(S)$ such that $a^n, a^nba^n \in G_e$. Thus $xa^nba^n = e$ for some $x \in G_e$, whence

$$a^n = a^ne = a^nxa^nba^n \in a^nSba^n.$$

(v)⇒(iv) If (v) holds, then it is evident that S is a π-regular semigroup. Assume $e, f \in E(S)$ such that $ef = fe = f$. From (v) we have that $e \in efSe = fSe$, whence $e = fe = f$. Thus, all the idempotents from S are primitive.

(iv)⇒(ii) Based on Lemma 3.15, S is completely π-regular and all the maximal subgroups of S are of the form $G_e = eSe$, $e \in E(S)$. According to Lemma 1.17, a subgroup G_e, $e \in E(S)$ is a minimal bi-ideal of S. Now, by Theorem 1.17 the union K of all the minimal bi-ideals of S i.e. $K = \cup_{e \in E(S)} G_e$, is the kernel of S. Based on Corollary 1.9, K is a simple semigroup and since K is a union of groups, then by Corollary 2.4 K is completely simple. In the end, since S is completely π-regular, then S is a nil-extension of K.

(i)⇒(vi) Let S be a completely Archimedean semigroup. Based on (i)⇔(ii) S is a nil-extension of a completely simple semigroup K. Since $\langle E(S)\rangle \subseteq K$ we then have by Lemma 2.10 that $\langle E(S)\rangle$ is completely simple. It is clear that S is completely π-regular.

(vi)⇒(i) If S is completely π-regular and $\langle E(S)\rangle$ is a simple semigroup, then by Lemma 2.11, $\langle E(S)\rangle$ is completely π-regular. According to Theorem 2.5, $\langle E(S)\rangle$ is completely simple, from where it follows that idempotents are primitive, so S is completely Archimedean. □

Corollary 3.12 *A semigroup S is a nil-extension of rectangular group if and only if S is π-regular and $E(S)$ is a rectangular band.*

Proof. Let S be a nil-extension of a rectangular group K. Then $E(S) = E(K)$ and by Lemma 3.8, $E(S)$ is a rectangular band.

Conversely, let S be π-regular and let $E(S)$ be a rectangular band. Then all the idempotents from S are primitive and by Theorem 3.16, S is a nil-extension of a completely simple semigroup K. Since $E(K) = E(S)$ based on Theorem 3.6, K is a rectangular group. □

A semigroup S is *left* (*right*) *Archimedean* if $a \xrightarrow{l} b$ ($a \xrightarrow{r} b$), for all $a, b \in S$. Left (right) Archimedean semigroups are the generalizations of left (right) simple semigroups.

In the following theorem we describe a left Archimedean semigroup which has an idempotent.

Theorem 3.17 *The following conditions on a semigroup S are equivalent:*

(i) *S is left Archimedean and it has an idempotent;*
(ii) *S is π-regular and $E(S)$ is a left zero band;*
(iii) *S is a nil-extension of a left group;*
(iv) $(\forall a, b \in S)(\exists m \in \mathbf{Z}^+)\ a^m \in a^m S a^m b$;
(iv') $(\forall a, b \in S)(\exists m \in \mathbf{Z}^+)\ a^m \in b a^m S a^m$.

Proof. (i)⇒(ii) Let S be a left Archimedean semigroup and let $e \in E(S)$. Assume $a \in S$. Then from $a \xrightarrow{l} e$ and $e \xrightarrow{l} a$ we have that $e \in Sa$ and $a^n \in Se$ for some $n \in \mathbf{Z}^+$, whence $a^n = a^n e \in a^n S a^n$. Thus, S is π-regular. Assume $f, g \in E(S)$. Then from $g \xrightarrow{l} f$ we have that $f \in Sg$, whence $fg = f$. Therefore, $E(S)$ is a left zero band.

(ii)⇒(iii) Let S be π-regular and let $E(S)$ be a left zero band. Then all the idempotents from S are primitive and by Theorem 3.16, S is a nil-extension of a completely simple semigroup K. It is clear that $E(S) = E(K)$, i.e. $E(K)$ is a left zero band and since K is a regular semigroup then by Theorem 3.7 K is a left group.

(iii)⇒(iv) Let S be a nil-extension of a left group K. Assume $a, b \in S$. Then there exists $n \in \mathbf{Z}^+$ such that $a^n \in K$, whence $a^n b \in K$ and by Theorem 3.7 we have that $a^n \in a^n K a^n b \subseteq a^n S a^n b$. Thus, (iv) holds.

(iv)⇒(i) If (iv) holds then it is evident that S is a left Archimedean semigroup. Since from (iv) it immediately follows that S is a π-regular, then S has an idempotent. □

A semigroup S is a *two-sided Archimedean*, *t-Archimedean* for short, if S is both a left and right Archimedean semigroup. A semigroup S is a *π-group* if S is π-regular and if it has only one idempotent.

Theorem 3.18 *The following conditions on a semigroup S are equivalent:*

(i) *S is t-Archimedean and it has an idempotent;*

(ii) *S is a π-group;*
(iii) *S is a nil-extension of a group;*
(iv) $(\forall a, b \in S)(\exists m \in \mathbf{Z}^+)\ a^m \in ba^mSa^mb$.

Proof. (i)⇒(ii) Let S be a t-Archimedean semigroup and let S have an idempotent. Then by Theorem 3.17 and it dual we have that S is a π-regular semigroup, $E(S)$ is a left zero band and $E(S)$ is a right zero band. Thus, $E(S)$ contains only one element, so S is a π-group.

(ii)⇒(iii) If S is a π-group then by Theorem 3.17, S is a nil-extension of a left group K. Since K has only one idempotent then K is a group.

(iii)⇒(iv) Let S be a nil-extension of a group G. Assume $a, b \in S$. Then there exists $n \in \mathbf{Z}^+$ such that $a^n \in G$ whence $ba^n, a^nb \in G$ and since G is a group, then we have $a^n \in ba^nGa^nb \subseteq ba^nSa^nb$.

(iv)⇒(i) If (iv) holds, then it is evident that S is a t-Archimedean semigroup. Also, it is clear that S is π-regular, so S has an idempotent. □

A semigroup S is *power-joined* if for all $a, b \in S$ there exist $m, n \in \mathbf{Z}^+$ such that $a^m = b^n$. It is clear that every power-joined semigroup is t-Archimedean.

Corollary 3.13 *The following conditions on a semigroup S are equivalent:*

(i) *S is power-joined and it has an idempotent;*
(ii) *S is t-Archimedean and periodic;*
(iii) *S is periodic and it has only one idempotent;*
(iv) *S is a nil-extension of a periodic group.*

Exercises

1. A semigroup S is completely Archimedean if and only if S is Archimedean and S contains at least one minimal left and at least one minimal right ideal.

2. The following conditions on a semigroup S are equivalent:

(a) S is periodic and Archimedean;
(b) S is π-regular and for all $a, b \in S$, $ab = ba$ implies $a^n = b^n$, for some $n \in \mathbf{Z}^+$;
(c) S is a nil-extension of a periodic simple semigroup.

3. A semigroup S is a nil-extension of a left simple semigroup if and only if S is a left Archimedean and left π-regular.

4. The following conditions on a semigroup S are equivalent:

(a) S is π-inverse Archimedean;
(b) S is a nil-extension of a simple π-inverse semigroup;
(c) S is a nil-extension of a simple inverse semigroup.

5. A semigroup S is a nil-extension of a rectangular band if and only if for every $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n = a^n b a^n$.

6. If for every element $a \in S$ there exists $n \in \mathbf{Z}^+$ and there exists exactly one $x \in S$ such that $a^n = xa^{n+1}$, then S is a nil-extension of a left group. Does the converse hold?

7. A semigroup S is a nil-extension of a periodic left group if and only if for every $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n = a^n b^n$.

8. A semigroup S is a π-group if and only if S is Archimedean with only one idempotent.

9. Let ξ be a congruence on a π-regular semigroup S. Then $e\xi f$, for all $e, f \in E(S)$ if and only if S/ξ is a π-group.

10. The following conditions on a semigroup S are equivalent:

(a) S is a group;
(b) S is regular and has only one idempotent;
(c) $(\forall a \in S)(\exists_1 x \in S)\ a = axa$;
(d) $(\forall a, b \in S)\ a \in bSb$.

11. A semigroup S is a subdirect product of nilpotent semigroups if and only if $|\bigcap_{n\in\mathbf{Z}^+} S^n| \leq 1$.

12. A semigroup S is a subdirect product of nil-semigroups if and only if $\bigcap_{n\in\mathbf{Z}^+} J(a^n) = \emptyset$, for all $a \in S$.

13. Let S be a subsemigroup of an Archimedean semigroup without intra-regular elements. Then S is a subdirect product of countable many nil-semigroups.

14. The following conditions on a semigroup S are equivalent:

(a) S is a π-group;
(b) S is a subdirect product of a group by a nil-semigroup;
(c) S is completely π-regular with the identity $x^0 = y^0$.

15. A semigroup S is a nil-extension of a left group if and only if S is an epigroup with the identity $x^0 y^0 = x^0$.

16. The following conditions on a semigroup S are equivalent:

(a) S is completely Archimedean;
(b) S is completely π-regular satisfying some heterotypical identity;
(c) S is completely π-regular with the identity $(a^0 b^0 a^0)^0 = a^0$.

17. The following conditions on a semigroup S are equivalent:

(a) $\mathcal{P}(S)$ is Archimedean;

(b) $\mathcal{P}(S)$ is a nilpotent extension of a rectangular band;
(c) S is a nilpotent extension of a rectangular band.

18. A semigroup S is a nilpotent extension of a left zero band if and only if $\mathcal{P}(S)$ is left Archimedean.

19. A semigroup S is nilpotent if and only if $\mathcal{P}(S)$ is t-Archimedean.

20. A semigroup S is Archimedean if and only if any its bi-ideal is Archimedean.

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3.4 Semigroups in Which Proper Ideals are Archimedean

Denote by $\mathcal{A}$ $(\mathcal{LA}, \mathcal{RA}, \mathcal{TA}, \mathcal{PJ})$ the class of Archimedean (left Archimedean, right Archimedean, t-Archimedean, power-joined) semigroups. As we have already noticed, the following relations between these classes hold

$$\mathcal{PJ} \subset \mathcal{TA} = \mathcal{LA} \cap \mathcal{RA} \subset \mathcal{LA} \cup \mathcal{RA} \subset \mathcal{A}.$$

Let $I(S)$ $(L(S))$ denote the *union of all proper two-sided* (*left*) *ideals of a semigroup* S.

Theorem 3.19 *Every proper ideal of a semigroup S is an Archimedean subsemigroup of S if and only if $I(S)$ is an Archimedean subsemigroup of S.*

Proof. Let all proper ideals of S be Archimedean semigroups and let $a, b \in I(S)$. Then there exists a proper ideal A of S such that $a, aba \in A$ and there exists $n \in \mathbf{Z}^+$ such that

$$a^n \in AabaA \subseteq I(S)bI(S).$$

Thus $I(S)$ is an Archimedean semigroup.

Conversely, let $I(S)$ be an Archimedean semigroup and let A be a proper ideal of S. Then for $a, b \in A$ there exists $n \in \mathbf{Z}^+$ such that $a^n = xby$ for some $x, y \in I(S)$. Thus $a^{n+2} = axbya$ where $ax, ya \in A$, so A is an Archimedean semigroup. □

Lemma 3.16 *Every left ideal of an Archimedean (left Archimedean, right Archimedean, t-Archimedean, power-joined) semigroup S is an Archimedean (left Archimedean, right Archimedean, t-Archimedean, power-joined) subsemigroup of S.*

Proof. We will only prove the case when S is an Archimedean semigroup, the other cases are proved similarly. Let L be an arbitrary left ideal of S and let $a, b \in L$. Then there exist $x, y \in S$ and $n \in \mathbf{Z}^+$ such that $a^n = xb^2y$. Hence, it follows that $a^{n+1} = xbbya$ and $xb, ya \in L$. □

In the following theorem we will give the characterization of a semigroup whose every proper left ideal is an Archimedean semigroup.

Theorem 3.20 *The following conditions on a semigroup S are equivalent:*

(i) *every proper left ideal of S is an Archimedean subsemigroup of S;*
(ii) *$L(S)$ is an Archimedean subsemigroup of S;*
(iii) *S satisfies one of the following conditions:*
   (a) *S is Archimedean;*
   (b) *S has a maximal left ideal M which is an Archimedean semigroup and $M \subseteq Ma$, for every $a \in S - M$.*

Proof. (i)⇒(ii) If S is a left simple semigroup then S is Archimedean. Assume that S is not left simple. For arbitrary $a, b \in L(S)$ there exists a proper left ideal L of S such that $a, ba \in L$ whence

$$a^n \in LbaL \subseteq L(S)bL(S),$$

for some $n \in \mathbf{Z}^+$, and thus $L(S)$ is an Archimedean subsemigroup of S.

(ii)$\Rightarrow$(iii) If $L(S) \neq S$ then $M = L(S)$ is a maximal left ideal of S and by Theorem 1.15, $S - M = \{a\}$, $a^2 \in M$, or $S - M \subseteq Sa$, for every $a \in S - M$. If $S - M = \{a\}$, $a^2 \in M$, then S is Archimedean. If $S - M \subseteq Sa$, for every $a \in S - M$, then by Theorem 1.16 $T = S - M$ is a subsemigroup of S. From $Sa = S, a \in T$ it follows that $S = Ma \cup Ta \subseteq Ma \cup T \subseteq S$, i.e. $S = Ma \cup T$. Thus, $M \subseteq Ma$, for every $a \in S - M$.

(iii)$\Rightarrow$(i) If (a) holds, then by Lemma 3.16 every left ideal of S is an Archimedean subsemigroup of S. Let (ii) hold and let L be a proper left ideal of S. If $L \subseteq M$ then by Lemma 3.4, L is an Archimedean subsemigroup of S. If $L \nsubseteq M$ then $L \cap (S - M) \neq \emptyset$ and for $a \in L \cap (S - M)$ is $M \subseteq Ma \subseteq L$ which is impossible. $\square$

Theorem 3.21 *Every proper left ideal of S is a left Archimedean subsemigroup of S if and only if S satisfies one of the following conditions:*

(a) *S is left Archimedean;*

(b) *S contains only two left ideals L_1 and L_2 which are left simple semigroups and $S = L_1 \cup L_2$;*

(c) *S has a maximal left ideal M which is a left Archimedean semigroup and $M \subseteq Ma$, for every $a \in S - M$.*

Proof. Let all proper left ideals of S be left Archimedean. If $L(S) \neq S$ then $M = L(S)$ is a maximal left ideal of S which is a left Archimedean semigroup. Based on Theorem 1.15, we have that $S - M = \{a\}, a^2 \in M$ or $S - M \subseteq Sa$ for every $a \in S - M$. If $S - M = \{a\}, a^2 \in M$ then S is a left Archimedean semigroup. If $S - M \subseteq Sa$ for every $a \in S - M$, then as in the proof of Theorem 3.20, we have that S is type (c).

If $L(S) = S$ and for every two proper left ideals L_1 and L_2 of S, $L_1 \cap L_2 \neq \emptyset$, then S is left Archimedean. On the other hand, there exist left ideals L_1 and L_2 of S such that $L_1 \cap L_2 = \emptyset$. In that case, $L_1 \cup L_2 = S$, because $L_1 \cup L_2$ is not a left Archimedean semigroup (since $L_1 \cap L_2 = \emptyset$). Let L_3 be a left ideal of S such that $L_3 \subset L_1$, $L_3 \neq L_1$. Then $L_2 \cup L_3$ is a proper left ideal of S and for $a \in L_3$, $b \in L_2$ we have that $a^n = xb \in Sb \subseteq L_2$, for some $n \in \mathbf{Z}^+$ and $x \in S$. Thus, $L_1 \cap L_2 \neq \emptyset$ which is not possible. Hence, L_1 is a minimal left ideal of S and by Theorem 1.14, it is left simple. Also, L_2 is a left simple semigroup of S. Thus, if every proper left ideal of S is a left Archimedean semigroup, then one of the conditions (a), (b) or (c) holds.

The converse follows immediately. $\square$

Also, on a semigroup S we define the relations $\uparrow$, $\uparrow_l$, $\uparrow_r$ and $\uparrow_t$ by

$$\begin{aligned} a \uparrow b &\Leftrightarrow (\exists n \in \mathbf{Z}^+)\ b^n \in \langle a,b\rangle\, a\, \langle a,b\rangle\,, \\ a \uparrow_l b &\Leftrightarrow (\exists n \in \mathbf{Z}^+)\ b^n \in \langle a,b\rangle\, a, \\ a \uparrow_r b &\Leftrightarrow (\exists n \in \mathbf{Z}^+)\ b^n \in a\, \langle a,b\rangle\,, \\ a \uparrow_t b &\Leftrightarrow (\exists n \in \mathbf{Z}^+)\ a \uparrow_l b\ \&\ a \uparrow_r b. \end{aligned}$$

Clearly, $a \uparrow_t b$ if and only if $b^n \in a\, \langle a,b\rangle\, a$, for some $n \in \mathbf{Z}^+$.

A semigroup S is a *hereditary Archimedean* if $a \uparrow b$ for all $a,b \in S$. By a *hereditary left Archimedean semigroup* we mean a semigroup S satisfying the condition: $a \uparrow_l b$, for all $a,b \in S$. A *hereditary right Archimedean semigroup* is defined dually. A semigroup S is called *hereditary t-Archimedean* if it is both hereditary left Archimedean and hereditary right Archimedean, i.e. if $a \uparrow_t b$ for all $a,b \in S$.

The next lemma gives an explanation of why we are use the term "hereditary Archimedean".

Lemma 3.17 *A semigroup S is hereditary Archimedean (hereditary left Archimedean, hereditary right Archimedean, hereditary t-Archimedean) if and only if every subsemigroup of S is Archimedean (left Archimedean, right Archimedean, t-Archimedean).*

By $\mathbf{C}_2$ we denote the two-element chain and for a prime p, $\mathbf{G}_p$ will denote the group of order p.

The class $\mathbf{Her}(\mathcal{A})$ of all hereditary Archimedean semigroups will be characterized in terms of forbidden divisors as follows:

Theorem 3.22 *A semigroup S is hereditary Archimedean if and only if $\mathbf{C}_2$ does not divide S.*

Proof. The class $\mathbf{Her}(\mathcal{A})$ is closed under the formation of divisors and it does not contain $\mathbf{C}_2$, while we have that $\mathbf{C}_2$ does not divide any semigroup from $\mathbf{Her}(\mathcal{A})$.

Conversely, let $\mathbf{C}_2$ not divide S. Suppose that S is not hereditary Archimedean. Then there exist $a,b \in S$ such that $a \uparrow b$ does not hold, i.e. such that $b^n \notin T^1 a T^1$, for any $n \in \mathbf{Z}^+$, where $T = \langle a,b\rangle$. But, now we have that the set $A_0 = T^1 a T^1$ and $A_1 = \langle b\rangle$ form a partition of T which determines a congruence relation on S whose related factor is isomorphic to $\mathbf{C}_2$. This means that $\mathbf{C}_2$ divides S, which contradicts our starting hypothesis. Therefore, we conclude that $S \in \mathbf{Her}(\mathcal{A})$. This completes the proof of the theorem. $\square$

In terms of forbidden divisors we also characterize nil-extensions of rectangular bands.

Theorem 3.23 *A semigroup S is a nil-extension of a rectangular band if and only if $\mathbf{C}_2$ and $\mathbf{G}_p$, for any prime p, do not divide S.*

Proof. The class of all semigroups which are nil-extensions of rectangular bands is closed under the formation of divisors and it does not contain semigroups $\mathbf{C}_2$ and $\mathbf{G}_p$, for any prime p, so $\mathbf{C}_2$ and $\mathbf{G}_p$ do not divide any semigroup from this class.

Conversely, let $\mathbf{C}_2$ and $\mathbf{G}_p$, for any prime p, not divide S. According to Theorem 3.22, $S \in \mathbf{Her}(\mathcal{A})$. Assume an arbitrary $a \in S$. If $\langle a \rangle$ is infinite, then it is isomorphic to the additive semigroup of positive integers, and any of the $\mathbf{G}_p$ groups is a homomorphic image of $\langle a \rangle$. Thus, $\mathbf{G}_p$ divides S, which contradicts our starting hypothesis. Hence, $\langle a \rangle$ is finite, for any $a \in S$, so S is periodic, and it is a nil-extension of a periodic completely simple semigroup K (by Theorem 3.16). In view of this hypothesis, K does not have non-trivial subgroups. So K is a rectangular band. $\square$

Lemma 3.18 *A semigroup S is left simple hereditary left Archimedean if and only if S is a periodic left group.*

Proof. Let S be a left simple semigroup. Then by Corollary 1.5 for $a \in S$ there exists $x \in S$ such that $a = xa$. Since S is a hereditary left Archimedean semigroup then there exists $n \in \mathbf{Z}^+$ and $u \in \langle a, x \rangle$ such that $x^n = ua$ so

$$a = x^n a = uaa = a^{i+1},$$

for some $i \in \mathbf{Z}^+$, because $u \in \langle a, x \rangle$. Thus, S is a periodic semigroup, so $E(S) \neq \emptyset$. Now by Theorem 3.7 we have that S is a periodic left group.

The converse follows from Lemma 3.17 and from Theorem 3.7. $\square$

Theorem 3.24 *Every proper subsemigroup of a semigroup S is left Archimedean if and only if S is hereditary left Archimedean or $|S| = 2$.*

Proof. Let every proper subsemigroup of S be left Archimedean. Then by Theorem 3.21 there are three cases:

(a) S is a left Archimedean. In that case by Lemma 3.17, S is a hereditary left Archimedean semigroup.

(b) S has only two left ideals L_1 and L_2 which are left simple semigroups and $S = L_1 \cup L_2$. In that case, since $L_1, L_2 \neq S$ based on the hypothesis and by Lemma 3.17, L_1 and L_2 are hereditary left Archimedean semigroups, so by Lemma 3.18, L_1 and L_2 are left groups. Now, according to Theorem 3.8, S is a union of a group, i.e. S is completely regular, so, since S is a simple semigroup then by Corollary 2.4, S is completely simple. Using the notation from Theorem 3.6, S is a left zero band I of a right group R_i, $i \in I$. If $|I| \geq 2$, then for $i \in I$ R_i is a hereditary left Archimedean semigroup and based on the dual of Theorem 3.7, and by Theorem 3.17, $E(R_i)$ is both a right and left zero band, whence $|E(R_i)| = 1$, i.e. R_i is a group. So, in that case, by Theorem 3.7, S is a left group, i.e. $E(S)$ is a left zero band, which is impossible, as for $e \in E(L_1)$ and $f \in E(L_2)$ we have $ef \in L_2$ because L_2 is a left ideal of S, and $e \notin L_2$. Thus $|I| = 1$, so S is a right group and $E(S)$ is a right zero band. Then $\langle e, f \rangle = \{e, f\}$ cannot be a left Archimedean semigroup, so $S = \{e, f\}$, i.e. $|S| = 2$.

(c) S has a maximal left ideal $M = L(S)$ which is a hereditary left Archimedean semigroup and $M \subseteq Ma$, for every $a \in T = S - M$. Based on Theorem 1.16, T is a subsemigroup of S. Assume that T is not a left simple semigroup. Then there exist $a \in T$ such that $Ta \neq T$. So, in that case, $M \neq Ma$ whence $S = Ma$. Let $a = xa$ for some $x \in M$. Then $(ax)^n = a^n x \in M$, for every $n \in \mathbf{Z}^+$, $n \geq 2$ and $\langle ax \rangle \cup \langle a \rangle$ is a subsemigroup of S. It is evident that $S = \langle ax \rangle \cup \langle a \rangle$ because $\langle ax \rangle \cup \langle a \rangle$ is not a hereditary left Archimedean semigroup (if it is, then $a^k \in \langle a, ax \rangle ax \in M$ that is impossible). Now we have that $x \in \langle ax \rangle$, i.e. $x = a^k x$ for some $k \in \mathbf{Z}^+$, so $a = xa = a^k xa = a^{k+1}$, whence $T = \langle a \rangle$ is a group, that is a contradict by hypothesis that T is not a left simple semigroup. Thus T is a left simple semigroup and by Lemma 3.18 T is a left group. For $e \in E(T)$ we have that $M \subseteq Me$ and for arbitrary $x \in M$ we have $x = ye$, for some $y \in M$. Hence $x = ye = yee = xe$ and $(ex)^n = ex^n \in M$ for every $n \in \mathbf{Z}^+$. Now, if $A = \{(ex)^2, (ex)^3, \ldots\} \cup \{e\}$ is a proper subsemigroup of S, then A is a hereditary left Archimedean semigroup, so $e \in \langle e, ex^2 \rangle ex^2 \subseteq M$ that is impossible. Thus, $S = A$, whence $ex = (ex)^k$ for some $k \in \mathbf{Z}^+$, $k \geq 2$, i.e. $\{(ex)^2, (ex)^3, \ldots\}$ is a group. For identity $(ex)^{k-1} = ex^{k-1}$ of these group we have that $\{ex^{k-1}, e\}$ is not hereditary left Archimedean. Thus, $S = \{ex^{k-1}, e\}$, i.e. $|S| = 2$.

The converse follows immediately. □

Lemma 3.19 *Every t-Archimedean semigroup contains at most one idempotent.*

Proof. Let e, f be idempotents of a t-Archimedean semigroup S. Then $e = xf$ and $f = ey$, for some $x, y \in S$, whence $e = xf = xf^2 = ef = e^2y = ey = f$. $\square$

Lemma 3.20 *A semigroup S is left simple* (*right simple, simple*) *t-Archimedean if and only if S is a group.*

Proof. We only give the proof when S is a left simple t-Archimedean semigroup. For an element $a \in S$ there exists $x \in S$ such that $a = xa^2$. Since S is t-Archimedean then for a and x there exist $y \in S$ and $n \in \mathbf{Z}^+$ such that $x^n = ay$. Now we have

$$a = xa^2 = x^2a^3 = \cdots = x^na^{n+1} = aya^{n+1}.$$

Thus, S is a regular semigroup and by Lemma 3.19, S has only one idempotent, so according to Theorem 3.18, S is a group.

The converse follows immediately. $\square$

Theorem 3.25 *Let a semigroup S be not left simple. Then every proper left ideal of S is a t-Archimedean semigroup if and only if one of the following conditions holds:*

(a) *S is t-Archimedean;*
(b) *S contains only two left ideals G_1 and G_2 which are groups and $S = G_1 \cup G_2$;*
(c) *S has a maximal left ideal M which is a t-Archimedean semigroup and $M \subseteq Ma$, for every $a \in S - M$.*

Proof. Let every proper left ideal of S be a t-Archimedean semigroup. Then by Theorem 3.21 and Lemma 3.20, we have that one of the conditions (b) or (c) holds, or S is a left Archimedean semigroup. If S is left Archimedean and $L(S) \neq S$, then $L(S)$ is a maximal left ideal of S and it is a t-Archimedean semigroup. Based on Theorems 1.15 and 1.16, there are two cases: $S - L(S)$ is a subsemigroup of S, and then we get a contradiction, or $S - L(S) = \{a\}$, $a^2 \in L(S)$, then S is t-Archimedean. If $L(S) = S$ then it is easy to prove that S is of the type (a).

The converse follows immediately. $\square$

Theorem 3.26 *Every proper subsemigroup of S is t-Archimedean if and only if S is a hereditary t-Archimedean semigroup or S is a two element band.*

Proof. Let every proper subsemigroup of S be t-Archimedean. If S is left simple, then by Lemma 3.20, S is a group. Suppose that S is not left simple. Then one of the conditions (a), (b) and (c) of Theorem 3.25 holds.

If (a) hold then S is a hereditary t-Archimedean semigroup.

Let (b) hold and let e, f be units of the groups G_1 and G_2 respectively. Then based on the proof of Theorem 3.24, we have that $S = \{e, f\}$.

Let (c) hold. Then M is an ideal of S and by Theorem 3.24, $S - M$ is a left simple semigroup, so by Lemma 3.20, $S - M$ is a group. Let $x \in M$ be an arbitrary element and let e be an identity of a group $S - M$. Then $ex, x^k e \in M$, for every $k \in \mathbf{Z}^+$, so $S = \langle e, ex \rangle = \langle x, xe \rangle$. Hence, we have that $x = ey$ for some $y \in S$, so $ex = e(ey)ey = x$. Thus $(xe)^k = x(ex)^{k-1}e$, so $S = \{e, xe, x^2e, \ldots\}$ and $A = \{e, x^2e, x^3e, \ldots\}$ is a subsemigroup of S. If A is t-Archimedean, then $e \in x^k eA \subseteq M$, which is impossible. Therefore, $S = A$, so $M = \{x^2e, x^3e, \ldots\}$ whence we have that $xe = x^k e = (xe)^k$, for some $k \in \mathbf{Z}^+$, so M is a group with the identity $(xe)^{k-1} = x^{k-1}e$. Thus, $S = \{(xe)^{k-1}, e\} = \{x^{k-1}, e\}$ is a band and $|S| = 2$.

The converse follows immediately. $\square$

Exercises

1. If S is not left simple, then every proper left ideal of S is a power-joined subsemigroup of S if and only if one of the following conditions holds:

(a) S is power-joined;
(b) S contains only two left ideals G_1 and G_2 which are periodic groups and $S = G_1 \cup G_2$;
(c) S has a maximal left ideal M which is a power joined subsemigroup of S and $M \subseteq Ma$, for all $a \in S - M$.

2. Every proper subsemigroup of a semigroup S is power-joined if and only if $|S| = 2$ or S is power-joined.

3. A semigroup S is a nilpotent extension of a rectangular band if and only if $\mathbf{C}_2$ does not divide $\mathcal{P}(S)$.

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Chapter 4

The Greatest Semilattice Decompositions of Semigroups

Semilattice decompositions of semigroups were first defined and studied by A. H. Clifford, in 1941. After that, several authors have worked on this very important topic. The existence of the greatest semilattice decomposition of a semigroup was established by M. Yamada, in 1955, and by T. Tamura and N. Kimura, in 1955. The smallest semilattice congruence on a semigroup, in notation σ, has been considered many times. T. Tamura, in 1964, described the congruence σ with the use of the concept of contents. M. Petrich, in 1964, described σ by means of completely prime ideals and filters. Another connection between σ and completely prime ideals and filters was given by R. Šulka, in 1970. T. Tamura, in 1972, and 1975, proved that $\sigma = \longrightarrow^{\infty} \cap (\longrightarrow^{\infty})^{-1}$ and M. S. Putcha, in 1974, proved that σ is the transitive closure of the relation $\longrightarrow \cap \longrightarrow^{-1}$. M. Ćirić and S. Bogdanović, in 1996, gave a new characterization of the greatest semilattice decomposition, i.e. of the least semilattice congruence on a semigroup, by using principal radicals, i.e. completely semiprime ideals of semigroups. Also, they described some special types of semilattice decompositions: semilattices and chains of σ_n- (λ-, λ_n-, τ-, τ_n-) simple semigroups.

Two relations that were introduced by M. S. Putcha and T. Tamura, denoted by $\longrightarrow$ and $\relbar\relbar$, play a crucial role in semilattice decompositions of semigroups. General properties of the graphs that correspond to these relations were studied by M. S. Putcha, in 1974, and the structure of semigroups in which the minimal paths in the graph corresponding to $\longrightarrow$ are bounded was described by M. Ćirić and S. Bogdanović, in 1996.

The celebrated theorem of T. Tamura, in 1956, asserts that every semigroup has the greatest semilattice decomposition and each of its components is a semilattice indecomposable semigroup. But, if we intend to study the structure of a semigroup through its greatest semilattice decomposition, we face the following problem: How to construct this decomposition? Another more convenient version of this problem is: How do we construct the smallest semilattice congruence σ on a semigroup?

One of the best construction methods for σ was also given by T. Tamura, in 1972. He devised the following procedure: We start from the division relation on a semigroup. In the way shown below we define a relation denoted by $\longrightarrow$. Finally, making the transitive closure of $\longrightarrow$ we obtain a quasi-order whose symmetric opening (that is, its natural equivalence) equals σ.

On the other hand, M. S. Putcha, in 1974, proved that the action of the transitive closure and the symmetric opening operators in Tamura's procedure can be permuted. In other words, on the relation $\longrightarrow$ we can apply the symmetric opening operator first, to obtain a relation denoted by $\relbar\relbar$, and applying the transitive closure operator on $\relbar\relbar$, we obtain σ again.

The hardest step in these procedures is the application of the transitive closure operator to relations $\longrightarrow$ and $\relbar\relbar$. As we know, one obtains the transitive closure on a relation by using an iteration procedure. In the general case, the number of iterations applied may be infinite. A natural problem that imposes itself here is the following: Under what conditions on a semigroup S, can the smallest semilattice congruence on S be obtained by applying only a finite number of iterations to $\longrightarrow$ or $\relbar\relbar$?

Problems of this type were first treated in the above mentioned paper of M. S. Putcha. The results which will be presented in this chapter were taken from the papers by M. Ćirić and S. Bogdanović (1996), and by S. Bogdanović, M. Ćirić and Ž. Popović (2000).

4.1 Principal Radicals and Semilattice Decompositions of Semigroups

In this section we introduce the notion of the principal radicals of semigroups, we introduce relations which generalize the well known Green's relations and we describe their basic characteristics.

Let a be an element of a semigroup S and let $n \in \mathbf{Z}^+$. We will use the following notations:

$$\Sigma(a) = \{x \in S \mid a \longrightarrow^{\infty} x\}, \quad \Sigma_n(a) = \{x \in S \mid a \longrightarrow^{n} x\}.$$

First we will give some basic characteristics of these sets.

Lemma 4.1 *Let a be an element of a semigroup S. Then*

$$\Sigma_1(a) = \sqrt{SaS}, \Sigma_n(a) \subseteq \Sigma_{n+1}(a) = \sqrt{S\Sigma_n(a)S}, n \in \mathbf{Z}^+, \Sigma(a) = \bigcup_{n \in \mathbf{Z}^+} \Sigma_n(a).$$

Lemma 4.2 *Let a be an element of a semigroup S. Then $\Sigma(a)$ is the least completely semiprime ideal of S containing a.*

Proof. Let $x \in \Sigma(a)$ and let $b \in S$. Then $a \longrightarrow^{\infty} x$ and since $x \longrightarrow bx$ and $x \longrightarrow xb$, then $a \longrightarrow^{\infty} xb$ and $a \longrightarrow^{\infty} bx$, so $xb, bx \in \Sigma(a)$. Thus, $\Sigma(a)$ is an ideal of S. Let $x \in S$ such that $x^2 \in \Sigma(a)$, i.e. $a \longrightarrow^{\infty} x^2$. Since $x^2 \longrightarrow x$, then $a \longrightarrow^{\infty} x$, so $x \in \Sigma(a)$. Therefore, $\Sigma(a)$ is a completely semiprime ideal of S containing a.

Let I be a completely semiprime ideal of S containing a. Then $SaS \subseteq SIS \subseteq I$, so $\Sigma_1(a) = \sqrt{SaS} \subseteq \sqrt{I} \subseteq I$. Assume that $\Sigma_n(a) \subseteq I$. Then $S\Sigma_n(a)S \subseteq SIS \subseteq I$, so $\Sigma_{n+1}(a) = \sqrt{S\Sigma_n(a)S} \subseteq \sqrt{I} \subseteq I$. Thus, by induction we obtain that $\Sigma_n(a) \subseteq I$ for every $n \in \mathbf{Z}^+$, whence $\Sigma(a) \subseteq I$. Hence $\Sigma(a)$ is the least completely semiprime ideal of S containing a. $\square$

Corollary 4.1 *Let A be a nonempty subset of a semigroup S. Then:*

$$\Sigma(A) \stackrel{\text{def}}{=} \bigcup_{a \in A} \Sigma(a)$$

is the least completely semiprime ideal of S containing A.

If S is a semigroup, then the set $\Sigma(a)$, $a \in S$, will be called the *principal radical* of S. The set of all principal radicals of S will be denoted by Σ_S.

Remark 4.1 If a is an element of a semigroup S, then it is easy to see that $\Sigma_n(a) = \Sigma_n(J(a))$ for every $n \in \mathbf{Z}^+$, whence $\Sigma(a) = \Sigma(J(a))$.

Let S be a semigroup and let $a, b \in S$. Then $a \xrightarrow{h} b$ if $a|_h b^i$, for some $i \in \mathbf{Z}^+$, $a \xrightarrow{h}{}^{n+1} b$ if there exists $x \in S$ such that $a \xrightarrow{h}{}^{n} x \xrightarrow{h} b$, $n \in \mathbf{Z}^+$, and $a \xrightarrow{h} {}^{\infty} b$ if $a \xrightarrow{h} {}^{n} b$ for some $n \in \mathbf{Z}^+$, where h is l or r.

For an element a of a semigroup S and for $n \in \mathbf{Z}^+$ we introduce the following notations

$$\Lambda(a) = \{x \in S \mid a \xrightarrow{l} {}^{\infty} x\}, \quad \Lambda_n(a) = \{x \in S \mid a \xrightarrow{l} {}^{n} x\},$$

$$P(a) = \{x \in S \mid a \xrightarrow{r} {}^{\infty} x\}, \quad P_n(a) = \{x \in S \mid a \xrightarrow{r} {}^{n} x\}.$$

Based on the following results we will present some of the basic characteristics of these sets.

Lemma 4.3 *Let a be an element of a semigroup S. Then:*

$$\Lambda_1(a) = \sqrt{Sa}, \Lambda_n(a) \subseteq \Lambda_{n+1}(a) = \sqrt{S\Lambda_n(a)}, n \in \mathbf{Z}^+, \Lambda(a) = \bigcup_{n \in \mathbf{Z}^+} \Lambda_n(a),$$

$$P_1(a) = \sqrt{aS}, P_n(a) \subseteq P_{n+1}(a) = \sqrt{P_n(a)S}, n \in \mathbf{Z}^+, P(a) = \bigcup_{n \in \mathbf{Z}^+} P_n(a).$$

Lemma 4.4 *Let a be an element of a semigroup S. Then $\Lambda(a)$ ($P(a)$) is the least completely semiprime left (right) ideal of S containing a.*

Proof. Let $x \in \Lambda(a)$ and let $b \in S$. Then $a \xrightarrow{l} {}^{\infty} x$ and since $x \xrightarrow{l} bx$, then $a \xrightarrow{l} {}^{\infty} bx$. Thus $bx \in \Lambda(a)$, so $\Lambda(a)$ is a left ideal of S.

Let $x \in S$ such that $x^2 \in \Lambda(a)$, i.e. such that $a \xrightarrow{l} {}^{\infty} x^2$. Since $x^2 \xrightarrow{l} x$, then $a \xrightarrow{l} {}^{\infty} x$, i.e. $x \in \Lambda(a)$. Therefore, $\Lambda(a)$ is a completely semiprime left ideal of S.

Let L be a completely semiprime left ideal of S containing a. Then $Sa \subseteq L$ whence $\Lambda_1(a) = \sqrt{Sa} \subseteq \sqrt{L} \subseteq L$. Assume that $\Lambda_n(a) \subseteq L$. Then $\Lambda_{n+1}(a) = \sqrt{S\Lambda_n(a)} \subseteq \sqrt{SL} \subseteq \sqrt{L} \subseteq L$. Therefore, by induction we obtain that $\Lambda_n(a) \subseteq L$ for all $a \in S$, whence $\Lambda(a) = \bigcup_{n \in \mathbf{Z}^+} \Lambda_n(a) \subseteq L$. Thus, $\Lambda(a)$ is the least completely semiprime left ideal of S containing a. $\square$

Corollary 4.2 *Let A be a nonempty subset of a semigroup S. Then:*

$$\Lambda(A) \stackrel{\text{def}}{=} \bigcup_{a\in A}\Lambda(a) \quad (\ P(A) \stackrel{\text{def}}{=} \bigcup_{a\in A} P(a)\ )$$

is the least completely semiprime left (*right*) *ideal of S containing A.*

If S is a semigroup, then the sets $\Lambda(a)$ ($P(a)$), $a \in S$, will be called the *principal left* (*right*) *radicals* of S.

Remark 4.2 If a is an element of a semigroup S, then it is easy to see that $\Lambda_n(a) = \Lambda_n(L(a))$ and $P_n(a) = P_n(R(a))$ for every $n \in \mathbf{Z}^+$, whence $\Lambda(a) = \Lambda(L(a))$ and $P(a) = P(R(a))$.

We introduce the following equivalences on a semigroup S:

$$\begin{array}{ll} a\ \sigma\ b \Leftrightarrow \Sigma(a) = \Sigma(b), & a\ \sigma_n\ b \Leftrightarrow \Sigma_n(a) = \Sigma_n(b), \\ a\ \lambda\ b \Leftrightarrow \Lambda(a) = \Lambda(b), & a\ \lambda_n\ b \Leftrightarrow \Lambda_n(a) = \Lambda_n(b), \\ a\ \rho\ b \Leftrightarrow P(a) = P(b), & a\ \rho_n\ b \Leftrightarrow\ P_n(a) = P_n(b), \end{array}$$

$$\tau = \lambda \cap \rho, \qquad \tau_n = \lambda_n \cap \rho_n,$$

$a, b \in S$. Based on the following lemma we prove that these equivalences are generalizations of the well-known Green's equivalences.

Lemma 4.5 *On every semigroup*

$$\begin{array}{ccccccccccccc}
\mathcal{H} & \subseteq & \tau_1 & \subseteq & \tau_2 & \subseteq & \cdots & \subseteq & \tau_n & \subseteq & \cdots & \subseteq & \tau \\
|\cap & & |\cap & & |\cap & & & & |\cap & & & & |\cap \\
\mathcal{L} & \subseteq & \lambda_1 & \subseteq & \lambda_2 & \subseteq & \cdots & \subseteq & \lambda_n & \subseteq & \cdots & \subseteq & \lambda \\
|\cap & & |\cap & & & & & & & & & & |\cap \\
\mathcal{J} & \subseteq & \sigma_1 & \subseteq & \sigma_2 & \subseteq & \cdots & \subseteq & \sigma_n & \subseteq & \cdots & \subseteq & \sigma \\
|\cup & & |\cup & & & & & & & & & & |\cup \\
\mathcal{R} & \subseteq & \rho_1 & \subseteq & \rho_2 & \subseteq & \cdots & \subseteq & \rho_n & \subseteq & \cdots & \subseteq & \rho
\end{array}$$

Proof. The inclusions in the third row of the previous diagram follow from Lemma 4.1. The inclusions in the second and fourth row from Lemma 4.3 and from this the inclusions in the first row follow. The inclusion $\lambda \subseteq \sigma$ follows from Lemmas 4.2 and 4.4.

Assume that $(a, b) \in \lambda_1$, i.e. that $\Lambda_1(a) = \Lambda_1(b)$. Let $x \in \Sigma_1(a)$, i.e. let $x^n = uav$ for some $n \in \mathbf{Z}^+$, $u, v \in S$. Then $vua \in \Lambda_1(a) =$

$\Lambda_1(b)$, whence there exists $k \in \mathbf{Z}^+$, $w \in S$ such that $(vua)^k = wb$. Thus $x^{n(k+1)} = (uav)^{k+1} = ua(vua)^k v = uawbv \in SbS$. Therefore, $x \in \Sigma_1(b)$, i.e. $\Sigma_1(a) \subseteq \Sigma_1(b)$. Similarly we prove that $\Sigma_1(b) \subseteq \Sigma_1(a)$. Hence, $(a,b) \in \Sigma_1$, so $\lambda_1 \subseteq \sigma_1$.

The rest of the proof follows immediately. □

If π is one of the equivalences from the diagram of Lemma 4.5, defined on a semigroup S, then S is π-*simple* if $\pi = S \times S$. It is clear that $\mathcal{J}$ ($\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$)-simple semigroups are simple semigroups (left simple semigroups, right simple semigroups, groups) and that σ_1 (λ_1, ρ_1, τ_1)-simple semigroups are Archimedean (left Archimedean, right Archimedean, t-Archimedean semigroups).

Lemma 4.6 *On every semigroup*

(i) $\sigma_n \subseteq \longrightarrow^n \cap (\longrightarrow^n)^{-1}$ *for every* $n \in \mathbf{Z}^+$;

(ii) $\lambda_n \subseteq \xrightarrow{l}{}^n \cap (\xrightarrow{l}{}^n)^{-1}$ *for every* $n \in \mathbf{Z}^+$;

(iii) $\sigma = \longrightarrow^\infty \cap (\longrightarrow^\infty)^{-1}$;

(iv) $\lambda = \xrightarrow{l}{}^\infty \cap (\xrightarrow{l}{}^\infty)^{-1}$.

Proof. (i) and (ii) This follows immediately.

(iii) Follows from the definition of a principal radical and by Lemma 4.2.

(iv) Follows from the definition of a principal left radical and by Lemma 4.4. □

Lemma 4.7 *Let* a, b, c *be elements of a semigroup* S. *Then:*

(i) $\Sigma(a) = \Sigma(a^2)$,

(ii) $\Sigma(ab) \subseteq \Sigma(a) \cap \Sigma(b)$,

(iii) $\Sigma(abc) = \Sigma(acb)$,

(iv) $\Sigma(ba) = \Sigma(a^n b^n)$,

for every $n \in \mathbf{Z}^+$.

Proof. (i) According to Lemma 4.2 we have $a^2 \in \Sigma(a)$ and $\Sigma(a^2) \subseteq \Sigma(a)$. Since $\Sigma(a^2)$ is a completely semiprime ideal of S and $a^2 \in \Sigma(a^2)$ we then have that $a \in \Sigma(a^2)$ and by Lemma 4.2 we obtain $\Sigma(a) \subseteq \Sigma(a^2)$. Thus, (i) holds.

(ii) Since $\Sigma(a)$ and $\Sigma(b)$ are the ideals of S, then $ab \in \Sigma(a)$ and $ab \in \Sigma(b)$ and from Lemma 4.2 we have (ii).

(iii) From (i) and (ii) we have

$$\Sigma(abc) = \Sigma(abcabc) \subseteq \Sigma(bcabc) = \Sigma(bcabcbcabc) \subseteq$$

$$\subseteq \Sigma(cbca) = \Sigma(cbcacbca) \subseteq \Sigma(acb).$$

Thus, $\Sigma(abc) \subseteq \Sigma(acb)$. Since the opposite inclusion also holds, we then have that (iii) holds.

(iv) From (i) and (ii) we have

$$\Sigma(ab) = \Sigma(abab) \subseteq \Sigma(ba) = \Sigma(baba) \subseteq \Sigma(ab),$$

i.e. $\Sigma(ab) = \Sigma(ba)$.

Assume that $\Sigma(ba) = \Sigma(a^k b^k), k \in \mathbf{Z}^+$. Then based on (i), (ii) and (iii) we have

$$\Sigma(ba) = \Sigma(a^k b^k) = \Sigma(a^k b^k a^k b^k) = \Sigma(a^{2k} b^{2k}) \subseteq \Sigma(a^{k+1} b^{k+1}) \subseteq \Sigma(ab) = \Sigma(ba).$$

Thus $\Sigma(ba) = \Sigma(a^{k+1} b^{k+1})$ and by induction we have that (iv) holds. □

Lemma 4.8 *Let a, b, c be elements of a semigroup S. Then:*

$$a \longrightarrow^n b \;\Rightarrow\; \Sigma(bc) \subseteq \Sigma(ac).$$

Proof. Let $n = 1$, i.e. $b^m = xay$ for some $x, y \in S$ and $m \in \mathbf{Z}^+$. Then from Lemma 4.7 we have that

$$\Sigma(bc) = \Sigma(c^m b^m) = \Sigma(c^m xay) = \Sigma(xa^m ay) \subseteq \Sigma(ca) = \Sigma(ac).$$

Thus, the assertion holds for $n = 1$.

Assume that the assertion holds for some $n \in \mathbf{Z}^+$ and assume that $a \longrightarrow^{n+1} b$, i.e. $a \longrightarrow^n x \longrightarrow b$ for some $x \in S$. Then $\Sigma(bc) \subseteq \Sigma(xc) \subseteq \Sigma(ac)$. By induction we obtain that the assertion of the lemma holds. □

Lemma 4.9 *Let ξ be a semilattice congruence on a semigroup S and let $n \in \mathbf{Z}^+$.*

(i) *Let $a, b \in S$ and $a \longrightarrow^n b$. Then $b\xi \leq a\xi$ in the semilattice S/ξ.*

(ii) *Let A be a ξ-class of S and $a, b \in A$. Then $a \longrightarrow^n b$ in S if and only if $a \longrightarrow^n b$ in A.*

Proof. (i) Let $n = 1$. Then $b^m = xay$ for some $x, y \in S$ and $m \in \mathbf{Z}^+$, whence $b\xi = (b^m)\xi = (xay)\xi = (xy)\xi a\xi \le a\xi$.

Assume that (i) holds for $n \in \mathbf{Z}^+$ and that $a \longrightarrow^{n+1} b$. Then $a \longrightarrow^n x \longrightarrow b$ for some $x \in S$, whence $b\xi \le x\xi \le a\xi$. By induction we have that (i) holds.

(ii) Let $n = 1$. Then $b^m = xay$, for some $x, y \in S$ and $m \in \mathbf{Z}^+$. From this it follows that

$$b\xi = (b^m)\xi = (xay)\xi = (x\xi)(a\xi)(y\xi).$$

Thus $(b\xi)(x\xi) = (y\xi)(b\xi) = b\xi$, i.e. $bx, yb \in A$. Hence $b^{m+2} = (bx)a(yb) \in AaA$, i.e. $a \longrightarrow b$ in A. Thus, (ii) holds for $n = 1$.

Assume that (ii) holds for $n \in \mathbf{Z}^+$ and let $a \longrightarrow^{n+1} b$ in S. Then $a \longrightarrow^n x \longrightarrow b$ for some $x \in S$, and from (i) we obtain $a\xi \le x\xi \le a\xi = b\xi$, i.e. $x\xi = b\xi$, i.e. $x \in A$. Thus, (ii) holds for $n = 1$ and based on the hypothesis we have that $a \longrightarrow^n x$ in A and $x \longrightarrow b$ in A, whence $a \longrightarrow^{n+1} b$ in A. Therefore, by induction we obtain (ii). □

Recall that on a semigroup S we have the following equivalence relation:

$$a\sigma b \quad \Leftrightarrow \quad \Sigma(a) = \Sigma(b).$$

Theorem 4.1 *On a semigroup S equivalence σ is the smallest semilattice congruence and every σ-class is semilattice indecomposable.*

Proof. From Lemmas 4.7 and 4.8 we have that σ is a semilattice congruence on S.

Let ξ be a semilattice conguence on S and let $a\sigma b$. Then $a \longrightarrow^\infty b$ and $b \longrightarrow^\infty a$. According to Lemma 4.9 (i) we have that $a\xi \le b\xi$ and $b\xi \le a\xi$ in S/ξ, i.e. $a\xi = b\xi$. Thus, $a\xi b$, whence $\sigma \subseteq \xi$. Hence, σ is the smallest semilattice congruence on S.

Let A be a σ-class of S, let σ^* be a relation of the type σ on A and let $a, b \in A$. Then $a\sigma b$ in S, i.e. $a \longrightarrow^\infty b$ and $b \longrightarrow^\infty a$ in S. From Lemma 4.9 (ii) we have that $a \longrightarrow^\infty b$ and $b \longrightarrow^\infty a$ in A, whence $a\sigma^* b$. Thus, σ^* is a universal relation on A and since σ^* is the smallest semilattice congruence on A, we then have that A is a semilattice indecomposable semigroup. □

The following theorem is one of the main results of this section. By means of this we describe the structure of the partially ordered set of principal radicals of a semigroup.

Theorem 4.2 *For elements a, b of a semigroup S $\Sigma(ab) = \Sigma(a) \cap \Sigma(b)$. Furthermore, the set Σ_S of all the principal radicals of S, partially ordered by inclusion, is the greatest semilattice homomorphic image of S.*

Proof. From Lemma 4.7 we obtain that $\Sigma(ab) \subseteq \Sigma(a) \cap \Sigma(b)$. Assume $x \in \Sigma(a) \cap \Sigma(b)$. Then $a \longrightarrow^{\infty} x$ and $b \longrightarrow^{\infty} x$, so by Lemma 4.8 we obtain that

$$\Sigma(ab) \supseteq \Sigma(xb) \supseteq \Sigma(x^2) = \Sigma(x).$$

Thus $x \in \Sigma(ab)$, so $\Sigma(a) \cap \Sigma(b) \subseteq \Sigma(ab)$. Hence $\Sigma(a) \cap \Sigma(b) = \Sigma(ab)$. Therefore Σ_S is a semilattice and $a \mapsto \Sigma(a)$, $(a \in S)$ is a homomorphism of S onto Σ_S with the kernel σ. From Lemma 4.6 (iii) and based on Theorem 4.1, σ is the smallest semilattice congruence on S, whence Σ_S is the greatest semilattice homomorphic image of S. $\square$

Lemma 4.10 *Let ξ be an equivalence relation on a semigroup S such that $xy\xi xyx\xi yx$ for all $x, y \in S^1$ and $1\xi 1$. Then*

(a) *$xay\xi xa^k y$, for all $x, a, y \in S^1$ and $k \in \mathbf{Z}^+$;*
(b) *$xyz\xi xzy$, for all $x, y, z \in S$.*

Proof. Assume that $x, a, y \in S^1$. Then $xay\xi yxa\xi ayxa\xi xa^2 y$. Then, (a) holds for $k = 2$. Assume that $xay\xi xa^k y$, for some $k \in \mathbf{Z}^+, k \geq 2$. Then based on the hypothesis we have that

$$xay\xi xa^k y = (xa^{k-1})ay\xi(xa^{k-1})a^2 y = xa^{k+1}y.$$

So, by induction we obtain that (a) holds.

Assume that $x, y, z \in S$. Then based on the hypothesis and from (a) we have

$$\begin{aligned} xyz\ & \xi x(yz)^2\xi(xyzyz)^2 = (xyzyzx)(yz)^2\xi(xyzyzx)(yz) \\ & = (xy)(zyzx)(yz)\xi(xy)(zyzx)^2(yz) = x(yz)^2(xzyzxyz) \\ & \xi x(yz)(xzyzxyz) = (xyzxzy)(zxyz)\xi(xyzxzy)^2(zxyz) \\ & = (xyzxzyxy)(yz)(xzyzxyz)\xi(xyzxzyx)(yz)^2(xzyzxyz) \\ & \xi(xyzxzyxy)(zyzx)^2(yz)\xi(xyzxzyxy)(zyzx)(yz) \\ & = (xyzxzyx)(yz)^2(xyz)\xi(xyzxzyx)(yz)(xyz) = (xyzxzy)(xyz)^2 \\ & \xi(xyzxzy)(xyz) = (xyz)(xzy)(xyz)\xi(xyz)(xzy). \end{aligned}$$

Thus, $xyz\xi(xyz)(xzy)$. Similarly, it can be proved that $xzy\xi(xzy)(xyz)$. Therefore, $xyz\xi xzy$. $\square$

Using the previous lemma, we can prove the following theorem:

Theorem 4.3 *On every semigroup* S, $\sigma = \text{---}^{\infty}$.

Proof. It is easy to see that $xy \text{---} xyx \text{---} yx$, for all $x, y \in S^1$, whence we obtain that ---^{∞} is an equivalence relation for which the conditions of Lemma 4.10 hold.

Assume $a, b \in S$ such that $a \longrightarrow b$, i.e. $b^m = uav$ for some $u, v \in S^1, m \in \mathbf{Z}^+$, and assume $x, y \in S^1$. From Lemma 4.10 we have

$$xaby \text{---}^{\infty} xab^m y = xauavy = (xa)u(avy) \text{---}^{\infty} (xa)(avy)u =$$

$$xa^2(vyu) \text{---}^{\infty} xa(vyu) = x(avy)u \text{---}^{\infty} xu(avy) =$$

$$x(uav)y = xb^m y \text{---}^{\infty} xby.$$

Thus, from $a \longrightarrow b$ it follows that $xaby \text{---}^{\infty} xby$, for all $x, y \in S^1$. Similarly it can be proved that $b \longrightarrow a$ implies $xaby \text{---}^{\infty} xay$, for all $x, y \in S^1$. Therefore, $a \text{---} b$ implies $xay \text{---}^{\infty} xby$, for all $x, y \in S^1$. By induction we obtain that for every $n \in \mathbf{Z}^+, a \text{---}^{n} b$ implies $xay \text{---}^{\infty} xby$ for all $x, y \in S^1$. Thus, ---^{∞} is a congruence relation on S. It is clear that ---^{∞} is a semilattice congruence and by Theorem 4.1 we have that $\sigma \subseteq \text{---}^{\infty}$. On the other hand, $\text{---}^{\infty} \subseteq \longrightarrow^{\infty} \cap (\longrightarrow^{\infty})^{-1} = \sigma$. Thus, $\text{---}^{\infty} = \sigma$. □

Using the previous theorem we describe the principal filters of a semigroup.

We remind the reader that, for an element a of a semigroup S, the intersection of all filters of S which contain a we call the *principal filter* of S *generated* by a, and denote by $N(a)$. It is the smallest filter containing an element a of a semigroup S.

Corollary 4.3 *Let* a *be an element of a semigroup* S. *Then:*

$$N(a) = \{x \in S \mid x \longrightarrow^{\infty} a\}.$$

Proof. Let $a \in S$ and let

$$A = \{x \in S \mid x \longrightarrow^{\infty} a\}.$$

Assume $x, y \in A$. Then $x \longrightarrow^{\infty} a$ and $y \longrightarrow^{\infty} a$, so $a \in \Sigma(x) \cap \Sigma(y) = \Sigma(xy)$. Thus $xy \longrightarrow^{\infty} a$, so $xy \in A$, i.e. A is a subsemigroup of S.

Let $x, y \in S$ and let $xy \in A$. Then $xy \longrightarrow^{\infty} a$ and since $x \longrightarrow xy$ and $y \longrightarrow xy$, then $x \longrightarrow^{\infty} a$ and $y \longrightarrow^{\infty} a$. Thus A is a filter.

Let $y \longrightarrow a$, i.e. let $a^n = uyv$, for some $n \in \mathbf{Z}^+$, $u, v \in S$. Since $a \in N(a)$, then $uyv = a^n \in N(a)$ and since $N(a)$ is a filter, then $u, y, v \in N(a)$, i.e. $y \in N(a)$. By induction we prove that $x \in N(a)$ for all $x \in A$, so $A \subseteq N(a)$. Since A is a filter, then $A = N(a)$. $\square$

Corollary 4.4 *Let a, b be elements of a semigroup S. Then:*

$$a\,\sigma\,b \quad \Leftrightarrow \quad N(a) = N(b).$$

Proof. Let $a\sigma b$. Then $b \in \Sigma(a)$ and $a \in \Sigma(b)$, i.e. $a \longrightarrow^\infty b$ and $b \longrightarrow^\infty a$, whence $a \in N(b)$ and $b \in N(a)$, so $N(b) \subseteq N(a)$ and $N(a) \subseteq N(b)$, i.e. $N(a) = N(b)$.

Conversely, let $N(a) = N(b)$. Then $a \in N(b)$ and $b \in N(a)$, i.e. $b \longrightarrow^\infty a$ and $a \longrightarrow^\infty b$. Thus, $a \in \Sigma(b)$ and $b \in \Sigma(a)$ whence $\Sigma(a) \subseteq \Sigma(b)$ and $\Sigma(b) \subseteq \Sigma(a)$, so $a\sigma b$. $\square$

We give the new proof for the known result concerning completely semiprime ideals of a semigroup, without Zorn's Lemma.

Corollary 4.5 *Let I be a completely semiprime ideal of a semigroup S and let $a \in S$ such that $a \notin I$. Then there exists a completely prime ideal P of S such that $I \subseteq P$ and $a \notin P$.*

Proof. Let $P = S - N(a)$. Then P is a completely prime ideal of S and $a \notin P$. Let $x \in I \cap N(a)$. Then from Corollary 4.3 it follows that $x \longrightarrow^\infty a$, so $a \in \Sigma(x) \subseteq I$ (from Lemma 4.2). Thus, we obtain that $a \in I$, which is not possible. Hence, $I \cap N(a) = \varnothing$, whence $I \subseteq P$. $\square$

Corollary 4.6 *Every completely semiprime ideal of a semigroup S is an intersection of completely prime ideals of S.*

Proof. This follows from Corollary 4.5. $\square$

Corollary 4.7 *On a semigroup S the following conditions are equivalent:*

(i) *S is semilattice indecomposable;*
(ii) *S is σ-simple;*
(iii) *S has no proper completely semiprime ideals;*
(iv) *S has no proper completely prime ideals.*

Proof. It follows from Theorem 4.1 and Corollary 4.6. □

As we have seen, every completely semiprime ideal of a semigroup is the intersection of all the completely prime ideals containing it. But, this is not true for completely semiprime left (right) ideals. For example, in the semigroup given by

$$\langle a, e \mid a^3 = a,\ e^2 = e,\ ae = ea^2 = e \rangle,$$

there exists a completely semiprime left ideal which is not an intersection of completely prime left ideals.

Based on the following theorem we give some characterizations of semigroups in which every completely semiprime left ideal is an intersection of completely prime left ideals.

Theorem 4.4 *The following conditions on a semigroup S are equivalent:*

(i) *every completely semiprime left ideal of S is an intersection of completely prime left ideals of S;*

(ii) $(\forall a, b, c \in S)\ a \xrightarrow{l}{}^{\infty} c \wedge b \xrightarrow{l}{}^{\infty} c \Rightarrow ab \xrightarrow{l}{}^{\infty} c$;

(iii) *for every $a \in S$, $\{x \in S \mid x \xrightarrow{l}{}^{\infty} a\}$ is the least right filter of S containing a.*

Proof. (ii)⇒(iii) Let $F = \{x \in S \mid x \xrightarrow{l}{}^{\infty} a\}$. Assume $x, y \in S$ such that $xy \in F$. Then $xy \xrightarrow{l}{}^{\infty} a$ and since $y \xrightarrow{l} xy$, then $y \xrightarrow{l}{}^{\infty} a$, so $y \in F$. Thus, F is a right consistent subset of S.

Let $x, y \in F$. Then $x \xrightarrow{l}{}^{\infty} a$ and $y \xrightarrow{l}{}^{\infty} a$, so by (ii) we obtain that $xy \xrightarrow{l}{}^{\infty} a$. Thus $xy \in F$, so F is a subsemigroup of S. Hence, F is a right filter of S containig a.

Let G be a right filter of S containing a. Assume $y \in S$ such that $y \xrightarrow{l} a$. Then $a^n = uy$ for some $n \in \mathbf{Z}^+$, $u \in S$, so by $uy = a^n \in G$ it follows that $y \in G$. By induction we show that $x \xrightarrow{l}{}^{\infty} a$ implies $x \in G$, whence $F \subseteq G$. Therefore, F is the smallest right filter of S containing a.

(iii)⇒(i) Let (iii) hold and let A be an arbitrary completely semiprime left ideal of S. Let M be the intersection of all completely prime left ideals of S containing A. Assume that $a \in M - A$. From (iii) it follows that the set $F = \{x \in S \mid x \xrightarrow{l}{}^{\infty} a\}$ is a right filter of S, so $L = S - F$ is a completely prime left ideal of S. Assume that $x \in A$. If $x \in F$, i.e. if $x \overset{l}{\longrightarrow}^{\infty} a$, then $a \in \Lambda(x) \subseteq A$, which is not possible. Thus $x \in L$, so $A \subseteq L$, whence $M \subseteq L$. But then $a \in L$ and $a \in F$, which is not possible. Therefore, $M = A$, so A is an intersection of completely prime left ideals.

(i)$\Rightarrow$(ii) Let every completely semiprime left ideal of S be an intersection of completely prime left ideals of S. Let $a, b \in S$ and let L be an arbitrary completely prime left ideal containing $\Lambda(ab)$. Then $ab \in L$ whence $a \in L$ or $b \in L$, since L is completely prime. Since L also is completely semiprime, then $\Lambda(a) \subseteq L$ or $\Lambda(b) \subseteq L$, whence $\Lambda(a) \cap \Lambda(b) \subseteq L$, so from the hypothesis we obtain that $\Lambda(a) \cap \Lambda(b) \subseteq \Lambda(ab)$, so (ii) holds. □

Exercises

1. Let $\mathfrak{C}$ be a class of semigroups. A congruence ξ on a semigroup S is the smallest $\mathfrak{C}$-*congruence* on S if ξ is the smallest element in the set of all $\mathfrak{C}$-congruences on S. The decomposition and the factor which corresponds to the smallest $\mathfrak{C}$-congruence on S we call the greatest $\mathfrak{C}$-*decomposition* and the greatest $\mathfrak{C}$-*homomorphic image* of S, respectively.

Let $\mathcal{V}$ be a variety of semigroups. Prove that every semigroup has the smallest $\mathcal{V}$-congruence, i.e. the greatest $\mathcal{V}$-decomposition.

2. Let A be a non-empty subset of a semigroup S. Then $\Sigma(A) = \cup_{a \in A}\Sigma(a)$ is the smallest completely semiprime ideal of S which contains A.

3. If a is an element of a semigroup S, then $\Sigma(a) = \Sigma(J(a))$ and $\Sigma_n(a) = \Sigma_n(J(a))$, for every $n \in \mathbf{Z}^+$.

4. Let $a_1, a_2, \dots, a_n$ be elements of a semigroup S, $n \in \mathbf{Z}^+$. Then $\Sigma(a_1a_2\cdots a_n) = \Sigma(a_{1\pi}a_{2\pi}\cdots a_{n\pi})$, for every permutation π of the set $\{1, 2, \dots, n\}$.

5. Let C be a σ-class of an element a of a semigroup S. Then $C = \Sigma(a) \cap N(a)$.

6. If A^+ is a free semigroup over an alphabet A, then:

(a) $\Sigma(u) = \{w \in A^+ \mid c(u) \subseteq c(w)\},\ u \in A^+$;
(b) $N(u) = \{w \in A^+ \mid c(u) \supseteq c(w)\},\ u \in A^+$;
(c) $u\sigma v \ \Leftrightarrow\ c(u) = c(v),\ u, v \in A^+$.

7. A rectangular band of semilattice indecomposable semigroups is a semilattice indecomposable semigroup.

8. Let $a_1, a_2, \dots, a_n \in S^1$, where S is a semigroup. By $\mathcal{C}(a_1, a_2, \dots, a_n)$ we denote the subsemigroup of S^1 which consists of the products of elements $a_1, a_2, \dots, a_n$ in which every element a_i is notified at least once. Prove that $\mathcal{C}(a_1, a_2, \dots, a_n)$ is an indecomposable subsemigroup of S^1.

9. Let a and b be elements of a semigroup S. Then $a\sigma b$ if and only if for all $x, y \in S^1$ there exists a semilattice indecomposable subsemigroup T of S such that $xay, xby \in T$.

10. The following conditions on a semigroup S are equivalent:

(a) for all $a, b \in S$, from $ab, ba \in E(S)$ it follows that $ab = ba$;
(b) every $\mathcal{J}$-class of S contains at most one idempotent;
(c) S is a semilattice of semilattice indecomposable semigroups such that every semigroup contains at most one idempotent and group ideal whenever it contains an idempotent;
(d) S is a semilattice of semigroups such that every semigroup contains at most one idempotent.

11. A semigroup S is *separative* if for all $a, b \in S$, $a^2 = ab$ and $b^2 = ba$ implies $a = b$, and $a^2 = ba$ and $b^2 = ab$ implies $a = b$. Prove that a semigroup S is separative if and only if S is a semilattice of cancellative semigroups.

12. The following conditions on a semigroup S are equivalent:

(a) $\Sigma(0) = 0$;
(b) S has no non-zero nilpotents;
(c) S is a subdirect product of semigroups without a divisor of zero.

13. Let S be a regular semigroup. Then $\sigma = \mathcal{D}^{\#} = \mathcal{J}^{\#}$, and if β is the smallest band congruence on S, then $\mathcal{H}^{\#} \subseteq \beta \subseteq \mathcal{L}^{\#} \cap \mathcal{R}^{\#}$. If S is an inverse semigroup, then $\mathcal{H}^{\#} \subseteq \sigma = \mathcal{R}^{\#} = \mathcal{L}^{\#} = \mathcal{D}^{\#} = \mathcal{J}^{\#}$.

References

S. Bogdanović and M. Ćirić [13]; S. Bogdanović, M. Ćirić and N. Stevanović [1]; I. E. Burmistrovich [1]; M. Ćirić and S. Bogdanović [3], [8]; M. Ćirić, S. Bogdanović and J. Kovačević [1]; E. Hewit and H. S. Zuckerman [1]; D. G. Johnson and J. E. Kist [1]; J. Luh [1]; M. Petrich [1]; M. S. Putcha [5]; M. S. Putcha and J. Weissglass [1]; R. Šulka [1]; T. Tamura [2], [3], [6], [13], [15]; T. Tamura and N. Kimura [2]; G. Thierrin [8]; M. Yamada [1]; R. Yoshida and M. Yamada [1].

4.2 Semilattices of σ_n-simple Semigroups

In Section 6.1 we proved that the relation σ is the smallest semilattice congruence on every semigroup. In this section we will study the conditions under which the relation σ_n is a congruence, i.e. we will consider the semilattices of σ_n-simple semigroups.

Lemma 4.11 *Let a, b be elements of a semigroup S. Then:*

$$\Sigma_n(ab) \subseteq \Sigma_n(a) \cap \Sigma_n(b).$$

Proof. This follows since $ab \longrightarrow x$ implies that $a \longrightarrow x$ and $b \longrightarrow x$. $\square$

Let ϱ be an arbitrary relation on a semigroup S. Recall that the *radical* $R(\varrho)$ of a binary relation ϱ on a semigroup S is defined by:

$$(a,b) \in R(\varrho) \Leftrightarrow (\exists m, n \in \mathbf{Z}^+)\, a^m\ \varrho b^n.$$

Based on the following theorem we characterize the semilattices of σ_n-simple semigroups.

Theorem 4.5 *Let $n \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of σ_n-simple semigroups;*
(ii) *S is a band of σ_n-simple semigroups;*
(iii) *every σ_n-class of S is a subsemigroup;*
(iv) $(\forall a \in S)\ a\ \sigma_n a^2$;
(v) $(\forall a, b \in S)\ a \longrightarrow^n b \Rightarrow a^2 \longrightarrow^n b$;
(vi) $(\forall a, b, c \in S)\ a \longrightarrow^n c \wedge b \longrightarrow^n c \Rightarrow ab \longrightarrow^n c$;
(vii) *for every $a \in S$, $\Sigma_n(a)$ is an ideal of S;*
(viii) $(\forall a, b \in S)\ \Sigma_n(ab) = \Sigma_n(a) \cap \Sigma_n(b)$;
(ix) *for every $a \in S$, $N(a) = \{x \in S \mid x \longrightarrow^n a\}$;*
(x) $\longrightarrow^n$ *is a quasi-order on S;*
(xi) $\sigma_n = \longrightarrow^n \cap (\longrightarrow^n)^{-1}$ *on S;*
(xii) $\text{---}^n \subseteq \sigma_n$;
(xiii) $\overset{l}{\text{---}}^n \subseteq \sigma_n$;
(xiv) $R(\sigma_n) = \sigma_n$.

Proof. (i)$\Rightarrow$(ii) and (iii)$\Rightarrow$(iv) This follows immediately.

(ii)$\Rightarrow$(i) If S is a band of σ_n-simple semigroups, then by Corollary 3.7 S is a semilattice of semigroups which are rectangular bands of σ_n-simple semigroups. Since a rectangular band of σ_n-simple semigroups is σ_n-simple, we obtain (i).

(iv)$\Leftrightarrow$(v) This follows from the definition of the relation σ_n.

(v)$\Rightarrow$(x) Let $a, b \in S$ and let $a \longrightarrow^{n+1} b$. Then $a \longrightarrow x \longrightarrow^n b$ for some $x \in S$. From (v) it follows that $x^k \longrightarrow^n b$ for every $k \in \mathbf{Z}^+$. On the other hand, there exists $k \in \mathbf{Z}^+$ such that $x^k \in SaS$. Let $y \in S$ such that $x^k \longrightarrow y \longrightarrow^{n-1} b$, if $n \geq 2$, and $y = b$, if $n = 1$. Then there exists $m \in \mathbf{Z}^+$

such that $y^m \in Sx^kS \subseteq SaS$. Thus $a \longrightarrow y$, whence $a \longrightarrow^n b$. Therefore $\longrightarrow^n = \longrightarrow^{n+1}$, whence $\longrightarrow^n = \longrightarrow^\infty$, so $\longrightarrow^n$ is transitive.

(x)$\Rightarrow$(vii) If $\longrightarrow^n$ is a transitive relation, then $\longrightarrow^n = \longrightarrow^\infty$, whence $\Sigma_n(a) = \Sigma(a)$, for every $a \in S$, so $\Sigma_n(a)$ is an ideal of S.

(vii)$\Rightarrow$(viii) If $\Sigma_n(a)$ is an ideal for every $a \in S$, then $\Sigma_n(a) = \Sigma(a)$ for every $a \in S$, so by Theorem 4.2 it follows that (viii) holds.

(viii)$\Rightarrow$(i) From (viii) it follows that the relation σ_n is a semilattice congruence on S, so S is a semilattice Y of semigroups $S_\alpha,\ \alpha \in Y$, where S_α are σ_n-classes of S. Let $\alpha \in Y$ and let $a, b \in S_\alpha$. Then $a\sigma_n b$, so $a \longrightarrow^n b$ in S. According to Lemma 4.9 we obtain that $a \longrightarrow^n b$ in S_α, whence S_α is a σ_n-simple semigroup.

(i)$\Rightarrow$(v) Let S be a semilattice Y of σ_n-simple semigroups $S_\alpha,\ \alpha \in Y$. Let $a, b \in S$ such that $a \longrightarrow^n b$. Then $a \in S_\alpha$ and $b \in S_\beta$ for some $\alpha, \beta \in Y$, and from Lemma 4.9 it follows that $\alpha \geq \beta$. Now we have that $a^2b \in S_\beta$ whence $a^2b \longrightarrow^n b$ in S_β, so $a^2 \longrightarrow^n b$ in S.

(viii)$\Rightarrow$(vi) This follows immediately.

(vi)$\Rightarrow$(viii) From (vi) it follows that $\Sigma_n(a) \cap \Sigma_n(b) \subseteq \Sigma_n(ab)$ for all $a, b \in S$, so by Lemma 4.11 we obtain that (viii) holds.

(x)$\Rightarrow$(ix) If $\longrightarrow^n$ is a transitive relation, then $\longrightarrow^n = \longrightarrow^\infty$, so by Corollary 4.3 we obtain (ix).

(ix)$\Rightarrow$(vi) Let $a, b, c \in S$ such that $a \longrightarrow^n c$ and $b \longrightarrow^n c$. Then $a, b \in N(c)$ and since $N(c)$ is a subsemigroup of S, then $ab \in N(c)$, i.e. $ab \longrightarrow^n c$.

(viii)$\Rightarrow$(iii) Let A be a σ_n-class of S and let $a, b \in A$. Then $a\sigma_n b$ so $\Sigma_n(a) = \Sigma_n(b)$. From this and from (viii) it follows that $\Sigma_n(ab) = \Sigma_n(a) \cap \Sigma_n(b) = \Sigma_n(a)$. Thus $ab\sigma_n a$, i.e. $ab \in A$, so (iii) holds.

(x)$\Rightarrow$(xi). Since (x)$\Leftrightarrow$(vii), then we obtain that $\sigma_n = \sigma$ and $\longrightarrow^n = \longrightarrow^\infty$, so by Lemma 4.6 we obtain (xi).

(xi)$\Rightarrow$(iv) This follows immediately.

(x)$\Rightarrow$(xii) Let S be a semilattice Y of σ_n-simple semigroups S_α, $\alpha \in Y$. Assume $a, b \in S$ such that $a \overline{}^{\,n} b$. Based on Lemma 4.9 we have that $a, b \in S_\alpha$, for some $\alpha \in Y$, whence $(a, b) \in \sigma_n$. Therefore, (xii) holds.

(xii)$\Rightarrow$(xiii) This is an immediate consequence of the inclusion $\overset{l}{\overline{}}{}^{n} \subseteq \overline{}^{\,n}$.

(xiii)$\Rightarrow$(iv) Since $a \overset{l}{\overline{}}{}^{n} a^2$, for each $a \in S$, then (xiii) yields (iv).

(x)$\Rightarrow$(xiv) The inclusion $\sigma_n \subseteq R(\sigma_n)$ always holds, so it we have to prove the opposite inclusion. Assume $a, b \in S$ such that $(a, b) \in R(\sigma_n)$.

Then $a^k \,\sigma_n\, b^m$, for some $k, m \in \mathbf{Z}^+$, and since σ_n is a semilattice congruence on S (based on the hypothesis) we have $a \,\sigma_n\, a^k \,\sigma_n\, b^m \,\sigma_n\, b$. Thus $(a, b) \in \sigma_n$, which was to be proved.

(xiv)$\Rightarrow$(iv) This follows from (v) and the fact that $(a, a^2) \in R(\varrho)$, for every reflexive relation ϱ on S. $\square$

If S is a finite semigroup, then there exists $n \in \mathbf{Z}^+$, $n \leq |S|$, such that $\longrightarrow^\infty = \longrightarrow^n$, so from Theorem 4.5 (iii) we obtain

Corollary 4.8 *Let S be a finite semigroup. Then there exists $n \in \mathbf{Z}^+$, $n \leq |S|$, such that S is a semilattice of σ_n-simple semigroups.*

Now we give the following important examples of semilattices of σ_2-simple semigroups.

Example 4.1 Let X be a *finite* set and let $\mathcal{T}_r(X)$ be the *full transformation semigroup* on X. If $|X| = 2$, then $\mathcal{T}_r(X)$ is a union of groups (and therefore, $\mathcal{T}_r(X)$ is a semilattice of completely simple semigroups). If $|X| > 2$, then based on the results of R.Croisot ([1], Example 3) $\mathcal{T}_r(X)$ is not a union of simple semigroups (and therefore, $\mathcal{T}_r(X)$ is not a semilattice of simple semigroups). Let $\mathcal{V}(X) = \mathcal{T}_r(X) - \mathcal{S}(X)$, where $\mathcal{S}(X)$ is the group of permutations on X. Then $\mathcal{V}(X)$ is a completely prime ideal of $\mathcal{T}_r(X)$. As M. S. Putcha ([5], Example 4.6) mentioned, there exists a fixed $a \in \mathcal{V}(X)$ such that for all $b \in \mathcal{V}(X)$ $a \longrightarrow b \longrightarrow a$. From this we conclude that $\mathcal{V}(X)$ is a σ_2-simple semigroup. Therefore, $\mathcal{T}_r(X)$ *is a chain of a σ_2-simple semigroup and of a group.*

Example 4.2 Let X be an *infinite* set and let $\mathcal{T}_r(X)$ be the *full transformation semigroup* on X. As M.S.Putcha ([5], Example 4.6) mentioned, there exists a fixed $a \in \mathcal{T}_r(X)$ such that for all $b \in \mathcal{T}_r(X)$ $a \longrightarrow b \longrightarrow a$. From this it follows that $\mathcal{T}_r(X)$ *is a σ_2-simple semigroup.*

Next we prove two auxiliary lemmas.

Lemma 4.12 *Let a be a completely π-regular element of a semigroup S. Then for every $b \in S$ and every $n \in \mathbf{Z}^+$,*

$$a^0 \ \longrightarrow^n \ b \ \Rightarrow \ a \ \longrightarrow^n \ b.$$

In other words, for every $n \in \mathbf{Z}^+$,

$$\Sigma_n(a^0) \subseteq \Sigma_n(a).$$

Proof. Let $m \in \mathbf{Z}^+$ such that $a^m \in G_{a_0}$, and let $(a^m)^{-1}$ be the inverse of a^m in the group G_{a_0}. Then $a^0 = (a^m(a^m)^{-1})^2 \in SaS$, which yields $Sa^0S \subseteq SaS$, and hence

$$\Sigma_1(a^0) = \sqrt{Sa^0S} \subseteq \sqrt{SaS} = \Sigma_1(a).$$

Now, by induction we easily verify that $\Sigma_n(a^0) \subseteq \Sigma_n(a)$, for every $n \in \mathbf{Z}^+$.□

Lemma 4.13 *Let b be a completely π-regular element of a semigroup S. Then for every $a \in S$ and every $n \in \mathbf{Z}^+$,*

$$a \ \longrightarrow^n \ b \ \Leftrightarrow \ a \ \longrightarrow^n \ b^0.$$

Proof. Let $m \in \mathbf{Z}^+$ such that $b^m \in G_{b_0}$. Consider an arbitrary $a \in S$. Suppose that $a \longrightarrow b$. Then $b^k \in SaS$, for some $k \in \mathbf{Z}^+$, and hence $b^{mk} \in G_{b_0} \cap SaS$. Let $(b^{mk})^{-1}$ be the inverse of b^{mk} in the group G_{b_0}. Now $b^0 = (b^{mk}(b^{mk})^{-1})^2 \in SaS$ so we obtain that $a \,|\, b^0$, which is equivalent to $a \longrightarrow b^0$, because b^0 is an idempotent. Conversely, let $a \longrightarrow b^0$, i.e. $a \,|\, b^0$. Then $b^m = b^0b^m \in SaSb^m \subseteq SaS$, and hence $a \longrightarrow b$.

Therefore, we have proved that our assertion holds for $n = 1$. By induction we easily verify that this assertion holds for every $n \in \mathbf{Z}^+$. □

Note that if b is a completely π-regular element then we have that $a \longrightarrow b^0$ if and only if $a \,|\, b^0$. Therefore, in such a case we obtain

$$a \ \longrightarrow \ b \text{ if and only if } a \ | \ b^0.$$

Now we are prepared for the next result.

Theorem 4.6 *Let S be a completely π-regular semigroup and $n \in \mathbf{Z}^+$. Then the following conditions are equivalent:*

(i) *S is a semilattice of σ_n-simple semigroups;*
(ii) $(\forall a \in S)$ $a\sigma_n a^0$;
(iii) $(\forall a, b \in S)$ $a \longrightarrow^n b \Rightarrow a^0 \longrightarrow^n b$;
(iv) $(\forall a \in S)(\forall f \in E(S))$ $a \longrightarrow^n f \Rightarrow a^2 \longrightarrow^n f$;
(v) $(\forall a, b \in S)(\forall g \in E(S))$ $a \longrightarrow^n g \,\&b \longrightarrow^n g \Rightarrow ab \longrightarrow^n g$;
(vi) $(\forall e, f \in E(S))(\forall c \in S)$ $e \longrightarrow^n c \,\&f \longrightarrow^n c \Rightarrow ef \longrightarrow^n c$;
(vii) $(\forall e, f, g \in E(S))$ $e \longrightarrow^n g \,\&f \longrightarrow^n g \Rightarrow ef \longrightarrow^n g$.

If $n \geq 2$, then any of the above conditions are equivalent to

(viii) $(\forall e, f, g \in E(S))\ e \longrightarrow^n f \ \& f \longrightarrow^n g \Rightarrow e \longrightarrow^n g.$

Proof. (i)⇒(ii) For an arbitrary $a \in S$, $a^0 \longrightarrow a$ and $a \,|\, a^0$, which implies $a \longrightarrow a^0$, and if (i) holds, then based on (xi) of Theorem 4.5 it follows that $a\sigma_n a^0$.

(ii)⇒(iii) The condition (ii) is equivalent to $\Sigma_n(a) = \Sigma_n(a^0)$, whereas (iii) is equivalent to $\Sigma_n(a) \subseteq \Sigma_n(a^0)$, so it is evident that (ii) implies (iii).

(iii)⇒(i) Let $a, b \in S$ such that $a \longrightarrow^n b$. Based on the assumption (iii) $a^0 \longrightarrow^n b$, and since $(a^2)^0 = a^0$, we have that $(a^2)^0 \longrightarrow^n b$, so based on Lemma 4.12, $a^2 \longrightarrow^n b$. Hence, from Theorem 4.5, S is a semilattice of σ_n-simple semigroups.

(i)⇒(iv) This is an immediate consequence of Theorem 4.5.

(iv)⇒(i) Consider $a, b \in S$ such that $a \longrightarrow^n b$. Based on Lemma 4.13, $a \longrightarrow^n b$ implies $a \longrightarrow^n b^0$, and from (iv), $a \longrightarrow^n b^0$ implies $a^2 \longrightarrow^n b^0$, so again by Lemma 4.13, $a^2 \longrightarrow^n b$. From this and from Theorem 4.5 it follows that (i) holds.

(i)⇒(vii) This is an immediate consequence of Theorem 4.5.

(vii)⇒(v) Let $a, b \in S$ and $g \in E(S)$ such that $a \longrightarrow^n g$ and $b \longrightarrow^n g$. This means that $a \longrightarrow x \longrightarrow^{n-1} g$ and $b \longrightarrow y \longrightarrow^{n-1} g$, for some $x, y \in S$. Based on the hypothesis, S is a completely π-regular semigroup, so $x \in T_{e_0}$ and $f \in T_{f_0}$, for some $e_0, f_0 \in E(S)$, and by Lemma 4.13, we have that $a \longrightarrow x$ is equivalent to $a \,|\, e_0$ and $b \longrightarrow y$ is equivalent to $b \,|\, f_0$. But, $a \,|\, e_0$ and $b \,|\, f_0$ yield $e_0 = uav$ and $f_0 = pbq$, for some $u, v, p, q \in S$. Set $e = (vua)^2$ and $f = (bqp)^2$. Then $e, f \in E(S)$ and

$$e_0 = e_0^3 = ua(vua)^2 v = uaev,$$

so we have that $e \,|\, e_0$, and similarly, $f \,|\, f_0$. Again from Lemma 4.13, $e \,|\, e_0$ is equivalent to $e \longrightarrow x$ and $f \,|\, f_0$ is equivalent to $f \longrightarrow y$, which yields

$$e \longrightarrow x \longrightarrow^{n-1} g \text{ and } f \longrightarrow y \longrightarrow^{n-1} g,$$

i.e. $e \longrightarrow^n g$ and $f \longrightarrow^n g$. Now, based on the assumption (vii), we obtain that $ef \longrightarrow^n g$, i.e. $ef \longrightarrow z \longrightarrow^{n-1} g$, for some $z \in S$, and hence

$$z^k \in SefS = S(vua)^2(bqp)^2 S \subseteq SabS,$$

which means that $ab \longrightarrow z$. Therefore, $ab \longrightarrow z \longrightarrow^{n-1} g$, so $ab \longrightarrow^n g$. Hence, we have proved that (v) holds.

(v)⇒(iv) This implication is obvious.

(vi)⇒(vii) This implication is obvious.

(vii)⇒(vi) Let $e, f \in E(S)$ and $c \in S$ such that $e \longrightarrow^n c$ and $f \longrightarrow^n c$. Based on Lemma 4.13, $e \longrightarrow^n c^0$ and $f \longrightarrow^n c^0$, and (vii) yields $ef \longrightarrow^n c^0$, so again from Lemma 4.13 we obtain $ef \longrightarrow^n c$, which was to be proved.

Further, let $n \geq 2$.

(i)⇒(viii) This is an immediate consequence of Theorem 4.5.

(viii)⇒(i) According to Theorem 4.5, in order to prove (i), it suffices to prove that $\longrightarrow^n$ is a transitive relation, and we will consider $a, b, c \in S$ such that $a \longrightarrow^n b$ and $b \longrightarrow^n c$.

First, according to Lemma 4.13 we have that $a \longrightarrow^n b^0$ and $b \longrightarrow^n c^0$. Furthermore, $a \longrightarrow^n b^0$ yields $a \longrightarrow y \longrightarrow^{n-1} b^0$, for some $y \in S$, and since $y \in T_{e_0}$, for some $e_0 \in E(S)$, from Lemma 4.13 it follows that $a \longrightarrow y$ if and only if $a \,|\, e_0$, i.e. $e_0 = uav$, for some $u, v \in S$. If we set $e = (vua)^2$, then $e \in E(S)$ and $e_0 = uaev$ so $e \,|\, e_0$. But, based on Lemma 4.13, $e \,|\, e_0$ is equivalent to $e \longrightarrow y$, so we have that $e \longrightarrow y \longrightarrow^{n-1} b^0$, i.e. $e \longrightarrow^n b^0$.

On the other hand, $b \longrightarrow^n c^0$ gives $b \longrightarrow z \longrightarrow^{n-1} c^0$, for some $z \in S$, and $z \in T_{h_0}$, for some $h_0 \in E(S)$. Now, based on Lemma 4.13, $b \longrightarrow z$ if and only if $b \,|\, h_0$, i.e. $h_0 = pbq$, for some $p, q \in S$. Set $h = (bqp)^2$. Then $h \in E(S)$ and $h \,|\, h_0$, which is equivalent to $h \longrightarrow z$, again from Lemma 4.13. Thus, $h \longrightarrow z \longrightarrow^{n-1} c^0$, which means $h \longrightarrow^n c^0$.

Finally we have $b_0 \longrightarrow b$, and also $b \,|\, h$, so $b \longrightarrow h$. Hence, $b^0 \longrightarrow^2 h$, so $b^0 \longrightarrow^n h$, because $n \geq 2$. Therefore,

$$e \longrightarrow^n b^0, \quad b_0 \longrightarrow^n h \quad \text{and} \quad h \longrightarrow^n c^0,$$

so based on the assumption (viii) we conclude that $e \longrightarrow^n c^0$.

Now, in order to prove that $a \longrightarrow^n c$, we start with the relation $e \longrightarrow^n c^0$, and from Lemma 4.13 we obtain that $e \longrightarrow^n c$. But this means that $e \longrightarrow t \longrightarrow^{n-1} c$, for some $t \in S$. Furthermore, $e \longrightarrow t$ implies

$$t^k \in SeS = S(vua)^2 S \subseteq SaS,$$

for some $k \in \mathbf{Z}^+$, so $a \longrightarrow t$. Therefore, $a \longrightarrow t \longrightarrow^{n-1} c$, and we have that $a \longrightarrow^n c$, which was to be proved. □

Remark 4.3 The requirement $n \geq 2$ is crucial for the equivalence of (i) and (viii) in the previous theorem. Namely, every completely π-regular semigroup S satisfies the condition

$$(\forall e, f, g \in E(S))\ e \longrightarrow f \ \& f \longrightarrow g \Rightarrow e \longrightarrow g,$$

because it is clearly equivalent to the condition

$$(\forall e, f, g \in E(S))\ e \mid f \ \& f \mid g \Rightarrow e \mid g,$$

and the division relation is transitive. But, S is not necessarily a semilattice of σ_1-simple semigroups. For example, the five-element Brandt semigroup

$$\mathbf{B}_2 = \langle a, b \mid a^2 = b^2 = 0, aba = a, bab = b\rangle$$

is completely π-regular, and hence satisfies the above mentioned conditions. But S is not a semilattice of σ_1-simple (Archimedean) semigroups.

Exercises

1. Let S be a completely π-regular semigroup and $n \in \mathbf{Z}^+$. Then the following conditions are equivalent:

(a) S is a semilattice of σ_n-simple semigroups;
(b) for every $e \in E(S)$, $\Sigma_n(e)$ is an ideal of S;
(c) $(\forall e, f \in E(S))\ \Sigma_n(ef) = \Sigma_n(e) \cap \Sigma_n(f)$;
(d) for every $e \in E(S)$, $N(e) = \{x \in S \mid x \longrightarrow^n e\}$.

References

S. Bogdanović, M. Ćirić and Ž. Popović [1], R. Croisot [1]; M. Ćirić and S. Bogdanović [3], [5]; V. L. Mannepalli and C. S. H. Nagore [1]; D. B. McAlister [1]; Ž. Popović, S. Bogdanović and M. Ćirić [1]; M. S. Putcha [5].

4.3 Semilattices of λ-simple Semigroups

In this section we consider semilattices of λ, λ_n, τ- and τ_n-simple semigroups. The results obtained here are generalizations of well known results concerning unions and semilattices of left simple semigroups and semilattices of groups and of results concerning semilattices of left and t-Archimedean semigroups.

First we will prove the following lemma:

Lemma 4.14 *Let S be a semilattice Y of semigroups $S_\alpha, \alpha \in Y$ and let $n \in \mathbf{Z}^+$.*

(a) *Let $\alpha \in Y$ with $a, b \in S_\alpha$. If $a \xrightarrow{l} b$ in S, then $a \xrightarrow{l} b$ in S_α.*

(b) *Let* $a \in S_\alpha,\ b \in S_\beta,\ \alpha, \beta \in Y$. *If* $a \xrightarrow{l} b$, *then* $\alpha \geq \beta$.

(c) *Let* $\alpha \in Y$ *with* $a, b \in S_\alpha$. *If* $a \xrightarrow{l}{}^n\ b$ *in* S, *then* $a \xrightarrow{l}{}^n\ b$ *in* S_α.

Proof. (a) Let $a \xrightarrow{l} b$ in S, i.e. let $b^m = ua$ for some $m \in \mathbf{Z}^+,\ u \in S$. If $u \in S_\beta$ for some $\beta \in Y$, then $\alpha\beta = \alpha$ whence

$$b^{m+1} = (bu)a \in S_{\alpha\beta}a = S_\alpha a.$$

Thus $a \xrightarrow{l} b$ in S_α.

(b) This follows from Lemma 4.9 (i), since $\xrightarrow{l} \subseteq \longrightarrow$.

(c) This can be proved in a way similar as Lemma 4.9 (ii). $\square$

Theorem 4.7 *The following conditions on a semigroup* S *are equivalent:*

(i) S *is a semilattice of* λ*-simple semigroups;*

(ii) $(\forall a, b \in S)\ a \xrightarrow{l}{}^\infty\ ab$*;*

(iii) *for every* $a \in S$, $\Lambda(a)$ *is an ideal;*

(iv) *every completely semiprime left ideal of* S *is two-sided;*

(v) $(\forall a, b \in S)\ \Lambda(ab) = \Lambda(a) \cap \Lambda(b)$*;*

(vi) *for every* $a \in S$, $N(a) = \{x \in S \mid x \xrightarrow{l}{}^\infty\ a\}$*;*

(vii) $(\forall a, b \in S)\ a \longrightarrow{}^\infty b \Rightarrow a \xrightarrow{l}{}^\infty b$*;*

(viii) $(\forall a, b \in S)\ a \text{---}{}^\infty b \Rightarrow a \xrightarrow{l}{}^\infty b$.

Proof. (i)⇒(ii) Let S be a semilattice Y of λ-simple semigroups $S_\alpha,\ \alpha \in Y$, and let $a, b \in S$. Then $ab, ba \in S_\alpha$ for some $\alpha \in Y$, so $ba \xrightarrow{l}{}^\infty\ ab$ in S_α, since S_α is λ-simple. Thus $ba \xrightarrow{l}{}^\infty\ ab$ in S, and since $a \xrightarrow{l} ba$ in S, then $a \xrightarrow{l}{}^\infty\ ab$ in S. Hence, (ii) holds.

(ii)⇒(iii) Let $a \in S$, let $x \in \Lambda(a)$ and let $b \in S$. Then from (ii) we obtain

$$a \xrightarrow{l}{}^\infty\ x \xrightarrow{l}{}^\infty\ xb,$$

whence $xb \in \Lambda(a)$. Thus, $\Lambda(a)$ is a right ideal of S, so from Lemma 4.4 it follows that $\Lambda(a)$ is an ideal of S.

(iii)⇒(iv) This follows immediately.

(iv)⇒(v) This follows from Theorem 4.2, since $\Lambda(a) = \Sigma(a)$ for all $a \in S$.

(v)⇒(i) From (v) it follows that S is a semilattice Y of semigroups $S_\alpha, \alpha \in Y$, such that S_α are λ-classes of S. Assume $\alpha \in Y$ and $a, b \in S_\alpha$.

Then $a\ \lambda b$, so $a \xrightarrow{l}{}^{\infty}\ b$ in S. According to Lemma 4.14 we obtain that $a \xrightarrow{l}{}^{\infty}\ b$ in S_α. Thus, S_α is a λ-simple semigroup.

(v)$\Rightarrow$(vi) Let (v) hold, let $a \in S$ and let $A = \{x \in S | x \xrightarrow{l}{}^{\infty}\ a\}$. Based on (v) and Theorem 4.4 we obtain that A is the smallest right filter of S containing a, so $A \subseteq N(a)$. Let $x, y \in S$ be such that $xy \in A$, i.e. such that $xy \xrightarrow{l}{}^{\infty}\ a$. Since from (v) we obtain that $x \xrightarrow{l}{}^{\infty}\ xy$, then $x \xrightarrow{l}{}^{\infty}\ a$, whence $x \in A$. Therefore, A is left consistent, i.e. A is a filter, whence $A = N(a)$.

(vi)$\Rightarrow$(ii) Let (vi) hold and let $a, b \in S$. Since $\{x \in S\,|\, x \xrightarrow{l}{}^{\infty}\ ab\} = N(ab)$ is a filter and $ab \in N(ab)$, then $a \in N(ab)$, i.e. $a \xrightarrow{l}{}^{\infty}\ ab$. Hence, (ii) holds.

(i)$\Rightarrow$(vii) Let S be a semilattice Y of λ-simple semigroups S_α, $\alpha \in Y$. Assume $a, b \in S$ such that $a \longrightarrow{}^{\infty} b$. Then based on Lemma 4.9 (i), for $n = 1$, $a \in S_\alpha$, $b \in S_\beta$, for some $\alpha, \beta \in Y$ and $\beta \leq \alpha$, whence $ba, b \in S_\beta$. So $ba \xrightarrow{l}{}^{\infty} b$. Since $a \xrightarrow{l} ba \xrightarrow{l}{}^{\infty} b$, we then have that $a \xrightarrow{l}{}^{\infty} b$.

(vii)$\Rightarrow$(i) Let (vii) hold. According to Theorem 4.2 every semigroup S is a semilattice Y of σ-simple semigroups S_α, $\alpha \in Y$. Then for $a, b \in S_\alpha$, $\alpha \in Y$, from Theorem 4.1 we have that $a \,—\,{}^{\infty} b$, and from Lemma 4.9 (ii), for $n = 1$, $a \,—\,{}^{\infty} b$ in S_α, $\alpha \in Y$, whence $a \longrightarrow{}^{\infty} b$ in S_α, $\alpha \in Y$. So based on the hypothesis $a \xrightarrow{l}{}^{\infty} b$ and Lemma 4.14 (a) $a \xrightarrow{l}{}^{\infty} b$ in S_α, $\alpha \in Y$, since $a, b \in S_\alpha$. Thus $a \xrightarrow{l}{}^{\infty} b$ in S_α, $\alpha \in Y$, for all $a, b \in S_\alpha$ and from Lemma 4.6 (iv) S_α, $\alpha \in Y$ is a λ-simple semigroup. Therefore, S is a semilattice of λ-simple semigroups.

(i)$\Rightarrow$(viii) Let S be a semilattice Y of λ-simple semigroups S_α, $\alpha \in Y$. Assume $a, b \in S$ such that $a \,—\,{}^{\infty} b$. Then based on Lemma 4.9 (ii), for $n = 1$, $a, b \in S_\alpha$ and $a \,—\,{}^{\infty} b$ in S_α, for some $\alpha \in Y$, whence $a\lambda b$ and based on Lemma 4.6 (iv) $a \xrightarrow{l}{}^{\infty}\ b$.

(viii)$\Rightarrow$(i) Let (viii) hold. Since every semigroup S is a semilattice Y of σ-simple semigroups S_α, $\alpha \in Y$, then for $a, b \in S_\alpha$, $\alpha \in Y$, based on Theorem 4.1 we have that $a \,—\,{}^{\infty} b$, whence $a \xrightarrow{l}{}^{\infty} b$ and $a(\xrightarrow{l}{}^{\infty})^{-1} b$ in S_α. Thus $a \xrightarrow{l}{}^{\infty} \cap (\xrightarrow{l}{}^{\infty})^{-1} b$ and based on Lemma 4.6 (iv) S_α is a λ-simple semigroup. $\square$

Theorem 4.8 *Let $n \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) $\xrightarrow{l}{}^{n}$ *is a quasi-order on S;*

(ii) $(\forall a \in S)\ a\ \lambda_n\ a^2$;

(iii) $(\forall a, b \in S)\ a \xrightarrow{l}{}^n b \Rightarrow a^2 \xrightarrow{l}{}^n b$;

(iv) *for all* $a \in S$, $\Lambda_n(a)$ *is a left ideal of* S.

Proof. (ii)⇔(iii) This follows immediately.

(iii)⇒(i) Let $a \xrightarrow{l}{}^{n+1} b$, i.e. let $a \xrightarrow{l} x \xrightarrow{l}{}^n b$ for some $x \in S$. From (iii) it follows that $x^k \xrightarrow{l}{}^n b$ for all $k \in \mathbf{Z}^+$. Let $k \in \mathbf{Z}^+$ such that $x^k \in Sa$. Let $y \in S$ be such that $x^k \xrightarrow{l} y \xrightarrow{l}{}^{n-1} b$, if $n \geq 2$, or $y = b$, if $n = 1$. Then there exists $m \in \mathbf{Z}^+$ such that $y^m \in Sx^k \subseteq Sa$. Thus $a \xrightarrow{l} y$, so $a \xrightarrow{l}{}^n b$. Therefore, $\xrightarrow{l}{}^n = \xrightarrow{l}{}^{n+1}$, so $\xrightarrow{l}{}^n = \xrightarrow{l}{}^\infty$, i.e. $\xrightarrow{l}{}^n$ is a transitive relation.

(i)⇒(iv) This follows from Lemma 4.4, since in this case $\xrightarrow{l}{}^\infty = \xrightarrow{l}{}^n$.

(iv)⇒(i) Let $\Lambda_n(a)$ be a left ideal for every $a \in S$. Then based on Lemma 4.4 we obtain that $\Lambda_n(a) = \Lambda(a)$ for all $a \in S$, whence $\xrightarrow{l}{}^n = \xrightarrow{l}{}^\infty$, so (i) holds. □

Theorem 4.9 *Let* $n \in \mathbf{Z}^+$. *Then the following conditions on a semigroup* S *are equivalent:*

(i) S *is a semilattice of* λ_n*-simple semigroups;*

(ii) $a\ \lambda_n\ a^2$ *for all* $a \in S$ *and* $a \xrightarrow{l}{}^n ab$ *for all* $a, b \in S$;

(iii) *for all* $a \in S$, $\Lambda_n(a)$ *is an ideal;*

(iv) $(\forall a, b \in S)\ \Lambda_n(ab) = \Lambda_n(a) \cap \Lambda_n(b)$;

(v) *for all* $a \in S$, $N(a) = \{x \in S \mid x \xrightarrow{l}{}^n a\}$.

Proof. (i)⇒(ii) Let S be a semilattice Y of λ_n-simple semigroups S_α, $\alpha \in Y$. Assume $a, b \in S$ such that $a \xrightarrow{l}{}^n b$, i.e. $a \in S_\alpha$, $b \in S_\beta$, $\alpha, \beta \in Y$. Then by Lemma 4.14 we obtain that $\alpha \geq \beta$, so $ba^2 \in S_{\alpha\beta} = S_\beta$. Since S_β is λ_n-simple, then $ba^2 \xrightarrow{l}{}^n b$ in S_β, whence $ba^2 \xrightarrow{l}{}^n b$ in S, so $a^2 \xrightarrow{l}{}^n b$ in S. Thus, by Theorem 4.8 we obtain that $a\lambda_n a^2$ for all $a \in S$.

Assume $a, b \in S$, i.e. $a \in S_\alpha$, $b \in S_\beta$ for some $\alpha, \beta \in Y$. Then $ab, ba \in S_{\alpha\beta}$, and since $S_{\alpha\beta}$ is a λ_n-simple semigroup, then $ba \xrightarrow{l}{}^n ab$ in $S_{\alpha\beta}$, whence $ba \xrightarrow{l}{}^n ab$ in S, so $a \xrightarrow{l}{}^n ab$ in S. Thus, (ii) holds.

(ii)⇒(iii) Let (ii) hold. Based on Theorem 4.8 we obtain that $\Lambda_n(a)$ is a left ideal of S, so $\Lambda_n(a) = \Lambda(a)$, for all $a \in S$. Now, according to Theorem 4.7 we obtain that $\Lambda_n(a) = \Lambda(a)$ is an ideal of S, for all $a \in S$.

(iii)⇒(iv) If for all $a \in S$, $\Lambda_n(a)$ is an ideal of S, then from Lemma 4.4 it follows that $\Lambda_n(a) = \Lambda(a)$ for all $a \in S$, so based on Theorem 4.7 we obtain that (iv) holds.

(iv)⇒(i) Let (iv) hold. Then S is a semilattice Y of semigroups S_α, $\alpha \in Y$, such that S_α are λ_n-classes of S. Assume $\alpha \in Y$ and $a, b \in S_\alpha$. Then $a\lambda_n b$, whence $a \xrightarrow{l} {}^n b$ in S, so from Lemma 4.14 we obtain that $a \xrightarrow{l} {}^n b$ in S_α. Thus, S_α is a λ_n-simple semigroup, so (i) holds.

(iii)⇒(v) From (iii) it follows that $\Lambda_n(a) = \Lambda(a)$ for all $a \in S$, so $\xrightarrow{l}{}^n = \xrightarrow{l}{}^\infty$. Thus, according to Theorem 4.7 we obtain that (v) holds.

(v)⇒(iv) Let (v) hold. Then for $a, b, x \in S$ we obtain that

$$x \in \Lambda_n(a) \cap \Lambda_n(b) \Leftrightarrow a, b \in N(x) \Leftrightarrow ab \in N(x) \Leftrightarrow x \in \Lambda_n(ab).$$

Thus, (iv) holds. □

Problem 4.1 For $n = 1$, in (ii) of Theorem 4.9 the condition $a\lambda_n a^2$ can be omitted. We can state a problem: Can this hypothesis also be omitted for $n \geq 2$?

Theorem 4.10 *The following conditions on a semigroup S are equivalent:*

(i) λ *is a matrix congruence on S;*
(ii) λ *is a right zero band congruence on S;*
(iii) $(\forall a, b, c \in S)\ abc \xrightarrow{l} {}^\infty ac$;
(iv) $(\forall a, b \in S)\ aba \xrightarrow{l} {}^\infty a$;
(v) $(\forall a, b \in S)\ ab \xrightarrow{l} {}^\infty b$;
(vi) *S is a disjoint union of all its principal left radicals;*
(vii) $\xrightarrow{l} {}^\infty$ *is a symmetric relation on S.*

Proof. (i)⇒(iii), (iii)⇒(iv) and (ii)⇒(i) This follows immediately.

(iv)⇒(v) For all $a, b \in S$, $ab \xrightarrow{l} bab$, so from (iv), $ab \xrightarrow{l} {}^\infty b$.

(v)⇒(ii) Let $a, b \in S$ such that $a\lambda b$, and $x \in S$. By (v), $\Lambda(ax) = \Lambda(x) = \Lambda(bx)$ and $\Lambda(xa) = \Lambda(a) = \Lambda(b) = \Lambda(xb)$. Therefore, λ is a congruence. Clearly, it is a right zero band congruence.

(ii)⇒(vi) Let S be a right zero band B of semigroups S_i, $i \in B$, which are λ-classes of S. Assume $a \in S$. Then $a \in S_i$, for some $i \in B$, and since S_i

is a completely semiprime left ideal of S (Lemma 4.4), then $\Lambda(a) \subseteq S_i$. On the other hand, if $b \in S_i$, then $b\lambda a$, so $b \in \Lambda(b) = \Lambda(a)$, whence $S_i \subseteq \Lambda(a)$. Therefore, $\Lambda(a) = S_i$, so (vi) holds.

(vi)⇒(vii) Let $a, b \in S$ such that $a \xrightarrow{l} {}^{\infty}b$. Then $b \in \Lambda(a)$, whence $\Lambda(a) \cap \Lambda(b) \neq \emptyset$, so from (vi), $\Lambda(a) = \Lambda(b)$. Therefore, $b \xrightarrow{l} {}^{\infty}a$.

(vii)⇒(v) For all $a, b \in S$, $b \xrightarrow{l} ab$, so from (vii), $ab \xrightarrow{l} {}^{\infty}b$. □

Corollary 4.9 *The following conditions on a semigroup S are equivalent:*

(i) λ_n *is a matrix congruence on S;*

(ii) λ_n *is a right zero band congruence on S;*

(iii) $(\forall a, b \in S)\ \Lambda_n(a) \subseteq \Lambda_n(aba)$;

(iv) $(\forall a, b \in S)\ \Lambda_n(b) \subseteq \Lambda_n(ab)$;

(v) $\xrightarrow{l} {}^{n}$ *is a symmetric relation on S.*

Based on the well-known result of A. H. Clifford, any band of λ-simple semigroups is a semillatice of matrices of λ-simple semigroups. These semigroups will be characterized by the following theorem.

Theorem 4.11 *A semigroup S is a semilattice of matrices of λ-simple semigroups if and only if*

$$a \longrightarrow {}^{\infty}b \ \Rightarrow\ ab \xrightarrow{l} {}^{\infty}b,$$

for every $a, b \in S$.

Proof. Let S be a semilattice Y of matrices of λ-simple semigroups S_α, $\alpha \in Y$. Assume that $a \longrightarrow {}^{\infty}b$, for $a \in S_\alpha$, $b \in S_\beta$, $\alpha, \beta \in Y$. Then based on Lemma 4.9, $\beta \leq \alpha$, whence $b, ba \in S_\beta$ and based on Theorem 4.10 we have that $ba \cdot b \xrightarrow{l} {}^{\infty}b$, i.e. $ab \xrightarrow{l} {}^{\infty}b$.

Conversely, since every semigroup S is a semilattice Y of semilattice indecomposable semigroups S_α, $\alpha \in Y$, then for $a, b \in S_\alpha$, $\alpha \in Y$ we have that $a\sigma b$ (where σ corresponds to the greatest semilattice congruence on S), whence from Lemma 4.6, $a \longrightarrow {}^{\infty}b$. Based on Lemma 4.9 we have that $a \longrightarrow {}^{\infty}b$ in S_α, $\alpha \in Y$. From this and from the hypothesis it follows that $ab \xrightarrow{l} {}^{\infty}b$. From Lemma 4.14 we have that $ab \xrightarrow{l} {}^{\infty}b$ in S_α, $\alpha \in Y$ and based Theorem 4.10, S_α is a matrix of λ-simple semigroups, for all $\alpha \in Y$.□

If a is an element of a semigroup S and if $n \in \mathbf{Z}^+$, then we will use the following notations:

$$Q(a) = \Lambda(a) \cap P(a), \quad Q_n(a) = \Lambda_n(a) \cap P_n(a).$$

Using the previous theorems, we obtain the following results:

Corollary 4.10 *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of τ-simple semigroups;*

(ii) $(\forall a, b \in S)\ a \xrightarrow{l} {}^{\infty}ab \wedge b \xrightarrow{r} {}^{\infty}ab$;

(iii) *for all $a \in S$, $Q(a)$ is an ideal;*

(iv) $(\forall a, b \in S)\ Q(ab) = Q(a) \cap Q(b)$;

(v) *$L \cap R$ is an ideal, for every completely semiprime left ideal L and for every completely semiprime right ideal R of S;*

(vi) *for all $a \in S$, $N(a) = \{x \in S \mid x \xrightarrow{l} {}^{\infty}a \wedge x \xrightarrow{r} {}^{\infty}a\}$.*

Proof. This follows from Theorem 4.7 and its dual. □

Corollary 4.11 *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of τ_n-simple semigroups;*

(ii) *for all $a \in S$, $Q_n(a)$ is an ideal;*

(iii) $(\forall a, b \in S)\ Q_n(ab) = Q_n(a) \cap Q_n(b)$;

(iv) *$a\tau_n a^2$ for all $a \in S$ and*

$$a \xrightarrow{l} {}^{n}ab \wedge b \xrightarrow{r} {}^{n}ab,$$

for all $a, b \in S$;

(v) *for all $a \in S$, $N(a) = \{x \in S \mid x \xrightarrow{l} {}^{n}a \wedge x \xrightarrow{r} {}^{n}a\}$.*

Proof. This follows from Theorem 4.9 and its dual. □

Exercises

1. The relation $\tau^{\#}$ on a semigroup S is the smallest band congruence on S.

References

S. Bogdanović and M. Ćirić [15]; S. Bogdanović, Ž. Popović and M. Ćirić [2]; M. Ćirić and S. Bogdanović [3], [5].

4.4 Chains of σ-simple Semigroups

Here we will give some characterizations of chains of σ-simple semigroups and the related consequences to chains of σ_n-simple and λ-simple semigroups.

Lemma 4.15 *Let a, b be elements of a semigroup S. Then:*

$$N(a) \cup N(b) \subseteq N(ab).$$

Lemma 4.16 *The following conditions for elements a, b of a semigroup S are equivalent:*

(i) $N(ab) = N(a) \cup N(b)$;
(ii) $N(b) \subseteq N(a)$ *or* $N(a) \subseteq N(b)$;
(iii) $N(ab) = N(a)$ *or* $N(ab) = N(b)$.

Proof. (i)$\Rightarrow$(ii) From $ab \in N(ab) = N(a) \cup N(b)$ it follows that $ab \in N(a)$ or $ab \in N(b)$. If $ab \in N(a)$, then $a, b \in N(a)$, since $N(a)$ is a filter, i.e. $b \in N(a)$, whence $N(b) \subseteq N(a)$. Similarly we show that from $ab \in N(b)$ it follows that $N(a) \subseteq N(b)$.

(ii)$\Rightarrow$(iii) Assume that $N(a) \subseteq N(b)$. Then $a, b \in N(b)$ so $ab \in N(b)$, since $N(b)$ is a subsemigroup. Thus $N(ab) \subseteq N(b)$. On the other hand, since $N(ab)$ is a filter, then $a, b \in N(ab)$, i.e. $b \in N(ab)$, so $N(b) \subseteq N(ab)$. Therefore, $N(ab) = N(b)$. In a similar way we prove that from $N(b) \subseteq N(a)$ it follows that $N(ab) = N(a)$.

(iii)$\Rightarrow$(i) From (iii) it follows that $N(ab) \subseteq N(a) \cup N(b)$, so based on Lemma 4.15 we obtain (i). $\square$

Lemma 4.17 *The union of every nonempty family of consistent subsets of a semigroup S is a consistent subset of S.*

Theorem 4.12 *The following conditions on a semigroup S are equivalent:*

(i) Σ_S *is a chain;*
(ii) S *is a chain of σ-simple semigroups;*
(iii) *the partially ordered set of all completely prime ideals of S is a chain;*
(iv) *every completely semiprime ideal of S is completely prime;*
(v) *principal radicals of S are completely prime;*

(vi) *the union of every nonempty family of filters of S is a filter of S;*

(vii) $(\forall a, b \in S)\ ab \longrightarrow^{\infty} a \vee ab \longrightarrow^{\infty} b$*;*

(viii) $\longrightarrow^{\infty} \cup (\longrightarrow^{\infty})^{-1}$ *is the universal relation on S.*

Proof. (i)⇔(ii) This follows immediately.

(i)⇒(vi) Let Σ_S be a chain, let $F_i,\ i \in I$, be a family of filters of S and let F be the union of this family. From Lemma 4.17 it follows that it is sufficient to prove that F is a subsemigroup of S. Let $a, b \in F$, i.e. let $a \in F_i,\ b \in F_j$ for some $i, j \in I$. Since Σ_S is a chain, then $ab\sigma a$ or $ab\sigma b$, so from Corollary 4.3 and Lemma 4.16 it follows that $N(ab) = N(a) \cup N(b)$. Since $N(a) \subseteq F_i$ and $N(b) \subseteq F_j$, then

$$ab \in N(ab) = N(a) \cup N(b) \subseteq F_i \cup F_j \subseteq F.$$

Thus, F is a subsemigroup.

(vi)⇒(vii) Let the union of every nonempty family of filters of S be a filter of S. Then $N(a) \cup N(b)$ is a filter for every $a, b \in S$. Thus $N(a) \cup N(b)$ is a subsemigroup of S, whence $ab \in N(a) \cup N(b)$, i.e. $ab \in N(a)$ or $ab \in N(b)$, so based on Corollary 4.3 we obtain (vii).

(vii)⇒(viii) This follows from the fact that $a \longrightarrow ab$ and $b \longrightarrow ab$.

(viii)⇒(i) Let $a, b \in S$. Then from (viii) it follows that $b \in \Sigma(a)$ or $a \in \Sigma(b)$, whence $\Sigma(b) \subseteq \Sigma(a)$ or $\Sigma(a) \subseteq \Sigma(b)$. Thus, Σ_S is a chain.

(i)⇒(iii) Let A and B be completely semiprime ideals of S. Assume that $A - B \neq \emptyset$ and $B - A \neq \emptyset$, i.e. assume that $a \in A - B$ and $b \in B - A$. Then $\Sigma(a) \subseteq A$ and $\Sigma(b) \subseteq B$, so from (i) we obtain that $\Sigma(a) \subseteq \Sigma(b) \subseteq B$ or $\Sigma(b) \subseteq \Sigma(b) \subseteq A$, whence $a \in B$ or $b \in A$, which is a contradiction according to the hypothesis. Thus, $A - B = \emptyset$ or $B - A = \emptyset$, i.e. $A \subseteq B$ or $B \subseteq A$. Therefore, (iii) holds.

(iii)⇒(viii) Assume $a, b \in S$. Let $A = S - N(a)$ and $B = S - N(b)$. Based on Lemma 1.21, A and B are completely prime ideals of S, so based on (iii), $A \subseteq B$ or $B \subseteq A$, whence $N(b) \subseteq N(a)$ or $N(a) \subseteq N(b)$, so according to Corollary 4.3, $b \longrightarrow^{\infty} a$ or $a \longrightarrow^{\infty} b$. Therefore, (viii) holds.

(vii)⇒(iv) Let A be a completely semiprime ideal of S. Assume $a, b \in S$ such that $ab \in A$. Then $\Sigma(ab) \subseteq A$, so from (vii) we obtain that $a \in \Sigma(ab) \subseteq A$ or $b \in \Sigma(ab) \subseteq A$. Hence, A is completely prime.

(iv)⇒(v) This follows immediately.

(v)⇒(vii) If $a, b \in S$, then $\Sigma(ab)$ is completely prime, whence $a \in \Sigma(ab)$ or $b \in \Sigma(ab)$, so (vii) holds. □

Corollary 4.12 *Let $n \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is a chain of σ_n-simple semigroups;*

(ii) *for every $a \in S$, $\Sigma_n(a)$ is a completely prime ideal of S;*

(iii) *S is a semilattice of σ_n-simple semigroups and for every $a \in S$, $\Sigma_n(a)$ is a completely prime subset of S;*

(iv) *S is a semilattice of σ_n-simple semigroups and $ab \longrightarrow^n a$ or $ab \longrightarrow^n b$ for all $a, b \in S$;*

(v) *S is a semilattice of σ_n-simple semigroups and $b \longrightarrow^n a$ or $a \longrightarrow^n b$ for all $a, b \in S$.*

Proof. (i)⇒(ii) Based on the hypothesis and Theorem 4.5 we obtain that $\Sigma_n(a)$ is an ideal of S, and based on Theorem 4.12 we obtain that $\Sigma_n(a)$ is completely prime, for all $a \in S$.

(ii)⇒(iii) This follows from Theorem 4.5.

(iii)⇒(iv) Assume $a, b \in S$. Since $\Sigma_n(ab)$ is completely prime and $ab \in \Sigma_n(ab)$, then we obtain that $a \in \Sigma_n(ab)$ or $b \in \Sigma_n(ab)$, so (iv) holds.

(iv)⇒(v) This follows immediately.

(v)⇒(i) This follows from Theorem 4.12. □

Problem 4.2 In [11] S. Bogdanović and M. Ćirić proved that for $n = 1$ the previous theorem can be proved without the hypothesis in (iii),(iv) and (v) that S is a semilattice of σ_n-simple semigroups. We can state the following problem: Can this hypothesis also be omitted for $n \geq 2$?

Corollary 4.13 *The following conditions on a semigroup S are equivalent:*

(i) *S is a chain of λ-simple semigroups;*

(ii) *principal left radicals of S are completely prime ideals of S;*

(iii) *S is a semilattice of λ-simple semigroups and $ab \xrightarrow{l} {}^{\infty}a$ or $ab \xrightarrow{l} {}^{\infty}b$ for all $a, b \in S$;*

(iv) *S is a semilattice of λ-simple semigroups and $b \xrightarrow{l} {}^{\infty}a$ or $a \xrightarrow{l} {}^{\infty}b$ for all $a, b \in S$.*

Proof. It follows from Theorems 4.7 and 4.12. □

The similar characterizations we can obtain for chains of λ_n-, τ- and τ_n-simple semigroups.

References

O. Anderson [1]; S. Bogdanović [16], [17]; S. Bogdanović and M. Ćirić [4], [11]; M. Ćirić and S. Bogdanović [3]; A. H. Clifford [1]; R. Croisot [1]; M. Petrich [1], [6]; M. S. Putcha [2], [5], [8]; M. Satyanarayana [2]; R. Šulka [1]; T. Tamura [1], [2], [6], [12], [13], [15]; T. Tamura and N. Kimura [2]; M. Yamada [1].

4.5 Semilattices of $\widehat{\sigma}_n$-simple Semigroups

Given $a, b \in S$. If a path exists from a to b in $(S, \longrightarrow)$ (resp. a path between a and b in $(S, \text{---})$), then paths exist from a to b (resp. between a and b) of minimal length. They will be called the *minimal paths* from a to b (resp. between a and b). Let $\mathcal{S}_n$ and $\widehat{\mathcal{S}}_n$, $n \in \mathbf{Z}^+$, denote respectively the classes of all semigroups S in which the lengths of all the minimal paths in the graphs $(S, \longrightarrow)$ and $(S, \text{---})$ are bounded by n. Equivalently, $\mathcal{S}_n$ and $\widehat{\mathcal{S}}_n$ are respectively the classes of all semigroups in which the n-th powers $\longrightarrow^n$ and ---^n of $\longrightarrow$ and --- are transitive relations. It is known that $\mathcal{S}_1 = \widehat{\mathcal{S}}_1$. This class consists of semigroups which are decomposable into a semilattice of Archimedean semigroups.

However, for $n \geq 2$ we have $\mathcal{S}_n \neq \widehat{\mathcal{S}}_n$, that is $\widehat{\mathcal{S}}_n \subset \mathcal{S}_n$. An example that confirms this inequality, obtained through the combination of two construction methods of M. S. Putcha from [5], will be given here. The purpose of this section is to study semigroups belonging to the class $\widehat{\mathcal{S}}_n$. These semigroups will be described by Theorem 4.13. This result is from a paper by S. Bogdanović, M. Ćirić and Ž. Popović [1]. By means of other theorems we characterize their various special types.

By the *rank* of a semigroup S, in notation ran(S), we mean the supremum of the lengths of all the minimal paths in the graph $(S, \text{---})$, and by the *semirank* of S, in notation sran(S), we mean the supremum of the lengths of all the minimal paths in the graph $(S, \longrightarrow)$. Equivalently, ran(S) is the smallest $n \leq \infty$ for which ---^n is transitive, and sran(S) is the smallest $n \leq \infty$ for which $\longrightarrow^n$ is transitive, where $\longrightarrow^n$ and ---^n denote the n-th powers of $\longrightarrow$ and ---, respectively. These notions were introduced by M. S. Putcha in [5], but our definition differs from his, since he denoted by $\longrightarrow^n$

and $\relbar\joinrel\relbar^{n}$ not the n-th powers of $\longrightarrow$ and $\relbar\joinrel\relbar$, but their $(n+1)$-th powers. Therefore, our definition increases Putcha's rank and semirank by 1, if they are finite.

The main goal of this section is to describe the structure of semigroups from the class $\widehat{\mathcal{S}}_n$.

We define the set $\widehat{\Sigma}_n(a)$ and the relation $\widehat{\sigma}_n$ on S by

$$\widehat{\Sigma}_n(a) = \{x \in S \mid a \relbar\joinrel\relbar^{n} x\}, \qquad (a,b) \in \widehat{\sigma}_n \ \Leftrightarrow\ \widehat{\Sigma}_n(a) = \widehat{\Sigma}_n(b).$$

Since σ_n is contained in the symmetric opening of $\longrightarrow^{n}$ (Lemma 4.6) and $\widehat{\sigma}_n \subseteq \relbar\joinrel\relbar^{n}$, then S is σ_n-simple if and only if $a \longrightarrow^{n} b$, for all $a, b \in S$, and S is $\widehat{\sigma}_n$-simple if and only if $a \relbar\joinrel\relbar^{n} b$, for all $a, b \in S$. Thus, for every $n \in \mathbf{Z}^+$, each $\widehat{\sigma}_n$-simple semigroup is σ_n-simple. We will show that for $n \geq 2$ the opposite statement does not hold. But, all σ_1-simple semigroups are $\widehat{\sigma}_1$-simple, and these are exactly the Archimedean semigroups.

Now we are ready to describe the semigroups from the class $\widehat{\mathcal{S}}_n$. The following theorem gives the relation between the class $\mathcal{S}_n$ and the class $\widehat{\mathcal{S}}_n$.

Theorem 4.13 *Let $n \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) $S \in \widehat{\mathcal{S}}_n$ *(i.e. $\relbar\joinrel\relbar^{n}$ is transitive);*
(ii) $\relbar\joinrel\relbar^{n}$ *is a semilattice congruence on S;*
(iii) $\widehat{\sigma}_n$ *is a band congruence on S;*
(iv) *S is a semilattice of $\widehat{\sigma}_n$-simple semigroups;*
(v) $\relbar\joinrel\relbar^{n} = \sigma_n$*;*
(vi) $(\forall a, b, c \in S)\ a \relbar\joinrel\relbar^{n+1} c \ \&\ b \relbar\joinrel\relbar^{n+1} c \ \Rightarrow\ ab \relbar\joinrel\relbar^{n} c$*;*
(vii) $(\forall a, b \in S)\ a \relbar\joinrel\relbar^{n+1} b \ \Rightarrow\ a^2 \relbar\joinrel\relbar^{n} b$*;*
(viii) *$S \in \mathcal{S}_n$ and $\relbar\joinrel\relbar^{n}$ equals the symmetric opening of $\longrightarrow^{n}$;*
(ix) $\relbar\joinrel\relbar^{n}$ *equals the symmetric opening of $\longrightarrow^{n+1}$.*

Proof. (i)$\Rightarrow$(ii) Let $S \in \widehat{\mathcal{S}}_n$, that is let $\relbar\joinrel\relbar^{n}$ be a transitive relation. Then $\relbar\joinrel\relbar^{n} = \relbar\joinrel\relbar^{\infty}$, and $\relbar\joinrel\relbar^{\infty}$ equals the smallest semilattice congruence on S, based on Theorem 4.3. Therefore, (ii) holds.

(ii)$\Rightarrow$(iii) Using the transitivity of $\relbar\joinrel\relbar^{n}$ we easily check that $\relbar\joinrel\relbar^{n} = \widehat{\sigma}_n$, whence we have that $\widehat{\sigma}_n$ is a semilattice congruence on S.

(iii)$\Rightarrow$(iv) Let $\widehat{\sigma}_n$ be a band congruence on S. Let $a, b \in S$ be arbitrary elements. Then $ab\, \widehat{\sigma}_n\, (ab)^2$, that is $\widehat{\Sigma}_n(ab) = \widehat{\Sigma}_n((ab)^2)$. Now, let $x \in \widehat{\Sigma}_n(ab)$

be an arbitrary element. Then $(ab)^2 \longrightarrow^n x$, whence $(ab)^2 \longrightarrow y \longrightarrow^{n-1} x$. But, $(ab)^2 \longrightarrow y$ implies $ba \longrightarrow y$, so we have $ba \longrightarrow^n x$, i.e. $x \in \widehat{\Sigma}_n(ba)$. Analogously we prove the opposite inclusion. Therefore, $ab \, \widehat{\sigma}_n \, ba$, so $\widehat{\sigma}_n$ is a semilattice congruence on S.

Let C be an arbitrary $\widehat{\sigma}_n$-class of S and let $a, b \in C$. Then $a \longrightarrow^n b$ in S, and based on Lemma 4.9, $a \longrightarrow^n b$ in C. Hence, we have proved that each $\widehat{\sigma}_n$-class of S is an $\widehat{\sigma}_n$-simple semigroup.

(iv)⇒(v) Let S be a semilattice of $\widehat{\sigma}_n$-simple semigroups. As we have already mentioned, every $\widehat{\sigma}_n$-simple semigroup is σ_n-simple, so S is also a semilattice of σ_n-simple semigroups, $\widehat{\sigma}_n = \sigma_n$ and it is the smallest semilattice congruence on S.

According to Theorem 4.5, $\longrightarrow^n \subseteq \sigma_n$. On the other hand, assume an arbitrary pair $(a, b) \in \sigma_n$. Then $(a, b) \in \widehat{\sigma}_n$, whence $a \longrightarrow^n b$, which was to be proved. Therefore, (v) holds.

(v)⇒(vi) Let $\longrightarrow^n = \sigma_n$. Based on Theorem 4.5, S is a semilattice Y of σ_n-simple semigroups S_α, $\alpha \in Y$. Assume $a, b, c \in S$ such that $a \longrightarrow^{n+1} c$ and $b \longrightarrow^{n+1} c$. Based on Lemma 4.9, $a, b, c \in S_\alpha$, for some $\alpha \in Y$, whence $ab, c \in S_\alpha$ and so $ab \, \sigma_n \, c$. But, $\sigma_n = \longrightarrow^n$, according to the hypothesis, so we have that $ab \longrightarrow^n c$, which was to be proved.

(vi)⇒(vii) This implication is trivial.

(vii)⇒(i) We always have $\longrightarrow^n \subseteq \longrightarrow^{n+1}$. To prove the opposite inclusion, assume $a, b \in S$ such that $a \longrightarrow^{n+1} b$. Then $a^2 \longrightarrow^n b$, by (vii), and so $a \longrightarrow^n b$, which we had to prove. Hence, $\longrightarrow^n = \longrightarrow^{n+1}$, so $\longrightarrow^n$ is transitive.

(v)⇒(viii) Let $\longrightarrow^n = \sigma_n$. Based on Theorem 4.5, S is a semilattice of σ_n-simple semigroups, and based on Theorem 4.5 we have that $\longrightarrow^n$ is transitive, that is $S \in \mathcal{S}_n$, and σ_n equals the symmetric opening of $\longrightarrow^n$. Therefore, we have proved (viii).

(viii)⇒(ix) Since $S \in \mathcal{S}_n$ means that $\longrightarrow^n$ is transitive, that is $\longrightarrow^n = \longrightarrow^{n+1}$, then (viii) yields (ix).

(ix)⇒(i) We have that

$$\longrightarrow^n \subseteq \longrightarrow^{n+1} \subseteq \longrightarrow^{n+1} \cap (\longrightarrow^{n+1})^{-1} = \longrightarrow^n,$$

so $\longrightarrow^n = \longrightarrow^{n+1}$, whence it follows that $\longrightarrow^n$ is transitive. □

Remark 4.4 A binary relation ξ on a semigroup S is said to satisfy the *power property* if $a \, \xi \, b$ implies $a^2 \, \xi \, b$, for all $a, b \in S$, and to satisfy the *common multiple property*, the *cm-property* in short, if $a\,\xi\,c$ and $b\,\xi\,c$ implies $ab\,\xi\,c$.

Remark 4.5 Let $\{S_k\}_{k\in\mathbf{Z}^+}$ be a sequence of semigroups such that for each $k\in\mathbf{Z}^+$ the following conditions are satisfied:

(1) S_k is a 0-simple semigroup with the zero 0_k;
(2) there exists a nonzero square-zero element a_k in S_k;
(3) there exists a nonzero idempotent e_k in S_k;
(4) $S_k\cap S_{k+1}=\{e_k\}=\{0_{k+1}\}$ and $S_k\cap S_i=\varnothing$ for $i\geq k+2$.

By induction we define a new sequence $\{T_n\}_{n\in\mathbf{Z}^+}$ of semigroups as follows: We set $T_1=S_1$. If, for $n\in\mathbf{Z}^+$, T_n is defined, then we set $T_{n+1}=T_n\cup S_{n+1}$ and we define a multiplication on T_{n+1} to coincide with the multiplications on T_n and S_{n+1}, and for $x\in T_n$ and $y\in S_{n+1}$ we set $xy=xe_n$ and $yx=e_nx$, where the right-hand side multiplications are from T_n. Since $\{T_n\}_{n\in\mathbf{Z}^+}$ is a chain of semigroups, then $T=\bigcup_{n\in\mathbf{Z}^+}T_n$ is also a semigroup and each T_n is an ideal of T. Let us denote $0=0_1$. We see that 0 is the zero of T.

As was proved by M. S. Putcha in [5], $\mathrm{ran(T_n)}=\mathrm{sran(T_n)}=\mathrm{n}+1$, for each $n\in\mathbf{Z}^+$, and $\mathrm{ran(T)}=\mathrm{sran(T)}=\infty$. Moreover, he proved that $0 — a_1 — a_2 — \cdots — a_n — e_n$ is a minimal sequence between 0 and e_n in T_n and T.

For $n\in\mathbf{Z}^+$, $n\geq 2$, let P_n be the orthogonal sum (0-direct union) of T_n and a 0-simple semigroup S having a nonzero square-zero element a and a nonzero idempotent e. Then $\mathrm{ran(P_n)}=\mathrm{n}+2$ and $\mathrm{sran(P_n)}=\mathrm{n}+1$. In particular, a minimal sequence between e and e_n is $e — a — a_1 — a_2 — \cdots — a_n — e_n$, and a minimal sequence from e into e_n is $e\longrightarrow a_1\longrightarrow a_2\longrightarrow\cdots\longrightarrow a_n\longrightarrow e_n$.

References

S. Bogdanović, M. Ćirić and Ž. Popović [1]; M. Ćirić and S. Bogdanović [3]; Ž. Popović, S. Bogdanović and M. Ćirić [1]; Ž. Popović [2].

4.6 Semilattices of $\widehat{\lambda}$-simple Semigroups

The problems, that were treated in the previous section for the relation $\relbar\relbar$, will here be considered for the left-hand analogue of this relation.

For $n \in \mathbf{Z}^+$ and an element a of a semigroup S, we define the sets $\widehat{\Lambda}_n(a)$ and $\widehat{\Lambda}(a)$ by

$$\widehat{\Lambda}_n(a) = \{x \in S \mid a \overset{l}{\relbar\relbar}{}^n x\}, \qquad \widehat{\Lambda}(a) = \{x \in S \mid a \overset{l}{\relbar\relbar}{}^\infty x\},$$

and the relations $\widehat{\lambda}_n$ and $\widehat{\lambda}$ on S by:

$$(a,b) \in \widehat{\lambda}_n \;\Leftrightarrow\; \widehat{\Lambda}_n(a) = \widehat{\Lambda}_n(b), \qquad\qquad (a,b) \in \widehat{\lambda} \;\Leftrightarrow\; \widehat{\Lambda}(a) = \widehat{\Lambda}(b).$$

The semilattices of λ-simple semigroups were described in the previous subsection. Here we study the semilattices of $\widehat{\lambda}$-simple semigroups. A semigroup S is $\widehat{\lambda}$-*simple* if $a \,\widehat{\lambda}\, b$, for all $a, b \in S$.

Theorem 4.14 *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of $\widehat{\lambda}$-simple semigroups;*
(ii) $\overset{l}{\relbar\relbar}{}^\infty = \relbar\relbar{}^\infty$*;*
(iii) $\overset{l}{\relbar\relbar}{}^\infty$ *is a semilattice congruence on S.*

Proof. (i)⇒(ii) Let S be a semilattice Y of $\widehat{\lambda}$-simple semigroups S_α, $\alpha \in Y$. Assume $a, b \in S$ such that $a \relbar\relbar{}^\infty b$. Based on Lemma 4.9, $a, b \in S_\alpha$, for some $\alpha \in Y$, whence $a \overset{l}{\relbar\relbar}{}^\infty b$. Therefore, $\relbar\relbar{}^\infty \subseteq \overset{l}{\relbar\relbar}{}^\infty$. The opposite inclusion is clear.

(ii)⇒(iii) This is an immediate consequence of Theorem 4.3.

(iii)⇒(i) This follows from Lemma 4.14. □

For $n \in \mathbf{Z}^+$, let us denote by $\mathcal{L}_n$ the class of all semigroups from $\mathcal{S}_n$ on which $\longrightarrow^n = \overset{l}{\longrightarrow}{}^n$, and let $\widehat{\mathcal{L}}_n$ denote the class of all semigroups from $\widehat{\mathcal{S}}_n$ on which $\relbar\relbar{}^n = \overset{l}{\relbar\relbar}{}^n$. Semigroups belonging to the class $\mathcal{L}_n$ were described in the previous subsection. In particular, it was to be proved that $S \in \mathcal{L}_n$ if and only if it is a semilattice of λ_n-simple semigroups. It can be also checked that $S \in \mathcal{L}_n$ if and only if $\overset{l}{\longrightarrow}{}^n = \longrightarrow^{n+1}$. Here we investigate the structure of semigroups from the class $\widehat{\mathcal{L}}_n$.

Theorem 4.15 *Let $n \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) $S \in \widehat{\mathcal{L}}_n$;

(ii) $\overset{l}{\relbar\!\!\relbar}{}^{n} = \relbar\!\!\relbar^{n+1}$ *on* S;

(iii) $\overset{l}{\relbar\!\!\relbar}{}^{n}$ *is a semilattice congruence on* S;

(iv) S *is a semilattice of* $\widehat{\lambda}_n$*-simple semigroups;*

(v) $\overset{l}{\relbar\!\!\relbar}{}^{n} = \sigma_n$ *on* S;

(vi) $(\forall a, b, c \in S)\ a \relbar\!\!\relbar^{n+1} b \ \Rightarrow\ a^2 \overset{l}{\relbar\!\!\relbar}{}^{n} b$;

(vii) $(\forall a, b, c \in S)\ a \relbar\!\!\relbar^{n} b \ \&\ b \relbar\!\!\relbar^{n} c \ \Rightarrow\ a \overset{l}{\relbar\!\!\relbar}{}^{n} c$;

(viii) $(\forall a, b, c \in S)\ a \relbar\!\!\relbar^{n+1} c \ \&\ b \relbar\!\!\relbar^{n+1} c \ \Rightarrow\ ab \overset{l}{\relbar\!\!\relbar}{}^{n} c$;

(ix) $S \in \mathcal{L}_n$ *and* $\overset{l}{\relbar\!\!\relbar}{}^{n}$ *equals the symmetric opening of* $\overset{l}{\longrightarrow}{}^{n}$.

Proof. (i)$\Rightarrow$(ii) This is evident.

(ii)$\Rightarrow$(iii) From (ii) it follows that $\relbar\!\!\relbar^{n+1} = \overset{l}{\relbar\!\!\relbar}{}^{n} \subseteq \relbar\!\!\relbar^{n} \subseteq \relbar\!\!\relbar^{n+1}$, whence $\overset{l}{\relbar\!\!\relbar}{}^{n} = \relbar\!\!\relbar^{n}$, and so $\overset{l}{\relbar\!\!\relbar}{}^{\infty} = \overset{l}{\relbar\!\!\relbar}{}^{n} = \relbar\!\!\relbar^{n} = \relbar\!\!\relbar^{\infty}$. Therefore, in view of Theorem 4.14, $\overset{l}{\relbar\!\!\relbar}{}^{n}$ is a semilattice congruence on S.

(iii)$\Rightarrow$(iv) This follows from Lemma 4.14.

(iv)$\Rightarrow$(v) Let S be a semilattice Y of $\widehat{\lambda}_n$-simple semigroups S_α, $\alpha \in Y$. Then S_α is a σ_n-simple semigroup, for each $\alpha \in Y$, so based on Theorem 4.5 we have that $\overset{l}{\relbar\!\!\relbar}{}^{n} \subseteq \sigma_n$. On the other hand, if $(a, b) \in \sigma_n$, then there exists $\alpha \in Y$ such that $a, b \in S_\alpha$, based on Lemma 4.9, whence $a \overset{l}{\relbar\!\!\relbar}{}^{n} b$, so $\sigma_n \subseteq \overset{l}{\relbar\!\!\relbar}{}^{n}$. Therefore, (v) holds.

(v)$\Rightarrow$(i) Based on (v), in view of Theorem 4.5, $\sigma_n = \overset{l}{\relbar\!\!\relbar}{}^{n} \subseteq \relbar\!\!\relbar^{n} \subseteq \sigma_n$, that is $\overset{l}{\relbar\!\!\relbar}{}^{n} = \relbar\!\!\relbar^{n} = \sigma_n$, and from Thorem 4.13 it follows that $S \in \widehat{\mathcal{S}}_n$, so we have proved (i).

(i)$\Rightarrow$(vi) Let $S \in \widehat{\mathcal{L}}_n$. Then $S \in \widehat{\mathcal{S}}_n$ and $\overset{l}{\relbar\!\!\relbar}{}^{n} = \relbar\!\!\relbar^{n} = \relbar\!\!\relbar^{n+1}$. Assume now $a, b \in S$ such that $a \relbar\!\!\relbar^{n+1} b$. According to Theorem 4.13 we have that $a^2 \relbar\!\!\relbar^{n} b$, and since $\overset{l}{\relbar\!\!\relbar}{}^{n} = \relbar\!\!\relbar^{n}$, then $a^2 \overset{l}{\relbar\!\!\relbar}{}^{n} b$, which was to be proved.

(vi)$\Rightarrow$(vii) Based on (vi) it follows that $a \relbar\!\!\relbar^{n+1} b$ implies $a^2 \relbar\!\!\relbar^{n} b$, for all $a, b \in S$, so based on Theorem 4.13 we have that $\relbar\!\!\relbar^{n}$ is a transitive relation on S. Assume now $a, b, c \in S$ such that $a \relbar\!\!\relbar^{n} b$ and $b \relbar\!\!\relbar^{n} c$. Then $a \relbar\!\!\relbar^{n} c$, that is $a \relbar\!\!\relbar^{n+1} c$, whence $a^2 \overset{l}{\relbar\!\!\relbar}{}^{n} c$, by (vi), and hence $a \overset{l}{\relbar\!\!\relbar}{}^{n} c$, which was to be proved.

(vii)$\Rightarrow$(i) From (vii) it follows that $\relbar\!\!\relbar^{n}$ is transitive, that is $S \in \widehat{\mathcal{S}}_n$, and also $\relbar\!\!\relbar^{n} = \overset{l}{\relbar\!\!\relbar}{}^{n}$, whence we obtain $S \in \widehat{\mathcal{L}}_n$.

(iv)⇒(ix) Let S be a semilattice Y of $\widehat{\lambda}_n$-simple semigroups S_α, $\alpha \in Y$. Assume $a, b \in S$ such that $a \longrightarrow^{n+1} b$. Let $a \in S_\alpha$, $b \in S_\beta$, for some $\alpha, \beta \in Y$. Based on Lemma 4.9, $\beta \leq \alpha$ in Y, that is $\alpha\beta = \beta$, whence $b, ba \in S_\beta$. Now we have that $ba \overset{l}{-\!\!-}{}^{n} b$ and hence $ba \overset{l}{\longrightarrow}{}^{n} b$. But, $ba \overset{l}{\longrightarrow}{}^{n} b$ implies $a \overset{l}{\longrightarrow}{}^{n} b$. Therefore, we have proved that $\longrightarrow^{n+1} \subseteq \overset{l}{\longrightarrow}{}^{n}$. Since the opposite inclusion always holds, we have that $\longrightarrow^{n+1} = \overset{l}{\longrightarrow}{}^{n}$, that is $S \in \mathcal{L}_n$.

We also have that S_α is a σ_n-simple semigroup, for each $\alpha \in Y$, so σ_n equals the symmetric opening of $\longrightarrow^n$, based on Theorem 4.5. But, we have proved that $\overset{l}{\longrightarrow}{}^{n} = \longrightarrow^n$, and in the part (iv)⇒(v) of this theorem we proved that $\overset{l}{-\!\!-}{}^{n} = \sigma_n$. Therefore, $\overset{l}{-\!\!-}{}^{n}$ equals the symmetric opening of $\longrightarrow^n$. This completes the proof of this implication.

(ix)⇒(v) From $S \in \mathcal{L}_n$ it follows that $\overset{l}{\longrightarrow}{}^{n} = \longrightarrow^n$ and $\overset{l}{\longrightarrow}{}^{n}$ and $\longrightarrow^n$ are transitive relations on S. On the other hand, based on Theorem 4.5, σ_n is the transitive closure of $\longrightarrow^n$, and now, in view of (ix), we have that $-\!\!-^{n} = \sigma_n$.

(i)⇒(viii) Let $S \in \widehat{\mathcal{L}}_n$. Assume $a, b, c \in S$ such that $a -\!\!-^{n+1} c$ and $b -\!\!-^{n+1} c$. Then $S \in \widehat{\mathcal{S}}_n$, and based on Theorem 4.13 we have that $ab -\!\!-^{n} c$. But, $-\!\!-^{n} = \overset{l}{-\!\!-}{}^{n}$, so we obtain $ab \overset{l}{-\!\!-}{}^{n} c$, which was to be proved.

(viii)⇒(vi) This implication is evident. □

References

S. Bogdanović, M. Ćirić and Ž. Popović [1]; M. Ćirić and S. Bogdanović [3]; Ž. Popović [2].

4.7 The Radicals of Green's $\mathcal{J}$-relation

The radicals $R(\varrho)$ and $T(\varrho)$ which we will use in this section are defined on page 28.

As was proved by M. S. Putcha in [1], in 1973, the smallest semilattice congruence on a completely π-regular semigroup equals the transitive closure of $R(\mathcal{J})$. But, this assertion does not hold in a general case, and we investigate some conditions under which the transitive closures and powers of the relations $R(\mathcal{J})$ and $T(\mathcal{J})$ are semilattice congruences.

A relation ϱ on S will be called *T-closed* if $T(\varrho) = \varrho$, and it is *R-closed* if $R(\varrho) = \varrho$. It is easy to check that $\relbar\joinrel\relbar^{n}$, for each $n \in \mathbf{Z}^+$, and $\relbar\joinrel\relbar^{\infty}$ are both T-closed and R-closed relations. Thus, for Green's $\mathcal{J}$-relation on S, $R(\mathcal{J})$ and $T(\mathcal{J})$ are contained in $\relbar\joinrel\relbar$.

Here we consider the semigroups on which $R(\mathcal{J})^\infty$ is a semilattice congruence.

Theorem 4.16 *On a semigroup S, $R(\mathcal{J})^\infty$ is a semilattice congruence if and only if $R(\mathcal{J})^\infty = \relbar\joinrel\relbar^{\infty}$.*

Proof. This is an immediate consequence of Theorem 4.3 and the fact that $R(\mathcal{J})$ is contained in $\relbar\joinrel\relbar$. $\square$

Similarly we have

Theorem 4.17 *On a semigroup S, $T(\mathcal{J})^\infty$ is a semilattice congruence if and only if $T(\mathcal{J})^\infty = \relbar\joinrel\relbar^{\infty}$.*

Further we study the conditions under which the powers of $R(\mathcal{J})$ are semilattice congruences.

Theorem 4.18 *Let $n \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) $R(\mathcal{J})^n$ *is a semilattice congruence;*
(ii) $R(\mathcal{J})^n = \sigma_n$;
(iii) $R(\mathcal{J})^n = \relbar\joinrel\relbar^{n+1}$;
(iv) $(\forall a, b \in S)\ a \relbar\joinrel\relbar^{n+1} b \ \Rightarrow\ (a^2, b) \in R(\mathcal{J})^n$;
(v) $(\forall a, b, c \in S)\ a \relbar\joinrel\relbar^{n} b\ \&\ b \relbar\joinrel\relbar^{n} c \ \Rightarrow\ (a, c) \in R(\mathcal{J})^n$;
(vi) $(\forall a, b, c \in S)\ a \relbar\joinrel\relbar^{n+1} c\ \&\ b \relbar\joinrel\relbar^{n+1} c \ \Rightarrow\ (ab, c) \in R(\mathcal{J})^n$.

Proof. (i)$\Rightarrow$(iii), (iv), (v), and (vi). Let S be a semilattice Y of semigroups S_α, $\alpha \in Y$, such that each S_α is an $R(\mathcal{J})^n$-class of S.

Assume $a, b \in S$ such that $a \relbar\joinrel\relbar^{n+1} b$. Then from Lemma 4.9 we have that $a, b \in S_\alpha$, for some $\alpha \in Y$, so $(a, b) \in R(\mathcal{J})^n$. Therefore,

$$\relbar\joinrel\relbar^{n+1} \subseteq R(\mathcal{J})^n \subseteq \relbar\joinrel\relbar^{n} \subseteq \relbar\joinrel\relbar^{n+1},$$

so we have obtained (iii). On the other hand, we also have that $a^2, b \in S_\alpha$, so $(a^2, b) \in R(\mathcal{J})^n$, whence it follows (iv).

Assume $a, b, c \in S$ such that $a \text{---}^n b$ and $b \text{---}^n c$. Then $a, b, c \in S_\alpha$, for some $\alpha \in Y$, in view of Lemma 4.9, and $a, c \in S_\alpha$ implies $(a, c) \in R(\mathcal{J})^n$. Therefore, we have proved (v). Similarly, if $a, b, c \in S$ such that $a \text{---}^{n+1} c$ and $b \text{---}^{n+1} c$, then $a, b, c \in S_\alpha$, for some $\alpha \in Y$, so $ab, c \in S_\alpha$, whence $(ab, c) \in R(\mathcal{J})^n$. This proves (vi).

(iii)⇒(ii) If (iii) holds, then

$$\text{---}^{n+1} \subseteq R(\mathcal{J})^n \subseteq \text{---}^n \subseteq \text{---}^{n+1},$$

so we have that $\text{---}^n = \text{---}^{n+1}$, that is ---^n is transitive, and from Theorem 4.13 it follows that $\sigma_n = \text{---}^n = R(\mathcal{J})^n$.

(ii)⇒(i) If $R(\mathcal{J})^n = \sigma_n$, then $(a^2, a) \in R(\mathcal{J}) \subseteq R(\mathcal{J})^n = \sigma_n$, for each $a \in S$, and based on Theorem 4.5 we have that $\sigma_n = R(\mathcal{J})^n$ is a semilattice congruence.

(vi)⇒(iv) This is obvious.

(iv)⇒(iii) Note that $(a^2, b) \in R(\mathcal{J})$ implies $(a, b) \in R(\mathcal{J})$, so $(a^2, b) \in R(\mathcal{J})^n$ implies $(a, b) \in R(\mathcal{J})^n$. Therefore, (iv) yields $\text{---}^{n+1} \subseteq R(\mathcal{J})^n$, whence it follows (iii).

(v)⇒(iii) First, from (v) it follows that ---^n is transitive, that is $\text{---}^n = \text{---}^{n+1}$. It also follows from (v) that $\text{---}^n = R(\mathcal{J})^n$, so we have proved (iii).□

In the case of the radical $T(\mathcal{J})$ we have the following:

Theorem 4.19 *Let $n \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) $T(\mathcal{J})^n$ *is a semilattice congruence;*
(ii) $T(\mathcal{J})^n = \sigma_n = R(\mathcal{J})^n$*;*
(iii) $T(\mathcal{J})^n = \text{---}^{n+1}$*;*
(iv) $(\forall a, b \in S)\ a \text{---}^{n+1} b \ \Rightarrow\ (a^2, b) \in T(\mathcal{J})^n$*;*
(v) $(\forall a, b, c \in S)\ a \text{---}^n b \ \&\ b \text{---}^n c \ \Rightarrow\ (a, c) \in T(\mathcal{J})^n$*;*
(vi) $(\forall a, b, c \in S)\ a \text{---}^{n+1} c \ \&\ b \text{---}^{n+1} c \ \Rightarrow\ (ab, c) \in T(\mathcal{J})^n$*.*

Proof. (iii)⇒(ii) If (iii) holds then

$$\text{---}^{n+1} = T(\mathcal{J})^n \subseteq R(\mathcal{J})^n \subseteq \text{---}^n \subseteq \text{---}^{n+1},$$

so $R(\mathcal{J})^n = \text{---}^{n+1}$, and from Theorem 4.18 we have that $\sigma_n = R(\mathcal{J})^n = \text{---}^{n+1} = T(\mathcal{J})^n$, which was to be proved.

The implication (ii)⇒(i) follows from Theorem 4.18. The implications between the remaining conditions can be proved in a similar way as the corresponding parts of Theorem 4.18. □

Problem 4.3 For an arbitrary Green's relation $\mathcal{X} \in \{\mathcal{J}, \mathcal{D}, \mathcal{L}, \mathcal{R}, \mathcal{H}\}$, we say that a semigroup S is $R(\mathcal{X})$-*simple* if $a, b \in R(\mathcal{X})$, for all $a, b \in S$.

At the end we state the following problems:

(i) Describe the bands of $R(\mathcal{J})$-simple ($R(\mathcal{L})$-simple) semigroups;
(ii) Describe the semigroups in which $R(\mathcal{X})$, $\mathcal{X} \in \{\mathcal{J}, \mathcal{D}, \mathcal{L}, \mathcal{R}, \mathcal{H}\}$, is a congruence.

Exercises

1. Let $\mathcal{X} \in \{\mathcal{J}, \mathcal{L}, \mathcal{H}\}$. Prove that the following conditions on a semigroup S are equivalent:

(i) S is a semilattice of Archimedean semigroups;
(ii) $R(\sigma_1)$ is a congruence on S;
(iii) $T(\sigma_1)$ is a semilattice (*band*) congruence on S;
(iv) $R(\sigma_1) = \sigma_1$;
(v) $R(\mathcal{X}) \subseteq \sigma_1$.

2. Let $\mathcal{X} \in \{\mathcal{J}, \mathcal{L}, \mathcal{H}\}$. Prove that the following conditions on a semigroup S are equivalent:

(i) $R(\mathcal{X})$ is a semilattice congruence;
(ii) $R(\mathcal{X}) = \sigma_1$;
(iii) S is a semilattice of $R(\mathcal{X})$-simple semigroups.

3. Show that the following conditions on a semigroup S are equivalent:

(i) $R(\mathcal{J})$ is a semilattice congruence;
(ii) $R(\mathcal{J}) = \sigma_1$;
(iii) $R(\mathcal{J}) = \,\text{—}^2$;
(iv) $(\forall a, b \in S)\ a \text{—} b \;\Rightarrow\; (a^2, b) \in R(\mathcal{J})$;
(v) $(\forall a, b, c \in S)\ a \text{—} b \;\&\; b \text{—} c \;\Rightarrow\; (a, c) \in R(\mathcal{J})$;
(vi) $(\forall a, b, c \in S)\ a \text{—} c \;\&\; b \text{—} c \;\Rightarrow\; (ab, c) \in R(\mathcal{J})$.

References

S. Bogdanović [16], [17]; S. Bogdanović and M. Ćirić [4], [10], [12], [21]; S. Bogdanović, M. Ćirić and Ž. Popović [1]; M. Ćirić and S. Bogdanović [3], [5]; J. M. Howie [3]; F. Kmeť [1]; M. Petrich [6]; Ž. Popović [2]; M. S. Putcha [1], [2], [5], [8]; L. N. Shevrin [4]; T. Tamura [2], [12], [13], [15].

Chapter 5

Semilattices of Archimedean Semigroups

Note that the semilattices of Archimedean semigroups have been studied by a number of authours. M. S. Putcha, in 1973, gave the first complete description of such semigroups. Other characterizations of semilattices of Archimedean semigroups have been given by T. Tamura, 1972, S. Bogdanović and M. Ćirić, 1992, and M. Ćirić and S. Bogdanović, 1993.

In this chapter we investigate the semigroups whose any subsemigroup is Archimedean, called hereditary Archimedean, and the semilattices of such semigroups.

Bands of left (also right and two-sided) Archimedean semigroups form important classes of semigroups studied by a number of authors. General characterizations of these semigroups were given by M. S. Putcha, in 1973, and in the completely π-regular case by L. N. Shevrin, in 1994. Some characterizations of bands of left Archimedean semigroups and of bands of nil-extensions of left simple semigroups have been given recently by S. Bogdanović and M. Ćirić, 1997. Based on the well-known results of A. H. Clifford, in 1954, any band of left Archimedean semigroups is a semilattice of matrices (rectangular bands) of left Archimedean semigroups. The converse of this assertion does not hold, i.e. the class of semilattices of matrices of left Archimedean semigroups is larger than the class of bands of left Archimedean semigroups. In this chapter we give a complete characterization of semigroups having a semilattice decomposition whose components are matrices of left Archimedean semigroups. Moreover, we describe such components in general and in some special cases.

As we all know, semilattices of completely Archimedean semigroups form an important class of semigroups studied by a number of authors. Several characterizations of these semigroups were given by M. S. Putcha, in 1973, and 1981, L. N. Shevrin, in 2005, M. L. Veronesi, in 1984, and S. Bogdanović, in 1987. We emphasize the results of L. N. Shevrin (see also M. L. Veronesi [1] and L. N. Shevrin [4]), which give a powerful tool for checking whether a π-regular semigroup is a semilattice of completely Archimedean semigroups. Based on this result, a π-regular semigroup has this property if and only if any of its regular elements are completely regular. In this chapter we generalize the notion of a completely Archimedean semigroup, introducing the notion of a left completely Archimedean semigroup. Several characterizations of these semigroups will be given in Theorem 5.26. The main results of this section are Theorem 5.27, which gives some characterizations of semilattices of left completely Archimedean semigroups, and Theorem 7.4, in which we give some new results concerning semilattices of completely Archimedean semigroups.

Semigroups which can be decomposed into a band of left Archimedean semigroups have been studied in many papers. M. S. Putcha, in 1973, proved a general theorem that characterizes such semigroups. This result we give here as the equivalence of conditions (i) and (ii) in Theorem 5.29. Some special decompositions of this type have also been treated in a number of papers. S. Bogdanović, in 1984, P. Protić, in 1991, and 1994, and S. Bogdanović and M. Ćirić, in 1992, and 1995, studied bands of left Archimedean semigroups whose related band homomorphic images belong to several very important varieties of bands. L. N. Shevrin, in 1994, investigated bands of nil-extensions of left groups, and S. Bogdanović and M. Ćirić, in 1992, investigated bands of nil-extensions of groups. Finally, bands of left simple semigroups, in the general and some special cases, were investigated by P. Protić, in 1995, and S. Bogdanović and M. Ćirić, in 1996.

5.1 The General Case

The semilattice of σ_n-simple and λ_n-simple semigroups were described in Sections 4.2 and 4.3. Here we give some new characterizations for the semilattices of σ_1-simple and λ_1-simple semigroups, i.e. for the semilattices of Archimedean semigroups and the semilattices of left Archimeden semigroups.

Theorem 5.1 *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of Archimedean semigroups;*
(ii) $(\forall a, b \in S)\;\; a \longrightarrow b \;\Rightarrow\; a^2 \longrightarrow b$;
(iii) $(\forall a, b \in S)\;\; a \,|\, b \;\Rightarrow\; a^2 \longrightarrow b$;
(iv) $(\forall a, b \in S)(\forall k \in \mathbf{Z}^+)(\exists n \in \mathbf{Z}^+)\quad (ab)^n \in Sa^kS$;
(v) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\quad (ab)^n \in Sa^2S$;
(vi) $(\forall a, b \in S)(\forall k \in \mathbf{Z}^+)(\exists n \in \mathbf{Z}^+)\quad (ab)^n \in Sb^kS$;
(vii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\quad (ab)^n \in Sb^2S$;
(viii) *the radical of every ideal of S is an ideal.*

Proof. (i)$\Leftrightarrow$(ii) This equivalence holds based on (i)$\Leftrightarrow$(v) of Theorem 4.5, for $n = 1$.

(ii)$\Rightarrow$(iii) Assume that $a|\, b$, then $a \longrightarrow b$, whence $a^2 \longrightarrow b$. Thus (iii) holds.

(iii)$\Rightarrow$(ii) Assume that $a \longrightarrow b$, i.e. $a \,|\, b^n$ for some $n \in \mathbf{Z}^+$. Then $a^2 \longrightarrow b^n$. Thus $a^2 \longrightarrow b$.

(i)$\Rightarrow$(iv) Let S be a semilattice Y of Archimedean semigroups $S_\alpha, \alpha \in Y$. Let $a \in S_\alpha, b \in S_\beta$ for some $\alpha, \beta \in Y$. Then we have that $ab, a^kb \in S_{\alpha\beta}$ for all $k \in \mathbf{Z}^+$, so there exists $n \in \mathbf{Z}^+$ such that

$$(ab)^n \in Sa^kbS \subseteq Sa^kS.$$

(iv)$\Rightarrow$(v) This follows immediately.

(v)$\Rightarrow$(i) Let $a, b \in S$ be elements such that $a \mid b$. Then there exists $u, v \in S^1$ such that $b = uav$, so $b^{n+1} = u(avu)^n av$ for every $n \in \mathbf{Z}^+$. From (v) we have that there exists $n \in \mathbf{Z}^+$ such that $(avu)^n \in Sa^2S$, whence

$$b^{n+1} = u(avu)^n av \in uSa^2Sav \subseteq Sa^2S.$$

Therefore, $a^2 \mid b^{n+1}$, and based on the equivalence (ii)$\Leftrightarrow$(iii) and from Theorem 4.5 it follows that S is a semilattice of Archimedean semigroups.

(i)$\Rightarrow$(vi)$\Rightarrow$(vii)$\Rightarrow$(i) This we prove in a similar way, as (i)$\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(i).

(i)$\Rightarrow$(viii) Let A be an ideal of S and let $a \in \sqrt{A},\; b \in S$. Then $a^k \in A$, for some $k \in \mathbf{Z}^+$. Since (i)$\Leftrightarrow$(iv)$\Leftrightarrow$(vi), we then have that there exist $m, n \in \mathbf{Z}^+$ such that $(ab)^n, (ba)^m \in Sa^kS \subseteq SAS \subseteq A$. Therefore, $ab, ba \in \sqrt{A}$, so $\sqrt{A}$ is an ideal of S.

(viii)⇒(v) Let (viii) hold. Let $a, b \in S$ and let $A = Sa^2S$. It is clear that A is an ideal of S and that $a \in \sqrt{A}$. From (viii) it follows that $\sqrt{A}$ is an ideal of S, so $ab \in \sqrt{A}$, i.e. there exists $n \in \mathbf{Z}^+$ such that $(ab)^n \in Sa^2S$. □

Let $m, n \in \mathbf{Z}^+$. On a semigroup S we define a relation $\rho_{(m,n)}$ by

$$(a, b) \in \rho_{(m,n)} \Leftrightarrow (\forall x \in S^m)(\forall y \in S^n)\ xay \,\text{---}\, xby,$$

i.e.

$$a\rho_{(m,n)}b \Leftrightarrow (\forall x\in S^m)(\forall y\in S^n)(\exists i, j\in \mathbf{Z}^+)(xay)^i \in SxbyS \wedge (xby)^j \in SxayS.$$

The relation $\rho_{(1,1)}$ we simply denote by ρ.

If instead of the relation — we assume the equality relation, then we obtain the relation which was introduced and discussed by S. J. L. Kopamu in [1], 1995. So, the relation $\rho_{(m,n)}$ is a generalization of Kopamu's relation.

Based on the following theorem we give a very important characteristic of the $\rho_{(m,n)}$ relation.

Theorem 5.2 *Let $m, n \in \mathbf{Z}^+$. On a semigroup S the relation $\rho_{(m,n)}$ is a congruence relation.*

Proof. It is evident that $\rho_{(m,n)}$ is a reflexive and symmetric relation on S.

Assume $a, b, c \in S$ such that $a\rho_{(m,n)}b$ and $b\rho_{(m,n)}c$. Then

$$a\rho_{(m,n)}b \Leftrightarrow (\forall x\in S^m)(\forall y\in S^n)(\exists i, j\in \mathbf{Z}^+)\ (xay)^i \in SxbyS \wedge (xby)^j \in SxayS,$$

$$b\rho_{(m,n)}c \Leftrightarrow (\forall x\in S^m)(\forall y\in S^n)(\exists p, q\in \mathbf{Z}^+)\ (xby)^p \in SxcyS \wedge (xcy)^q \in SxbyS.$$

So, $(xay)^i = uxbyv$ and $(xcy)^q = wxbyz$, for some $u, v, w, z \in S$. Since $b\rho_{(m,n)}c$, then for $x \in S^m$ and $yvu \in S^{n+2} \subseteq S^n$ we have that there exists $t \in \mathbf{Z}^+$ such that $(xbyvu)^t \in SxcyvuS$ and

$$((xay)^i)^{t+1} = (uxbyv)^{t+1} = u(xbyvu)^t xbyv \in uSxcyvuSxbyv \subseteq SxcyS.$$

Thus $(xay)^{i(t+1)} \in SxcyS$.

Similarly, we prove that $(xcy)^k \in SxayS$, for some $k \in \mathbf{Z}^+$. Hence, $a\rho_{(m,n)}c$. Therefore, $\rho_{(m,n)}$ is a transitive relation on S.

Now, assume $a, b, c \in S$ are such that $a\rho_{(m,n)}b$. Then for $x \in S^m$, $y \in S^n$ we have $cy \in S^{n+1} \subseteq S^n$, so, there exist $p, q \in \mathbf{Z}^+$ such that

$$(x(ac)y)^p = (xa(cy))^p \in Sxb(cy)S = Sx(bc)yS,$$

$$(x(bc)y)^q = (xb(cy))^q \in Sxa(cy)S = Sx(ac)yS.$$

Hence $ac\rho_{(m,n)}bc$. Similarly, we prove that $ca\rho_{(m,n)}cb$. Thus, $\rho_{(m,n)}$ is a congruence relation on S. $\square$

Remark 5.1 *Let μ be an equivalence relation on a semigroup S and let $m, n \in \mathbf{Z}^+$. Then a relation $\mu_{(m,n)}$ defined on S by*

$$(a, b) \in \mu_{(m,n)} \;\Leftrightarrow\; (\forall x \in S^m)(\forall y \in S^n)\ (xay, xby) \in \mu$$

is a congruence relation on S. But, there exists a relation μ which is not equivalence, for example $\mu = \mathrel{-\!\!-}$, for which the relation $\mu_{(m,n)}$ is a congruence on S.

The following two lemmas are useful for further work. Their proofs are elementary and they will be omitted.

Lemma 5.1 *Let ξ be an equivalence on a semigroup S. Then ξ is a congruence relation on S if and only if $\xi = \xi^\flat$.*

Lemma 5.2 *Let ξ be an equivalence relation on a semigroup S. Then $\xi^\flat$ is a band congruence if and only if*

$$(\forall a \in S)(\forall x, y \in S^1)\ (xay, xa^2y) \in \xi.$$

Now we give some new characterizations of the semilattices of Archimedean semigroups.

Theorem 5.3 *Let $m, n \in \mathbf{Z}^+$. The following conditions on a semigroup S are equivalent:*

(i) *$\rho_{(m,n)}$ is a band congruence;*
(ii) *$(\forall a \in S)(\forall x \in S^m)(\forall y \in S^n)\ xay \mathrel{-\!\!-} xa^2y$;*
(iii) *S is a semilattice of Archimedean semigroups;*
(iv) *$R(\rho_{(m,n)}) = \rho_{(m,n)}$;*
(v) *$\rho^\flat_{(m,n)}$ is a band congruence;*
(vi) *$(\forall a \in S)(\forall u, v \in S^1)\ (uav, ua^2v) \in \rho_{(m,n)}$;*
(vii) *ρ is a band congruence.*

Proof. (i)⇒(ii) This implication follows immediately.

(ii)⇒(iii) Let (ii) hold. Then for every $a, b \in S$, if $x = (ab)^l$ and $y = (ba)^k b$, for some $k, l \in \mathbf{Z}^+$, $l \geq m$ and $k \geq n$, there exists $i \in \mathbf{Z}^+$ such that

$$((ab)^l a(ba)^k b)^i \in S(ab)^l a^2 (ba)^k bS \subseteq Sa^2S,$$

i.e.

$$(ab)^{(2k+l+1)i} \in Sa^2S.$$

Thus, based on Theorem 5.1 S is a semilattice of Archimedean semigroups.

(iii)⇒(i) Let S be a semilattice Y of Archimedean semigroups S_α, $\alpha \in Y$ and let $m, n \in \mathbf{Z}^+$ be fixed elements. Based on Theorem 5.2 $\rho_{(m,n)}$ is a congruence relation on S. It remains to be proven that $\rho_{(m,n)}$ is a band congruence on S. Assume $a \in S$, $x \in S^m$ and $y \in S^n$. Then $xay, xa^2y \in S_\alpha$, for some $\alpha \in Y$. Since S_α, $\alpha \in Y$, is Archimedean, then there exist $p, q \in \mathbf{Z}^+$ such that

$$(xay)^p \in S_\alpha xa^2yS_\alpha \subseteq Sxa^2yS,$$
$$(xa^2y)^q \in S_\alpha xayS_\alpha \subseteq SxayS,$$

hence $a\rho_{(m,n)}a^2$, i.e. $\rho_{(m,n)}$ is a band congruence on S. Thus (i) holds.

(i)⇒(iv) The inclusion $\rho_{(m,n)} \subseteq R(\rho_{(m,n)})$ always holds, so it remains for us to prove the opposite inclusion. Since $\rho_{(m,n)}$ is a band congruence on S, then we have that

$$(\forall a \in S)(\forall k \in \mathbf{Z}^+) \;\; a\rho_{(m,n)}a^k.$$

Now assume $a, b \in S$ such that $aR(\rho_{(m,n)})b$. Then $a^i\rho_{(m,n)}b^j$, for some $i, j \in \mathbf{Z}^+$, and from the previous statement we have that $a\rho_{(m,n)}a^i\rho_{(m,n)}b^j\rho_{(m,n)}b$. Thus $a\rho_{(m,n)}b$. So $R(\rho_{(m,n)}) \subseteq \rho_{(m,n)}$. Therefore, (iv) holds.

(iv)⇒(i) Since $\rho_{(m,n)}$ is reflexive, then based on the hypothesis for every $a \in S$ we have that

$$a^2\rho_{(m,n)}a^2 \Leftrightarrow (a^1)^2\rho_{(m,n)}(a^2)^1 \Leftrightarrow aR(\rho_{(m,n)})a^2 \Leftrightarrow a\rho_{(m,n)}a^2.$$

Thus, (i) holds.

(i)⇒(v) This implication follows from Lemma 5.1.

(v)⇒(vi) This implication follows from Lemma 5.2.

(vi)⇒(i) Let (vi) hold. Based on Theorem 5.2 $\rho_{(m,n)}$ is a congruence and based on (vi) for $u = v = 1$ we obtain that $(a, a^2) \in \rho_{(m,n)}$, for every $a \in S$, i.e. $\rho_{(m,n)}$ is a band congruence. Thus, (i) holds.

(i)⇔(vii) This equivalence follows immediately from the equivalence (i)⇔(iii). □

The following result shows the connections between relations ρ and $\sigma_1^\flat$.

Theorem 5.4 *Let S be an arbitrary semigroup. Then $\rho = \sigma_1^\flat$.*

Proof. Assume $a, b \in S$ such that $a\rho b$. If $c \in \Sigma_1(a)$, then $c^k = uav$, for some $u, v \in S$ and some $k \in \mathbf{Z}^+$. Since $a\rho b$ then we obtain that $(uav)^i \in SubvS \subseteq SbS$, for some $i \in \mathbf{Z}^+$. Thus

$$c^{ki} = (c^k)^i = (uav)^i \in SbS,$$

whence $c \in \Sigma_1(b)$. So, we proved that $\Sigma_1(a) \subseteq \Sigma_1(b)$. Similarly we prove that $\Sigma_1(b) \subseteq \Sigma_1(a)$. Therefore, $\Sigma_1(a) = \Sigma_1(b)$, i.e. $a\sigma_1 b$. Thus, $\rho \subseteq \sigma_1$.

Let ξ be an arbitrary congruence relation on S contained in σ_1 and let $a, b \in S$ be elements such that $a\xi b$. Since ξ is a congruence, then for every $x, y \in S$ we have that

$$(xay, xby) \in \xi \subseteq \sigma_1 \subseteq \frac{\quad}{},$$

so it follows that $(\forall x, y \in S)\ xay \frac{\quad}{} xby$, i.e. $a\rho b$. Therefore, $\xi \subseteq \rho$. Since $\sigma_1^\flat$ is the greatest congruence contained in σ_1, then from the previous statement it is evident that $\rho = \sigma_1^\flat$. $\square$

On an arbitrary semigroup S, it is clear that the following inclusion holds

$$\rho^\flat = \rho = \sigma_1^\flat \subseteq \sigma_1.$$

Theorem 5.5 *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of nil-extensions of simple semigroups;*
(ii) *S is intra π-regular and each $\mathcal{J}$-class of S containing an intra regular element is a subsemigroup;*
(iii) *S is intra π-regular and a semilattice of Archimedean semigroups;*
(iv) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n \in S(ba)^n(ab)^nS$;
(v) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)(\forall k \in \mathbf{Z}^+)\ a^k \mid (ab)^n$;
(vi) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^{4n} \mid (ab)^n$.

Proof. (i)$\Rightarrow$(ii) Let S be a semilattice Y of semigroups S_α, $\alpha \in Y$, and for each $\alpha \in Y$, let S_α be a nil-extension of a simple semigroup K_α. Let J be a $\mathcal{J}$-class of S containing an intra regular element a, and let $a \in S_\alpha$, for some $\alpha \in Y$. Then $a = xa^2y$, for some $x, y \in S$, whence $a = (xa)^n a y^n$, for each

$n \in \mathbf{Z}^+$. It is easy to verify that $xa \in S_\alpha$, so $(xa)^n \in K_\alpha$, for some $n \in \mathbf{Z}^+$, and also, $ay^n \in S_\alpha$. Now, $a = (xa)^n ay^n \in K_\alpha S_\alpha \subseteq K_\alpha$. Thus, $a \in K_\alpha$. Since K_α is simple, then every element of K_α is $\mathcal{J}$-related with a in S, so $K_\alpha \subseteq J$. Further, assume $b \in J$. Then $(a, b) \in \mathcal{J} \subseteq \sigma$, so $b \in S_\alpha$, and since $b = uav$, for some $u, v \in S^1$, then $b = uxa^2yv = u(xa)^2ay^2v = (uxax)a(ay^2v)$. It is not difficult to check that $uxax, ay^2v \in S_\alpha$, so $b \in S_\alpha K_\alpha S_\alpha \subseteq K_\alpha$, whence $J \subseteq K_\alpha$. Therefore, $J = K_\alpha$, so it is a subsemigroup of S.

(ii)⇒(iv) Assume $a, b \in S$. Since S is intra π-regular, then $(ab)^n = x(ab)^{2n}y$, for some $n \in \mathbf{Z}^+$, $x, y \in S$. Without a loss of generality we can assume that $n \geq 2$, so $(ab)^n = x(ab)^{2n}y \in S(ba)^{n+1}S$, and clearly, $(ba)^{n+1} \in S(ab)^nS$, whence $(ba)^{n+1}\mathcal{J}(ab)^n$, i.e., $(ba)^{n+1} \in J$, where J is the $\mathcal{J}$-class of $(ab)^n$. Similarly, $(ab)^{n+1} \in J$. Based on the hypothesis, J is a subsemigroup of S, so $(ba)^{n+1}(ab)^{n+1} \in J$, i.e., $(ba)^{n+1}(ab)^{n+1}\mathcal{J}(ab)^n$. Therefore,

$$(ab)^n \in S^1(ba)^{n+1}(ab)^{n+1}S^1 \subseteq S(ba)^n(ab)^nS.$$

(iv)⇒(iii) Assume $a \in S$, then $a^{2n} \in Sa^{2n}a^{2n}S = S(a^{2n})^2S$ for some $n \in \mathbf{Z}^+$, i.e. S is an intra π-regular semigroup. From (iv) we have that for every $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $(ab)^n \in S(ba)^n(ab)^nS \subseteq Sa^2S$ and based on Theorem 5.1 S is a semilattice of Archimedean semigroups.

(iii)⇒(i) This follows from Theorem 3.14 and Lemma 2.7.

(i)⇒(v) Let (i) hold and let ξ be a corresponding semilattice congruence. Assume $a, b \in S$ and let A be a ξ-class of element ab. Then A is a nil-extension of a simple semigroup K, so there exist $n \in \mathbf{Z}^+$ such that $(ab)^n \in K$. Assume $k \in \mathbf{Z}^+$. Since $a^kb \in A$ then $(a^kb)^m \in K$, for some $m \in \mathbf{Z}^+$. Thus,

$$(ab)^n \in K(a^kb)^mK \subseteq Sa^kS,$$

because K is a simple semigroup. Therefore, (v) holds.

(v)⇒(vi) This is evident.

(vi)⇒(iii) Based on (vi) for every $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^{4n} \mid a^{2n}$, so S is intra π-regular. According to Theorem 5.1 we have that S is a semilattice of Archimedean semigroups. Thus, (iii) holds. □

A subset A of a semigroup S is *semiprimary* iff

$$(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ ab \in A \ \Rightarrow\ a^n \in A \ \vee\ b^n \in A.$$

A semigroup S is *semiprimary* if all of its ideals are semiprimary subsets of S. Based on the following theorem we prove that the class of semiprimary semigroups is equal to the class of chains of Archimedean semigroups.

Theorem 5.6 *The following conditions on a semigroup S are equivalent:*

(i) *S is a chain of Archimedean semigroups;*
(ii) $(\forall a, b \in S)\ ab \longrightarrow a\ \vee\ ab \longrightarrow b$;
(iii) *S is semiprimary;*
(iv) *$\sqrt{A}$ is a completely prime ideal, for every ideal A of S;*
(v) *$\sqrt{A}$ is a completely prime subset of S, for every ideal A of S.*

Proof. (i)⇒(ii) This follows from Corollary 4.12.

(ii)⇒(iii) Let A be an ideal of S and let $a, b \in S$. From (ii), $ab \longrightarrow a$ or $ab \longrightarrow b$, so there exists $n \in \mathbf{Z}^+$ such that $a^n \in SabS$ or $b^n \in SabS$. Now, if $ab \in A$, then $a^n \in SabS \subseteq SAS \subseteq A$ or $b^n \in SabS \subseteq SAS \subseteq A$. Thus, S is a semiprimary semigroup.

(iii)⇒(iv) Let S be a semiprimary semigroup and let $a, b \in S$. Since $(ba)(ab) \in J((ba)(ab))$, then there exists $n \in \mathbf{Z}^+$ such that

$$(ba)^n \in S(ba)(ab)S \text{ or } (ab)^n \in S(ba)(ab)S,$$

whence $(ab)^{n+1} \in Sa^2S$. Now, from Theorems 5.1 and 4.5 it follows that $\sqrt{A}$ is an ideal, for every ideal A of S. Assume an arbitrary ideal A of S and assume $a, b \in S$ such that $ab \in \sqrt{A}$. Based on (iii) there exists $n \in \mathbf{Z}^+$ such that $a^n \in SabS \subseteq \sqrt{A}$ or $b^n \in SabS \subseteq \sqrt{A}$, so, it follows that $a \in \sqrt{A}$ or $b \in \sqrt{A}$. Therefore, $\sqrt{A}$ is a completely prime ideal.

(iv)⇒(v) This implication follows immediately.

(v)⇒(ii) Assume $a, b \in S$. Based on (v), $\sqrt{SabS}$ is a completely prime subset of S. Since $a^2b^2 \in SabS \in \sqrt{SabS}$, we then have that $a^2 \in \sqrt{SabS}$ or $b^2 \in \sqrt{SabS}$, whence it follows that (ii) holds.

(ii)⇒(i) Assume $a, b \in S$. Then, from (ii), $(ba)(ab) \longrightarrow ba$ or $(ba)(ab) \longrightarrow ab$, whence it is easy to prove that $a^2 \longrightarrow ab$, so based on Theorem 5.1 S is a semilattice Y of Archimedean semigroups $S_\alpha, \alpha \in Y$. Let $\alpha, \beta \in Y$. Assume that $a \in S_\alpha$ and $b \in S_\beta$. Then based on (ii) there exists $n \in \mathbf{Z}^+$ such that $a^n \in SabS$ or $b^n \in SabS$, whence $\alpha \leq \beta$ or $\beta \leq \alpha$. Thus Y is a chain. □

Theorem 5.7 *The following conditions on a semigroup S are equivalent:*

(i) $(\forall a, b \in S)\ a \xrightarrow{l} b \Rightarrow a^2 \xrightarrow{l} b$;
(ii) $(\forall a, b \in S)(\forall k \in \mathbf{Z}^+)\ b^k \xrightarrow{l} ab$;
(iii) $(\forall a, b \in S)\ b^2 \xrightarrow{l} ab$.

Proof. (i)⇒(ii) Assume $a, b \in S$ and $k \in \mathbf{Z}^+$. Then $b \xrightarrow{l} ab$, so based on (i) it is easy to prove that $b^k \xrightarrow{l} ab$. Thus, (ii) holds.

(ii)⇒(iii) This is evident.

(iii)⇒(i) Assume $a, b \in S$ such that $a \xrightarrow{l} b$, i.e. $b^n = xa$, for some $n \in \mathbf{Z}^+$, $x \in S$. Based on (iii), $a^2 \xrightarrow{l} xa$, i.e. $(xa)^m = ya^2$, for some $m \in \mathbf{Z}^+$, $y \in S$. Thus, $b^{mn} = ya^2$, so $a^2 \xrightarrow{l} b$. Whence, (i) holds. □

Theorem 5.8 *Let S be a semigroup. Then*

(i) *S is a semilattice of right Archimedean semigroups if and only if for all $a, b \in S$, $a \mid b \;\Rightarrow\; a \mid_r b^n$, for some $n \in \mathbf{Z}^+$;*

(ii) *S is a semilattice of left Archimedean semigroups if and only if for all $a, b \in S$, $a \mid b \;\Rightarrow\; a \mid_l b^n$, for some $n \in \mathbf{Z}^+$;*

(iii) *S is a semilattice of t-Archimedean semigroups if and only if for all $a, b \in S$, $a \mid b \;\Rightarrow\; a \mid_t b^n$, for some $n \in \mathbf{Z}^+$.*

Proof. We prove (i). The proofs of (ii) and (iii) are similar.

Suppose that for all $a, b \in S$, $a \mid b \;\Rightarrow\; a \mid_r b^n$, for some $n \in \mathbf{Z}^+$. Let $a, b \in S$ such that $a \mid b$. Then $b = xay$, for some $x, y \in S^1$. Let $c = yxa$. Then $a \mid c$. So $a \mid_r c^n$, for some $n \in Z^+$. So $az = (yxa)^n$ for some $z \in S^1, n \in \mathbf{Z}^+$. Hence $a^2 \mid xa^2z = xa(yxa)^n \mid (xay)^{n+1} = b^{n+1}$. Based on Theorem 5.1, S is a semilattice of Archimedean semigroups $S_\alpha, \alpha \in Y$. Let $a, b \in S_\alpha$, for some $\alpha \in Y$. Then $a \mid b^n$ for some $n \in \mathbf{Z}^+$. So $a \mid_r b^m$ in S for some $m \in \mathbf{Z}^+$. Then $au = b^m$ for some $u \in S^1$. So $a(ub) = b^{m+1}$, $ub \in S_\alpha$. Thus $a \mid_r b^{m+1}$ in S_α. Hence S_α is right Archimedean. Now assume conversely that S is a semilattice of right Archimedean semigroups $S_\alpha, \alpha \in Y$. Let $a, b \in S, a \mid b$. Then $xay = b$ for some $x, y \in S^1$. Then $ayx, b \in S_\alpha$ for some $\alpha \in Y$. So $ayx \mid_r b^n$ for some $n \in \mathbf{Z}^+$. Then $a \mid_r b^n$. This proves the theorem. □

Theorem 5.9 *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of left Archimedean semigroups;*

(ii) *$(\forall a, b \in S)(\forall k \in \mathbf{Z}^+)\; a^k \xrightarrow{l} ab$;*

(iii) *$(\forall a, b \in S)\; a \xrightarrow{l} ab$;*

(iv) *the radical of every left ideal of S is a right ideal of S.*

Proof. (i)⇒(ii) This we prove in a way similar to (i)⇒(ii) in Theorem 5.1.

(ii)⇒(iii) This is evident.

(iii)⇒(i) Assume $a, b \in S$. Based on (iii) there exists $n \in \mathbf{Z}^+$ and $x \in S$ such that $(ba)^n = xb$. Now we have that $(ab)^{n+1} = axb^2$, so $b^2 \xrightarrow{l} ab$. Based on (iii), Theorems 4.7 and 4.8, for $n = 1$, and Theorem 5.1, we have (i).

(ii)⇒(iv) Let L be a left ideal of S. Assume that $a \in \sqrt{L}, b \in S$. Then $a^k \in L$, for some $k \in \mathbf{Z}^+$, and we have that $(ab)^n \in Sa^k \subseteq SL \subseteq L$, for some $n \in \mathbf{Z}^+$. Thus $ab \in \sqrt{L}$, i.e. $\sqrt{L}$ is a right ideal of S.

(iv)⇒(i) Let $a, b \in S, L = Sa$. Then $a \in \sqrt{L}$. Since $\sqrt{L}$ is a right ideal of S we then have that $ab \in \sqrt{L}$, i.e. there exists $n \in \mathbf{Z}^+$ such that $(ab)^n \in L = Sa$, whence from (ii) Theorem 5.8 we have that the condition (i) holds. □

From Corollary 4.13 and Theorems 4.8, 4.9 we have the following

Corollary 5.1 *The following conditions on a semigroup S are equivalent:*

(i) *S is a chain of left Archimedean semigroups;*
(ii) *for every left ideal A of S, $\sqrt{A}$ is a completely prime ideals of S;*
(iii) *S is a semilattice of left Archimedean semigroups and every left ideal of S is semiprime;*
(iv) *S is a semilattice of left Archimedean semigroups and $ab \xrightarrow{l} a$ or $ab \xrightarrow{l} b$ for all $a, b \in S$.*

As in the case of Theorem 5.5, we prove the following corollary:

Corollary 5.2 *The following conditions on a semigroup S are equivalent:*

(i) *S is semilattice of nil-extensions of left simple semigroups;*
(ii) *S is left π-regular and a semilattice of left Archimedean semigroups;*
(iii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)(\forall k \in \mathbf{Z}^+)\ a^k \mid_l (ab)^n$;
(iv) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^{2n+1} \mid_l (ab)^n$.

For the semilattice and chains of t-Archimedean semigroups it is easy to prove the following characterizations:

Corollary 5.3 *The following conditions on a semigroup S are equivalent:*

(i) *S is semilattice of t-Archimedean semigroups;*

(ii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n \in bSa$;

(iii) *for every bi-ideal A of S, $\sqrt{A}$ is an ideal of S.*

Corollary 5.4 *The following conditions on a semigroup S are equivalent:*

(i) *S is a chain of t-Archimedean semigroups;*

(ii) *S is a semilattice of t-Archimedean semigroups and $ab \xrightarrow{\text{t}} a$ or $ab \xrightarrow{\text{t}} b$ for all $a, b \in S$;*

(iii) *for every bi-ideal A of S, $\sqrt{A}$ is a completely prime ideals of S.*

Theorem 5.10 *For every subsemigroup A of S, $\sqrt{A}$ is a completely prime subset of S if and only if for all $a, b \in S$ is $ab \overset{p}{—} a$ or $ab \overset{p}{—} b$.*

Proof. Let $\sqrt{A}$ be completely prime for every subsemigroup A of S. Then, for all $a, b \in S$, from $ab \in \langle ab\rangle \subseteq \sqrt{\langle ab\rangle}$ we have that $a \in \sqrt{\langle ab\rangle}$ or $b \in \sqrt{\langle ab\rangle}$, i.e. $ab \overset{p}{—} a$ or $ab \overset{p}{—} b$.

Conversely, let $ab \overset{p}{—} a$ or $ab \overset{p}{—} b$, for every $a, b \in S$ and let A be a subsemigroup of S. Let $ab \in \sqrt{A}$, $a, b \in S$. Then $(ab)^k \in A$, for some $k \in \mathbf{Z}^+$. Since $a^n = (ab)^r$ or $b^n = (ab)^t$, for some $n, r, t \in \mathbf{Z}^+$, we then have that $a^{nk} = (ab)^{rk} \in A$ or $b^{nk} = (ab)^{tk} \in A$, whence $a \in \sqrt{A}$ or $b \in \sqrt{A}$. Therefore, $\sqrt{A}$ is a completely prime subset of S. □

Based on Theorem 4.5 we know that a semigroup S is a band of Archimedean semigroups if and only if S is a semilattice of Archimedean semigroups. If the term "Archimedean" we replace with "left (right) Archimedean" the same statements does not hold. That is confirmed by every completely simple semigroup which is not a left group (see Corollary 3.8). By means of the following theorem we describe a band of left Archimedean semigroups.

Theorem 5.11 *A semigroup S is a band of left Archimedean semigroups if and only if*

$$xay \overset{l}{—} xa^2y,$$

for all $a \in S$, $x, y \in S^1$.

Proof. Let S be a band of left Archimedean semigroups and let ξ be a corresponding band congruence. Assume $a \in S$, $x, y \in S^1$ and assume that A is a ξ-class of the element xay. Then $xay, xa^2y \in A$ and since A is a left Archimedean semigroup then we have that $xay \overset{l}{—} xa^2y$.

Conversely, assume $a, b \in S$, then based on the hypothesis we have that $ab \overset{l}{\text{---}} ab^2$, so $(ab)^n \in Sab^2 \subseteq Sb^2$. Whence, $b^2 \overset{l}{\longrightarrow} ab$, and according to Theorem 5.7 ((i)⇔(iii)) and Theorem 4.8 ((iii)⇔(v), for $n = 1$), we have that $\overset{l}{\text{---}} = \lambda_1$, so $\overset{l}{\text{---}}$ is an equivalence relation on S.

We define the relation ξ on S with

$$a\xi b \;\Leftrightarrow\; (\forall x, y \in S^1)\; xay \overset{l}{\text{---}} xby, \quad a, b \in S.$$

From Theorem 1.2 and the hypothesis we have that ξ is a band congruence on S. Let A be a ξ-class of S. Assume $a, b \in A$. Then $a^2 \xi b$, whence $b^n = xa^2$, for some $n \in \mathbf{Z}^+$, $x \in S$. Now, we have that $xa \xi xa^2 = b^n \xi b$, so $xa \in A$. Hence, $b^n = (xa)a \in Aa$, i.e. $a \overset{l}{\longrightarrow} b$ in A, so A is a left Archimedean semigroup. Thus, S is a band of left Archimedean semigroups. □

Corollary 5.5 *A semigroup S is a band of t-Archimedean semigroups if and only if*

$$xay \overset{t}{\text{---}} xa^2y,$$

for all $a \in S$, $x, y \in S^1$.

Proof. This follows from Theorem 5.11 and its dual. □

Otherwise, it is easy to prove that t-Archimedean semigroups are band indecomposable, i.e. the universal relation on a t-Archimedean semigroup S is an unique band congruence on S.

A band B is *left* (*right*) *seminormal* if $axy = axyay$ ($yxa = yayxa$), for all $a, x, y \in B$. A band B is *normal* if $axya = ayxa$, or all $a, x, y \in B$. A band B is *left* (*right*) *regular* if $xy = yxy$ ($xy = yay$), for all $a, x, y \in B$.

Theorem 5.12 *On a semigroup S the following conditions are equivalent:*

(i) *S is a normal band;*
(ii) $(\forall a, x, y, b \in S)$ *$axyb = ayxb$;*
(iii) *S is a left and right seminormal band.*

Proof. (i)⇒(ii) Assume $a, x, y, b \in S$. Then

$$axyb = axybaxyb = aybxaxyb = aybaxyxb = ayxbayxb = ayxb.$$

(ii)⇒(iii) Assume $a, x, y \in S$. Then

$$axy = axyaxy = axxyay = axyay.$$

Similarly we prove that $yxa = yayxa$. Therefore, S is left and right seminormal, so (iii) holds.

(iii)⇒(i) Assume $a, x, y \in S$. Then

$$\begin{aligned} axya &= ayaaxya = ayaxya = ayxyaayaxya \\ &= ayx(ya)^2xya = ayxyaxya = ay(xya)^2 = ayxya, \\ ayxa &= ayxaaya = ayxaya = ayxayaayxya \\ &= ayx(ay)^2xya = ayxayxya = (ayx)^2ya = ayxya. \end{aligned}$$

Thus, $axya = ayxa$, so (i) holds. □

Corollary 5.6 *A semigroup S is a left seminormal band of left Archimedean semigroups if and only if for all $a, b, c \in S$, $ac \xrightarrow{l} abc$.*

Proof. Let S be a left seminormal band of left Archimedean semigroups and let ξ be a corresponding band congruence. Assume $a, b, c \in S$. Since S/ξ is a left seminormal band, then $abc\xi abcac$. Assume that A is a ξ-class of elements abc and $abcac$. Since A is a left Archimedean semigroup, then $(abc)^n \in Sabcac \subseteq Sac$, so $ac \xrightarrow{l} abc$.

Conversely, let $ac \xrightarrow{l} abc$, for all $a, b, c \in S$. Assume $x, y, a \in S$. Then

$$xa^2y = (xa)(ay) \xrightarrow{l} (xa)(yx)(ay) = (xay)^2,$$

$$xay \xrightarrow{l} (xa)(ayxa^2)y = (xa^2y)^2,$$

whence $xa^2y \xrightarrow{l} xay$ and $xay \xrightarrow{l} xa^2y$, i.e. $xay \overset{l}{\text{—}} xa^2y$. Thus, based on Theorem 5.11, S is a band B of left Archimedean semigroups. Since B is a homomorphic image of S, then $ik \xrightarrow{l} ijk$ in B for all $i, j, k \in B$, i.e. $ijk \in Bik$, whence $ijk = ijkik$. Therefore, B is a left seminormal band. □

Corollary 5.7 *A semigroup S is a normal band of t-Archimedean semigroups if and only if for all $a, b, c \in S$, $ac \xrightarrow{\mathrm{t}} abc$.*

Proof. This follows from Corollary 5.6, Theorem 5.12 and the fact that t-Archimedean semigroups are band indecomposable. □

Theorem 5.13 *The following conditions on a semigroup S are equivalent:*

(i) *S is a band of power-joined semigroups;*
(ii) $(\forall a, b \in S)\ ab \overset{p}{\relbar\relbar} a^2b \overset{p}{\relbar\relbar} ab^2$*;*
(iii) $(\forall a, b \in S)(\forall m, n \in \mathbf{Z}^+)\ ab \overset{p}{\relbar\relbar} a^m b^n$.

Proof. (i)$\Rightarrow$(ii) Let S be a band of power-joined semigroups and let ξ be a corresponding band congruence. Assume $a, b \in S$ and let A be a ξ-class of the element ab. Then $ab, a^2b, ab^2 \in A$, whence we have that (ii) holds.

(ii)$\Rightarrow$(iii) Let (ii) hold. Assume $a, b \in S$. From (ii) we have that $ab \overset{p}{\relbar\relbar} a^2b \overset{p}{\relbar\relbar} a^2b^2$, i.e. $ab \overset{p}{\relbar\relbar} a^2b^2$, because $\overset{p}{\relbar\relbar}$ is an equivalence on S. Assume $ab \overset{p}{\relbar\relbar} a^m b^n$ for $m, n \in \mathbf{Z}^+$, $m, n \geq 2$. Then from (ii) we have that

$$\begin{aligned} ab \overset{p}{\relbar\relbar} a^m b^n &= (a^m b^{n-1})b \overset{p}{\relbar\relbar} (a^m b^{n-1})b^2 = a^m b^{n+1} = \\ &= a(a^{m-1}b^{n+1}) \overset{p}{\relbar\relbar} a^2(a^{m-1}b^{n+1}) = a^{m+1}b^{n+1}, \end{aligned}$$

i.e. $ab \overset{p}{\relbar\relbar} a^{m+1}b^{n+1}$. Thus, by induction we have that (iii) holds.

(iii)$\Rightarrow$(i) It is clear that $\overset{p}{\relbar\relbar}$ is an equivalence relation on S. Let $a \overset{p}{\relbar\relbar} b$, $a, b \in S$ and assume $x \in S$. Then $a^m = b^n$ for some $m, n \in \mathbf{Z}^+$, and from (iii) we have that

$$ax \overset{p}{\relbar\relbar} a^m x = b^n x \overset{p}{\relbar\relbar} bx, \qquad xa \overset{p}{\relbar\relbar} xa^m = xb^n \overset{p}{\relbar\relbar} xb.$$

Thus, $\overset{p}{\relbar\relbar}$ is a congruence on S. It is evident that $a \overset{p}{\relbar\relbar} a^2$, for every $a \in S$, so $\overset{p}{\relbar\relbar}$ is a band conguence on S. Also, it is clear that every $\overset{p}{\relbar\relbar}$-class is a power-joined semigroup. $\square$

Corollary 5.8 *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of power-joined semigroups;*
(ii) $(\forall a, b \in S)\ ab \overset{p}{\relbar\relbar} a^2b \overset{p}{\relbar\relbar} ab^2 \overset{p}{\relbar\relbar} ba$*;*
(iii) $(\forall a, b \in S)(\forall m, n \in \mathbf{Z}^+)\ ba \overset{p}{\relbar\relbar} a^m b^n$.

Exercises

1. A semigroup S is a semilattice of Archimedean semigroups if and only if the following relation ρ on S:

$$a \rho b \Leftrightarrow (\forall x, y \in S)(\exists m, n \in \mathbf{Z}^+)\ (xay)^m \in SxbyS,\ (xby)^n \in SxayS,$$

is a semilattice congruence.

2. The following conditions on a semigroup S are equivalent:

(i) S is a semilattice of Archimedean semigroups;
(ii) $\longrightarrow$ is transitive;
(iii) $(\forall a,b,c \in S)\ a \longrightarrow c\ \&\ b \longrightarrow c \ \Rightarrow\ ab \longrightarrow c$;
(iv) $(\forall a,b \in S)\ a \longrightarrow b \ \Rightarrow\ a^2 \longrightarrow b$;
(v) $(\forall a,b \in S)\ a \,|_r\, b \ \Rightarrow\ a^2 \longrightarrow b$;
(vi) $\sqrt{SaS}$ is an ideal of S, for all $a \in S$;
(vii) in every homomorphic image with a zero of S the set of all nilpotent elements is an ideal.

3. The following conditions on a semigroup S are equivalent:

(i) S is a semilattice of nil-extensions of groups;
(ii) $T(\mathcal{H})$ is a semilattice congruence;
(iii) $T(\mathcal{H}) = \sigma_1 = R(\mathcal{H})$;
(iv) $(\forall a,b \in S)\ (ab, ba^2) \in T(\mathcal{H})$.

4. Prove that the following conditions on a semigroup S are equivalent:

(i) $T(\mathcal{J})$ is a semilattice congruence;
(ii) $T(\mathcal{J}) = \sigma_1 = R(\mathcal{J})$;
(iii) $T(\mathcal{J}) = \longrightarrow^2$;
(iv) $(\forall a,b \in S)\ a \longrightarrow b \ \Rightarrow\ (a^2, b) \in T(\mathcal{J})$;
(v) $(\forall a,b,c \in S)\ a \longrightarrow b\ \&\ b \longrightarrow c \ \Rightarrow\ (a,c) \in T(\mathcal{J})$;
(vi) $(\forall a,b,c \in S)\ a \longrightarrow c\ \&\ b \longrightarrow c \ \Rightarrow\ (ab,c) \in T(\mathcal{J})$;
(vii) S is a semilattice of nil-extensions of simple semigroups.

5. $\mathcal{A} \circ \mathcal{S}$ is homomorphically closed.

6. $\mathcal{A} \circ \mathcal{S}$ is not subsemigroup closed.

7. $\mathcal{A} \circ \mathcal{S}$ is finite-direct product closed.

8. $\mathcal{A} \circ \mathcal{S}$ is not infinite-direct product closed.

9. A semigroup S is a rectangular band of power-joined semigroups if and only if

$$(\forall a,b,c \in S)(\exists m,n \in \mathbf{Z}^+)(\ (abc)^m = (ac)^n\ ).$$

10. A semigroup S is a left zero band of power-joined semigroups if and only if

$$(\forall a,b \in S)(\exists m,n \in \mathbf{Z}^+)(\ (ab)^m = a^n\ ).$$

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5.2 Semilattices of Hereditary Archimedean Semigroups

In this section we investigate semigroups whose any subsemigroup is a semilattice of Archimedean semigroups.

Theorem 5.14 *Any subsemigroup of a semigroup S is a semilattice of Archimedean semigroups if and only if*

$$(\forall a,b\in S)(\exists n\in \mathbf{Z}^+)\ (ab)^n\in \langle a,b\rangle\, a^2\, \langle a,b\rangle\,.$$

Proof. If $a,b\in S$ and $T=\langle a,b\rangle$, then from Theorem 5.1 it follows that

$$(ab)^m\in Ta^2T=\langle a,b\rangle\, a^2\, \langle a,b\rangle\,,$$

for some $m\in\mathbf{Z}^+$.

Conversely, if T is a subsemigroup of S and $a,b\in T$, then there exists $m\in\mathbf{Z}^+$ such that

$$(ab)^m\in \langle a,b\rangle\, a^2\, \langle a,b\rangle\subseteq Ta^2T,$$

so based on Theorem 5.1, T is a semilattice of Archimedean semigroups. □

The main result of this section is the following theorem which characterizes the semilattices of hereditary Archimedean semigroups.

Theorem 5.15 *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of hereditary Archimedean semigroups;*
(ii) $(\forall a,b\in S)\ a\longrightarrow b\ \Rightarrow\ a^2\uparrow b$;
(iii) $(\forall a,b,c\in S)\ a\longrightarrow c\ \&\ b\longrightarrow c\ \Rightarrow\ ab\uparrow c$;
(iv) $(\forall a,b,c\in S)\ a\longrightarrow b\ \&\ b\longrightarrow c\ \Rightarrow\ a\uparrow c$.

Proof. (i)⇒(ii) Let S be a semilattice Y of hereditary Archimedean semigroups S_α, $\alpha \in Y$. Assume $a, b \in S$ such that $a \longrightarrow b$. Then $b, a^2b \in S_\alpha$, for some $\alpha \in Y$, so based on the hypothesis we obtain that

$$b^n \in \left\langle b, a^2b \right\rangle a^2b \left\langle b, a^2b \right\rangle \subseteq \left\langle a^2, b \right\rangle a^2 \left\langle a^2, b \right\rangle .$$

Thus $a^2 \uparrow b$, so (ii) holds.

(ii)⇒(iii) Assume $a, b, c \in S$ such that $a \longrightarrow c$ & $b \longrightarrow c$. Then based on Theorem 4.5 $ab \longrightarrow c$. Now, from (ii) it follows $(ab)^2 \uparrow c$, whence $ab \uparrow c$.

(iii)⇒(iv) Based on (iii) and Theorem 4.5, for $n = 1$, $\longrightarrow$ is transitive. Assume $a, b, c \in S$ such that $a \longrightarrow b$ and $b \longrightarrow c$. Then $a \longrightarrow c$, so $a^2 \uparrow c$, by (iii), whence $a \uparrow c$.

(iv)⇒(i) Based on (iv), $\longrightarrow$ is transitive, so according to Theorem 4.5, for $n = 1$, S is a semilattice Y of Archimedean semigroups S_α, $\alpha \in Y$.

Assume $\alpha \in Y$ and $a, b \in S_\alpha$. Then $a \longrightarrow b$ and $b \longrightarrow b$, whence $a \uparrow b$, by (iv). Therefore, S_α is hereditary Archimedean. Hence, (i) holds. □

The next theorem gives a characterization of semigroups which are chains of hereditary Archimedean semigroups.

Theorem 5.16 *A semigroup S is a chain of hereditary Archimedean semigroups if and only if*

$$ab \uparrow a \quad \textit{or} \quad ab \uparrow b.$$

for all $a, b \in S$.

Proof. Let S be a chain Y of hereditary Archimedean semigroups S_α, $\alpha \in Y$. If $a \in S_\alpha$, $b \in S_\beta$, for some $\alpha, \beta \in Y$, then $a, ab \in S_\alpha$ or $b, ab \in S_\beta$, whence

$$a^n \in \langle a, ab \rangle\, ab\, \langle a, ab \rangle \qquad \text{or} \qquad b^n \in \langle b, ab \rangle\, ab\, \langle b, ab \rangle$$

for some $n \in \mathbf{Z}^+$.

Conversely, based on the hypothesis and Theorem 5.6, S is a chain Y of Archimedean semigroups $S_\alpha, \alpha \in Y$. If $\alpha \in Y$ and $a, b \in S_\alpha$, then then there exists $n \in \mathbf{Z}^+$ such that $b^n \in S_\alpha a S_\alpha$, and based on Theorem 5.15, $a^2 \uparrow b^n$, whence $a \uparrow b$. Thus, S_α is hereditary Archimedean. Hence, S is a chain of hereditary Archimedean semigroups. □

We proceed on to study the semilattices of hereditary left Archimedean semigroups.

Theorem 5.17 *A semigroup S is a semilattice of hereditary left Archimedean semigroups if and only if for all $a, b \in S$,*

$$a \longrightarrow b \;\Rightarrow\; a \uparrow_l b.$$

Proof. Let S be a semilattice Y of hereditary left Archimedean semigroups S_α, $\alpha \in Y$. Assume $a, b \in S$ such that $a \longrightarrow b$. Since $a \in S_\alpha$, $b \in S_\beta$, for some $\alpha, \beta \in Y$, then we have that $\beta \leq \alpha$, so $b, ba \in S_\beta$. Now $ba \uparrow_l b$, whence $a \uparrow_l b$, which proves the direct part of the theorem.

Conversely, based on the hypothesis and Theorem 5.8, S is a semilattice Y of the left Archimedean semigroups S_α, $\alpha \in Y$. Assume $\alpha \in Y$ and $a, b \in S_\alpha$. Then $a \longrightarrow b$, whence $a \uparrow_l b$, based on the hypothesis. Therefore, any S_α is hereditary left Archimedean, so S is a semilattice of hereditary left Archimedean semigroups. □

Corollary 5.9 *A semigroup S is a semilattice of hereditary t-Archimedean semigroups if and only if for all $a, b \in S$,*

$$a \longrightarrow b \;\Rightarrow\; a \uparrow_t b.$$

Theorem 5.18 *The folowing conditions on a semigroup S are equivalent:*

(i) *S is hereditary Archimedean and π-regular;*
(ii) *S is hereditary Archimedean and has a primitive idempotent;*
(iii) *S is a nil-extension of a periodic completely simple semigroup;*
(iv) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\; a^n = (a^n b^n a^n)^n$.

Proof. (i)⇒(ii) First we prove that

$$(\forall a \in S)(\forall e \in E(S))(\exists n \in \mathbf{Z}^+)\; e = (eae)^n. \tag{1}$$

Indeed, for $a \in S$, $e \in E(S)$, $ea \uparrow e$, by (i), whence $e = (ea)^n$ or $e = (ea)^n e$, for some $n \in \mathbf{Z}^+$. However, in both of cases it follows that $e = (ea)^n e = (eae)^n$. Thus, (1) holds.

Further, assume $a \in S$. Let $m \in \mathbf{Z}^+$ such that $a^m \in \mathrm{Reg}(S)$ and let x be an inverse of a^m. Then $a^m x, xa^m \in E(S)$, so from (1) we obtain that

$$a^m x = (a^m x \cdot a \cdot a^m x)^n = (a^{m+1} x)^n,$$

for some $n \in \mathbf{Z}^+$, whence

$$\begin{aligned} a^m = a^m x a^m &= (a^{m+1}x)^n a^m = (a^{m+1}x)^{n-1} a^{m+1} x a^m = \\ &= (a^{m+1}x)^{n-1} a a^m x a^m = (a^{m+1}x)^{n-1} a^{m+1} = \\ &= (a^{m+1}x)^{n-2} a^{m+1} x a^{m+1} = (a^{m+1}x)^{n-2} a a^m x a^m a = \\ &= (a^{m+1}x)^{n-2} a a^m a = (a^{m+1}x)^{n-2} a^{m+2} = \cdots = \\ &= (a^{m+1}x)^{n-(n-1)} a^{m+(n-1)} = \\ &= a^{m+1} x a^{m+n-1} = a a^m x a^m a^{n-1} = \\ &= a a^m a^{n-1} = a^{m+n}. \end{aligned}$$

Thus, S is periodic, and by Theorem 3.16, S has a primitive idempotent.

(ii)⇒(iii) Based on Theorem 3.16, S is a nil-extension of a completely simple semigroup K. But, K is hereditary Archimedean and regular, so it is periodic, based on the proof of (i)⇒(ii).

(iii)⇒(iv) Assume $a, b \in S$. Then $a^k = e$ and $b^n = f$, for some $e, f \in E(S)$, $k \in \mathbf{Z}^+$. Further, $efe \in eSe = G_e$, by Lemma 3.15, whence $(efe)^m = e$, for some $m \in \mathbf{Z}^+$. Now, for $n = km$ we obtain that $a^n = (a^n b^n a^n)^n$.

(iv)⇒(i) This follows immediately. □

Theorem 5.19 *The following conditions on a semigroup S are equivalent:*

(i) *S is π-regular and a semilattice of hereditary Archimedean semigroups;*
(ii) *S is a semilattice of nil-extensions of periodic completely simple semigroups;*
(iii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n = (ab)^n((ba)^n(ab)^n)^n$;
(iv) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n = ((ab)^n(ba)^n(ab)^n)^n$.

Proof. (i)⇒(ii) This follows immediately from Theorem 5.18.

(ii)⇒(iii) and (ii)⇒(iv) This follows from Theorem 5.18.

(iii)⇒(i) and (iv)⇒(i) This follows immediately. □

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5.3 Semilattices of Weakly Left Archimedean Semigroups

Based on the well-known results of A. H. Clifford, from 1954, any band of left Archimedean semigroups is a semilattice of matrices (rectangular bands) of left Archimedean semigroups (see Corollary 3.7). The converse of this assertion does not hold, i.e. the class of semilattices of matrices of left Archimedean semigroups is larger than the class of bands of left Archimedean semigroups. In this section we characterize the semilattices of matrices of left Archimedean semigroups, and especially matrices of left Archimedean semigroups.

Recall that a semigroup S is called *left Archimedean* if $a \stackrel{l}{\longrightarrow} b$, for all $a, b \in S$. Here we introduce a more general notion: a semigroup S will be called *weakly left Archimedean* if $ab \stackrel{l}{\longrightarrow} b$, for all $a, b \in S$. By $\mathcal{WLA}$ we denote the class of all weakly left Archimedean semigroups. *Weakly right Archimedean semigroups* are defined dually. A semigroup S is *weakly t-Archimedean* (or *weakly two-sided Archimedean*) if it is both weakly left and weakly right Archimedean, i.e. if for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in abSba$.

First we prove the following important lemma:

Lemma 5.3 *Let ξ be a band congruence on a semigroup S. Then the following conditions are equivalent:*

(i) $\xi \subseteq \stackrel{l}{\longrightarrow}$;
(ii) $\xi \subseteq \lambda_1$;
(iii) *any ξ-class is a left Archimedean semigroup.*

Proof. (i)$\Rightarrow$(iii) Let A be a ξ-class of S and let $a, b \in A$. Then $a^2 \xi b$, whence $a^2 \stackrel{l}{\longrightarrow} b$, that is $b^n = xa^2$, for some $n \in \mathbf{Z}^+$, $x \in S^1$. Seeing that ξ is a band congruence, $xa \xi xa^2 = b^n \xi b$, so $xa \in A$ and $b^n = (xa)a \in Aa$. Therefore, A is left Archimedean.

(iii)$\Rightarrow$(ii) Assume an arbitrary pair $(a, b) \in \xi$. Let $c \in \Lambda_1(a)$, that is $a \stackrel{l}{\longrightarrow} c$. Then $c^n = xa$, for some $n \in \mathbf{Z}^+$ and $x \in S^1$, and $xa, xb \in A$, where A is a ξ-class of S. Since A is left Archimedean, then there exists $m \in \mathbf{Z}^+$ and $y \in S^1$ such that $(xa)^m = yxb$. Therefore, $c^{mn} = (xa)^m = yxb$, so $b \stackrel{l}{\longrightarrow} c$

and $c \in \Lambda_1(b)$. Thus, $\Lambda_1(a) \subseteq \Lambda_1(b)$. Similarly we prove $\Lambda_1(b) \subseteq \Lambda_1(a)$. Hence, $\Lambda_1(a) = \Lambda_1(b)$, so $(a, b) \in \lambda_1$ This proves (ii).

(ii)$\Rightarrow$(i) This is obvious. $\square$

Now, we give the following characterization of semilattices of weakly left Archimedean semigroups:

Theorem 5.20 *A semigroup S is a semilattice of weakly left Archimedean semigroups if and only if*

$$a \longrightarrow b \;\Rightarrow\; ab \xrightarrow{l} b,$$

for all $a, b \in S$.

Proof. Let S be a semillatice Y of weakly left Archimedean semigroups $S_\alpha,\ \alpha \in Y$. Assume $a, b \in S$ such that $a \longrightarrow b$. If $a \in S_\alpha,\ b \in S_\beta$, for some $\alpha, \beta \in Y$, then $\beta \leq \alpha$, whence $b, ba \in S_\beta$. Now, $b^n \in S_\beta bab \subseteq Sab$, for some $n \in \mathbf{Z}^+$, since S_β is weakly left Archimedean. Therefore, $ab \xrightarrow{l} b$.

Conversely, let for all $a, b \in S$, $a \longrightarrow b$ implies $ab \xrightarrow{l} b$. Assume $a, b \in S$. Since $a \longrightarrow ab$, then based on the hypothesis, $a^2b \xrightarrow{l} ab$, i.e. $(ab)^n \in Sa^2b \subseteq Sa^2S$, for some $n \in \mathbf{Z}^+$. Now, based on Theorem 5.1, S is a semilattice Y of Archimedean semigroups $S_\alpha,\ \alpha \in Y$. Further, assume $\alpha \in Y,\ a, b \in S_\alpha$. Then $a \longrightarrow b$, so based on the hypothesis, $ab \xrightarrow{l} b$ in S, and Lemma 4.14 (c), $ab \xrightarrow{l} b$ in S_α. Therefore, S_α is weakly left Archimedean. $\square$

Corollary 5.10 *A semigroup S is a semilattice of weakly t-Archimedean semigroups if and only if*

$$a \longrightarrow b \;\Rightarrow\; ab \xrightarrow{l} b \;\;\&\;\; ba \xrightarrow{r} b,$$

for all $a, b \in S$.

The components of the semilattice decomposition treated in Theorem 5.20 will be characterized in the next theorem. Namely, we will give a description of weakly left Archimedean semigroups.

Theorem 5.21 *The following conditions on a semigroup S are equivalent:*

(i) *S is weakly left Archimedean;*
(ii) *S is a matrix of left Archimedean semigroups;*
(iii) *S is a right zero band of left Archimedean semigroups;*
(iv) *$\xrightarrow{l}$ is a symmetric relation on S.*

Proof. (i)⇒(iv) Let $a, b \in S$ such that $a \xrightarrow{l} b$, i.e. $b^n = xa$, for some $n \in \mathbf{Z}^+,\ x \in S$. Based on (i), $a^m = yxa = yb^n$, for some $m \in \mathbf{Z}^+,\ y \in S$, whence $b \xrightarrow{l} a$.

(iv)⇒(i) This follows from the proof for (vii)⇒(v) of Theorem 4.10.

(iv)⇒(iii) Let $a, b, c \in S$ such that $a \xrightarrow{l} b$ and $b \xrightarrow{l} c$. From (iv), $c \xrightarrow{l} b$, so $b^n = xa = yc$, for some $n \in \mathbf{Z}^+,\ x, y \in S$. Since (iv)⇔(i), then there exists $m \in \mathbf{Z}^+,\ z \in S$ such that $c^m = z(yc) = zb^n = zxa \in Sa$. Therefore, $a \xrightarrow{l} c$, so $\xrightarrow{l}$ is transitive, i.e. $\xrightarrow{l} = \xrightarrow{l}{}^{\infty}$. Now, based on Theorem 4.10, $\lambda_1 = \lambda$ is a right zero band congruence. According to Lemma 5.3, λ_1-classes are left Archimedean semigroups.

(iii)⇒(ii) This follows immediately.

(ii)⇒(i) Let S be a matrix B of left Archimedean semigroups $S_i,\ i \in B$. Then for $a, b \in S$, $a, aba \in S_i$, for some $i \in B$, whence $a^n \in S_i aba \subseteq Sba$, for some $n \in \mathbf{Z}^+$. □

Recall that, the relation $\xrightarrow{t}$ on a semigroup S is defined by $\xrightarrow{t} = \xrightarrow{l} \cap \xrightarrow{r}$. Now, from Theorem 5.21 and its dual we obtain the following corollary:

Corollary 5.11 *The following conditions on a semigroup S are equivalent:*

(i) *S is weakly t-Archimedean;*
(ii) *S is a matrix of t-Archimedean semigroups;*
(iii) *$\xrightarrow{t}$ is a symmetric relation on S;*
(iv) *$\xrightarrow{l}$ and $\xrightarrow{r}$ are symmetric relations on S.*

By means of the following theorem we characterize the matrices of nil-extensions of left simple semigroups.

Theorem 5.22 *The following conditions on a semigroup S are equivalent:*

(i) *S is weakly left Archimedean and left π-regular;*
(ii) *S is weakly left Archimedean and intra-π-regular;*
(iii) *S is a matrix of nil-extensions of left simple semigroups;*
(iv) *S is a right zero band of nil-extensions of left simple semigroups;*
(v) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in S(ba)^n$*;*
(vi) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in Sb^n a$*.*

Proof. (i)⇒(iv) This follows from Theorem 5.21 and Theorem 3.14, since the components of any band decomposition of a left π-regular semigroup are also left π-regular.

(iv)⇒(iii) This follows immediately.

(iii)⇒(ii) Based on follows from Theorem 5.21, since a nil-extension of a left simple semigroup is intra-π-regular.

(ii)⇒(i) By Theorem 5.21, S is a right zero band B of left Archimedean semigroups S_i, $i \in B$. Let $a \in \mathrm{Intra}(S)$, i.e. $a = xa^2y$, for some $x, y \in S$. Then $a = (xa)^k ay^k$, for each $k \in \mathbf{Z}^+$. Further, $a \in S_i$, for some $i \in B$, and clearly, $y \in S_i$, so $y^k = za^2$, for some $k \in \mathbf{Z}^+$, $z \in S$, since S_i is left Archimedean. Therefore, $a = (xa)^k ay^k = (xa)^k aza^2$, whence $a \in \mathrm{LReg}(S)$, so based on Theorem 2.4, S is left π-regular.

(iv)⇒(vi) Let S be a right zero band B of semigroups S_i, $i \in B$, and for each $i \in B$, let S_i be a nil-extension of a left simple semigroup K_i. Since (v)⇔(i), then S is a nil-extension of a left completely simple semigroup K. Clearly, $K = \mathrm{LReg}(S) = \bigcup_{i \in B} K_i$. Now, for $a, b \in S$, $a \in S_i$, $b \in S_j$, for some $i, j \in B$, and $a^n \in K_i$, $b^n \in K_j$, for some $n \in \mathbf{Z}^+$, whence $b^n a \in S_i \cap K = K_i$, so $a^n \in K_i b^n a \subseteq Sb^n a$.

(vi)⇒(v) Assume $a, b \in S$. By (vii), there exists $n \in \mathbf{Z}^+$ such that $a^n \in S(ab)^n a \subseteq S(ba)^n$.

(v)⇒(i) This follows immediately. □

Let T be a subsemigroup of a semigroup S. A mapping φ of S onto T is a *right retraction* of S onto T if $a\varphi = a$, for each $a \in T$, and $(ab)\varphi = a(b\varphi)$, for all $a, b \in S$. *Left retraction* is defined dually. A mapping φ of S onto T is a *retraction* of S onto T if it is a homomorphism and $a\varphi = a$, for each $a \in T$. If T is an ideal of S, then φ is a retraction of S onto T if and only if it is both a left and right retraction of S onto T. An ideal extension S of a semigroup T is a (*left, right*) *retractive* extension of T if there exists a (*left, right*) *retraction* of S onto T.

By means of the next theorem we prove that such semigroups are exactly right retractive nil-extensions of completely simple semigroups.

Theorem 5.23 *The following conditions on a semigroup S are equivalent:*

(i) *S is a right retractive nil-extension of a completely simple semigroup;*
(ii) *S is weakly left Archimedean and has an idempotent;*
(iii) *S is a matrix of nil-extensions of left groups;*
(iv) *S is a right zero band of nil-extensions of left groups;*
(v) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in a^n S(ba)^n$;
(vi) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in a^n S b^n a$.

Proof. (iv)⇒(iii) and (iii)⇒(ii) This follows immediately.

(ii)⇒(i) Based on Theorem 3.14, S is a nil-extension of a simple semigroup K, so it is intra π-regular and based on Theorem 2.4, S is left π-regular, it is a right zero band B of semigroups S_i, $i \in B$, and for each $i \in B$, S_i is a nil-extension of a left simple semigroup K_i. Further, $K = \text{Intra}(S) = \text{LReg}(S) = \bigcup_{i\in B} K_i$, based on Theorem 2.4, since the components of any band decomposition of a left π-regular semigroup are also left π-regular. Thus, K is left completely simple, so it is completely simple, since it has an idempotent. Thus, for each $i \in B$, K_i is a left group, so based on Theorem 3.7, it has a right identity e_i. Define a mapping φ of S onto K by:

$$a\varphi = ae_i \qquad\qquad \text{if } a \in S_i,\ i \in B.$$

Clearly, $a\varphi = a$, for each $a \in K$. Further, for $a, b \in S$, $a \in S_i$, $b \in S_j$, for some $i, j \in B$, and $ab \in S_j$, whence $(ab)\varphi = (ab)e_j = a(be_j) = a(b\varphi)$. Therefore, φ is a right retraction of S onto K.

(i)⇒(vi) Let S be a right retractive nil-extension of a completely simple semigroup K, and let K be a right zero band B of left groups K_i, $i \in B$. Let $a, b \in S$. Then $a^n, b^n \in K$, for some $n \in \mathbf{Z}^+$, and $a^n \in K_i$, $b^n \in K_j$, for some $i, j \in B$. If $a\varphi \in K_l$, for some $l \in B$, since $a^{n+1} \in K_i$, then $a^{n+1} = a^{n+1}\varphi = a^n(a\varphi) \in K_i K_l \subseteq K_l$, whence $l = i$. Thus, $a\varphi \in K_i$, so $b^n a = (b^n a)\varphi = b^n(a\varphi) \in K_j K_i \subseteq K_i$. Therefore, $a^n, b^n a \in K_i$, so based on Theorem 3.7, $a^n \in a^n K_i b^n a \subseteq a^n S b^n a$.

(vi)⇒(v) For $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in a^n S(ab)^n a = a^n S a(ba)^n \subseteq a^n S(ba)^n$.

(v)⇒(iv) This follows from Theorem 5.22. □

Corollary 5.12 *The following conditions on a semigroup S are equivalent:*

(i) *S is a retractive nil-extension of a completely simple semigroup;*
(ii) *S is weakly t-Archimedean and intra-π-regular;*
(iii) *S is weakly t-Archimedean and has an idempotent;*
(iv) *S is a matrix of π-groups;*
(v) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in (ab)^n S(ba)^n$.

A semigroup S is *hereditary weakly left Archimedean* if

$$(\forall a, b \in S)(\exists i \in \mathbf{Z}^+)\ b^i \in \langle a, b\rangle ab.$$

The next theorem gives an explanation why the notion "hereditary weakly left Archimedean" is used.

Theorem 5.24 *The following conditions on a semigroup S are equivalent:*

(i) *S is hereditary weakly left Archimedean;*
(ii) *any subsemigroup of S is weakly left Archimedean;*
(iii) *$\uparrow_l$ is a symmetric relation on S.*

Proof. (i)⇒(ii) Let T be a subsemigroup of S. For $a, b \in T$ we have that $b^i \in \langle a, b\rangle ab \subseteq Tab$, for some $i \in \mathbf{Z}^+$. Hence, T is a weakly left Archimedean semigroup and therefore S is a hereditary weakly left Archimedean semigroup.

(ii)⇒(i) Assume $a, b \in S$, then $\langle ba, b\rangle$ is a weakly left Archimedean semigroup, whence

$$b^i \in \langle ba, b\rangle ba \cdot b \subseteq \langle a, b\rangle ab,$$

for some $i \in \mathbf{Z}^+$.

(i)⇒(iii) Let $a, b \in S$ such that $a \uparrow_l b$, i.e. $b^n \in \langle a, b\rangle a$, for some $n \in \mathbf{Z}^+$. Then $b^n = xa$, for some $x \in \langle a, b\rangle$. For x and a there exists $m \in \mathbf{Z}^+$, $y \in \langle x, a\rangle \subseteq \langle a, b\rangle$ such that $a^m = yax = yb^n$, i.e. $b \uparrow_l a$.

(iii)⇒(i) Let $a, b \in S$, then $b \uparrow_l ab$, whence $ab \uparrow_l b$, i.e. $b^i \in \langle ab, b\rangle ab \subseteq \langle a, b\rangle ab$, for some $i \in \mathbf{Z}^+$. □

T. Tamura [15] proved that in the general case semilattices of Archimedean semigroups are not subsemigroup closed. Here, we prove that semilattices of hereditary weakly Archimedean semigroups are subsemigroup closed. Based on the following theorem we generalize some results obtained by S. Bogdanović, M. Ćirić and M. Mitrović [1].

Theorem 5.25 *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of hereditary weakly left Archimedean semigroups;*

(ii) $(\forall a, b \in S)\ a \longrightarrow b \Rightarrow (\exists i \in \mathbf{Z}^+)\ b^i \in \langle a, b\rangle ab$;

(iii) *every subsemigroup of S is a semilattice of hereditary weakly left Archimedean semigroups.*

Proof. (i)⇒(ii) Let S be a semilattice Y of hereditary weakly left Archimedean semigroups S_α, $\alpha \in Y$. Assume $a, b \in S$ such that $a \longrightarrow b$. If $a \in S_\alpha$, $b \in S_\beta$ for some $\alpha, \beta \in Y$, then $\beta \leq \alpha$, whence $b, ba \in S_\beta$. Now

$$b^n \in \langle ba, b\rangle bab \subseteq \langle a, b\rangle ab,$$

for some $n \in \mathbf{Z}^+$. Hence, (ii) holds.

(ii)⇒(i) Assume $a, b \in S$. Since $a \longrightarrow ab$, then based on the hypothesis $a \cdot ab \uparrow_l ab$, i.e. $(ab)^n \in \langle a, ab\rangle a^2 b$, for some $n \in \mathbf{Z}^+$. Now based on Theorem 5.1 S is a semilattice Y of Archimedean semigroups S_α, $\alpha \in Y$. Further, assume $\alpha \in Y$, $a, b \in S_\alpha$. Then $a \longrightarrow b$, so according to the hypothesis $b^n \in \langle a, b\rangle ab$, for some $n \in \mathbf{Z}^+$. Therefore, S_α, $\alpha \in Y$ is an hereditary weakly left Archimedean semigroup.

(ii)⇒(iii) Let T be a subsemigroup of S and $a, b \in T$ such that $a \longrightarrow b$ in T, then $a \longrightarrow b$ in S and based on (ii), $b^n \in \langle a, b\rangle ab \subseteq Tab$, for some $n \in \mathbf{Z}^+$. Thus, T is a semilattice of hereditary weakly left Archimedean semigroups.

(iii)⇒(i) This implication follows immediately. □

Let us introduce the following notations for some classes of semigroups:

Notation	Class of semigroups
$\mathcal{B}$	*bands*
$\mathcal{RB}$ ($\mathcal{M}$)	*rectangular bands* (*matrix*)
$\mathcal{S}$	*semilattices*

and by $\mathcal{X}_1 \circ \mathcal{X}_2$ we denote the Mal'cev product (see page 189.) of classes $\mathcal{X}_1$ and $\mathcal{X}_2$ of the semigroups. Let

$$\mathcal{LA} \circ \mathcal{M}^{k+1} = \left(\mathcal{LA} \circ \mathcal{M}^k\right) \circ \mathcal{M}, \quad k \in \mathbf{Z}^+.$$

Now we can state the following:

Problem 5.1 Describe the structure of semigroups from the following classes

$$\mathcal{LA} \circ \mathcal{M}^{k+1}, \quad \left(\mathcal{LA} \circ \mathcal{M}^{k+1}\right) \circ \mathcal{B}, \quad \left(\mathcal{LA} \circ \mathcal{M}^{k+1}\right) \circ \mathcal{S}.$$

The previous problem can be formulated in the same way if instead the class $\mathcal{LA}$ we take the class of all power-joined semigroups, the class of all λ-simple semigroups or the class of all λ_n-simple semigroups.

Exercises

1. The following conditions on a semigroup S are equivalent:

(a) S is a matrix of π-groups;
(b) S is π-regular and S satisfies the identities $a^0 = (a^0ba^0)^0$, $(ab)^0 = (a^0b^0)^0$;
(c) S is a subdirect product of a completely simple semigroup and a nil-semigroup.

References

S. Bogdanović [17]; S. Bogdanović and M. Ćirić [9], [10], [17]; S. Bogdanović, M. Ćirić and B. Novikov [1]; S. Bogdanović and S. Milić [1]; S. Bogdanović, Ž. Popović and M. Ćirić [2]; S. Bogdanović and B. Stamenković [1]; M. Ćirić and S. Bogdanović [3], [4], [5], [9]; J. L. Galbiati and M. L. Veronesi [1]; A. Mărkuş [1]; A. Nagy [1]; M. Petrich [10]; M. S. Putcha [2], [3]; M. S. Putcha and J. Weissglass [4]; L. N. Shevrin [5]; M. Siripitukdet and A. Iampan [1].

5.4 Semilattices of Left Completely Archimedean Semigroups

In this section we introduce the notion of a left completely Archimedean semigroup, which is a generalization of the notion of a completely Archimedean semigroup. We give certain characterizations of semilattices of left completely Archimedean semigroups and some results concerning semilattices of completely Archimedean semigroups.

A semigroup S is *left completely Archimedean* if it is Archimedean and left π-regular. *Right completely Archimedean* semigroups are defined dually. Clearly, a semigroup is *completely Archimedean* if and only if it is both left and right completely Archimedean.

Certain characterizations of left completely Archimedean semigroups will be given in the following theorem:

Theorem 5.26 *The following conditions on a semigroup S are equivalent:*

(i) *S is left completely Archimedean;*

(ii) *S is a nil-extension of a left completely simple semigroup;*

(iii) *S is Archimedean and has a minimal left ideal;*

(iv) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in Sba^n$.

Proof. (i)⇒(ii) Based on Theorem 3.14, S is a nil-extension of a simple semigroup K. Clearly, K is left π-regular, so based on Theorem 2.18, K is left completely simple.

(ii)⇒(iv) Let S be a nil-extension of a left completely simple semigroup K. Assume $a, b \in S$. Then $a^n, b^m \in K$, for some $n, m \in \mathbf{Z}^+$, so based on Theorem 2.18, $a^n \in Kb^m a^n \subseteq Sba^n$.

(iv)⇒(i) This follows immediately.

(ii)⇒(iii) Let S be a nil-extension of a left completely simple semigroup K. According to Theorem 2.18, K has a minimal left ideal L. Clearly, $L^2 = L$, whence $SL = SLL \subseteq KL \subseteq L$. Therefore, L is a left ideal of S, and clearly, a minimal left ideal of S.

(iii)⇒(ii) It is known that the union of all minimal left ideals of S, if it is non-empty, is the kernel of S, so based on (iii), S has a kernel K, which is the union of all minimal left ideals of S, and hence, a union of left simple semigroups, so it is left regular. Moreover, K is simple, so it is left completely simple. Finally, since S is Archimedean, it is a nil-extension of K. □

Theorem 5.27 *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of left completely Archimedean semigroups;*

(ii) *S is left π-regular and each $\mathcal{L}$-class of S containing a left regular element is a subsemigroup;*

(iii) *S is left π-regular and each $\mathcal{J}$-class of S containing a left regular element is a subsemigroup;*

(iv) *S is left π-regular and a semilattice of Archimedean semigroups;*

(v) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n \in Sa(ab)^n$.

Proof. (i)⇒(ii) Let S be a semilattice Y of left completely Archimedean semigroups S_α, $\alpha \in Y$, and for each $\alpha \in Y$, let S_α be a nil-extension of a left completely simple semigroup K_α, and let K_α be a right zero band B_α of left simple semigroups K_i, $i \in B_\alpha$. Clearly, S is left π-regular. As in the proof for (i)⇒(ii) of Theorem 5.5 we obtain that for each $\mathcal{L}$-class L of S containing a left regular element, there exists $\alpha \in Y$, $i \in B_\alpha$, such that $L = K_i$, so it is a subsemigroup of S.

(ii)$\Rightarrow$(v) Assume $a, b \in S$. Then $(ba)^n = x(ba)^{2n}$, for some $n \in \mathbf{Z}^+$, $x \in S$, whence $(ba)^n \in Sa(ba)^n$, and clearly, $a(ba)^n \in S(ba)^n$, whence $(ba)^n \mathcal{L} a(ba)^n$, i.e., $a(ba)^n \in L$, where L is the $\mathcal{L}$-class of S containing $(ba)^n$. Based on the hypothesis, L is a subsemigroup of S, whence $(ba)^n a(ba)^n \in L$, i.e., $(ba)^n \mathcal{L} (ba)^n a(ba)^n$. Therefore,

$$(ab)^{n+1} = a(ba)^n b \in aS^1(ba)^n a(ba)^n b \subseteq Sa(ab)^{n+1}.$$

(v)$\Rightarrow$(iv) For every $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $(ab)^n \in Sa(ab)^n \subseteq Sa^2S$ and based on Theorem 5.1 S is a semilattice of Archimedean semigroups. It is clear that S is left π-regular.

(iv)$\Rightarrow$(i) This follows from Theorem 5.26, since in every semilattice decomposition of a left π-regular semigroup, each of its components is also left π-regular.

(iii)$\Leftrightarrow$(iv) This follows from Theorem 2.4 and Theorem 5.5. $\square$

References

S. Bogdanović [8], [17], [19]; S. Bogdanović and M. Ćirić [5], [9], [17]; M. Ćirić and S. Bogdanović [3]; A. H. Clifford and G. B. Preston [2]; W. D. Munn [4]; M. S. Putcha [2], [8]; L. N. Shevrin [4]; M. L. Veronesi [1].

5.5 Bands of Left Archimedean Semigroups

In this section we give some new results concerning decompositions into a band of left Archimedean semigroups, in general and some special cases. Based on Theorem 5.29 we give some new characterizations of these decompositions in general. Then we study the bands of nil-extensions of left simple semigroups (Theorem 5.30) and bands of nil-extensions of left groups (Theorem 5.31). We investigate the decompositions which correspond to various varieties of bands. All such decompositions will be characterized in Theorems 5.32 and 5.34. Some of the results obtained in this section generalize many results from the above mentioned papers, and some of them simplify some known results.

In the following table we outline the notations for some classes of semigroups and some varieties of bands which will be used later.

Notation	Class of semigroups	Notation	Class of semigroups
$\mathcal{LS}$	*left simple*	$\pi\mathcal{R}$	*π-regular*
$\mathcal{LG}$	*left groups*	$\mathcal{I}\pi\mathcal{R}$	*intra π-regular*
$\mathcal{G}$	*groups*	$\mathcal{L}\pi\mathcal{R}$	*left π-regular*
$\mathcal{N}$	*nil-semigroups*	$\mathcal{R}\pi\mathcal{R}$	*right π-regular*
Λ	*λ-simple*	$\mathcal{C}\pi\mathcal{R}$	*completely π-regular*

Notation	Variety of bands	Notation	Variety of bands
$\mathcal{O}$	*one-element bands*	$\mathcal{LN}$	*left normal bands*
$\mathcal{LZ}$	*left zero bands*	$\mathcal{RN}$	*right normal bands*
$\mathcal{RZ}$	*right zero bands*		

For two classes $\mathcal{X}_1$ and $\mathcal{X}_2$ of semigroups, $\mathcal{X}_1 \circ \mathcal{X}_2$ will denote the *Mal'cev product* of $\mathcal{X}_1$ and $\mathcal{X}_2$, i.e. the class of all semigroups S on which there exists a congruence ϱ such that S/ϱ belongs to $\mathcal{X}_2$ and each ϱ-class of S which is a subsemigroup of S belongs to $\mathcal{X}_1$. If $\mathcal{X}_2$ is a subclass of $\mathcal{B}$, then $\mathcal{X}_1 \circ \mathcal{X}_2$ is the class of all semigroups having a band decomposition whose related factor band belongs to $\mathcal{X}_2$ and the components belong to $\mathcal{X}_1$. Such decompositions will be called *$\mathcal{X}_1 \circ \mathcal{X}_2$-decompositions*. On the other hand, if $\mathcal{X}_2$ is a subclass of $\mathcal{N}$, then $\mathcal{X}_1 \circ \mathcal{X}_2$ is the class of all semigroups that are ideal extensions of semigroups from $\mathcal{X}_1$ by semigroups from $\mathcal{X}_2$.

Here we describe some other properties of relations $\xrightarrow{l}$, $\overset{l}{\text{---}}$, λ_1 and λ.

Lemma 5.4 *If a semigroup S satisfies*

$$(\forall a, b \in S)\ ab \xrightarrow{l} ab^2, \tag{1}$$

then for any $k \in \mathbf{Z}^+$, it satisfies

$$(\forall a, b \in S)\ ab \xrightarrow{l} ab^k. \tag{2}$$

Proof. Suppose that S satisfies (2) for some $k \in \mathbf{Z}^+$. Assume $a, b \in S$. Based on (1) it follows that $ab^k = ab^{k-1}b \xrightarrow{l} ab^{k-1}b^2 = ab^{k+1}$, that is $(ab^{k+1})^m = xab^k$, for some $m \in \mathbf{Z}^+$, $x \in S^1$. Based on the hypothesis, $xab \xrightarrow{l} xab^k$, that is $(xab^k)^n = yxab$, for some $n \in \mathbf{Z}^+$, $y \in S^1$, so $(ab^{k+1})^{mn} = yxab$. Hence, S satisfies (2) for $k+1$. Now, by induction we have that S satisfies (2) for any $k \in \mathbf{Z}^+$. $\square$

Lemma 5.5 *If a semigroup S satisfies*

$$(\forall a, b \in S)\ b^2 \xrightarrow{l} ab, \tag{3}$$

then it also satisfies

$$(\forall a, b \in S)\ a^2 b \xrightarrow{l} ab. \tag{4}$$

Proof. Assume $a, b \in S$. Based on (3) we have $a^2 \xrightarrow{l} ba$, that is $(ba)^n = xa^2$, for some $n \in \mathbf{Z}^+$, $x \in S^1$, whence $(ab)^{n+1} = a(ba)^n b = axa^2 b$, which gives $a^2 b \xrightarrow{l} ab$. □

Theorem 5.28 *The following conditions on a semigroup S are equivalent:*

(i) $(\forall a, b \in S)\ a\,|_r\, b \ \Rightarrow\ a^2 \xrightarrow{r} b$;
(ii) $(\forall a, b \in S)(\forall k \in \mathbf{Z}^+)\ a^k \xrightarrow{r} ab$;
(iii) $(\forall a, b \in S)\ a^2 \xrightarrow{r} ab$;
(iv) $\sqrt{aS}$ *is a right ideal of S, for every $a \in S$;*
(v) $\sqrt{R}$ *is a right ideal of S, for every right ideal R of S.*

Proof. (i)⇒(iii) Since $ab \in aS$ for every $a, b \in S$, we then have that $(ab)^n \in a^2 S$. Thus $a^2 \xrightarrow{r} ab$.

(iii)⇒(ii) By induction.

(ii)⇒(i) Let $b = au$ for some $u \in S$. Then there exists $n \in \mathbf{Z}^+$ such that $b^n = (au)^n \in a^2 S$. Thus $a^2 \xrightarrow{r} b$.

(ii)⇒(iv) Let $x \in \sqrt{aS}$ and let $b \in S$. Then $x^k \in aS$ for some $k \in \mathbf{Z}^+$. Since $(xb)^n \in x^k S \subseteq aSS \subseteq aS$, for some $n \in \mathbf{Z}^+$ we then have that $xb \in \sqrt{aS}$. Thus $\sqrt{aS}$ is a right ideal of S.

(iv)⇒(iii) Let $a, b \in S$. Then $a \in \sqrt{a^2 S}$. Since $\sqrt{a^2 S}$ is a right ideal of S, then $ab \in \sqrt{a^2 S}$, and therefore (iii) holds.

(v)⇒(iv) Since aS is a right ideal of S, from (v) we then have that $\sqrt{aS}$ is also a right ideal of S.

(ii)⇒(v) Let R be a right ideal of S. Let $a \in \sqrt{R}$, $b \in S$. Then $a^k \in R$ for some $k \in \mathbf{Z}^+$. Now, $(ab)^n \in a^k S \subseteq RS \subseteq R$, for some $n \in \mathbf{Z}^+$ and thus $ab \in \sqrt{R}$, i.e. $\sqrt{R}$ is a right ideal of S. □

Lemma 5.6 *The following conditions on a semigroup S are equivalent:*

(i) $\xrightarrow{l}$ *is a transitive relation on S;*

(ii) $\xrightarrow{l}$ *is a right compatible quasi-order on* S*;*

(iii) $\xrightarrow{l}$ $= \lambda_1$ *on* S*;*

(iv) $(\forall a \in S)\ a\lambda_1 a^2$*;*

(v) $(\forall a, b \in S)\ a \xrightarrow{l} b \ \Rightarrow\ a^2 \xrightarrow{l} b$*;*

(vi) $(\forall a, b \in S)(\forall k \in \mathbf{Z}^+)\ b^k \xrightarrow{l} ab$*;*

(vii) $(\forall a, b \in S)\ b^2 \xrightarrow{l} ab$*;*

(viii) *any* λ_1*-class of* S *is a subsemigroup;*

(ix) $\sqrt{Sa}$ *is a left ideal of* S*, for any* $a \in S$*;*

(x) $\sqrt{L}$ *is a left ideal of* S*, for any left ideal* L *of* S*.*

Proof. Note that the equivalence of conditions (i), (iv), (v) and (ix) is a particular case of Theorem 4.8, for $n = 1$, and the equivalence of (v), (vi), (vii), (ix) and (x) is the dual of Theorem 5.28. Therefore, it remains for us to prove that the conditions (ii), (iii) and (viii) are equivalent to the remaining ones.

We will establish the following sequences of implications: (i)⇒(iii)⇒(iv) and (vii)⇒(ii)⇒(viii)⇒(iv).

(i)⇒(iii). This follows from Lemma 4.6.

(iii)⇒(iv). This is obvious.

(vii)⇒(ii). Based on the equivalence of conditions (vii) and (i) we have that $\xrightarrow{l}$ is a quasi-order. Assume that $a \xrightarrow{l} b$, for $a, b \in S$, and assume an arbitrary $c \in S$. Then $b^n = xa$, for some $n \in \mathbf{Z}^+$, $x \in S^1$, and based on (vii) and Lemma 5.5 we have that $b^{2k}c \xrightarrow{l} bc$, for any $k \in \mathbf{Z}^+$. Assume $k \in \mathbf{Z}^+$ such that $2k > n$. Then $(bc)^m = yb^{2k}c = yb^{2k-n}xac$, for some $m \in \mathbf{Z}^+$, $y \in S^1$, whence $ac \xrightarrow{l} bc$. Hence, $\xrightarrow{l}$ is right compatible.

(ii)⇒(viii). Clearly, λ_1 is a right congruence on S. Let A be a λ_1-class of S and let $a, b \in A$. Then $b\lambda_1 a$, whence $b\lambda_1 b^2 \lambda_1 ab$, since λ_1 is a right congruence, and hence $ab \in A$.

(viii)⇒(iv). This is obvious. □

Lemma 5.7 *The following conditions on a semigroup* S *are equivalent:*

(i) $(\forall a, b \in S)\ ab^2 \xrightarrow{l} ab$*;*

(ii) $(\forall a, b, c \in S)\ a \,|_l\, c \ \wedge\ b \,|_l\, c \ \Rightarrow\ ab \xrightarrow{l} c$*.*

(iii) $(\forall a, b \in S)\ a \xrightarrow{l} b \Rightarrow ba \xrightarrow{l} b$*;*

(iv) $\xrightarrow{l}$ *satisfies the cm-property on* S;

(v) *for any left ideal* L *of* S, $\sqrt{L}$ *is an intersection of completely prime left ideals of* S.

Proof. (i)⇒(ii) Let $c = ua = vb$ for some $u, v \in S$, whence $c^2 = (vb)^2$. Now, there exists $i \in \mathbf{Z}^+$ such that

$$c^{2i} = ((vbv)b)^i \in S(vbv)b^2 \subseteq Svb^2 = S(vb)b = S(ua)b \subseteq Sab.$$

Thus $ab \xrightarrow{l} c$.

(ii)⇒(i) It is clear that $ab \,|_l\, ab,\ b \,|_l\, ab$, for all $a, b \in S$, and based on (ii) we have that $(ab)b = ab^2 \xrightarrow{l} ab$.

(i)⇒(iv) Let $a, b, c \in S$, $a \xrightarrow{l} c$ and $b \xrightarrow{l} c$. Then $c^n = xa = yb$, for some $n \in \mathbf{Z}^+$, $x, y \in S^1$, and based on (i), $(yb)^m = zyb^2$, for some $m \in \mathbf{Z}^+$, $z \in S^1$, whence

$$c^{nm} = (yb)^m = zyb^2 = z(yb)b = zuab \in Sab,$$

so $ab \xrightarrow{l} c$.

(iv)⇒(iii) Let $a, b \in S$ and $a \xrightarrow{l} b$. Then $b \xrightarrow{l} b$ and $a \xrightarrow{l} b$, whence $ba \xrightarrow{l} b$, by (iv).

(iii)⇒(i) Let $a, b \in S$. Then $b \xrightarrow{l} ab$, so by (iii), $ab^2 \xrightarrow{l} ab$.

(iv)⇒(v) Since (i)⇔(ii), then according to Lemma 5.6 we have that $\xrightarrow{l}$ is transitive, that is $\xrightarrow{l} = \xrightarrow{l}{}^{\infty}$, so based on Theorem 4.8, for each left ideal L of S, $\sqrt{L}$ is a completely semiprime left ideal of S, and based on Theorem 4.4, it is an intersection of completely prime left ideals of S.

(v)⇒(iv) Let $a \in S$. Based on (iv), $\sqrt{Sa}$ is a completely semiprime left ideal of S, so according to Theorem 4.8, $\xrightarrow{l}$ is transitive, i.e. $\xrightarrow{l} = \xrightarrow{l}{}^{\infty}$. Now, based on Theorem 4.4, $\xrightarrow{l}$ satisfies the *cm*-property. □

Lemma 5.8 *On a semigroup* S *the relation* η *defined by*

$$a\eta b \Leftrightarrow (\forall x \in S^1)\ xa \overset{l}{\text{---}} xb,$$

is a congruence relation.

Proof. It is evident that η is a reflexive and symmetric relation.

Now, assume $a, b, c \in S$ such that $a\eta b$ and $b\eta c$, i.e. $xa \overset{l}{\text{---}} xb$ and $xb \overset{l}{\text{---}} xc$, for every $x \in S^1$. Then, there exist $i, j, p, q \in \mathbf{Z}^+$ such that

$$(xa)^i \in Sxb, \quad (xb)^j \in Sxa, \quad (xb)^p \in Sxc, \quad (xc)^q \in Sxb.$$

By this we have that

$$(xa)^i = uxb, \quad (xb)^j = vxa, \quad (xb)^p = wxc, \quad (xc)^q = zxb,$$

for some $u, v, w, z \in S$ and for every $x \in S^1$. Now, we obtain that

$$(xa)^{ip} = \big((xa)^i\big)^p = (uxb)^p = ((ux)b)^p = w(ux)c \in Sxc,$$

and

$$(xc)^{qj} = ((xc)^q)^j = (zxb)^j = ((zx)b)^j = v(zx)a \in Sxa.$$

Hence, $xa \overset{l}{\text{---}} xc$, for every $x \in S^1$, i.e. $a\eta c$. So, η is transitive. Thus, η is an equivalence relation on S.

Furthermore, assume $a, b, c \in S$ such that $a\eta b$, i.e. $xa \overset{l}{\text{---}} xb$, for every $x \in S^1$. Then, there exist $i, j \in \mathbf{Z}^+$ such that

$$(xa)^i \in Sxb, \qquad (xb)^j \in Sxa,$$

for every $x \in S^1$. Based on this, we have that

$$(x(ca))^i = ((xc)a)^i \in S(xc)b = Sx(cb),$$

and

$$(x(cb))^j = ((xc)b)^j \in S(xc)a = Sx(ca).$$

Hence, $x(ca) \overset{l}{\text{---}} x(cb)$, for every $x \in S^1$, i.e. $ca\eta cb$.

Also, we have that

$$(x(ac))^{i+1} = xa(cxa)^i c = xa((cx)a)^i c \in xa \cdot S(cx)b \cdot c \in Sx(bc),$$

and

$$(x(bc))^{j+1} = xb(cxb)^j c = xb((cx)b)^j c \in xb \cdot S(cx)a \cdot c \in Sx(ac).$$

Hence, $x(ac) \overset{l}{\text{---}} x(bc)$, for every $x \in S^1$, i.e. $ac\eta bc$. Thus, η is a congruence relation on S. $\square$

Now we prove the following lemma.

Lemma 5.9 *On any semigroup S, $\eta = \lambda_1^\flat$.*

Proof. Assume an arbitrary pair $(a,b) \in \eta$. If $c \in \Lambda_1(a)$, that is $c^n = xa$, for some $x \in S^1$, $n \in \mathbf{Z}^+$, then from $a\eta b$ we have that $xa \overset{l}{\longrightarrow} xb$, so $(xa)^m \in Sxb$, for some $m \in \mathbf{Z}^+$, which yields $c^{nm} \in Sb$, so $c \in \Lambda_1(b)$. Thus we proved $\Lambda_1(a) \subseteq \Lambda_1(b)$. Similarly we prove $\Lambda_1(b) \subseteq \Lambda_1(a)$. Therefore, $a\lambda_1 b$, which means that $\eta \subseteq \lambda_1$.

Let ϱ be an arbitrary congruence relation on S contained in λ_1. Assume an arbitrary pair $(a,b) \in \varrho$. Then for any $x \in S^1$ we have that

$$(xa, xb) \in \varrho \subseteq \lambda_1 \subseteq \overset{l}{\longrightarrow},$$

whence it follows that $(a,b) \in \eta$. Therefore, $\varrho \subseteq \eta$, which was to be proved. This completes the proof of the lemma. □

As we noted before, the first characterization of bands of left Archimedean semigroups was given by M. S. Putcha in [3], and this result we quote in the next theorem as the equivalence of conditions (i) and (ii). Moreover, we give several new characterizations of semigroups having such a decomposition.

Theorem 5.29 *The following conditions on a semigroup S are equivalent:*

(i) *$S \in \mathcal{LA} \circ \mathcal{B}$;*
(ii) *$(\forall a \in S)(\forall x, y \in S^1)\ xay \overset{l}{\longrightarrow} xa^2y$;*
(iii) *η is a band congruence on S;*
(iv) *$(\forall a, b \in S)\ a^2b \overset{l}{\longrightarrow} ab\ \&\ ab \overset{l}{\longrightarrow} ab^2$;*
(v) *$(\forall a, b \in S)\ ab \overset{l}{\longrightarrow} ab^2$.*

Proof. (i)⇔(ii). This is Theorem 5.11.

(ii)⇒(v) This is clear.

(v)⇒(ii) Clearly, $b^2 \overset{l}{\longrightarrow} ab$, for all $a, b \in S$, so based on Lemma 5.6, $\overset{l}{\longrightarrow}$ is a right congruence. Assume $a, b, c \in S$. Based on (v) and (iv) we have $ab \overset{l}{\longrightarrow} ab^2$ and $ab \overset{l}{\longrightarrow} a^2b$, and since $\overset{l}{\longrightarrow}$ is a right congruence, then $abc \overset{l}{\longrightarrow} ab^2c$. Hence, (ii) holds.

(iv)⇒(v) Assume $a, b \in S$ such that $a \longrightarrow b$, that is $b^m = xay$, for some $m \in \mathbf{Z}^+$, $x, y \in S^1$. Based on (iv) we have $(xa)^2y \xrightarrow{l} xay$, that is $(xay)^n = z(xa)^2y = zxab^m$, for some $n \in \mathbf{Z}^+$, $z \in S^1$. On the other hand, according to Lemma 5.4, $zxab \xrightarrow{l} zxab^m$, that is $(zxab^m)^k = uzaxb$, for some $k \in \mathbf{Z}^+$, $u \in S^1$, which gives $b^{mnk} = uzxab$, that is $ab \xrightarrow{l} b$. Now, according to Theorem 5.20, S is a semilattice Y of weakly left Archimedean semigroups S_α, $\alpha \in Y$.

Assume $a, b \in S$ Then $ab \xrightarrow{l} ab^2$ in S, and $ab, ab^2 \in S_\alpha$, for some $\alpha \in Y$, so based on Lemma 4.14 (c), $ab \xrightarrow{l} ab^2$ in S_α. According to Theorem 5.21, $\xrightarrow{l}$ is a symmetric relation on S_α, whence $ab^2 \xrightarrow{l} ab$.

(v)⇒(iv) This follows from Lemma 5.5.

(v)⇒(iii) This follows from Lemma 5.9.

(iii)⇒(i) This follows from Lemma 5.3. □

As a consequence of the previous theorem we obtain the next corollary.

Corollary 5.13 *A semigroup S belongs to $\mathcal{TA} \circ \mathcal{B}$ if and only if $a^2b \overset{r}{—} ab \overset{l}{—} ab^2$ for all $a, b \in S$.*

The concept of π-regularity, in its various forms, appeared first in ring theory, as a natural generalization of the regularity. In semigroup theory this concept attracts great attention both as a generalization of the regularity and a generalization of finiteness and periodicity. On the other hand, there are specific relations between the π-regularity and the Archimedeanness, as was shown by M. S. Putcha in [2]. That motivates us to investigate $\mathcal{LA} \circ \mathcal{B}$-decompositions of π-regular semigroups.

We do it first for intra π-regular and left π-regular semigroups. It is interesting to note that for left π-regular semigroups only one half of the condition (v) of Theorem 5.29 is enough.

Theorem 5.30 *The following conditions on a semigroup S are equivalent:*

(i) $S \in \mathcal{L}\pi\mathcal{R} \cap \mathcal{LA} \circ \mathcal{B}$;

(ii) $S \in \mathcal{I}\pi\mathcal{R} \cap \mathcal{LA} \circ \mathcal{B}$;

(iii) $S \in (\mathcal{LS} \circ \mathcal{N}) \circ \mathcal{B}$;

(iv) $S \in \mathcal{L}\pi\mathcal{R}$ *and* $ab^2 \xrightarrow{l} ab$, *for all* $a, b \in S$;

(v) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n \in S(ab^2)^n$.

Proof. (iii)⇒(i) and (i)⇒(ii) This is trivial.

(ii)⇒(i) Since $\mathcal{I}\pi\mathcal{R}\cap\mathcal{LA}\circ\mathcal{B}\subseteq\mathcal{I}\pi\mathcal{R}\cap\mathcal{WLA}\circ\mathcal{S}=(\mathcal{I}\pi\mathcal{R}\cap\mathcal{WLA})\circ\mathcal{S}=(\mathcal{L}\pi\mathcal{R}\cap\mathcal{WLA})\circ\mathcal{S}=\mathcal{L}\pi\mathcal{R}\cap\mathcal{WLA}\circ\mathcal{S}$, based on Theorems 5.20, 5.21 and 5.22, then (iii) implies (ii).

(i)⇒(iii) As we all know, each component of a band decomposition of a left π-regular semigroup is also left π-regular. Based on this and Theorem 3.14 we obtain (i).

(iii)⇒(v) Let S be a band I of semigroups $S_i,\ i\in I$, and for each $i\in I$, let S_i be a nil-extension of a left simple semigroup K_i. Then for all $a,b\in S$, $ab, ab^2\in S_i$, for some $i\in I$, and $(ab)^n,(ab^2)^n\in K_i$, for some $n\in\mathbf{Z}^+$, whence $(ab)^n\in K_i(ab^2)^n\subseteq S(ab^2)^n$.

(v)⇒(iv) This is obvious.

(iv)⇒(i) Based on Theorem 5.1, S is a semilattice Y of Archimedean semigroups $S_\alpha,\ \alpha\in Y$. It was proved in Theorem 5.26 that $\mathcal{A}\cap\mathcal{L}\pi\mathcal{R}=(\mathcal{LS}\circ\mathcal{RZ})\circ\mathcal{N}$, so for any $\alpha\in Y$, S_α is a nil-extension of a semigroup K_α which is a right zero band I_α of left simple semigroups $K_i,\ i\in I_\alpha$.

Assume $\alpha\in Y,\ i\in I_\alpha$, and set $S_i=\sqrt{K_i}$. Further, let $i,j\in I_\alpha,\ a\in S_i,\ b\in S_j$, and assume $m\in\mathbf{Z}^+$ such that $b^m\in K_j$. By (iv) and based on Lemma 5.4, $ab^{m+1}\xrightarrow{l}ab$ in S, so based on Lemma 4.14 (c), $(ab)^n=xab^{m+1}$, for some $n\in\mathbf{Z}^+,\ x\in S_\alpha^1$. Assume $k\in\mathbf{Z}^+$ such that $(ab)^k\in K_\alpha$. Then

$$(ab)^{k+n}=(ab)^k(xab)b^m\in K_\alpha S_\alpha K_i\subseteq K_\alpha K_i\subseteq K_i,$$

so $ab\in S_j$. Hence, for any $\alpha\in Y$, S_α is a right zero band I_α of semigroups $S_i,\ i\in I_\alpha$, and for any $i\in I_\alpha$, S_i is a nil-extension of a left simple semigroup K_i. Now, according to Theorem 5.21, for any $\alpha\in Y$, $\xrightarrow{l}$ is a symmetric relation on S_α, and as in the proof of (iv)⇒(v) of Theorem 5.29 we obtain that $ab\overset{l}{—}ab^2$, for all $a,b\in S$. Hence, by Theorem 5.29 we obtain (ii). □

For π-regular semigroups we have the following:

Theorem 5.31 *The following conditions on a semigroup S are equivalent:*

(i) $S\in\mathcal{R}\pi\mathcal{R}\cap\mathcal{LA}\circ\mathcal{B}$;

(ii) $S\in\pi\mathcal{R}\cap\mathcal{LA}\circ\mathcal{B}$;

(iii) $S\in\mathcal{C}\pi\mathcal{R}\cap\mathcal{LA}\circ\mathcal{B}$;

(iv) $S\in(\mathcal{LG}\circ\mathcal{N})\circ\mathcal{B}$;

(v) $S\in\pi\mathcal{R}$ *and* $ab^2\xrightarrow{l}ab$, *for all* $a,b\in S$;

(vi) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n \in (ab)^n S(ab^2)^n$.

Proof. (iv)⇒(iii) and (vi)⇒(v) This is clear.

(i)⇔(iii) and (ii)⇔(iii) This can be proved in a similar way as (ii)⇒(i) of Theorem 5.30, using Theorems 5.22 and 5.23.

(iii)⇒(iv) This follows from the arguments similar to the ones used in (i)⇒(iii) of Theorem 5.30.

(iv)⇒(vi) This can be proved in a similar way as (iii)⇒(v) of Theorem 5.30, using Theorem 3.7.

(v)⇒(ii) Let $a \in \mathrm{Reg}(S)$, $a' \in V(a)$. Then $a'a^2 \xrightarrow{l} a'a$, whence $a \in \mathrm{LReg}(S)$, so S is left π-regular, and based on Theorem 5.30, $S \in \mathcal{LA} \circ \mathcal{B}$. □

Some other characterizations of semigroups from $(\mathcal{LG} \circ \mathcal{N}) \circ \mathcal{B}$ one can obtain by the results concerning their dual semigroups, given by L. N. Shevrin in [5].

Corollary 5.14 *The following conditions on a semigroup S are equivalent:*

(i) $S \in (\mathcal{G} \circ \mathcal{N}) \circ \mathcal{B}$;
(ii) $S \in \mathcal{I}\pi\mathcal{R} \cap \mathcal{TA} \circ \mathcal{B}$;
(iii) $S \in \pi\mathcal{R} \cap \mathcal{TA} \circ \mathcal{B}$;
(iv) $S \in \pi\mathcal{R}$ *and* $a^2b \xrightarrow{r} ab$ & $ab^2 \xrightarrow{l} ab$, *for all* $a, b \in S$.

Our next goal is to characterize the semigroups from $\mathcal{LA} \circ \mathcal{V}$, for an arbitrary variety of bands $\mathcal{V}$.

The lattice **LVB** of all varieties of bands was studied by P. A. Birjukov, C. F. Fennemore, J. A. Gerhard, M. Petrich and others. Here we use the characterization of **LVB** given by J. A. Gerhard and M. Petrich in [1]. Using induction they defined three systems of words as follows:

$$\begin{array}{lll} G_2 = x_2 x_1, & H_2 = x_2, & I_2 = x_2 x_1 x_2, \\ G_n = x_n \overline{G}_{n-1}, & H_n = x_n \overline{G}_{n-1} x_n \overline{H}_{n-1}, & I_n = x_n \overline{G}_{n-1} x_n \overline{I}_{n-1}, \end{array}$$

(for $n \geq 3$), and they shown that the lattice **LVB** can be represented by the graph given in Figure 1.

Let us give some additional explanations concerning the graph from Figure 1. Throughout this section, for a semigroup identity $u = v$, based on $[u = v]$ we will denote the variety of bands determined by this identity. In other words, this is a shortened notation for the semigroup variety

$[x^2 = x,\ u = v]$. For a word w, $\overline{w}$ denotes the *dual* of w, that is, the word obtained from w by reversing the order of the letters in w. In the graph from Figure 1 we have labelled only the nodes which represent varieties of bands that will appear in our further investigations.

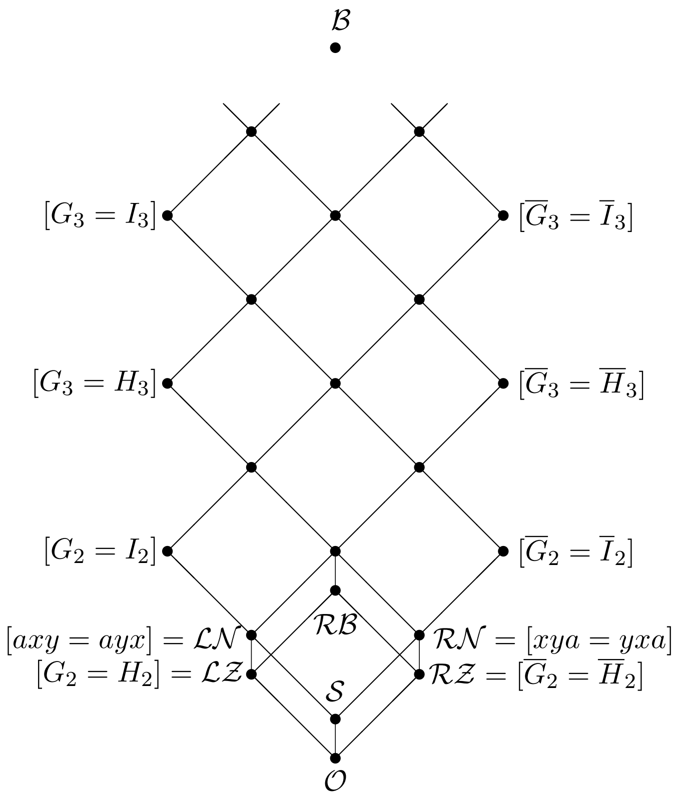

Figure 1.

The central point of this section is the following theorem:

Theorem 5.32 *Let $\mathcal{V}$ be an arbitrary variety of bands. Then*

$$\mathcal{LZ}\circ\mathcal{V} = \begin{cases} \mathcal{LZ}, & \text{if } \mathcal{V}\in[\mathcal{O},\mathcal{LZ}]; \\ \mathcal{RB}, & \text{if } \mathcal{V}\in[\mathcal{RZ},\mathcal{RB}]; \\ [G_2 = I_2]\,, & \text{if } \mathcal{V}\in[\mathcal{S},[G_2 = I_2]]\,; \\ [G_3 = I_3]\,, & \text{if } \mathcal{V}\in[\mathcal{RN},[G_3 = H_3]]\,; \\ [G_{n+1} = I_{n+1}]\,, & \text{if } \mathcal{V}\in\big[[\overline{G}_n = \overline{I}_n],[G_{n+1} = I_{n+1}]\big]\,, n\geq 2; \\ [G_{n+1} = H_{n+1}]\,, & \text{if } \mathcal{V}\in\big[[\overline{G}_n = \overline{H}_n],[G_{n+1} = H_{n+1}]\big]\,, n\geq 3. \end{cases}$$

Proof. Consider the congruence η on a band S. Since $\lambda_1 = \overset{l}{\text{---}} = \mathcal{L}$ on S, then $\eta = \mathcal{L}^\flat$. It is known that the Green relation $\mathcal{L}$ on S is defined by $(a,b) \in \mathcal{L} \Leftrightarrow ab = a \ \&\ b = ba$, whence we conclude that

$$(a,b) \in \eta \Leftrightarrow (\forall x \in S^1)\ xa = xaxb \ \&\ xb = xbxa. \tag{5}$$

But, if $xa = xaxb$ and $xb = xbxa$, for any $x \in S$, then for $x = a$ we have $a = ab$, and for $x = b$ we have $b = ba$, so the condition (5) is equivalent to

$$(a,b) \in \eta \Leftrightarrow (\forall x \in S)\ xa = xaxb \ \&\ xb = xbxa. \tag{6}$$

Let $[\mathcal{V}_1, \mathcal{V}_2]$ be some of the intervals of **LVB** which appear in the formulation of the theorem. We will prove:

$$S \in \mathcal{V}_2 \Leftrightarrow S/\eta \in \mathcal{V}_1, \tag{7}$$

for any band S.

Case 1: $[\mathcal{V}_1, \mathcal{V}_2] = [\mathcal{O}, \mathcal{LZ}]$. This case is trivial.

Case 2: $[\mathcal{V}_1, \mathcal{V}_2] = [\mathcal{RZ}, \mathcal{RB}]$. In this case the assertion (7) is an immediate consequence of the construction of a rectangular band.

Case 3: $[\mathcal{V}_1, \mathcal{V}_2] = [\mathcal{S}, [G_2 = I_2]]$.

Case 4: $[\mathcal{V}_1, \mathcal{V}_2] = [\mathcal{RN}, [G_3 = H_3]]$.

Case 5: $[\mathcal{V}_1, \mathcal{V}_2] = [\overline{G}_2 = \overline{I}_2, [G_3 = I_3]]$. [1]

Note that in all of these cases the Green relation $\mathcal{L}$ is a congruence, i.e. $\eta = \mathcal{L}$. In other words, for a band S we have that $\mathcal{L}$ is a congruence on S if and only if $S \in [G_3 = I_3]$.

Case 6: $[\mathcal{V}_1, \mathcal{V}_2] = [\overline{G}_n = \overline{I}_n, [G_{n+1} = I_{n+1}]]$, $n \geq 3$. Here we have that $V_2 = [x_{n+1}\overline{G}_n = x_{n+1}\overline{G}_n x_{n+1}\overline{I}_n]$.

Let S be an arbitrary band. Suppose first that $S \in \mathcal{V}_2$. For $1 \leq i \leq n$ let the letter x_i get a value a_i in S. Then the words $\overline{G}_n$ and $\overline{I}_n$ get some values u and v in S, respectively. To prove that $S/\eta \in \mathcal{V}_1 = [\overline{G}_n = \overline{I}_n]$, it is enough to prove that $(u,v) \in \eta$.

Assume an arbitrary $a \in S$. If the letter x_{n+1} assumes in S a value a, then from $S \in \mathcal{V}_2$ it follows that $au = auav$. Since the words $\overline{G}_n$ and $\overline{I}_n$ have the same letters, then $(u,v) \in \mathcal{D}$ and $(au, av) \in \mathcal{D}$. But, any $\mathcal{D}$-class of S is a rectangular band, whence by $au = auav$ it follows $avau = avauav = av$.

[1] For details of the proof for cases 3, 4 and 5 see Section II 3 of book [10] by M. Petrich.

Therefore, $au = auav$ and $av = avau$, for any $a \in S$, whence $(u, v) \in \eta$, which was to be proved.

Conversely, assume that $S/\eta \in \mathcal{V}_1$. For $1 \le i \le n+1$ let the letter x_i get an arbitrary value a_i in S. Then the words $\overline{G}_n$ and $\overline{I}_n$ get some values u and v in S, respectively, and $(u, v) \in \eta$, since $S/\eta \in \mathcal{V}_1 = [\overline{G}_n = \overline{I}_n]$. But, from $(u, v) \in \eta$ it follows that $a_{n+1}u = a_{n+1}ua_{n+1}v$, by (6), whence we conclude that $S \in [x_{n+1}\overline{G}_n = x_{n+1}\overline{I}_n] = \mathcal{V}_2$. This completes the proof of this case.

Case 7: $[\mathcal{V}_1, \mathcal{V}_2] = [\overline{G}_n = \overline{H}_n, [G_{n+1} = H_{n+1}]]$, $n \ge 3$. This case is analogous to the previous one.

Taking into consideration all the cases, we have completed the proof of the theorem. □

By means of a straightforward verification we give the following lemma:

Lemma 5.10 *Let $\mathcal{C}$ be a class of semigroups and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two classes of bands. Then $\mathcal{C} \circ (\mathcal{B}_1 \circ \mathcal{B}_2) \subseteq (\mathcal{C} \circ \mathcal{B}_1) \circ \mathcal{B}_2$.*

A particular case of the previous lemma is the well-known result of A. H. Clifford from 1954 (see Corollary 3.7) that asserts that $\mathcal{X} \circ \mathcal{B} = \mathcal{X} \circ (\mathcal{RB} \circ \mathcal{S}) \subseteq (\mathcal{X} \circ \mathcal{RB}) \circ \mathcal{S}$, for an arbitrary class $\mathcal{X}$ of semigroups. For the class $\mathcal{G}$ of all groups, $\mathcal{G} \circ \mathcal{B} = \mathcal{G} \circ (\mathcal{RB} \circ \mathcal{S})$ is the class of all semigroups that are bands of groups, and $(\mathcal{G} \circ \mathcal{RB}) \circ \mathcal{S}$ is the class of all semigroups that are unions of groups. As we all know, these classes are different, so $\mathcal{G} \circ (\mathcal{RB} \circ \mathcal{S}) \subsetneq (\mathcal{G} \circ \mathcal{RB}) \circ \mathcal{S}$. This proves that the inclusion in Lemma 5.10 can be proper.

The following theorem gives a very important result. It gives the conditions under which a band of semigroups from any class of semigroups coincides with a semilattice of semigroups from the same class.

Theorem 5.33 *Let $\mathcal{C}$ be a class of semigroups. Then*

$$\mathcal{C} \circ \mathcal{RB} \subseteq \mathcal{C} \quad \Leftrightarrow \quad \mathcal{C} \circ \mathcal{B} = \mathcal{C} \circ \mathcal{S}.$$

Proof. Let $\mathcal{C} \circ \mathcal{RB} \subseteq \mathcal{C}$. Then based on Lemma 5.10 and Corollary 3.6 we have that

$$\mathcal{C} \circ \mathcal{B} = \mathcal{C} \circ (\mathcal{RB} \circ \mathcal{S}) \subseteq (\mathcal{C} \circ \mathcal{RB}) \circ \mathcal{S} \subseteq \mathcal{C} \circ \mathcal{S} \subseteq \mathcal{C} \circ \mathcal{B}.$$

Hence, $\mathcal{C} \circ \mathcal{B} = \mathcal{C} \circ \mathcal{S}$.

Conversely, from the hypothesis we have that

$$\mathcal{C} = \mathcal{C} \circ \mathcal{O} = \mathcal{C} \circ \mathcal{S} = \mathcal{C} \circ \mathcal{RB}.$$

$\square$

Using the above theorem and lemma we prove the following:

Theorem 5.34 *Let V be an arbitrary variety of bands. Then*

$$\mathcal{LA} \circ \mathcal{V} = \begin{cases} \mathcal{LA}, & \text{if } \mathcal{V} \in [\mathcal{O}, \mathcal{LZ}]; \\ \mathcal{LA} \circ \mathcal{RZ}, & \text{if } \mathcal{V} \in [\mathcal{RZ}, \mathcal{RB}]; \\ \mathcal{LA} \circ \mathcal{S}, & \text{if } \mathcal{V} \in [\mathcal{S}, [G_2 = I_2]]; \\ \mathcal{LA} \circ \mathcal{RN}, & \text{if } \mathcal{V} \in [\mathcal{RN}, [G_3 = H_3]]; \\ \mathcal{LA} \circ [\overline{G}_n = \overline{I}_n] & \text{if } V \in [[\overline{G}_n = \overline{I}_n], [G_{n+1} = I_{n+1}]], n \geq 2; \\ \mathcal{LA} \circ [\overline{G}_n = \overline{H}_n] & \text{if } V \in [[\overline{G}_n = \overline{H}_n], [G_{n+1} = H_{n+1}]], n \geq 3. \end{cases}$$

Proof. One verifies easily that $\mathcal{LA} \circ \mathcal{LZ} = \mathcal{LA}$. Further, let $[\mathcal{V}_1, \mathcal{V}_2]$ be some of the intervals of the lattice **LVB** which appears in the formulation of the theorem, and let $\mathcal{V} \in [\mathcal{V}_1, \mathcal{V}_2]$. According to Theorem 5.32 we have that $\mathcal{V}_2 = \mathcal{LZ} \circ \mathcal{V}_1$, whence

$$\mathcal{LA} \circ \mathcal{V}_1 \subseteq \mathcal{LA} \circ \mathcal{V} \subseteq \mathcal{LA} \circ \mathcal{V}_2 = \mathcal{LA} \circ (\mathcal{LZ} \circ \mathcal{V}_1) \subseteq (\mathcal{LA} \circ \mathcal{LZ}) \circ \mathcal{V}_1 = \mathcal{LA} \circ \mathcal{V}_1,$$

using Lemma 5.10. Therefore, $\mathcal{LA} \circ \mathcal{V}_1 = \mathcal{LA} \circ \mathcal{V} = \mathcal{LA} \circ \mathcal{V}_2$, which was to be proved. $\square$

Finally, we prove the following:

Theorem 5.35 *Let $\mathcal{V}$ be an arbitrary variety of bands and let S be a semigroup. Then $S \in \mathcal{LA} \circ \mathcal{V}$ if and only if $S/\eta \in \mathcal{V}$.*

Proof. Let $S \in \mathcal{LA} \circ \mathcal{V}$. Then there exists a congruence ξ on S such that $S/\xi \in \mathcal{V}$ and any ξ-class of S is in $\mathcal{LA}$. Based on Lemma 5.3 we have $\xi \subseteq \lambda_1$, and Lemma 5.9, $\xi \subseteq \eta$. Therefore, S/η is a homomorphic image of S/ξ and $S/\xi \in \mathcal{V}$, whence $S/\eta \in \mathcal{V}$, which was to be proved.

Conversely, if $S/\eta \in \mathcal{V}$, then based on Lemma 5.3 we have that any η-class is in $\mathcal{LA}$, and hence, $S \in \mathcal{LA} \circ \mathcal{V}$. $\square$

Lemma 5.11 *Let S be a semigroup. Then*

$$\Lambda = \Lambda \circ \mathcal{LZ}.$$

Proof. Let S be a left zero band Y of λ-simple semigroups S_α, $\alpha \in Y$. Assume $a, b \in S$, then $a \in S_\alpha$, $b \in S_\beta$, for some $\alpha, \beta \in Y$, whence $ab \in S_\alpha S_\beta \subseteq S_{\alpha\beta} = S_\alpha$. Hence, $ab, a \in S_\alpha$. So $ab \xrightarrow{l} {}^\infty a$, whence $b \xrightarrow{l} {}^\infty a$. In a similar way we can prove that $a \xrightarrow{l} {}^\infty b$. Thus $a \xrightarrow{l} {}^\infty \cap (\xrightarrow{l} {}^\infty)^{-1} b$ and based on Lemma 4.6 we have that $a\lambda b$. Therefore, S is a λ-simple semigroup.

The converse follows immediately. $\square$

Our next goal is to characterize semigroups from $\Lambda \circ \mathcal{V}$, for an arbitrary variety of bands $\mathcal{V}$.

Theorem 5.36 *Let $\mathcal{V}$ be an arbitrary variety of bands. Then*

$$\Lambda \circ \mathcal{V} = \begin{cases} \Lambda, & \text{if } \mathcal{V} \in [\mathcal{O}, \mathcal{LZ}]; \\ \Lambda \circ \mathcal{RZ}, & \text{if } \mathcal{V} \in [\mathcal{RZ}, \mathcal{RB}]; \\ \Lambda \circ \mathcal{S}, & \text{if } \mathcal{V} \in [\mathcal{S}, [G_2 = I_2]]\,; \\ \Lambda \circ \mathcal{RN}, & \text{if } \mathcal{V} \in [\mathcal{RN}, [G_3 = H_3]]\,; \\ \Lambda \circ \left[\overline{G}_n = \overline{I}_n\right], & \text{if } \mathcal{V} \in \left[[\overline{G}_n = \overline{I}_n], [G_{n+1} = I_{n+1}]\right], n \geq 2; \\ \Lambda \circ \left[\overline{G}_n = H_n\right], & \text{if } \mathcal{V} \in \left[[\overline{G}_n = \overline{H}_n], [G_{n+1} = H_{n+1}]\right], n \geq 3. \end{cases}$$

Proof. Based on Lemma 5.11 we have that $\Lambda \circ \mathcal{LZ} = \Lambda$. Let $\mathcal{V} \in [\mathcal{V}_1, \mathcal{V}_2]$, whence $[\mathcal{V}_1, \mathcal{V}_2]$ is some of the intervals of the lattice **LVB** from the theorem. Based on Theorem 5.32 we have that $\mathcal{V}_2 = \mathcal{LZ} \circ \mathcal{V}_1$, whence

$$\Lambda\circ\mathcal{V}_1 \subseteq \Lambda\circ\mathcal{V} \subseteq \Lambda\circ\mathcal{V}_2 = \Lambda\circ(\mathcal{LZ}\circ\mathcal{V}_1) \subseteq (\Lambda\circ\mathcal{LZ})\circ\mathcal{V}_1 = \Lambda\circ\mathcal{V}_1 \text{ (by Lemma 5.11).}$$

Therefore, $\Lambda \circ \mathcal{V}_1 = \Lambda \circ \mathcal{V} = \Lambda \circ \mathcal{V}_2$. $\square$

Note that the corresponding results can be obtained for bands of left simple semigroups and bands of left groups.

Exercises

1. The following conditions on a semigroup S are equivalent:

(i) S is a right weakly commutative;
(ii) S is a semilattice of left Archimedean semigroups;
(iii) $(\forall a, b \in S)\, a \,|\, b \Rightarrow (\exists i \in \mathbf{Z}^+)\, a \,|_l\, b^i$;
(iv) $N(x) = \{y \in S \,|\, (\exists n \in \mathbf{Z}^+)\, x^n \in Sy\}$, for every $x \in S$;
(v) $(\forall a, b \in S)\; ab \overset{l}{\text{—}} ba$;
(vi) $(\forall a, b \in S)\; a \text{—} b \Rightarrow a^2 \overset{l}{\text{—}} b$;
(vii) $(\forall a, b, c \in S)\; a \text{—} b \;\&\; b \text{—} c \Rightarrow a \overset{l}{\text{—}} c$;
(viii) $(\forall a, b, c \in S)\; a \text{—} c \;\&\; b \text{—} c \Rightarrow ab \overset{l}{\text{—}} c$.

2. $\sqrt{R}$ is a subsemigroup of S, for every right ideal R of S if and only if

$$(\forall a, b \in S)(\forall k, l \in \mathbf{Z}^+)\ a^k \xrightarrow{r} ab \quad \vee \quad b^l \xrightarrow{r} ab.$$

3. The radical of every right ideal of a semigroup S is a bi-ideal of S if and only if

$$(\forall a, b, c \in S)(\forall k, l \in \mathbf{Z}^+)\ a^k \xrightarrow{r} abc \quad \vee \quad c^l \xrightarrow{r} abc. \tag{1}$$

4. The radical of every ideal of a semigroup S is a bi-ideal of S if and only if

$$(\forall a, b, c \in S)(\forall k, l \in \mathbf{Z}^+)\ a^k \longrightarrow abc \quad \vee \quad c^l \longrightarrow abc.$$

References

S. Bogdanović [16], [17]; S. Bogdanović and M. Ćirić [4], [6], [9], [15], [17], [20]; S. Bogdanović, M. Ćirić and B. Novikov [1]; S. Bogdanović and Ž. Popović [1]; S. Bogdanović, Ž. Popović and M. Ćirić [1], [2]; S. Bogdanović and B. Stamenković [1]; M. Ćirić and S. Bogdanović [3], [5]; J. A. Gerhard and M. Petrich [1]; F. Pastijn [1]; M. Petrich [10]; P. Protić [3], [5], [6]; M. S. Putcha [2], [3]; L. N. Shevrin [5]; A. Spoletini Cherubini and A. Varisco [1]; E. V. Sukhanov [1]; X. Y. Xie [1].

Chapter 6

Semilattice of k-Archimedean Semigroups

In this section, on an arbitrary semigroup we define a few different types of relations and its congruence extensions. Also, we describe the structure of semigroups in which these relations are band (semilattice) congruences. The components of such obtained band (semilattice) decompositions usually are in some sense simple semigroups.

L. N. Shevrin proved that a completely π-regular semigroup $R(\mathcal{D})$ is transitive if and only if it is a semilattice congruence. A more general result has been obtained by M. S. Putcha who proved that in a completely π-regular semigroup the transitive closure of $R(\mathcal{J})$ is the smallest semilattice congruence. Since $\mathcal{D} = \mathcal{J}$ on any completely π-regular semigroup, Shevrin's result can also be derived from the one of M. S. Putcha.

Various characterizations of semigroups in which the radical $R(\varrho)$ $(T(\varrho))$, where $\varrho \in \{\mathcal{J}, \mathcal{L}, \mathcal{R}, \mathcal{D}, \mathcal{H}\}$, is a band (semilattice) congruence have been investigated by S. Bogdanović and M. Ćirić, S. Bogdanović, M. Ćirić and Ž. Popović and S. Bogdanović, Ž. Popović and M. Ćirić.

In this section we define one new radical ϱ_k, $k \in \mathbf{Z}^+$, of a relation ϱ on a semigroup S and using it we describe the structure of a semigroup in which this radical is a band (semilattice) congruence for some Green's relation. For these descriptions of the structure of semigroups we consider some new types of k-regularity of semigroups and also some new types of k-Archimednness of semigroups. Also, here we characterize the semilattices of

k-Archimedean semigroups and describe the hereditary properties of semilattices of k-Archimedean semigroups.

Very interesting decompositions are band decompositions in which components are power-joined, periodic and both power-joined and periodic semigroups. These decompositions were studied by T. Tamura, T. Nordahl, K. Iseki and S. Bogdanović.

T. Tamura studied commutative Archimedean semigroups which have a finite number of power-joined components. Bands of power-joined semigroups were studied by T. Nordahl, in medial cases, and by S. Bogdanović, in general. K Iseki considered periodic semigroups which are the disjoint union of semigroups, each containing only one idempotent. S. Bogdanović considered bands of periodic power-joined semigroups.

In this section, on a semigroup S, for $k \in \mathbf{Z}^+$, we define some new equivalence relations η, η_k and τ. If these equivalences are band congruences then they makes band decompositions of η-simple (power-joined) semigroups, and band decompositions of two types of periodic power-joined semigroups (η_k-simple and τ-simple semigroups). The obtained results generalize the results of the above mentioned authors.

It is known that Lallement's lemma does not hold true in arbitrary semigroups. In fact, this lemma fails to hold in the semigroup of all positive integers under addition, since it does not have an idempotent element but the entire semigroup can be mapped onto a trivial semigroup, which of course is an idempotent.

Idempotent-consistent semigroups are defined by the property that each idempotent in a homomorphic image of a semigroup has an idempotent preimage. In a way this property is another formulation for the well known Lallement's lemma. Idempotent-consistent semigroups were studied by P. M. Higgins, P. M. Edwards, P. M. Edwards, P. M. Higgins and S. J. L. Kopamu, S. Bogdanović, H. Mitsch, S. J. L. Kopamu and S. Bogdanović, Ž. Popović and M. Ćirić.

Here on an arbitrary semigroup we introduce a system of congruence relations and using them we give a new version of the proof of Lallement's lemma. The results presented in this section are generalizations of results obtained by the above mentioned authors.

6.1 k-Archimedean Semigroups

Let $k \in \mathbf{Z}^+$ be a fix integer. A semigroup S is *k-nil* if $a^k = 0$ for every $a \in S$. This notion was introduced by T. Tamura in [17]. A semigroup S is *nilpotent* if $S^n = \{0\}$, for some $n \in \mathbf{Z}^+$. All finite nil-semigroups are nilpotent. An ideal extension S of a semigroup I is a *k-nil-extension* of I if S/I is a k-nil-semigroup.

In the following table we introduce the notations for some new classes of semigroups.

Notation	*Class of semigroups*	*Definition*
$k\mathcal{R}$	k-regular	$(\forall a \in S)\ a^k \in a^k S a^k$
$\mathcal{L}k\mathcal{R}$	left k-regular	$(\forall a \in S)\ a^k \in S a^{k+1}$
$\mathcal{R}k\mathcal{R}$	right k-regular	$(\forall a \in S)\ a^k \in a^{k+1} S$
$\mathcal{C}k\mathcal{R}$	completely k-regular	$(\forall a \in S)\ a^k \in a^{k+1} S a^{k+1}$
$\mathcal{I}k\mathcal{R}$	intra k-regular	$(\forall a \in S)\ a^k \in S a^{2k} S$
$k\mathcal{A}$	k-Archimedean	$(\forall a, b \in S)\ a^k \in S^1 b S^1$
$\mathcal{L}k\mathcal{A}$	left k-Archimedean	$(\forall a, b \in S)\ a^k \in S^1 b$
$\mathcal{R}k\mathcal{A}$	right k-Archimedean	$(\forall a, b \in S)\ a^k \in b S^1$
$\mathcal{T}k\mathcal{A}$	t-k-Archimedean	$(\forall a, b \in S)\ a^k \in b S^1 \cap S^1 b$

Semigroups from the class $k\mathcal{R}$ were introduced by K. S. Harinath in [2]. The other types of semigroups were introduce by S. Bogdanović, Ž. Popović and M. Ćirić in [1] for the first time.

We give here one very simple example.

Example 6.1 Let S be a semigroup defined by Cayley's table

$$\begin{array}{c|ccccc}
 & e & a & b & c & d \\ \hline
e & e & a & b & c & d \\
a & a & b & b & c & d \\
b & b & b & b & c & d \\
c & c & c & c & d & b \\
d & d & d & d & b & c
\end{array}.$$

It is easy to see that the subsemigroup $\{a, b, c, d\}$ of S is t-2-Archimedean. Also, $a \notin Reg(S)$, i.e. S is not regular. Since $a^2 = a^3 \in Reg(S)$ and $\{e, b^2, c^2, d^2\} \subseteq Reg(S)$, then S is a 2-regular semigroup.

Based on the following lemmas we describe the structure of k-Archimedean, left k-Archimedean and k-regular and Archimedean semigroups.

Lemma 6.1 *Let $k \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) $S \in k\mathcal{A}$;
(ii) $S \in \mathcal{A} \cap \mathcal{I}k\mathcal{R}$;
(iii) *S is a k-nil-extension of a simple semigroup.*

Proof. (i)⇒(ii) This implication follows immediately.

(ii)⇒(iii) Based on Theorem 3.14, S is a nil-extension of a simple semigroup I. Let $a \in S - I$, $b \in I$. Then $a^k = xa^{2k}y$, for some $x, y \in S$, whence

$$a^k = x^k a^k (a^k y)^k \in x^k a^k SbS \subseteq SbS.$$

Thus, S is a k-nil-extension of a simple semigroup I.

(iii)⇒(i) This implication follows immediately. □

Lemma 6.2 *Let $k \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) $S \in \mathcal{L}k\mathcal{A}$;
(ii) $S \in \mathcal{L}\mathcal{A} \cap \mathcal{L}k\mathcal{R}$;
(iii) *S is a k-nil-extension of a left simple semigroup.*

Lemma 6.3 *Let $k \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) $S \in k\mathcal{R} \cap \mathcal{A}$;
(ii) $S \in \pi\mathcal{R} \cap k\mathcal{A}$;
(iii) *S is a k-nil-extension of a simple regular semigroup.*

Proof. (i)⇒(ii) Let $a, b \in S$, then $a^k = a^k x a^k$, for some $x \in S$. Since S is Archimedean, then for $a^k x$ and b we have that $a^k x \in SbS$, whence

$$a^k = a^k x a^k \in SbSa^k \subseteq SbS.$$

Hence, $a^k \in SbS$, i.e. S is k-Archimedean.

(ii)$\Rightarrow$(iii) Based on Lemma 6.1, S is a k-nil-extension of a simple semigroup and based on Theorem 3.15, S is a k-nil-extension of a simple regular semigroup.

(iii)$\Rightarrow$(i) Let S be a k-nil-extension of a simple regular semigroup I. Assume $a \in S$, then $a^k \in I$. So, $a^k \in Reg(S)$. Clearly, S is an Archimedean semigroup. $\square$

Let $k \in \mathbf{Z}^+$ be a fix integer. A semigroup S is a *k-group* if S is k-regular and if it has only one idempotent. By means of the following theorem we describe the structure of the k-group.

Theorem 6.1 *Let $k \in \mathbf{Z}^+$. The following conditions on a semigroup S are equivalent:*

(i) $S \in \mathcal{T}k\mathcal{A}$;
(ii) $S \in \mathcal{T}\mathcal{A} \cap \mathcal{C}k\mathcal{R}$;
(iii) *S is a k-group;*
(iv) *S is a k-nil-extension of a group;*
(v) $(\forall a, b \in S)\ a^k \in bSb$.

Proof. (i)$\Rightarrow$(ii) Let S be a t-k-Archimedean semigroup. Then S is both left k-Archimedean and right k-Archimedean. So, based on Lemma 6.2 and its dual, we have that S is t-Archimedean and both left k-regular and right k-regular. Thus, it is evident that S is t-Archimedean and a completely k-regular semigroup. Hence, (ii) holds.

(ii)$\Rightarrow$(iii) Let (ii) hold. Then it is clear that S is k-regular and that S contains idempotent elements. Assume $e, f \in E(S)$. Since S is t-Archimedean, then $e = fx$ and $f = ye$, for some $x, y \in S^1$. So, we obtain that $e = fx = f(fx) = fe = (ye)e = ye = f$. Hence, S has only one idempotent element. Thus, S is a k-group.

(iii)$\Rightarrow$(iv) Let (iii) hold. It is clear that S is a π-group. So, based on Theorem 3.18, S is a nil-extension of a group G. Assume $a \in S - G$. Then $a^n \in G$, for some $n \in \mathbf{Z}^+$. Now, we make a distinction between two cases. If $k \geq n$, then $a^k = a^n a^{k-n} \in GS \subset G$, i.e. S is a k-nil-extension of a group G. If $k < n$, then since S is k-regular and since S has only one idempotent, from $a^k = a^k x a^k$, for some $x \in S$, and from $a^k x = x a^k \in E(S)$, we obtain that $a^k = a^{ik} x$, for every $i \in \mathbf{Z}^+$. Assume $j \in \mathbf{Z}^+$ such that $n < jk$. Then we have that $a^k = a^{jk} x = a^n a^{jk-n} x \in GS \subseteq G$, whence S is a k-nil-extension of a group G. Thus, (iv) holds.

(iv)$\Rightarrow$(v) Let S be a k-nil-extension of a group G. Assume $a, b \in S$. Then $a^k \in G$, whence $ba^k, a^kb \in G$ and since G is a group, then we have that $a^k \in ba^kGa^kb \subseteq ba^kSa^kb \subseteq bSb$.

(v)$\Rightarrow$(i) If (v) holds, then it is evident that S is a t-k-Archimedean semigroup. $\square$

By means of the following theorem we describe the structure of left k-Archimedean semigroups.

Theorem 6.2 *Let $k \in \mathbf{Z}^+$. The following conditions on a semigroup S are equivalent:*

(i) *$S \in \mathcal{L}k\mathcal{A}$ and it has an idempotent;*
(ii) *$S \in k\mathcal{R}$ and $E(S)$ is a left zero band;*
(iii) *S is a k-nil-extension of a left group;*
(iv) *$(\forall a, b \in S)$ $a^k \in a^kSa^kb$.*

Proof. (i)$\Rightarrow$(ii) Let S be a left k-Archimedean semigroup and let $e \in E(S)$. Assume $a \in S$. Then $e \in S^1a$ and $a^k \in S^1e$. Since $a^k = xe$, for some $x \in S^1$, then $a^ke = (xe)e = xe = a^k$. Also, since S is left k-Archimedean and $e, a^k \in S$, then $e = e^k \in S^1a^k$. Thus, $a^k = a^ke \in a^kS^1a^k \subseteq a^kSa^k$, for all $a \in S$, i.e. S is k-regular. Now, assume $f, g \in E(S)$. Then $f \in S^1g$, i.e. $f = yg$, for some $y \in S^1$. Hence $fg = (yg)g = yg = f$. Therefore, $E(S)$ is a left zero band.

(ii)$\Rightarrow$(i) Let S be k-regular and let $E(S)$ be a left zero band. According to Theorem 3.17, S has an idempotent. Assume $a, b \in S$. Then $a^k = a^kxa^k$ and $b^k = b^kyb^k$, for some $x, y \in S$. Let $e = xa^k$ and $f = yb^k$. Then $e^2 = ee = xa^kxa^k = xa^k = e$ and $f^2 = ff = yb^kyb^k = yb^k = f$, i.e. $e, f \in E(S)$. Since $E(S)$ is a left zero band then $ef = e$, i.e. $xa^kyb^k = xa^k$. Thus, we obtain that $a^k = a^kxa^k = a^kxa^kyb^k \in Sb$, for every $a, b \in S$, i.e. S is left k-Archimedean. Therefore, (i) holds.

(i)$\Rightarrow$(iii) Let S be a left k-Archimedean semigroup and let $e \in E(S)$. Based on Lemma 6.2, S is a k-nil-extension of a left simple semigroup K. Then $e = e^k \in K$ and based on Theorem 3.7, K is a left group. Thus, (iii) holds.

(iii)$\Rightarrow$(iv) Let S be a k-nil-extension of a left group K. Assume $a, b \in S$. Then $a^k \in K$, whence $a^kb \in K$ and based on Theorem 3.7 we obtain that $a^k \in a^kKa^kb \subseteq a^kSa^kb$. Thus, (iv) holds.

(iv)⇒(i) If (iv) holds then it is evident that S is a left k-Archimedean semigroup. Since from (iv) it immediately follows that S is a k-regular, then based on Theorem 3.17, S has an idempotent. □

References

S. Bogdanović and Ž. Popović [1], [2], [3]; S. Bogdanović, Ž. Popović and M. Ćirić [1]; K. S. Harinatah [2]; T. Tamura [17].

6.2 Bands of $\mathcal{J}_k$-simple Semigroups

Recall that by $\mathcal{J}$, $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ we denote Green's equivalences on a semigroup S. Here we define a new radical ϱ_k, $k \in \mathbf{Z}^+$ by

$$(a, b) \in \varrho_k \Leftrightarrow (a^k, b^k) \in \varrho.$$

It is clear that

$$\varrho_k \subseteq T(\varrho) \subseteq R(\varrho).$$

If $\varrho \in \{\mathcal{J}, \mathcal{L}, \mathcal{R}, \mathcal{D}, \mathcal{H}\}$, then it is easy to see that ϱ_k, $k \in \mathbf{Z}^+$ is an equivalence relation. So, in this case these equivalences are very similar to Green's equivalences and they can be considered its generalizations. The conditions under which the relations $R(\varrho)$ and $T(\varrho)$ are transitive (i.e. are equivalences) have been discussed by L. N. Shevrin in [4], by S. Bogdanović and M. Ćirić in [19], [21] and by S. Bogdanović, M. Ćirić and Ž. Popović in [1].

We start with a few lemmas in which we give some general characteristics of band congruences on an arbitrary semigroup.

Lemma 6.4 *Let ξ be a congruence relation on a semigroup S. Then $R(\xi) = \xi$ if and only if ξ is a band congruence on S.*

Proof. Let $R(\xi) = \xi$. Since ξ is reflexive, then for every $a \in S$ we have that

$$a^2 \,\xi\, a^2 \Leftrightarrow (a^1)^2 \,\xi\, (a^2)^1 \Leftrightarrow a \,R(\xi)\, a^2 \Leftrightarrow a \,\xi\, a^2.$$

Thus, ξ is a band congruence on S.

Conversely, let ξ be a band congruence on a semigroup S. Since the inclusion $\xi \subseteq R(\xi)$ always holds, then it remains for us to prove the opposite inclusion. Also, since ξ is a band congruence on S, then we have that

$$(\forall a \in S)(\forall k \in \mathbf{Z}^+) \;\; a\,\xi\,a^k.$$

Now assume $a, b \in S$ such that $a\,R(\xi)\,b$. Then $a^i\,\xi\,b^j$, for some $i, j \in \mathbf{Z}^+$, and based on the previously stated, we have that $a\,\xi\,a^i\,\xi\,b^j\,\xi\,b$. Thus $a\,\xi\,b$. Therefore, $R(\xi) \subseteq \xi$, i.e. $R(\xi) = \xi$. □

Lemma 6.5 *Let ξ be an equivalence relation on a semigroup S. Then the following conditions are equivalent:*

(i) *ξ is a band congruence;*
(ii) *$\xi = \xi^\flat = R(\xi)$;*
(iii) *$\xi = R(\xi)$ and ξ is a congruence on S.*

Proof. (i)⇔(ii) This equivalence follows from Lemmas 6.4 and 5.1.
(i)⇔(iii) This equivalence follows from Lemma 6.4. □

Let $k \in \mathbf{Z}^+$ be a fix integer. On S we define the following relations by

$$(a, b) \in \mathcal{J}_k \;\Leftrightarrow\; (a^k, b^k) \in \mathcal{J};$$

$$(a, b) \in \mathcal{J}_k^\flat \;\Leftrightarrow\; (\forall x, y \in S^1)\; (xay, xby) \in \mathcal{J}_k.$$

It is easy to verify that $\mathcal{J}_k$ is an equivalence relation on a semigroup S. But $R(\mathcal{J})$ and $T(\mathcal{J})$ are not equivalences (see L. N. Shevrin [4]).

A semigroup S is $\mathcal{J}_k$-*simple* if

$$(\forall a, b \in S)\; (a, b) \in \mathcal{J}_k.$$

It is clear that a semigroup S is $\mathcal{J}_k$-simple if and only if S is k-Archimedean. In the remainder of our study there is no distinction between these notions.

Example 6.2 It is not difficult to verify that on the semigroup S, as shown in the table in Example 6.1, we have that the relation

$$\mathcal{J} = \mathcal{J}_1 = \{(e,e),(a,a),(b,b),(c,c),(d,d),(b,c),(c,b),(b,d),(d,b),(c,d),(d,c)\},$$

is an equivalence and it is not a band congruence, since $(a, a^2) = (a, b) \notin \mathcal{J}$. Further, the relation

$$\mathcal{J}_2 = \{(e,e),(a,a),(b,b),(c,c),(d,d),(a,b),(a,c),(a,d),(b,a),(b,c),(b,d),$$
$$(c,a),(c,b),(c,d),(d,a),(d,b),(d,c)\},$$

is a band congruence on S, and S is a band of 2-Archimedean semigroups.

Example 6.3 Let $S = T^e$ be the semigroup T with an identity adjoined, where T is from the Example of T. Tamura in [17]. It is clear that S is a band (two-element chain) of two semigroups $\{e\}$ and T, and then the corresponding band congruence is $\mathcal{J}_2$.

The following lemma holds.

Lemma 6.6 *Let S be a semigroup and let $k \in \mathbf{Z}^+$. If $S \in k\mathcal{A} \circ \mathcal{RB}$, then $S \in k\mathcal{A}$.*

Proof. Let S be a rectangular band I of k-Archimedean semigroups S_i, $i \in I$. Assume $a, b \in S$, then there exist $i, j \in I$ such that $aba \in S_iS_jS_i \subseteq S_{iji} \subseteq S_i$. Thus $a, aba \in S_i$, whence $a^k \in S_iabaS_i \subseteq SbS$. Hence, S is k-Archimedean. □

Based on the following result we describe the structure of a semigroup which can be decomposed into a band (semilattice) of $\mathcal{J}_k$-simple semigroups.

Theorem 6.3 *Let $k \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) $\mathcal{J}_k$ *is a band congruence;*
(ii) $\mathcal{J}_k = \mathcal{J}_k^\flat = R(\mathcal{J}_k)$*;*
(iii) $S \in k\mathcal{A} \circ \mathcal{B}$*;*
(iv) $S \in k\mathcal{A} \circ \mathcal{S}$*;*
(v) $\mathcal{J}_k^\flat$ *is a band congruence;*
(vi) $(\forall a \in S)(\forall x, y \in S^1)\ xay\, \mathcal{J}_k\, xa^2y$*;*
(vii) $\mathcal{J}_k^\flat = R(\mathcal{J}_k^\flat)$*;*
(viii) $S \in \mathcal{A} \circ \mathcal{S} \cap \mathcal{I}k\mathcal{R}$*.*

Proof. (i)$\Leftrightarrow$(ii) This follows from Lemma 6.5.

(ii)$\Rightarrow$(iii) For all $a \in S$, $x, y \in S^1$, by (ii) we have that $xay \,\mathcal{J}_k\, xa^2y$. From this and based on Theorem 5.1 we have that S is a semilattice of Archimedean semigroups. Also, since $a \,\mathcal{J}_k\, a^2$ implies $a^k \,\mathcal{J}\, a^{2k}$, for every $a \in S$, then S is intra k-regular. Thus, based on Lemma 6.1, S is a semilattice of k-Archimedean semigroups. Thus, (iii) holds.

(iii)$\Rightarrow$(i) Let S be a semilattice Y of $\mathcal{J}_k$-simple semigroups S_α, $\alpha \in Y$. Assume $a, b, c \in S$, then $a \in S_\alpha$, $b \in S_\beta$ and $c \in S_\gamma$, for some $\alpha, \beta, \gamma \in Y$. Let $(a, b) \in \mathcal{J}_k$, then $(a^k, b^k) \in \mathcal{J}$, whence $\alpha = \beta$, i.e. $a, b \in S_\alpha$. Further, $ac, bc, ca, cb \in S_{\alpha\gamma}$. Hence, $ac \,\mathcal{J}_k\, bc$ and $ca \,\mathcal{J}_k, cb$, i.e. $\mathcal{J}_k$ is a congruence. Since $a, a^2 \in S_\alpha$, $\alpha \in Y$, we then have that $a \,\mathcal{J}_k\, a^2$, i.e. $\mathcal{J}_k$ is a band congruence on S.

(iii)$\Leftrightarrow$(iv) This equivalence follows from Theorem 5.33 and Lemma 6.6.

(iii)$\Rightarrow$(vi) Let S be a band Y of $\mathcal{J}_k$-simple semigroups S_α, $\alpha \in Y$. Assume $a \in S$ and $x, y \in S^1$. Then $xay, xa^2y \in S_\alpha$, for some $\alpha \in Y$. Since S_α, $\alpha \in Y$ is an $\mathcal{J}_k$-simple semigroup then $xay \,\mathcal{J}_k\, xa^2y$. Thus, (vi) holds.

(vi)$\Rightarrow$(iii) This implication is the same as (ii)$\Rightarrow$(iii).

(vi)$\Leftrightarrow$(v) This equivalence follows from Lemma 5.2.

(v)$\Leftrightarrow$(vii) This equivalence follows from Lemma 6.4.

(i)$\Leftrightarrow$(viii) This equivalence is the same as the equivalence (i)$\Leftrightarrow$(iii). $\square$

Theorem 6.4 *Let $k \in \mathbf{Z}^+$. A semigroup S is a semilattice of k-Archimedean semigroups if and only if*

$$(\forall a, b \in S)\ (ab)^k \in Sa^2S \quad \& \quad S \in \mathcal{I}k\mathcal{R}.$$

Proof. Let S be a semilattice Y of k-Archimedean semigroups S_α, $\alpha \in Y$. For $a, b \in S$ there exists $\alpha \in Y$ such that $ab, a^2b \in S_\alpha$, whence $(ab)^k \in S_\alpha a^2 b S_\alpha \subseteq Sa^2S$. Based on Theorem 6.3 we have that $S \in \mathcal{I}k\mathcal{R}$.

Conversely, from the first condition of Theorem 5.3 we have that S is a semilattice of Archimedean semigroups and since S is intra k-regular we have from Theorem 6.3 that the assertion follows. $\square$

T. Tamura [15] proved that the class of all semigroups which are semilattices of Archimedean semigroups is not subsemigroup closed. Based on the following theorem we determine the greatest subsemigroup closed subclass of the class of all semigroups which are semilattices of k-Archimedean semigroups.

Theorem 6.5 *Let $k \in \mathbf{Z}^+$. Then $k\mathcal{A} \circ \mathcal{S}$ is a subsemigroup closed if and only if*

$$(\forall a, b \in S)\ (ab)^k \in \langle a, b\rangle a^2 \langle a, b\rangle \quad \& \quad a^k \in \langle a, b\rangle a^{2k} \langle a, b\rangle.$$

Proof. Assume $a, b \in S$ and $T = \langle a, b\rangle$. Since T is a semilattice of k-Archimedean semigroups then based on Theorem 6.4 we obtain

$$(ab)^k \in Ta^2T = \langle a, b\rangle a^2 \langle a, b\rangle,$$

and

$$a^k \in Ta^{2k}T = \langle a, b\rangle a^{2k} \langle a, b\rangle.$$

Conversely, let T be an arbitrary subsemigroup of S. Assume $a, b \in T$. Based on the hypothesis we have that

$$(ab)^k \in \langle a, b\rangle a^2 \langle a, b\rangle \subseteq Ta^2T,$$

so based on Theorem 5.1, T is a semilattice of Archimedean semigroups. Also, according to the second part of hypothesis we have that

$$a^k \in \langle a, b\rangle a^{2k} \langle a, b\rangle \subseteq Ta^{2k}T,$$

thus T is an intra k-regular semigroup. Therefore, based on Theorem 6.4, T is a semilattice of k-Archimedean semigroups. $\square$

Let $k \in \mathbf{Z}^+$ be a fixed positive integer and let a and b be elements of a semigroup S. Then:

$$a \uparrow_k b \Leftrightarrow b^k \in \langle a, b\rangle a \langle a, b\rangle.$$

A semigroup S is *hereditary k-Archimedean* if $a \uparrow_k b$, for all $a, b \in S$. The class of all hereditary k-Archimedean semigroups we denote by $\mathbf{Her}(k\mathcal{A})$.

Theorem 6.6 *Let $k \in \mathbf{Z}^+$. Then $S \in \mathbf{Her}(k\mathcal{A})$ if and only if every one of its subsemigroups is k-Archimedean.*

Proof. Let S be a hereditary k-Archimedean semigroup and let T be a subsemigroup of S. Assume $a, b \in T$, then $\langle a, b\rangle \subseteq T$, and also based on the hypothesis we have that

$$b^k \in \langle a, b\rangle b \langle a, b\rangle \subseteq TaT.$$

Thus, T is k-Archimedean.

Conversely, assume $a, b \in S$. Then $a, b \in \langle a, b\rangle$ and since $\langle a, b\rangle$ is k-Archimedean, we obtain that

$$b^k \in \langle a, b\rangle a \langle a, b\rangle.$$

Thus, S is hereditary k-Archimedean. $\square$

Based on the following theorem we describe the semilettices of hereditary k-Archimedean semigroups which are subsemigroup closed.

Theorem 6.7 *Let $k \in \mathbf{Z}^+$. The following conditions on a semigroup S are equivalent:*

(i) $S \in \mathbf{Her}(k\mathcal{A}) \circ \mathcal{S}$;
(ii) $(\forall a, b \in S)\ a \longrightarrow b \Rightarrow a^2 \uparrow_k b$;
(iii) $(\forall a, b \in S)\ a \longrightarrow c\ \&\ b \longrightarrow c \Rightarrow ab \uparrow_k c$;
(iv) $(\forall a, b, c \in S)\ a \longrightarrow b\ \&\ b \longrightarrow c \Rightarrow a \uparrow_k c$;
(v) *the class* $\mathbf{Her}(k\mathcal{A}) \circ \mathcal{S}$ *is subsemigroup closed.*

Proof. (i)⇒(ii) Let S be a semilattice Y of hereditary k-Archimedean semigroups S_α, $\alpha \in Y$. Let $a, b \in S$, such that $a \longrightarrow b$. Then $b, a^2b \in S_\alpha$, for some $\alpha \in Y$ and based on the hypothesis we have that

$$b^k \in \langle b, a^2b\rangle a^2b \langle b, a^2b\rangle \subseteq \langle a^2, b\rangle a^2 \langle a^2, b\rangle,$$

i.e. $a^2 \uparrow_k b$. So, (ii) holds.

(ii)⇒(i) Based on Theorem 5.3, S is a semilattice Y of Archimedean semigroups S_α, $\alpha \in Y$. Assume $a, b \in S_\alpha$, $\alpha \in Y$. Then $a \longrightarrow b$ and from (ii) we have that $a^2 \uparrow_k b$, whence

$$b^k \in \langle a^2, b\rangle a^2 \langle a^2, b\rangle \subseteq \langle a, b\rangle a \langle a, b\rangle.$$

Hence, $a \uparrow_k b$ in S_α, i.e. S_α, $\alpha \in Y$ is a k-Archimedean semigroup.

(ii)⇒(iii) Assume $a, b, c \in S$ such that $a \longrightarrow c$ and $b \longrightarrow c$. Then based on (i)⇔(ii) and Theorem 5.1 and Theorem 4.5, for $n = 1$, we have that $ab \longrightarrow c$. Now, based on the hypothesis we have that $(ab)^2 \uparrow_k c$, whence $ab \uparrow_k c$.

(iii)⇒(iv) Based on Theorem 4.5, for $n = 1$, we have that $\longrightarrow$ is transitive. Assume $a, b, c \in S$ such that $a \longrightarrow b$ and $b \longrightarrow c$. Then $a^2 \uparrow_k c$, whence $a \uparrow_k c$.

(iv)⇒(i) Since $\longrightarrow$ is transitive, then based on Theorem 4.5, for $n = 1$, we have that S is a semilattice Y of Archimedean semigroups S_α, $\alpha \in Y$. Let $a, b \in S_\alpha$, $\alpha \in Y$. Then $a \longrightarrow b$ and $b \longrightarrow b$ and from (iv) we have that $a \uparrow_k b$. Hence, (i) holds.

(ii)⇒(v) Let T be a subsemigroup of S and let $a, b \in T$ such that $a \longrightarrow b$ in T. By (ii), $a^2 \uparrow_k b$, i.e.

$$b^k \in \langle a^2, b\rangle a^2 \langle a^2, b\rangle \subseteq Ta^2T.$$

Hence, $a^2 \uparrow_k b$ in T. Based on (i)⇔(ii) we have that T is a semilattice of hereditary k-Archimedean semigroups.

(v)⇒(i) This implication is obvious. □

References

S. Bogdanović and M. Ćirić [10]; S. Bogdanović and Ž. Popović [1], [2]; S. Bogdanović, Ž. Popović and M. Ćirić [1], [5], [6]; A. H. Cliford [5]; M. S. Putcha [3]; T. Tamura [15].

6.3 Bands of $\mathcal{L}_k$-simple Semigroups

Let $k \in \mathbf{Z}^+$ be a fix integer. Let $\mathcal{L}$ be a Green's relation on a semigroup S. On S we define the following relations by

$$(a, b) \in \mathcal{L}_k \;\Leftrightarrow\; (a^k, b^k) \in \mathcal{L};$$

$$(a, b) \in \mathcal{L}_k^\flat \;\Leftrightarrow\; (\forall x, y \in S^1)\ (xay, xby) \in \mathcal{L}_k.$$

It is easy to verify that $\mathcal{L}_k$ is an equivalence relation on a semigroup S.

A semigroup S is $\mathcal{L}_k$-*simple* or left k-Archimedean, if $a\,\mathcal{L}_k\,b$, for all $a, b \in S$. It is clear that a $\mathcal{L}_k$-simple semigroup is left π-regular and left Archimedean.

Lemma 6.7 *Let S be a semigroup and let $k \in \mathbf{Z}^+$. If $S \in \mathcal{L}k\mathcal{R} \cap \mathcal{L}\mathcal{A} \circ \mathcal{B}$, then $S \in \mathcal{L}k\mathcal{A} \circ \mathcal{B}$.*

Proof. Let S be a band of left Archimedean semigroups S_α, $\alpha \in Y$. Since S_α, $\alpha \in Y$ is left Archimedean and left k-regular, then based on Lemma 6.2 S_α, $\alpha \in Y$ is left k-Archimedean, i.e. an $\mathcal{L}_k$-simple semigroup. Thus, S is a band of $\mathcal{L}_k$-simple semigroups. □

Based on the following theorem we describe the structure of a semigroup which can be decomposed into a band of $\mathcal{L}_k$-simple semigroups. Also, we should emphasize that a band of left k-Archimedean semigroups is not coincident with a semilattice of left k-Archimedean semigroups.

Theorem 6.8 *Let $k \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) $S \in \mathcal{L}k\mathcal{A} \circ \mathcal{B}$;
(ii) $(\forall a, b \in S)\ (ab\,\mathcal{L}_k\,ab^2 \ \wedge\ a\,\mathcal{L}_k\,a^2)$;
(iii) $\mathcal{L}_k^\flat$ *is a band congruence on S;*
(iv) $(\forall a \in S)(\forall x, y \in S^1)\ xay\,\mathcal{L}_k\,xa^2y$;
(v) $R(\mathcal{L}_k^\flat) = \mathcal{L}_k^\flat$;
(vi) $S \in \mathcal{L}\mathcal{A} \circ \mathcal{B} \cap \mathcal{L}k\mathcal{R}$.

Proof. (i)⇒(ii) Let S be a band Y of $\mathcal{L}_k$-simple semigroups S_α, $\alpha \in Y$. Then for every $a, b \in S$ we have that $a \in S_\alpha$, $b \in S_\beta$, for some $\alpha, \beta \in Y$, whence $ab, ab^2 \in S_\alpha S_\beta \subseteq S_{\alpha\beta}$. Thus $ab\,\mathcal{L}_k\,ab^2$. Also, $a, a^2 \in S_\alpha$, for every $\alpha \in Y$ and thus $a\,\mathcal{L}_k\,a^2$. Hence, (ii) holds.

(ii)⇒(i) Let $a, b \in S$. From (ii) it follows that $ab \overset{l}{\text{—}} ab^2$, whence based on Theorem 5.29 we have that S is a band Y of left Archimedean semigroups S_α, $\alpha \in Y$. From the second condition of the hypothesis we have that S is left k-regular. Based on Lemma 6.7 we have that S_α is left k-regular, for all $\alpha \in Y$. Finally, from Lemma 6.2 we obtain that S_α, $\alpha \in Y$, is a left k-Archimedean ($\mathcal{L}_k$-simple) semigroup.

(i)⇒(iv) Let S be a band Y of $\mathcal{L}_k$-simple semigroups S_α, $\alpha \in Y$. Assume $a \in S$ and $x, y \in S^1$. Then $xay, xa^2y \in S_\alpha$, for some $\alpha \in Y$. Since S_α, $\alpha \in Y$ is an $\mathcal{L}_k$-simple semigroup then $xay\,\mathcal{L}_k\,xa^2y$. Thus, (iv) holds.

(iv)⇒(i) Based on Theorem 5.29, S is a band Y of left Archimedean semigroups S_α, $\alpha \in Y$. From (iv) it follows that S is a left k-regular. Assume $a \in S$, then $a \in S_\alpha$, for some $\alpha \in Y$ and $a^k = xa^{2k}$, for some $x \in S_\beta$, $\beta \in Y$. Since $\alpha = \beta\alpha$, we have that $a^k = x^2a^ka^{2k} \in S_\alpha a^{2k}$. Hence, S_α, $\alpha \in Y$, is left k-regular and since it is left Archimedean, then based on Lemma 6.2 we have that S_α, $\alpha \in Y$, is a left k-Archimedean ($\mathcal{L}_k$-simple) semigroup.

(iv)⇔(iii) This equivalence is evident.

(iii)⇔(v) This equivalence immediately follows from Lemma 6.4.

(i)⇔(vi) This equivalence follows from Lemmas 6.2 and 6.7, and from Theorem 5.29. □

Theorem 6.9 *Let $k \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) *$\mathcal{L}_k$ is a band congruence on S;*
(ii) *$\mathcal{L}_k = \mathcal{L}_k^\flat = R(\mathcal{L}_k)$;*
(iii) *$R(\mathcal{L}_k) = \mathcal{L}_k$ and $\mathcal{L}_k$ is a congruence on S.*

Proof. (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) These equivalences follow from Lemma 6.5. □

Proposition 6.1 *Let $k \in \mathbf{Z}^+$. If $\mathcal{L}_k$ is a band congruence on a semigroup S, then $S \in \mathcal{L}k\mathcal{A} \circ \mathcal{B}$.*

Proof. Let $a, b \in A$, where A is an arbitrary $\mathcal{L}_k$-class of S. Then $a^2 \,\mathcal{L}_k\, b$, whence $a^{2k} \,\mathcal{L}\, b^k$, i.e. $b^k = xa^{2k}$, for some $x \in S^1$. Since $a \,\mathcal{L}_k\, a^2$, for every $a \in S$, then for every $i \in \mathbf{Z}^+$ we have that $a \,\mathcal{L}_k\, a^i$, for every $a \in S$, whence $xa \,\mathcal{L}_k\, xa^i$, i.e. $xa \,\mathcal{L}_k\, b^k$, so $xa \in A$, and therefore, $xa^k \in A$. Now, we have that

$$b^k = xa^{2k} = xa^k \cdot a^k \in Aa^k.$$

Similarly we prove that $a^k \in Ab^k$. Therefore, $\mathcal{L}_k$-class A of S is a $\mathcal{L}_k$-simple semigroup. Thus, S is a band of $\mathcal{L}_k$-simple semigroups. □

Based on the following theorem we describe the structure of a semigroup which can be decomposed into a semilattice of $\mathcal{L}_k$-simple semigroups.

Theorem 6.10 *Let $k \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) *$S \in \mathcal{L}k\mathcal{A} \circ \mathcal{S}$;*
(ii) *$\mathcal{L}_k$ is a semilattice congruence on S;*
(iii) *$S \in \mathcal{L}\mathcal{A} \circ \mathcal{S} \cap \mathcal{L}k\mathcal{R}$.*

Proof. (i)$\Rightarrow$(ii) Let S be a semilattice Y of $\mathcal{L}_k$-simple semigroups S_α, $\alpha \in Y$. Assume $a, b, c \in S$ such that $(a, b) \in \mathcal{L}_k$. Since $a \in S_\alpha$, $b \in S_\beta$, $c \in S_\gamma$, for some $\alpha, \beta, \gamma \in Y$, and since $a^k = xb^k$ and $b^k = ya^k$, for some $x \in S_\delta$, $y \in S_\varepsilon$, where $\delta, \varepsilon \in Y$, then we obtain that $\alpha = \delta\beta$ and $\beta = \varepsilon\alpha$. Based on this we have that $\alpha\beta = (\delta\beta)\beta = \delta\beta = \alpha$ and $\beta\alpha = (\varepsilon\alpha)\alpha = \varepsilon\alpha = \beta$. Since Y is a semilattice then it follows that $\alpha = \alpha\beta = \beta\alpha = \beta$. Thus $a, b \in S_\alpha$, $\alpha \in Y$. So, $ac, bc \in S_{\alpha\gamma}$, $\alpha, \gamma \in Y$, and since $S_{\alpha\gamma}$, $\alpha, \gamma \in Y$, is an $\mathcal{L}_k$-simple semigroup, then $(ac, bc) \in \mathcal{L}_k$. Similarly we prove that $(ca, cb) \in \mathcal{L}_k$.

Thus $\mathcal{L}_k$ is a congruence relation on S. Further, $a, a^2 \in S_\alpha$, $\alpha \in Y$ and S_α, $\alpha \in Y$, is $\mathcal{L}_k$-simple, then $(a, a^2) \in \mathcal{L}_k$, for every $a \in S$, whence $\mathcal{L}_k$ is a band congruence on S. Also, $ab, ba \in S_\alpha$, $\alpha \in Y$ and S_α, $\alpha \in Y$, is $\mathcal{L}_k$-simple, then $(ab, ba) \in \mathcal{L}_k$, for all $a, b \in S$, whence $\mathcal{L}_k$ is a semilattice congruence on S.

(ii)⇒(i) Let (ii) hold. Then S is a semilattice of $\mathcal{L}_k$-classes. Let $a, b \in A$, where A is an arbitrary $\mathcal{L}_k$-class of S. Then $a^2 \,\mathcal{L}_k\, b$, whence $a^{2k} \,\mathcal{L}\, b^k$, i.e. $b^k = xa^{2k}$, for some $x \in S^1$. Since $\mathcal{L}_k$ is a semilattice congruence, then $a \,\mathcal{L}_k\, a^2$, for every $a \in S$. Based on this, for every $i \in \mathbf{Z}^+$ we have that $a \,\mathcal{L}_k\, a^i$, for every $a \in S$, whence $xa \,\mathcal{L}_k\, xa^i$, i.e. $xa \,\mathcal{L}_k\, b^k$, so $xa \in A$, and therefore, $xa^k \in A$. Now, we have that

$$b^k = xa^{2k} = xa^k \cdot a^k \in Aa^k.$$

Similarly, we prove that $a^k \in Ab^k$. Therefore, the $\mathcal{L}_k$-class A of S is a $\mathcal{L}_k$-simple semigroup. Thus, S is a semilattice of $\mathcal{L}_k$-simple semigroups.

(i)⇒(iii) Let S be a semilattice Y of $\mathcal{L}_k$-simple semigroups S_α, $\alpha \in Y$. Assume $a, b \in S$. Then $ab, ba \in S_\alpha$, for some $\alpha \in Y$. Since S_α, $\alpha \in Y$, is $\mathcal{L}_k$-simple, then $(ab, ba) \in \mathcal{L}_k$, whence $(ab)^k \in S(ba)^k \subseteq Sa$, i.e. $a \xrightarrow{l} ab$. Then based on Theorem 5.9, S is a semilattice of left Archimedean semigroups. Also, $a, a^2 \in S_\alpha$, for some $\alpha \in Y$, and since S_α, $\alpha \in Y$, is $\mathcal{L}_k$-simple, then $(a, a^2) \in \mathcal{L}_k$, whence $a^k \in Sa^{2k} \subseteq Sa^{k+1}$, for every $a \in S$. Thus, S is a left k-regular semigroup. Therefore, (iii) holds.

(iii)⇒(i) This implication immediately follows from Lemma 6.2. □

References

S. Bogdanović and M. Ćirić [10]; S. Bogdanović, M. Ćirić and B. Novikov [1]; S. Bogdanović and Ž. Popović [1], [2]; S. Bogdanović, Ž. Popović and M. Ćirić [1], [6]; A. H. Cliford [5]; M. S. Putcha [3]; T. Tamura [15].

6.4 Bands of $\mathcal{H}_k$-simple Semigroups

Let $k \in \mathbf{Z}^+$ be a fix integer. Let $\mathcal{H}$ be a Green's relation on a semigroup S. On S we define the following relations by

$$(a, b) \in \mathcal{H}_k \;\Leftrightarrow\; (a^k, b^k) \in \mathcal{H};$$

$$(a,b) \in \mathcal{H}_k^\flat \;\Leftrightarrow\; (\forall x,y \in S^1)\ (xay, xby) \in \mathcal{H}_k.$$

It is easy to verify that $\mathcal{H}_k$ is an equivalence relation on a semigroup S. Also, it is evident that $\mathcal{H}_k = \mathcal{L}_k \cap \mathcal{R}_k$.

A semigroup S is $\mathcal{H}_k$-*simple* (t-k-Archimedean), if $a\,\mathcal{H}_k\,b$, for all $a,b \in S$. Also, it is easy to verify that a semigroup S is $\mathcal{H}_k$-simple if it is both $\mathcal{L}_k$-simple and $\mathcal{R}_k$-simple, and conversely.

Based on the following theorem we describe the structure of a semigroup which can be decomposed into a band of $\mathcal{H}_k$-simple semigroups.

Theorem 6.11 *Let $k \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) $S \in \mathcal{T}k\mathcal{A} \circ \mathcal{B}$;
(ii) $\mathcal{H}_k^\flat$ *is a band congruence on* S;
(iii) $(\forall a \in S)(\forall x,y \in S^1)\ xay\,\mathcal{H}_k\,xa^2y$;
(iv) $R(\mathcal{H}_k^\flat) = \mathcal{H}_k^\flat$;
(v) $S \in \mathcal{T}\mathcal{A} \circ \mathcal{B} \cap \mathcal{C}k\mathcal{R}$.

Proof. (i)$\Rightarrow$(iii) Let S be a band Y of $\mathcal{H}_k$-simple semigroups S_α, $\alpha \in Y$. Assume $a \in S$ and $x,y \in S^1$. Then $xay, xa^2y \in S_\alpha$, for some $\alpha \in Y$. Since S_α, $\alpha \in Y$ is an $\mathcal{H}_k$-simple semigroup then $xay\,\mathcal{H}_k\,xa^2y$. Thus, (iii) holds.

(iii)$\Rightarrow$(i) Let $a \in S$ and $x,y \in S^1$. From (iii) it follows that $xay \overset{t}{\longrightarrow} xa^2y$, whence based on Corollary 5.5, S is a band Y of t-Archimedean semigroups S_α, $\alpha \in Y$. Also, based on (iii) S is both left k-regular and right k-regular. Just like (iv)$\Rightarrow$(i) of Theorem 6.8 we prove that every band component S_α, $\alpha \in Y$, of S is left k-regular, and, dually, that S_α, $\alpha \in Y$, is right k-regular, i.e. S_α, $\alpha \in Y$, is completely k-regular. Thus, S_α, $\alpha \in Y$, is t-Archimedean and completely k-regular. So, based on Theorem 6.1, S_α, $\alpha \in Y$ is t-k-Archimedean. Therefore, S is a band of t-k-Archimedean ($\mathcal{H}_k$-simple) semigroups.

(iii)$\Leftrightarrow$(ii) This equivalence follows from Lemma 5.2.

(ii)$\Leftrightarrow$(iv) This equivalence follows from Lemma 6.4

(i)$\Leftrightarrow$(v) This equivalence follows from Theorem 6.1. □

Theorem 6.12 *Let $k \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) $\mathcal{H}_k$ *is a band congruence on* S;

(ii) $\mathcal{H}_k = \mathcal{H}_k^\flat = R(\mathcal{H}_k)$;
(iii) $R(\mathcal{H}_k) = \mathcal{H}_k$ *and* $\mathcal{H}_k$ *is a congruence on* S.

Proof. (i)⇔(ii)⇔(iii) These equivalences follow from Lemma 6.5. □

Theorem 6.13 *Let* $k \in \mathbf{Z}^+$. *Then the following conditions on a semigroup* S *are equivalent:*

(i) $S \in \mathcal{T}k\mathcal{A} \circ \mathcal{S}$;
(ii) $\mathcal{H}_k$ *is a semilattice congruence on* S;
(iii) $S \in \mathcal{T}\mathcal{A} \circ \mathcal{S} \cap \mathcal{C}k\mathcal{R}$.

Proof. These equivalences follow from Theorem 6.10 and its dual. □

References

S. Bogdanović and M. Ćirić [10]; S. Bogdanović and Ž. Popović [1], [2]; S. Bogdanović, Ž. Popović and M. Ćirić [1], [6]; A. H. Cliford [5]; M. S. Putcha [3]; T. Tamura [15].

6.5 Bands of η-simple Semigroups

Recall that a semigroup S is called *power-joined* if for each pair of elements $a, b \in S$ there exist $m, n \in \mathbf{Z}^+$ such that $a^m = b^n$. These semigroups were first considered by P. Abellanas [1], in 1965, for cancellative semigroups only, and D. B. Mc Alister [1], in 1968, who called them *rational* semigroups. Every power-joined semigroup is Archimedean. An element a of a semigroup S is *periodic* if there exist $m, n \in \mathbf{Z}^+$ such that $a^m = a^{m+n}$. A semigroup S is *periodic* if every one of its element is periodic.

On a semigroup S we define the following relations:

$$(a, b) \in \eta \;\Leftrightarrow\; (\exists i, j \in \mathbf{Z}^+)\; a^i = b^j,$$

$$(a, b) \in \eta^\flat \;\Leftrightarrow\; (\forall x, y \in S^1)\; (xay, xby) \in \eta.$$

It is easy to verify that η is an equivalence relation on a semigroup S.

A semigroup S is η-*simple* if

$$(\forall a, b \in S)\; (a, b) \in \eta.$$

These semigroups are well-known in the literature as power-joined semigroups.

The important result is the following lemma.

Lemma 6.8 *If ξ is a band congruence on a semigroup S, then $\xi \subseteq \eta$ if and only if every ξ-class of S is an η-simple semigroup.*

Proof. Let A be a ξ-class of S. Then A is a subsemigroup of S, since $a\,\xi\,a^2$, for all $a \in S$. Let $a, b \in A$, then $a\,\xi\,b$, whence $a\,\eta\,b$ in A.

Conversely, let $(a, b) \in \xi$, then $a^i = b^j$, for some $i, j \in \mathbf{Z}^+$, since a and b are in the same ξ-class A of S. Thus $(a, b) \in \eta$. Therefore, $\xi \subseteq \eta$. $\square$

By means of the following theorem we describe the structure of semigroups in which the relation η is a congruence relation. These semigroups have been treated by S. Bogdanović in a different way in [9].

Theorem 6.14 *The following conditions on a semigroup S are equivalent:*

(i) *S is a band of η-simple semigroups;*
(ii) *η is a (band) congruence on S;*
(iii) *$\eta^\flat$ is a band congruence on S;*
(iv) $(\forall a \in S)(\forall x, y \in S^1)$ $xay\,\eta\,xa^2y$;
(v) $R(\eta^\flat) = \eta^\flat$.

Proof. (i)⇒(ii) Let S be a band B of η-simple semigroups S_α, $\alpha \in B$. Assume $a, b, c \in S$ such that $a\,\eta\,b$. Then $a, b \in S_\alpha$ and $c \in S_\beta$, for some $\alpha, \beta \in B$. Also, $ac, bc \in S_\alpha S_\beta \subseteq S_{\alpha\beta}$, $\alpha, \beta \in B$ and since $S_{\alpha\beta}$, $\alpha, \beta \in B$, is η simple, then $ac\,\eta\,bc$. Similarly we prove that $ca\,\eta\,cb$. Thus η is a congruence relation on S. Furthermore, since $a, a^2 \in S_\alpha$, $\alpha \in B$ and S_α, $\alpha \in B$, is η-simple, then $a\,\eta\,a^2$, i.e. η is a band congruence on S.

(ii)⇒(i) Let (ii) hold. Then S is a band of η-classes. Since $\eta \subseteq \eta$, then based on Lemma 5.2 we have that every η-class is an η-simple semigroup. Thus S is a band of η-simple semigroups.

(i)⇒(iv) Let S be a band B of η-simple semigroups S_α, $\alpha \in B$. Assume $a \in S$ and $x, y \in S^1$. Then $xay, xa^2y \in S_\alpha$, for some $\alpha \in Y$. Since S_α, $\alpha \in Y$ is η-simple, then $xay\,\eta\,xa^2y$. Thus, (iv) holds.

(iv)⇒(iii) Let (iv) hold. Then by definition, for $\eta^\flat$ it is evident that $a\,\eta^\flat\,a^2$, for every $a \in S$. Thus $\eta^\flat$ is a band congruence.

(iii)⇒(i) Let $\eta^\flat$ be a band congruence on S, then S is a band of $\eta^\flat$-classes. Since $\eta^\flat$ is the greatest congruence on S contained in η, then based on Lemma 5.2 we have that every $\eta^\flat$-class is an η-simple semigroup. Thus S is a band of η-simple semigroups.

(iii)⇔(v) This equivalence immediately follows from Lemma 6.4. □

Let $m, n \in \mathbf{Z}^+$. On a semigroup S we define a relation $\overline{\eta}_{(m,n)}$ by

$$(a, b) \in \overline{\eta}_{(m,n)} \;\Leftrightarrow\; (\forall x \in S^m)(\forall y \in S^n)\ (xay, xby) \in \eta.$$

If instead of η we assume the equality relation, then we obtain the relation which was discussed by S. J. L. Kopamu in [1] and [2]. The main characteristic of the previous defined relation gives the following theorem.

Theorem 6.15 *Let S be a semigroup and let $m, n \in \mathbf{Z}^+$. Then $\overline{\eta}_{(m,n)}$ is a congruence relation on S.*

Proof. It is clear that $\overline{\eta}_{(m,n)}$ is reflexive and symmetric. Assume that $a\,\overline{\eta}_{(m,n)}\,b$ and $b\,\overline{\eta}_{(m,n)}\,c$. Then for every $x \in S^m$ and $y \in S^n$ there exist $k, l, s, t \in \mathbf{Z}^+$ such that

$$(xay)^k = (xby)^l \quad \text{and} \quad (xby)^s = (xcy)^t$$

whence

$$(xay)^{ks} = (xby)^{ls} = (xcy)^{tl},$$

i.e. $xay\,\eta\,xcy$. Thus $\overline{\eta}_{(m,n)}$ is transitive and therefore it is a congruence on S. □

The complete description of $\overline{\eta}_{(m,n)}$ congruence, for $\eta = \text{—}$, was given by S. Bogdanović, Ž. Popović and M. Ćirić in [5].

Theorem 6.16 *Let $m, n \in \mathbf{Z}^+$. The following conditions on a semigroup S are equivalent:*

(i) *$\overline{\eta}_{(m,n)}$ is a band congruence on S;*
(ii) *$(\forall x \in S^m)(\forall y \in S^n)(\forall a \in S)\ xay\,\eta\,xa^2y$;*
(iii) *$\eta \subseteq \overline{\eta}_{(m,n)}$;*
(iv) *$R(\overline{\eta}_{(m,n)}) = \overline{\eta}_{(m,n)}$.*

Proof. (i)$\Rightarrow$(ii) This implication follows immediately.

(ii)$\Rightarrow$(iii) Assume that $a\,\eta\,b$. Then $a^i = b^j$, for some $i, j \in \mathbf{Z}^+$. Then for every $x \in S^m$, $y \in S^n$ and $i, j \in \mathbf{Z}^+$ we have that

$$xay\,\eta\,xa^2y\,\eta\,xa^iy = xb^jy\,\eta\,xby.$$

Since η is transitive, we have that $a\,\overline{\eta}_{(m,n)}\,b$. Thus $\eta \subseteq \overline{\eta}_{(m,n)}$.

(iii)$\Rightarrow$(i) Since $a\,\eta\,a^2$, for every $a \in S$, then we have that $a\,\overline{\eta}_{(m,n)}\,a^2$, for every $a \in S$, i.e. $\overline{\eta}_{(m,n)}$ is a band congruence.

(i)$\Leftrightarrow$(iv) This equivalence immediately follows from Lemma 6.4. $\square$

Proposition 6.2 *Let $m, n \in \mathbf{Z}^+$. If $\overline{\eta}_{(m,n)}$ is a band congruence on a semigroup S, then S is a band of $\overline{\eta}_{(m,n)}$-simple semigroups.*

Proof. Let A be an $\overline{\eta}_{(m,n)}$-class of a semigroup S. Assume $a, b \in A$, then $a\,\overline{\eta}_{(m,n)}\,b$ in S, i.e. $xay\,\eta\,xby$, for every $x \in S^m$ and every $y \in S^n$, whence we have that for every $x \in A^m$ and every $y \in A^n$ is $xay\,\eta\,xby$, i.e. $a\,\overline{\eta}_{(m,n)}\,b$ in A. Thus A is $\overline{\eta}_{(m,n)}$-simple. $\square$

Let $k \in \mathbf{Z}^+$ be a fix integer. On a semigroup S we define the following relations by

$$(a, b) \in \eta_k \;\Leftrightarrow\; a^k = b^k;$$

$$(a, b) \in \eta_k^\flat \;\Leftrightarrow\; (\forall x, y \in S^1)\ (xay, xby) \in \eta_k.$$

It is easy to verify that η_k is an equivalence relation on a semigroup S.

A semigroup S is η_k*-simple* if

$$(\forall a, b \in S)\ (a, b) \in \eta_k.$$

These semigroups are periodic.

Lemma 6.9 *Let $k \in \mathbf{Z}^+$. If ξ is a band congruence on a semigroup S, then $\xi \subseteq \eta_k$ if and only if every ξ-class of S is an η_k-simple semigroup.*

Proof. Let A be a ξ-class of S. Then A is a subsemigroup of S, since $a\,\xi\,a^2$, for all $a \in S$. Let $a, b \in A$, then $a\,\xi\,b$, whence $a\,\eta_k\,b$ in A.

Conversely, let $(a, b) \in \xi$. Since a and b are in the some ξ-class A of S and since A is η_k-simple,then $(a, b) \in \eta_k$. Therefore, $\xi \subseteq \eta_k$. $\square$

By means of the following theorem we give the structural characterization of bands of η_k-simple semigroups.

Theorem 6.17 *Let $k \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is a band of η_k-simple semigroups;*

(ii) $(\forall a, b \in S)\ (\,(ab)^k = (a^k b^k)^k \ \wedge\ a^k = a^{2k}\,)$;

(iii) *η_k is a band congruence on S;*

(iv) *$\eta_k^\flat$ is a band congruence on S;*

(v) $(\forall a \in S)(\forall x, y \in S^1)\ xay\,\eta_k\,xa^2y$;

(vi) *$R(\eta_k) = \eta_k$ and η_k is a congruence on S;*

(vii) $R(\eta_k^\flat) = \eta_k^\flat$.

Proof. (i)⇒(ii) Let S be a band Y of η_k-simple semigroups S_α, $\alpha \in Y$. For every $a, b \in S$ we have that $a \in S_\alpha$, $b \in S_\beta$, for some $\alpha, \beta \in Y$, whence $ab, a^k b^k \in S_{\alpha\beta}$ and so $(ab)^k = (a^k b^k)^k$. Clearly, $a^k = a^{2k}$.

(ii)⇒(iii) It is clear that η_k is an equivalence. Let $a\eta_k b$ and $x \in S$, then $a^k = b^k$ and based on the hypothesis we have that $(ax)^k = (a^k x^k)^k = (b^k x^k)^k = (bx)^k$, i.e. $ax\,\eta_k\,bx$. Similarly, $xa\,\eta_k, xb$. Thus η_k is a congruence relation on S, and since $a^k = a^{2k}$ we have that η_k is a band congruence on S.

(iii)⇒(i) Let η_k be a band congruence and A be an η_k-class of S. Assume $a, b \in A$, then $a\,\eta_k\,b$ in A and thus A is an η_k-simple semigroup. Therefore, S is a band of η_k-simple semigroups.

(i)⇒(v) Let S be a band Y of η_k-simple semigroups S_α, $\alpha \in Y$. Assume $a \in S$ and $x, y \in S^1$. Then $xay, xa^2y \in S_\alpha$, for some $\alpha \in Y$. Since S_α, $\alpha \in Y$ is an η_k-simple semigroup then $xay\,\eta_k\,xa^2y$. Thus, (v) holds.

(v)⇒(iv) Let (v) hold. Then based on the definition for $\eta_k^\flat$ it is evident that $a\,\eta_k^\flat\,a^2$, for every $a \in S$. Thus $\eta_k^\flat$ is a band congruence.

(iv)⇒(i) Let $\eta_k^\flat$ be a band congruence on S, then S is a band of $\eta_k^\flat$-classes. Since $\eta_k^\flat$ is the largest congruence on S contained in η_k, then based on Lemma 6.9 we have that every $\eta_k^\flat$-class is η_k-simple semigroup. Thus S is a band of η_k-simple semigroups.

(iii)⇔(vi) and (iv)⇔(vii) These equivalences immediately follows from Lemma 6.4. □

Let $k, m, n \in \mathbf{Z}^+$. On a semigroup S we define a relation $\overline{\eta}_{(k;m,n)}$ by

$$(a, b) \in \overline{\eta}_{(k;m,n)} \;\Leftrightarrow\; (\forall x \in S^m)(\forall y \in S^n)\ (xay, xby) \in \eta_k.$$

The following lemma holds.

Lemma 6.10 *Let S be a semigroup and let $k, m, n \in \mathbf{Z}^+$, then $\overline{\eta}_{(k;m,n)}$ is a congruence relation on S.*

Proof. It is clear that $\overline{\eta}_{(k;m,n)}$ is reflexive and symmetric. Assume $a, b, c \in S$ such that $a\,\overline{\eta}_{(k;m,n)}\,b$ and $b\,\overline{\eta}_{(k;m,n)}\,c$. Then for every $x \in S^m$ and every $y \in S^n$ we obtain that

$$(xay)^k = (xby)^k \quad \text{and} \quad (xby)^k = (xcy)^k$$

whence

$$(xay)^k = (xcy)^k,$$

i.e. $xay\,\eta_{(k;m,n)}\,xcy$. Thus $\overline{\eta}_{(k;m,n)}$ is transitive and therefore it is a congruence on S. □

Theorem 6.18 *Let $k, m, n \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) *$\overline{\eta}_{(k;m,n)}$ is a band congruence on S;*
(ii) *$(\forall x \in S^m)(\forall y \in S^n)(\forall a \in S)\ xay\,\eta_k\,xa^2y$;*
(iii) *$R(\overline{\eta}_{(k;m,n)}) = \overline{\eta}_{(k;m,n)}$.*

Proof. (i)⇔(ii) This equivalence is evident.

(i)⇔(iii) This equivalence immediately follows from Lemma 6.4. □

Proposition 6.3 *Let $k, m, n \in \mathbf{Z}^+$. If $\overline{\eta}_{(k;m,n)}$ is a band congruence on a semigroup S, then $\eta_k \subseteq \overline{\eta}_{(k;m,n)}$.*

Proof. Since $\overline{\eta}_{(k;m,n)}$ is a band congruence on S, then $xay\,\eta_k\,xa^iy$, for every $i \in \mathbf{Z}^+$ and for all $x \in S^m$, $y \in S^n$, $a \in S$. Assume $a, b \in S$ such that $a\,\eta_k\,b$. Then $a^k = b^k$. Thus for every $x \in S^m$ and $y \in S^n$ we have that

$$xay\,\eta_k\,xa^ky = xb^ky\,\eta_k\,xby.$$

Since η_k is transitive, we obtain that $a\,\overline{\eta}_{(k;m,n)}\,b$. Thus $\eta_k \subseteq \overline{\eta}_{(k;m,n)}$. □

Furthermore, based on the previously defined relations on a semigroup S, we define the following relations:

$$(a,b) \in \tau \;\Leftrightarrow\; (\exists k \in \mathbf{Z}^+)\; (a,b) \in \eta_k;$$

$$(a,b) \in \tau^\flat \;\Leftrightarrow\; (\forall x,y \in S^1)\; (xay, xby) \in \tau.$$

It is easy to verify that the relation τ is an equivalence on a semigroup S.

A semigroup S is τ-*simple* if

$$(\forall a,b \in S)\; (a,b) \in \tau.$$

By means of the following theorem we describe the structure of bands of τ-simple semigroups. S. Bogdanović in [10] gave some other characterizations of these semigroups.

Theorem 6.19 *The following conditions on a semigroup S are equivalent:*

(i) *S is a band of τ-simple semigroups;*
(ii) *τ is a band congruence on S;*
(iii) *$\tau^\flat$ is a band congruence on S;*
(iv) $(\forall a \in S)(\forall x,y \in S^1)\; xay\,\tau\, xa^2y$;
(v) *$R(\tau) = \tau$ and τ is a congruence on S;*
(vi) $R(\tau^\flat) = \tau^\flat$.

Proof. (i)⇒(ii) Let S be a band Y of τ-simple semigroups S_α, $\alpha \in Y$. Assume $a,b,c \in S$ such that $a\,\tau\, b$. Then $a^k = b^k$, for some $k \in \mathbf{Z}^+$. So, then $a,b \in S_\alpha$ and $c \in S_\beta$, for some $\alpha,\beta \in Y$. Thus $ac, bc \in S_\alpha S_\beta \subseteq S_{\alpha\beta}$, $\alpha,\beta \in Y$ and since $S_{\alpha\beta}$, $\alpha,\beta \in Y$, is τ-simple, then $ac\,\tau\, bc$. Similarly, $ca\,\tau\, cb$. Hence, τ is a congruence relation on S. Furthermore, since $a, a^2 \in S_\alpha$, $\alpha \in Y$ and S_α, $\alpha \in Y$, is τ-simple, then $a\,\tau\, a^2$, i.e. τ is a band congruence on S.

(ii)⇒(i) Let (ii) hold. Then S is a band of τ-classes. Let A be a τ-class of S. Then A is a subsemigroup of S. Assume $a,b \in A$, then $a\,\tau\, b$ in A and A is a τ-simple. Therefore, S is a band of τ-simple semigroups.

(i)⇒(iv) Let S be a band Y of τ-simple semigroups S_α, $\alpha \in Y$. Assume $a \in S$ and $x,y \in S^1$. Then $xay, xa^2y \in S_\alpha$, for some $\alpha \in Y$. Since S_α, $\alpha \in Y$ is τ-simple then $xay\,\tau\, xa^2y$. Thus, (iv) holds.

(iv)⇒(iii) This implication follows immediately.

(iii)$\Rightarrow$(i) Let (iii) hold. Then S is a band of $\tau^\flat$-classes. Let A be an arbitrary $\tau^\flat$-class of S. Then A is a subsemigroup of S. Assume $a, b \in A$, then $a\,\tau^\flat\, b$ in A and since $\tau^\flat \subseteq \tau$, then $a\,\tau\, b$ in A. Thus A is a τ-simple. Therefore, S is a band of τ-simple semigroups.

(ii)$\Leftrightarrow$(v) and (iii)$\Leftrightarrow$(vi) These equivalences follow from Lemma 6.4. $\square$

Let $m, n \in \mathbf{Z}^+$. On a semigroup S we define a relation $\overline{\tau}_{(m,n)}$ by

$$(a, b) \in \overline{\tau}_{(m,n)} \;\Leftrightarrow\; (\forall x \in S^m)(\forall y \in S^n)\ (xay, xby) \in \tau.$$

The following theorem holds.

Theorem 6.20 *Let S be a semigroup and let $m, n \in \mathbf{Z}^+$. Then $\overline{\tau}_{(m,n)}$ is a congruence relation on S.*

Proof. It is clear that $\overline{\tau}_{(m,n)}$ is reflexive and symmetric. Assume $a, b, c \in S$ such that $a\,\overline{\tau}_{(m,n)}\, b$ and $b\,\overline{\tau}_{(m,n)}\, c$. Then for every $x \in S^m$ and $y \in S^n$ there exist $k, l \in \mathbf{Z}^+$ such that

$$(xay)^k = (xby)^k \quad \text{and} \quad (xby)^l = (xcy)^l$$

whence

$$(xay)^{kl} = (xby)^{kl} = (xby)^{lk} = (xcy)^{lk}.$$

So, we have that $xay\,\eta_{lk}\, xcy$, i.e. $xay\,\tau\, xcy$. Thus $\overline{\tau}_{(m,n)}$ is transitive and therefore it is a congruence on S. $\square$

Theorem 6.21 *Let $m, n \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) *$\overline{\tau}_{(m,n)}$ is a band congruence on S;*
(ii) *$(\forall x \in S^m)(\forall y \in S^n)(\forall a \in S)\ xay\,\tau\, xa^2y$;*
(iii) *$R(\overline{\tau}_{(m,n)}) = \overline{\tau}_{(m,n)}$.*

Proof. (i)$\Leftrightarrow$(ii) This equivalence follows immediately.

(i)$\Leftrightarrow$(iii) This equivalence immediately follows from Lemma 6.4. $\square$

Proposition 6.4 *Let $m, n \in \mathbf{Z}^+$. If $\overline{\tau}_{(m,n)}$ is a band congruence on a semigroup S, then $\tau \subseteq \overline{\tau}_{(m,n)}$.*

Proof. Since $\overline{\tau}_{(m,n)}$ is a band congruence on S, then $xay\,\tau\,xa^iy$, for every $i \in \mathbf{Z}^+$ and for all $x \in S^m$, $y \in S^n$, $a \in S$. Assume $a, b \in S$ such that $a\,\tau\,b$. Then $a^k = b^k$, for some $k \in \mathbf{Z}^+$. Thus for every $x \in S^m$, $y \in S^n$ and $k \in \mathbf{Z}^+$ we have that

$$xay\,\tau\,xa^ky = xb^ky\,\tau\,xby.$$

Since τ is transitive, then $a\,\overline{\tau}_{(m,n)}\,b$. Therefore $\tau \subseteq \overline{\tau}_{(m,n)}$. □

References

P. Abellanas [1]; S. Bogdanović [9], [10]; S. Bogdanović and Ž. Popović [1], [2]; S. Bogdanović, Ž. Popović and M. Ćirić [1], [4], [5], [6]; K. Iseki [1]; S. J. L. Kopamu [1], [2]; D. B. Mc Alister [1]; T. Nordahl [3]; T. Tamura [10].

6.6 On Lallement's Lemma

Lallement's lemma for regular semigroups says that if ρ is a congruence on a regular semigroup S and $a\rho$ is an idempotent in the quotient S/ρ then $a\,\rho\,e$ for some idempotent $e \in S$. We can formulate this property in terms of homomorphic images. The property featured in the conclusion of the lemma therefore has merited a name of its own and so we say that a congruence relation ξ on a semigroup S is *idempotent-consistent* (or *idempotent-surjective*) if for every idempotent class $a\xi$ of S/ξ there exists $e \in E(S)$ such that $a\xi e$. This property is found in the conclusion of the well known Lallement's lemma. A semigroup is *idempotent-consistent* if all of its congruences enjoy this property. These notions were explored by P. M. Higgins [1], [4], P. M. Edwards [1], P. M. Edwards, P. M. Higgins and S. J. L. Kopamu [1], S. Bogdanović [14], and H. Mitsch [3], [4].

The class of regular semigroups certainly does not exhaust the class of idempotent-consistent semigroups as it is a simple matter to check that every periodic semigroup, or more generally every (completely) π-regular, is idempotent-consistent. A generalization of Lallement's lemma that includes all the cases mentioned so far was provided by P. M. Edwards [1], where it was shown that the class of idempotent-consistent semigroups includes all π-regular semigroups.

Although the class of π-regular semigroups does not contain all idempotent-consistent semigroups, any idempotent-consistent and weakly commutative semigroup is also π-regular. A semigroup S is *weakly commutative* if for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $(ab)^n \in bSa$.

The converse implication does not generally hold, however, not all idempotent-consistent semigroups are π-regular. This was first shown by S. J. L. Kopamu [2] through the introduction of the class of structurally regular semigroups which are defined using a special family of congruences. Some characterizations of semigroups, based on congruences which are more general then ones introduced by S. J. L. Kopamu in [1], are considered by S. Bogdanović, Ž. Popović and M. Ćirić in [1] and [4]. S. J. L. Kopamu proved that Lallement's lemma holds for the class of all structurally regular semigroups.

Let ξ be a congruence relation on a semigroup S. An element $a \in S$ is *ξ-regular* if there exists $b \in S$ such that $a\xi = (aba)\xi$. A semigroup S is *ξ-regular* if all its elements are ξ-regular, i.e. if S/ξ is a regular semigroup. An element $b \in S$ is such that $a\xi = (aba)\xi$ and $b\xi = (bab)\xi$ is a *ξ-inverse* of the element a.

Lemma 6.11 *For any ξ-regular element of a semigroup S there exists a ξ-inverse element.*

Proof. Let $a, b \in S$ such that $a\xi = (aba)\xi$, then it is easy to verify that

$$(a\xi)\,(bab)\xi\,(a\xi) = a\xi \text{ and } (bab)\xi\,(a\xi)\,(bab)\xi = (bab)\xi.$$

Thus $a\xi$ and $(bab)\xi$ are mutually inverses. $\square$

Before we present the main result of this section, we give the following helpful lemma.

Lemma 6.12 *Let $m, n \in \mathbf{Z}^+$. An element $a \in S$ is $\overline{\tau}_{(m,n)}$-regular if and only if a has a $\overline{\tau}_{(m,n)}$-inverse element.*

Proof. Let $a \in S$ is $\overline{\tau}_{(m,n)}$-regular. Then $a\overline{\tau}_{(m,n)}axa$, for some $x \in S$, i.e. $(uav)^p = (uaxav)^p$, for every $u \in S^m$ and every $v \in S^n$ and some $p \in \mathbf{Z}^+$. Put $x' = xax$. Since $xav \in S^{n+2} \subseteq S^n$ then we have that $(uax'av)^q = (uaxaxav)^q = (uaxav)^q$, for some $q \in \mathbf{Z}^+$. Hence,

$$(uax'av)^{qp} = ((uax'av)^q)^p = ((uaxav)^q)^p = ((uaxav)^p)^q = ((uav)^p)^q = (uav)^{pq}.$$

Thus, $a\overline{\tau}_{(m,n)}ax'a$. Since $ux \in S^{m+1} \subseteq S^m$ and $xaxv \in S^{n+3} \subseteq S^n$ we have that $(ux'ax'v)^k = (uxaxaxaxv)^k = (uxaxaxv)^k$, for some $k \in \mathbf{Z}^+$. Also, since $ux \in S^m$ and $xv \in S^n$ we have and $(uxaxaxv)^t = (uxaxv)^t = (ux'v)^t$, for some $t \in \mathbf{Z}^+$. Hence,

$$(ux'ax'v)^{kt} = ((ux'ax'v)^k)^t = ((uxaxaxv)^k)^t =$$

$$= ((uxaxaxv)^t)^k = ((ux'v)^t)^k = (ux'v)^{tk}.$$

Thus, $x'ax'\overline{\tau}_{(m,n)}x'$. Therefore, x' is a $\overline{\tau}_{(m,n)}$-inverse of a.

The converse follows immediately. □

By means of the following theorem we give a new result of the type of Lallement's lemma. This theorem is a generalization of the results obtained by P. M. Edwards, P. M. Higgins and S. J. L. Kopamu [1].

Theorem 6.22 *Let $m, n \in \mathbf{Z}^+$. Let ϕ be a homomorphism from a semigroup S onto T and let $S/\overline{\tau}_{(m,n)}$ be a π-regular semigroup. Then for every $f \in E(T)$ there exists $e \in E(S)$ such that $e\phi = f$.*

Proof. Since ϕ is surjective, then there exists $a \in S$ such that $a\phi = f$. Assume $a^{2(mn)} \in S$, then based on Lemma 6.12 we have that

$$(1)\quad a^{2(mn)i}\overline{\tau}_{(m,n)} = (a^{2(mn)i}xa^{2(mn)i})\overline{\tau}_{(m,n)}, \quad x\overline{\tau}_{(m,n)} = (xa^{2(mn)i}x)\overline{\tau}_{(m,n)},$$

for some $x \in S$ and $i \in \mathbf{Z}^+$, whence

$$\begin{aligned} ((a^{(mn)i}xa^{(mn)i})^j)^2 &= ((a^{(mn)i}xa^{(mn)i})^2)^j = (a^{(mn)i}(xa^{2(mn)i}x)a^{(mn)i})^j \\ &= (a^{(ni)m}(xa^{2(mn)i}x)a^{(mi)n})^j = (a^{(ni)m}xa^{(mi)n})^j \\ &= (a^{(mn)i}xa^{(mn)i})^j \in E(S), \end{aligned}$$

for some $j \in \mathbf{Z}^+$. Let $e = (a^{(mn)i}xa^{(mn)i})^j$, then

$$\begin{aligned} e\phi &= ((a^{(mn)i}xa^{(mn)i})^j)\phi = ((a^{(mn)i}\phi)(x\phi)(a^{(mn)i}\phi))^j \\ &= ((a\phi)^{(mn)i}(x\phi)(a\phi)^{(mn)i})^j \\ &= ((a\phi)^{3(mn)i}(x\phi)(a\phi)^{3(mn)i})^j, \qquad \text{(since } (a\phi)^2 = a\phi = f = f^2) \\ &= ((a^{3(mn)i}\phi)(x\phi)(a^{3(mn)i}\phi))^j = ((a^{3(mn)i}xa^{3(mn)i})^j)\phi \\ &= ((a^{(mn)i}(a^{2(mn)i}xa^{2(mn)i})a^{(mn)i})^j)\phi. \end{aligned}$$

Based on (1) there exists $k \in \mathbf{Z}^+$ such that

$$(a^{(mn)i}(a^{2(mn)i}xa^{2(mn)i})a^{(mn)i})^k = (a^{(ni)m}(a^{2(mn)i}xa^{2(mn)i})a^{(mi)n})^k =$$

$$(a^{(ni)m}a^{2(mn)i}a^{(mi)n})^k = a^{4(mn)ik}.$$

Finally,

$$\begin{aligned}(e\phi)^k &= (((a^{(mn)i}(a^{2(mn)i}xa^{2(mn)i})a^{(mn)i})^j)\phi)^k \\ &= (((a^{i(mn)}(a^{2(mn)i}xa^{2(mn)i})a^{(mn)i})^k)\phi)^j = ((a^{4(mn)ik})\phi)^j \\ &= (a^{4(mn)ikj})\phi = (a\phi)^{4(mn)ikj} = f^{4(mn)ikj} = f.\end{aligned}$$

Therefore, $e\phi = f$. □

The proof of the following corollary immediately follows from the previous theorem.

Corollary 6.1 *Let $m, n \in \mathbf{Z}^+$. Every semigroup S for which $S/\overline{\tau}_{(m,n)}$ is π-regular is idempotent-consistent.*

The relation $\overline{\tau}_{(1,1)}$ we simply denote by $\overline{\tau}$. On a semigroup S this relation is defined by

$$(a,b) \in \overline{\tau} \;\Leftrightarrow\; (\forall x, y \in S)\; (xay, xby) \in \tau.$$

According to Theorem 6.20 it is evident that:

Corollary 6.2 *Let S be an arbitrary semigroup, then $\overline{\tau}$ is a congruence relation on S.*

For $m = 1$ and $n = 1$ based on the previously obtained results we give the following corollaries which refer to the relation $\overline{\tau}$.

Corollary 6.3 *An element $a \in S$ is $\overline{\tau}$-regular if and only if a has a $\overline{\tau}$-inverse element.*

Corollary 6.4 *Let ϕ be a homomorphism from a semigroup S onto T and let $S/\overline{\tau}$ be a π-regular semigroup. Then for every $f \in E(T)$ there exists $e \in E(S)$ such that $e\phi = f$.*

Corollary 6.5 *Every semigroup S for which $S/\overline{\tau}$ is π-regular is idempotent-consistent.*

References

S. Bogdanović [10], [14]; S. Bogdanović and Ž. Popović [1]; S. Bogdanović, Ž. Popović and M. Ćirić [1], [3], [4]; P. M. Edwards [1]; P. M. Edwards, P. M. Higgins and S. J. L. Kopamu [1]; P. M. Higgins [1], [4]; S. J. L. Kopamu [1], [2]; H. Mitsch [3], [4].

Chapter 7

Semilattices of Completely Archimedean Semigroups

This chapter continues the previous study in a natural way. Here we give the theory of semilattice decompositions of completely π-regular semigroups on Archimedean components, i.e. we are going to talk about a completely π-regular semigroups whose every regular element is a group element. These semigroups were introduced by L. N. Shevrin, in 1977, but the first proof concerning them was given by M. L. Veronesi, in 1984. These semigroups will be described structurally in Theorem 7.4. Semilattices of completely Archimedean semigroups are of special interest. In the first section we will present the results regarding the semilattice of simple semigroups which are regular. Various structures and characterizations of these semigroups represent the results obtained by S. Bogdanović and M. Ćirić, in 1993, which will be shown in Theorem 7.6. In the last section of this chapter we will present the results regarding bands and semilattices of nil-extensions of groups.

7.1 Semilattices of Nil-extensions of Simple Regular Semigroups

The main purpose of this section is to study semigroups which are π-regular and are decomposable into a semilattice of Archimedean semigroups.

We characterize them as semilattices of nil-extensions of simple regular semigroups.

The following theorem is a helpful result for future work.

Theorem 7.1 *Let $E(S) \neq \varnothing$. Then the following conditions on a semigroup S are equivalent:*

(i) $(\forall a \in S)(\forall e \in E(S))\ a|e \Rightarrow a^2|e$;

(ii) $(\forall a, b \in S)(\forall e \in S)\ a|e \ \&\ b|e \ \Rightarrow\ ab|e$;

(iii) $(\forall e, f, g \in E(S))\ e|g \ \&\ f|g \ \Rightarrow\ ef|g$.

Proof. (i)⇒(ii) Let $a, b \in S$ and let $e \in E(S)$ such that $a|e$ and $b|e$, i.e. let $e = xay = ubv$, for some $x, y, u, v \in S^1$. Based on the hypothesis we have

$$e = ee = ubvxay \in S(bvxa)^2S \subseteq SabS.$$

Hence, $ab|e$.

(ii)⇒(i) and (ii)⇒(iii) This is obvious.

(iii)⇒(ii) Let $a, b \in S$ and let $e \in E(S)$ such that $a|e$ and $b|e$. Then $e = xay = ubv$ for some $x, y, u, v \in S^1$. It is easy to verify that $(yxa)^2, (bvu)^2 \in E(S)$ and $e = xa(yxa)^2y = u(bvu)^2bv$. Now, based on (iii) we obtain that $(yxa)^2\ (bvu)^2|e$ whence $ab|e$. □

Now, we are ready to prove the main result of this section.

Theorem 7.2 *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of nil-extensions of simple regular semigroups;*

(ii) *S is a band of nil-extensions of simple regular semigroups;*

(iii) *S is π-regular and S is a semilattice of Archimedean semigroups;*

(iv) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n \in (ab)^n Sa^2S(ab)^n$;

(v) *S is π-regular and* $(\forall a \in S)(\forall e \in E(S))\ a|e \Rightarrow a^2|e$;

(vi) *S is π-regular and* $(\forall a, b \in S)(\forall e \in E(S))\ a|e \ \&\ b|e \Rightarrow ab|e$;

(vii) *S is π-regular and* $(\forall e, f, g \in E(S))\ e|g \ \&\ f|g \Rightarrow ef|g$;

(viii) *S is π-regular and in every homomorphic image with zero of S, the set of all nilpotent elements is an ideal;*

(ix) *S is π-regular and every $\mathcal{J}$-class of S containing an idempotent is a subsemigroup of S;*

(x) *S is intra-π-regular and every $\mathcal{J}$-class of S containing an intra-regular element is a regular subsemigroup of S;*

(xi) *S is a semilattice of nil-extensions of simple semigroups and* Intra(S)= Reg(S).

Proof. (i)$\Leftrightarrow$(ii) This is evident.

(i)$\Rightarrow$(iii) Clearly, S is π-regular and based on Theorem 3.15 S is a semilattice of Archimedean semigroups.

(iii)$\Rightarrow$(i) Let S be a π-regular semigroup which is a semilattice Y of Archimedean semigroups S_α, $\alpha \in Y$. Then S_α is also π-regular and based on Theorem 3.15 we have that S_α is a nil-extension of a simple regular semigroup, for every $\alpha \in Y$.

(i)$\Rightarrow$(iv) Let S be a semilattice Y of nil-extensions of simple regular semigroups S_α, $\alpha \in Y$. Let $a, b \in S$. Then $ab, a^2b \in S_\alpha$, for some $\alpha \in Y$. Now according to Theorem 3.15 there exists $n \in \mathbf{Z}^+$ such that:

$$(ab)^n \in (ab)^n S_\alpha a^2 b S_\alpha (ab)^n \subseteq (ab)^n S a^2 S (ab)^n.$$

(iv)$\Rightarrow$(iii) Let $a, b \in S$. Then there exists $n \in \mathbf{Z}^+$ such that

$$(ab)^n \in (ab)^n S a^2 S (ab)^n \subseteq S a^2 S,$$

and based on Theorem 5.1, S is a semilattice of Archimedean semigroups. It is clear that S is π-regular.

(v)$\Leftrightarrow$(vi)$\Leftrightarrow$(vii) This follows from Theorem 7.1.

(iii)$\Rightarrow$(v) This follows form Theorem 5.1.

(v)$\Rightarrow$(iii) Let $a, b \in S$. Then $(ab)^n = (ab)^n x (ab)^n$, for some $x \in S$ and $n \in \mathbf{Z}^+$. Since $a \,|\, (ab)^n x$, we then have that $a^2 \,|\, (ab)^n x$, whence $(ab)^n = (ab)^n x (ab)^n \in S a^2 S$, and based on Theorem 5.1, S is a semilattice of Archimedean semigroups.

(iii)$\Leftrightarrow$(viii) This equivalence follows from Theorem 4.5, for $n = 1$.

(i)$\Rightarrow$(x) Let S be a semilattice Y of semigroups S_α, $\alpha \in Y$, and for each $\alpha \in Y$ let S_α be a nil-extension of a simple regular semigroup K_α. Based on Theorem 5.5, S is an intra π-regular semigroup and every $\mathcal{J}$-class containing an intra regular element is a subsemigroup of S. Let $a \in \mathrm{Intra}(S)$. Then $a = xa^2y$, for some $x, y \in S^1$, and $a \in S_\alpha$, for some $\alpha \in Y$, whence we have that $xa, ay \in S_\alpha$ and $a = (xa)^n a y^n$, for each $n \in \mathbf{Z}^+$. But $xa \in S_\alpha$ yields $(xa)^n \in K_\alpha$, for some $n \in \mathbf{Z}^+$, whence $a = (xa)^n a y^n \in K_\alpha S_\alpha \subseteq K_\alpha$.

This means that K_α is the $\mathcal{J}$-class of a, which completes the proof of the implication (i)⇒(x).

(x)⇒(iii) Let S be an intra π-regular semigroup whose every $\mathcal{J}$-class containing an intra regular element is a regular subsemigroup of S. According to Theorem 5.5, S is an intra π-regular semigroup and a semilattice of Archimedean semigroups. Let $a \in S$. Then there exists $n \in \mathbf{Z}^+$ such that $a^n \in \mathrm{Intra}(S)$. If we denote by J the $\mathcal{J}$-class of a^n, then J is a regular semigroup and we have that $a^n \in a^n J a^n \subseteq a^n S a^n$. Thus, S is a π-regular semigroup.

(iii)⇒(ix) Since S is a π-regular semigroup, then based on the proof of (iii)⇔(x) we have that each $\mathcal{J}$-class of S containing an idempotent is a regular subsemigroup.

(ix)⇒(iii) Let $a, b \in S$. Then there exist $x \in S$ and $n \in \mathbf{Z}^+$ such that $(ab)^n = (ab)^n x (ab)^n$ and $(ab)^n x, x(ab)^n \in E(S)$. It is also true that

$$(ab)^n = (ab)^n x (ab)^n = (ab)^n x (ab)^n x (ab)^n \in Sx(ab)^n S$$

and

$$x(ab)^n = x(ab)^n x(ab)^n \in S(ab)^n S.$$

Thus $(ab)^n \mathcal{J} x(ab)^n$, and in a similar way we show that $(ab)^n \mathcal{J} (ab)^n x \mathcal{J} (ab)^{2n}$. Therefore, $(ab)^n \in S(ab)^{2n} S \subseteq S(ba)^{n+1} S$ and $(ba)^{n+1} \in S(ab)^n S$, which implies $(ab)^n, (ba)^{n+1} \in J_{(ab)^n}$. Since the $\mathcal{J}$-class $J_{(ab)^n}$ contains an idempotent, then it is a subsemigroup of S. Now $(ba)^{n+1}(ab)^n \in J_{(ab)^n}$, whence

$$(ab)^n \in S(ba)^{n+1}(ab)^n S \subseteq Sa^2 S.$$

Based on Theorem 5.1, S is a semilattice of Archimedean semigroups.

(i)⇒(xi) Let S be a semilattice Y of semigroups S_α which are nil-extensions of simple regular semigroups K_α, $\alpha \in Y$. Consider an arbitrary $a \in \mathrm{Reg}(S)$. Then $a \in S_\alpha$, for some $\alpha \in Y$, and there exists $x \in S$ such that $a = axa$. Let $x \in S_\beta$, for some $\beta \in Y$. Then $\alpha = \alpha\beta = \beta\alpha$. Thus $xa \in S_\alpha$, and $xa \in E(S_\alpha) = E(K_\alpha)$, whence $(xa)x \in K_\alpha S_\beta \subseteq S_\alpha S_\beta \subseteq S_{\alpha\beta} = S_\alpha$. Now

$$a = a(xax)a \in aS_\alpha a$$

so

$$a \in \mathrm{Reg}(S_\alpha) \subseteq K_\alpha \subseteq \mathrm{Intra}(S).$$

Therefore

$$\mathrm{Reg}(S) \subseteq \mathrm{Intra}(S). \tag{1}$$

Conversely, let $a \in \mathrm{Intra}(S)$. Then there exists $\alpha \in Y$ such that $a \in S_\alpha$, and based on Lemma 2.7 we have that $a \in \mathrm{Intra}(S_\alpha)$, i.e. there exist $u, v \in S_\alpha$ such that

$$a = ua^2v = u^k a(av)^k,$$

for every $k \in \mathbf{Z}^+$. Since S_α is a nil-extension of a simple regular semigroup K_α, then there exists $n \in \mathbf{Z}^+$ such that $u^n, (av)^n \in K_\alpha$. Hence,

$$a = u^{n+1}a^2y(ay)^n \in K_\alpha a^2 K_\alpha \subseteq K_\alpha \subseteq \mathrm{Reg}(S).$$

Thus

$$\mathrm{Intra}(S) \subseteq \mathrm{Reg}(S). \tag{2}$$

Based on (1) and (2) we have that $\mathrm{Intra}(S) = \mathrm{Reg}(S)$.

(xi)$\Rightarrow$(i) Let S be a semilattice Y of semigroups S_α, $\alpha \in Y$, and for each $\alpha \in Y$, let S_α be a nil-extension of a simple semigroup K_α. For an arbitary $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in K_\alpha \subseteq \mathrm{Intra}(S) = \mathrm{Reg}(S)$. Thus, S is a π-regular semigroup, and using (i)$\Leftrightarrow$(iii) we have that S is a semlattice of nil-extensions of simple regular semigroups. $\square$

Later we will consider chains of nil-extensions of simple regular semigroups.

Theorem 7.3 *The following conditions on a semigroup S are equivalent:*

(i) *S is a chain of nil-extensions of simple regular semigroups;*
(ii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)$ $a^n \in a^nSabSa^n$ *or* $b^n \in b^nSabSb^n$;
(iii) *S is π-regular and $(\forall e, f \in E(S))$ $ef|e$ or $ef|f$;*
(iv) *S is π-regular and* $\mathrm{Reg}(S)$ *is a chain of simple regular semigroups.*

Proof. (i)$\Rightarrow$(ii) Let S be a chain Y of nil-extensions of simple regular semigroups S_α, $\alpha \in Y$. Let $a, b \in S$. Then $a \in S_\alpha$ and $b \in S_\beta$, for some $\alpha, \beta \in Y$. If $\alpha\beta = \alpha$ then $a, ab \in S_\alpha$, and based on Theorem 3.15, there exists $n \in \mathbf{Z}^+$ such that $a^n \in a^nSabSa^n$. In a similar way, from $\alpha\beta = \beta$ we obtain that $b^n \in b^nSabSb^n$, for some $n \in \mathbf{Z}^+$.

(ii)$\Rightarrow$(i) It is clear that S is π-regular. Let $a, b \in S$. Then, based on the hypothesis, there exists $n \in \mathbf{Z}^+$ such that $a^n \in SabS$ or $b^n \in SabS$. According to Theorem 5.6 we have that S is a chain of Archimedean semigroups. Since S is π-regular, then based on Theorem 7.2 we have that S is a chain of nil-extensions of simple regular semigroups.

(ii)⇒(iii) This is clear.

(iii)⇒(i) Let S be a π-regular semigroup and let $e, f, g \in E(S)$ such that $f|e$ and $g|e$. Then there exist $x, y, u, v \in S^1$ such that $e = xfy = ugv$, whence $(yxf)^2, (gvu)^2 \in E(S)$. Now we have that $(yxf)^2 \in S(yxf)^2(gvu)^2S$ or $(gvu)^2 \in S(yxf)^2(gvu)^2S$. If $(yxf)^2 \in S(yxf)^2(gvu)^2S$, then $(yxf)^2 \in SfgS$. Thus

$$e = eee = xfyxfyxfy = xf(yxf)^2y \in xfSfgSy \subseteq SfgS,$$

so $fg|e$ in S. If $(gvu)^2 \in S(yxf)^2(gvu)^2S$, then $fg|e$ in S. Now, based on Theorem 7.2, S is a semilattice Y of nil-extensions of simple regular semigroups S_α, $\alpha \in Y$.

Let $\alpha, \beta \in Y$ and $e, f \in E(S)$ be such that $e \in S_\alpha$, $f \in S_\beta$. If $ef|e$ in S, then $\alpha\beta = \alpha$, and if $ef|f$, then $\alpha\beta = \beta$. Therefore, Y is a chain and S is a chain of simple regular semigroups.

(i)⇒(iv) Let S be a chain Y of semigroups S_α, $\alpha \in Y$, and for $\alpha \in Y$, let S_α be a nil-extension of a simple regular semigroup K_α. Let $a, b \in \mathrm{Reg}(S)$. Then $a \in S_\alpha$, $b \in S_\beta$, for some $\alpha, \beta \in Y$. It is clear that $a \in K_\alpha$ and $b \in K_\beta$. Since Y is a chain, then $\alpha\beta = \alpha$ or $\alpha\beta = \alpha$. Suppose that $\alpha\beta = \alpha$. Then $ab \in S_\alpha$, whence $ab \in K_\alpha S_\alpha \subseteq K_\alpha$, i.e. $ab \in \mathrm{Reg}(S)$. Similarly, we prove that $\alpha\beta = \beta$ implies $ab \in \mathrm{Reg}(S)$. Hence, $\mathrm{Reg}(S)$ is a subsemigroup of S and clearly

$$\mathrm{Reg}(S) = \bigcup_{\alpha\in Y} \mathrm{Reg}(S_\alpha) = \bigcup_{\alpha\in Y} K_\alpha.$$

Therefore $\mathrm{Reg}(S)$ is a chain Y of simple regular semigroups K_α, $\alpha \in Y$.

(iv)⇒(iii) Let S be π-regular and let $\mathrm{Reg}(S)$ be a chain Y of simple regular semigroups K_α, $\alpha \in Y$. Consider arbitrary $e, f \in E(S)$. Then $e \in K_\alpha$ and $f \in K_\beta$, for some $\alpha, \beta \in Y$. Since Y is a chain, then $e, ef \in K_\alpha$ or $f, ef \in K_\beta$, whence $ef|e$ or $ef|f$. □

Exercises

1. A semigroup S is π-inverse and S is a semilattice of Archimedean semigroups if and only if S is a semilattice of nil-extensions of simple inverse semigroups.

References

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7.2 Uniformly π-regular Semigroups

In this section we will give some general structural characteristics of the semilattice of completely Archimedean semigroups, i.e. of *uniformly π-regular* semigroups which are defined as π-regular semigroups whose any regular element is completely regular, i.e. semigroups whose $\mathrm{Reg}(S) = \mathrm{Gr}(S)$. We remind the reader that semigroups $\mathbf{A}_2$ and $\mathbf{B}_2$, which we used in the following theorem, are defined by the presentations $\mathbf{A}_2 = \langle a, e \,|\, a^2 = 0,\, e^2 = e,\, aea = a,\, eae = e\rangle$ and $\mathbf{B}_2 = \langle a, b \,|\, a^2 = b^2 = 0,\, aba = a,\, bab = b\rangle$.

Theorem 7.4 *On a semigroup S the following conditions are equivalent:*

(i) *S is a semilattice of completely Archimedean semigroups;*

(ii) *S is a semilattice of Archimedean semigroups and completely π-regular;*

(iii) *S is uniformly π-regular;*

(iv) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n \in (ab)^n Sa(ab)^n$;

(iv') $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n \in (ab)^n bS(ab)^n$;

(v) *S is completely π-regular and every $\mathcal{D}$-class of S is its subsemigroup;*

(vi) *S is completely π-regular and between the factors of completely π-regular subsemigroups of S there are no $\mathbf{A}_2$ and $\mathbf{B}_2$ semigroups;*

(vii) *S is completely π-regular,* $\mathrm{Reg}(\langle E(S)\rangle) = \mathrm{Gr}(\langle E(S)\rangle)$ *and for all $e, f \in E(S)$, $f|e$ in S implies $f|e$ in $\langle E(S)\rangle$;*

(viii) *S is right π-regular and a semilattice of left completely Archimedean semigroups;*

(ix) *S is π-regular and a semilattice of left completely Archimedean semigroups;*

(x) *S is π-regular and every regular element of S is left regular;*

(xi) *S is π-regular and each $\mathcal{L}$-class of S containing an idempotent is a subsemigroup.*

Proof. (i)⇒(iv) Let S be a semilattice Y of completely Archimedean semigroups $S_\alpha, \alpha \in Y$. Assume $a, b \in S$. Then $ab, ba \in S_\alpha$, for some $\alpha \in Y$, so according to Theorem 3.16 we obtain that

$$(ab)^n \in (ab)^n S(ba)(ab)^n \subseteq (ab)^n Sa(ab)^n,$$

for some $n \in \mathbf{Z}^+$.

(iv)⇒(ii) From (iv) it immediately follows that S is completely π-regular. Assume $a, b \in S$. Based on (iv), $(ab)^n \in Sa^2S$, for some $n \in \mathbf{Z}^+$, so based on Theorem 5.1, S is a semilattice of Archimedean semigroups.

(ii)⇒(i) This follows from Lemma 2.7 and Theorem 3.16.

(i)⇒(iii) Let S be a semilattice Y of completely Archimedean semigroups S_α, $\alpha \in Y$. Assume $a \in \mathrm{Reg}(S)$. Then $a \in S_\alpha$, for some $\alpha \in Y$. For a there exists $x \in S_\beta$, $\beta \in Y$ such that $a = axa \in S_\alpha S_\beta S_\alpha \subseteq S_{\alpha\beta}$, so it follows that $\alpha\beta = \alpha$. Since $xax \in S_\alpha$, then based on Theorem 3.16 we obtain that $a \in \mathrm{Reg}(S_\alpha) = \mathrm{Gr}(S_\alpha) \subseteq \mathrm{Gr}(S)$. Whence, $\mathrm{Reg}(S) \subseteq \mathrm{Gr}(S) \subseteq \mathrm{Reg}(S)$, i.e. $\mathrm{Reg}(S) = \mathrm{Gr}(S)$. Thus, S is uniformly π-regular.

(iii)⇒(ii) From (iii) it immediately follows that S is completely π-regular. Assume $a, b \in S$. Then $(ab)^n \in G_e$, for some $n \in \mathbf{Z}^+$, $e \in E(S)$, so based on Theorem 1.8 it follows that $eab \in G_e$. Let x be an inverse of eab in the group G_e. Then $e = eabx = eabxe$, whence $ea = eabxea$. Thus, $ea \in \mathrm{Reg}(S) = \mathrm{Gr}(S)$, i.e. $ea = (ea)^2 y = (eae)(ay)$, for some $y \in S$. Now we have that $eae = eabxeae = (eae)(ay)(bx)(eae)$, so $eae \in \mathrm{Reg}(S) = \mathrm{Gr}(S)$, i.e. $eae \in G_f$, for some $f \in E(S)$. It is easy to see that $ef = fe = f$. On the other hand, $e = eabx = (eae)(ay)(bx) = f(eae)(ay)(bx)$, whence $fe = e$. Thus, $e = f$, i.e. $eae, eab \in G_e$, whence

$$ea^2be = (ea)(abe) = (ea)e(ab) = (eae)(eab) \in G_e.$$

Thus, $(ab)^n, ea^2be \in G_e$, whence

$$(ab)^n \in G_e ea^2be \subseteq Sa^2S,$$

so according to Theorem 5.1, S is semilattice of Archimedean semigroups.

(i)⇒(vi) Let S be a semilattice Y of completely Archimedean semigroups $S_\alpha, \alpha \in Y$. Assume a completely π-regular subsemigroup T of S. Then T is a semilattice Z of semigroups T_α, $\alpha \in Z$, where $Z = \{\alpha \in Y \mid T \cap S_\alpha \neq \emptyset\}$ and $T_\alpha = T \cap S_\alpha$, $\alpha \in Z$. It is evident that T_α, $\alpha \in Z$, is a completely π-regular semigroup and all its idempotents are primitive. Based on Theorem 3.16

semigroups T_α, $\alpha \in Z$ are completely Archimedean. Thus, T is a semilattice of completely Archimedean semigroups. Since (i)$\Leftrightarrow$(iv), then every factor of T is a semilattice of completely Archimedean semigroups. Hence, between the factors of T there are no semigroups $\mathbf{A}_2$ or $\mathbf{B}_2$.

(vi)$\Rightarrow$(v) Assume that there exists a regular $\mathcal{D}$-class D_a, $a \in S$, which is not a subsemigroup of S. Based on Lemma 1.32 $\mathcal{D} = \mathcal{J}$, so $D_a = J_a$. The ideal $J(a)$ of a semigroup S is a completely π-regular semigroup and it is also the principal factor $K = J(a)/I(a)$. Based on Theorem 1.22, K is a completely 0-simple semigroup, i.e. $K = \mathcal{M}^0(G; I, \Lambda, P)$, where P is a regular matrix. Since J_a is not a subsemigroup of S, then K has the zero divisor, i.e. there exists $i \in I$, $\lambda \in \Lambda$ such that $p_{i\lambda} = 0$. On the other hand, since P is regular, then there exists $j \in I$ and $\mu \in \Lambda$ such that $p_{\mu i} \neq 0$ and $p_{\lambda j} \neq 0$. Let $I_0 = \{i, j\}$, $\Lambda_0 = \{\lambda, \mu\}$ and let P_0 be a $P_0 \times \Lambda_0$ submatrix of P. There is a subsemigroup $M = \mathcal{M}^0(G; I_0, \Lambda_0, P_0)$ of K. Then $T = M^\bullet \cup I(a)$ is a completely π-regular subsemigroup of S, because and M and $I(a)$ are completely π-regular. Also, M is a factor of T, and since M is a completely 0-simple, then $\mathcal{H}$ is a congruence on M and $M/\mathcal{H} \cong \mathbf{A}_2$, for $p_{\mu j} \neq 0$, and $M/\mathcal{H} \cong \mathbf{B}_2$, for $p_{\mu j} = 0$, respectively. Thus, one of the semigroups $\mathbf{A}_2$ or $\mathbf{B}_2$ is a factor of T, which is a contradiction according to hypothesis in (vi). Therefore, (v) holds.

(v)$\Rightarrow$(ii) Assume $a, b \in S$. Based on Theorems 2.3 and 1.8 $(ab)^n, (ba)^n \in \mathrm{Gr}(S)$, for some $\mathbf{Z}^+$, whence $(an)^n \in (ab)^{n+1}S \subseteq (ab)^n aS$, $(ba)^n \in S(ba)^{n+1} \subseteq Sa(ba)^n$, so $(ab)^n \mathcal{R} (ab)^n a = a(ba)^n \mathcal{L} (ba)^n$. Thus, $(ab)^n \mathcal{D} (ba)^n$, and since every regular $\mathcal{D}$-class of S is a subsemigroup, then $(ab)^n \mathcal{D} (ba)^n (ab)^n$. On the other hand, from $\mathcal{D} \subseteq \mathcal{J}$ we obtain that $(ab)^n \mathcal{J} (ba)^n (ab)^n$. Whence, $(ab)^n \in S(ba)^n (ab)^n S \subseteq Sa^2S$. Now, according to Theorem 5.1, S is a semilattice of Archimedean semigroups.

(i)$\Rightarrow$(vii) Let S be a semilattice Y of completely Archimedean semigroups S_α, $\alpha \in Y$, and for every $\alpha \in Y$, S_α is a nil-extension of a completely simple semigroup K_α. Consider $e, f \in E(S)$ such that $e \in SfS$, and let $e \in S_\alpha$ and $f \in S_\beta$, for some $\alpha, \beta \in Y$. Then $\alpha = \alpha\beta = \beta\alpha$ and $ef \in S_\alpha$, whence $ef = eef \in K_\alpha S_\alpha \subseteq K_\alpha$. Now there exists $x \in K_\alpha$ such that $ef = efxef$. Thus $xef \in E(S_\alpha)$. Based on Theorem 3.16, $\langle E(K_\alpha)\rangle$ is (completely) simple, whence

$$e \in \langle E(K_\alpha)\rangle xef \langle E(K_\alpha)\rangle = \langle E(K_\alpha)\rangle xeff \langle E(K_\alpha)\rangle \subseteq \langle E(S)\rangle f \langle E(S)\rangle,$$

which was to be proved. Using Lemma 2.11 we have that $\langle E(S)\rangle$ is completely π-regular and based on Lemma 2.5 we have

$$\mathrm{Reg}\langle E(S)\rangle = S\cap \mathrm{Reg}(S) = S\cap \mathrm{Gr}(S) = \mathrm{Gr}\langle E(S)\rangle.$$

(vii)$\Rightarrow$(i) Conversely, let S be completely π-regular. Then based on Lemma 2.11, $\langle E(S)\rangle$ is completely π-regular and based on (i)$\Leftrightarrow$(iii) $\langle E(S)\rangle$ is a semilattice of completely Archimedean semigroups. Consider $e,f,g\in E(S)$ such that $e|g$ and $f|g$ in S. Then from the hypothesis we have that $e|g$ and $f|g$ in $\langle E(S)\rangle$. Now, based on Theorem 7.2, $ef|g$ in $\langle E(S)\rangle$ (and also in S). Again based on Theorem 7.2 we have that S is a semilattice of Archimedean semigroups. Since S is completely π-regular, we then have based on (i)$\Leftrightarrow$(ii) that S is a semilattice of completely Archimedean semigroups.

(i)$\Leftrightarrow$(viii) and (viii)$\Leftrightarrow$(ix) This follows from Theorem 5.27.

(i)$\Rightarrow$(xi) This follows from Theorem 5.27.

(xi)$\Rightarrow$(x) Assume $a\in \mathrm{Reg}(S)$, $x\in V(a)$. Let L be the $\mathcal{L}$-class of a. Clearly, $a\mathcal{L}xa$, i.e. $xa\in L$. Based on the hypothesis, L is a subsemigroup of S, so $xa^2=(xa)a\in L$, i.e. $a\mathcal{L}xa^2$, whence $a\in Sxa^2\subseteq Sa^2$, and $a\in \mathrm{LReg}(S)$.

(x)$\Rightarrow$(ii) Clearly, S is left π-regular, so according to Theorem 2.3, it is completely π-regular. Assume $a,b\in S$. Then $(ab)^n\in G_e$, for some $n\in \mathbf{Z}^+$, $e\in E(S)$, and based on Lemma 1.8, $abe\in G_e$. Let x be the inverse of abe in the group G_e. Then $e=xabe=exabe$, whence $be=bexabe$. Therefore, $be\in \mathrm{Reg}(S)\subseteq \mathrm{LReg}(S)$, so $be=y(be)^2=(yb)(ebe)$, for some $y\in S$. Clearly, $be=y^m(be)^{m+1}$, for each $m\in \mathbf{Z}^+$. Assume that $(ebe)^m\in G_f$, for some $m\in \mathbf{Z}^+$, $f\in E(S)$. Then it is easy to verify that $ef=fe=f$. On the other hand,

$$e=xabe=xay^m(be)^{m+1}=xay^mb(ebe)^m=xay^mb(ebe)^mf=ef.$$

Hence, $e=f$, i.e. $(ebe)^m\in G_e$, so again based on Lemma 1.8, $ebe=e(ebe)\in G_e$. Now, $eab^2e=(eab)(be)=(abe)(be)=(abe)(ebe)\in G_e$, whence $(ab)^n, eab^2e\in G_e$. Therefore, $(ab)^n\in G_eeab^2e\subseteq Sb^2S$, so based on Theorem 5.1, S is a semilattice of Archimedean semigroups. $\square$

Theorem 7.5 *The following conditions on a semigroup S are equivalent:*

(i) *S is a chain of completely Archimedean semigroup;*

(ii) *S is completely π-regular and for all $e,f\in E(S)$ is $e\in efSfe$ or $f\in feSef$;*

(iii) *S is completely π-regular and for all $e, f \in E(S)$ is $e \in efS$ or $f \in Sef$;*
(iv) *S is completely π-regular and* $\mathrm{Reg}(S)$ *is a chain of completely simple semigroups;*
(v) *S is completely π-regular and for all $e, f \in E(S)$, $e \in ef\langle E(S)\rangle\, fe$ or $f \in fe\langle E(S)\rangle ef$;*
(vi) *S is completely π-regular and for all $e, f \in E(S)$, $e \in ef\langle E(S)\rangle$ or $f \in \langle E(S)\rangle ef$;*
(vii) *S is completely π-regular and $\langle E(S)\rangle$ is a chain of completely simple semigrups.*

Proof. (i)$\Rightarrow$(ii) Let S be a chain Y of completely Archimedean semigroups S_α, $\alpha \in Y$. It is evident that S is completely π-regular. Assume $e, f \in E(S)$, and assume that $e \in S_\alpha$, $f \in S_\beta$, $\alpha, \beta \in Y$. Since Y is a chain, then $\alpha\beta = \alpha$ or $\alpha\beta = \beta$. If $\alpha\beta = \alpha$, then $e, ef \in S_\alpha$, so based on Theorem 3.16 and Lemma 3.15 we have that $efe = e(ef)e \in eS_\alpha e = G_e$. Thus, $e, efe \in G_e$, so

$$e \in efeG_eefe \subseteq efSfe.$$

Similarly, if $\alpha\beta = \beta$ it follows that $f \in feSef$.

(ii)$\Rightarrow$(iii) This follows immediately.

(iii)$\Rightarrow$(i) Assume $a, b \in S$. Then $(ab)^m, (ba)^n \in \mathrm{Reg}(S)$, for some $m, n \in \mathbf{Z}^+$. Assume $x \in V((ab)^m)$, $y \in V((ba)^n)$. Then $y(ba)^n, (ab)^m x \in E(S)$, so by (iii) we obtain that

$$y(ba)^n \in y(ba)^n(ab)^m xS \quad \text{or} \quad (ab)^m x \in Sy(ba)^n(ab)^m x,$$

so

$$y(ba)^n \in (ba)^n(ab)^m xS \quad \text{or} \quad (ab)^m \in Sy(ba)^n(ab)^m.$$

Thus, $(ab)^{n+1} \in Sa^2S$ or $(ab)^m \in Sa^2S$, so based on Theorems 5.1 and 7.4 we obtain that S is a semilattice Y of completely Archimedean semigroups S_α, $\alpha \in Y$. Assume $\alpha, \beta \in Y$, $e \in E(S_\alpha)$, $f \in E(S_\beta)$. Then $e \in efS$ or $f \in Sef$. If $e \in efS$, then $e = efu$, for some $u \in S$. If we assume that $u \in S_\gamma$, $\gamma \in Y$, then we obtain that $\alpha = \alpha\beta\gamma$, whence $\alpha\beta = \alpha$. Similar, if $f \in Sef$ then it follows that $\alpha\beta = \beta$. Thus, Y is a chain.

(ii)$\Rightarrow$(iv) Let $T = \mathrm{Reg}(S)$. Assume $a, b \in T$, $x \in V(a)$, $y \in V(b)$. Then $xa, by \in E(S)$, so from (ii) it follows that $xa \in xabySbyxa$ or $by \in byxaSxaby$. If $xa \in xabySbyxa$, then

$$ab = axabyb \in axabySbyxabyb = abySyxab \subseteq abSab,$$

so $ab \in T$. Similar, if $by \in byxaSxaby$, then $ab \in T$. Thus, $T = \mathrm{Reg}(S)$ is a subsemigroup of S. Since $\mathrm{Gr}(S) = \mathrm{Gr}(T) \subseteq T$ and since S is completely π-regular, then we obtain that T is also completely π-regular.

Assume $a \in T$, $x \in V(a)$. Then, from $ax, xa \in E(S)$, from (ii) we obtain that $ax \in ax^2aSxa^2x$ or $xa \in xa^2xSax^2a$, whence $a = axa \in Sa^2S$. Based on Theorem 2.6 we obtain that T is a semilattice Y of simple semigroups T_α, $\alpha \in Y$, so based on Lemma 2.7 and Theorem 2.5, T_α, $\alpha \in Y$ are completely simple semigroups. In the same way as in proof (iii)$\Rightarrow$(i) we obtain that T is a chain.

(iv)$\Rightarrow$(ii) This follows from the fact that is $E(S) = E(\mathrm{Reg}(S))$ and the fact is (i)$\Leftrightarrow$(ii).

(i)$\Rightarrow$(v) Let S be a chain Y of completely Archimedean semigroups S_α, $\alpha \in Y$. Clearly, S is a completely π-regular semigroup. Let $e, f \in E(S)$ and let $e \in S_\alpha$, $f \in S_\beta$, for some $\alpha, \beta \in Y$. Since Y is a chain, then $\alpha\beta = \alpha$ or $\alpha\beta = \beta$. If $\alpha\beta = \alpha$, then $e, ef \in S_\alpha$, so based on Theorem 3.16 and Lemma 3.15, $efe = e(ef)e \in eS_\alpha e = G_e$. Thus, $e, efe \in G_e$, whence $e \in efeG_eefe$, i.e. $e = efexefe = efxfe$, for some $x \in G_e$. Therefore, $e = ef(fxfe)(efxf)fe \in ef\langle E(S)\rangle fe$. Similarly we prove that $\alpha\beta = \beta$ implies $f \in fe\langle E(S)\rangle ef$.

(v)$\Rightarrow$(vi) This follows immediately.

(vi)$\Rightarrow$(i) Let $a, b \in S$. Then $(ab)^m, (ba)^n \in \mathrm{Reg}(S)$, for some $m, n \in \mathbf{Z}^+$. Let $x \in V((ab)^m)$, $y \in V((ba)^n))$. Then $y(ba)^n, (ab)^m x \in E(S)$, so based on (iii) we obtain that

$$y(ba)^n \in y(ba)^n(ab)^m x\langle E(S)\rangle$$

or

$$(ab)^m x \in \langle E(S)\rangle y(ba)^n(ab)^m x,$$

whence

$$(ba)^n \in (ba)^n(ab)^m xS$$

or

$$(ab)^m \in Sy(ba)^n(ab)^m.$$

Therefore, $(ab)^{n+1} \in Sa^2S$ or $(ab)^m \in Sa^2S$, so from Theorem 5.1 it follows that S is a semilattice Y of completely Archimedean semigroups S_α, $\alpha \in Y$. If $\alpha, \beta \in Y$ and $e \in E(S_\alpha)$, $f \in E(S_\beta)$, then $e \in ef\langle E(S)\rangle$ implies $\alpha\beta = \alpha$, and $f \in \langle E(S)\rangle ef$ implies $\alpha\beta = \beta$. Thus, based on (vi) we obtain that Y is a chain.

(vii)$\Rightarrow$(v) Since $\langle E(S)\rangle$ is a chain of completely simple semigroups, then based on (i)$\Leftrightarrow$(v) we have the assertion.

(i)$\Rightarrow$(vii) Based on (i)$\Leftrightarrow$(iv), $\mathrm{Reg}(S)$ is a chain of completely simple semigroups. Based on this and Theorem 2.16 we have that $\langle E(S)\rangle$ is a union of groups, whence from (i)$\Leftrightarrow$(v) we obtain that $\langle E(S)\rangle$ is a chain of completely simple semigroups. $\square$

Exercises

1. A semigroup S is a semilattice of completely Archimedean semigroups if and only if S is completely π-regular with the identity $(ab)^0 = ((ab)^0(ba)^0(ab)^0)^0$.

References

S. Bogdanović [16], [19]; S. Bogdanović and M. Ćirić [6], [10]; M. Ćirić and S. Bogdanović [2], [3], [6]; J. L. Galbiati and M. L. Veronesi [1], [2], [3], [4]; B. L. Madison, T. K. Mukherjee and M. K. Sen [1], [2]; M. S. Putcha [1], [2], [8]; M. V. Sapir and E. V. Suhanov [1]; L. N. Shevrin [4], [5], [6]; L. N. Shevrin and E. V. Suhanov [1]; L. N. Shevrin and M. V. Volkov [1]; M. L. Veronesi [1].

7.3 Semilattices of Nil-extensions of Rectangular Groups

In the previous section we observed a decomposition of (completely) π-regular semigroups into a semilattice of completely Archimedean semigroups, i.e. a semilattice of nil-extension of completely simple semigroups (Theorem 3.16). In this section we will discuss one special case of these decompositions, i.e. we will discuss semillatice decompositions in which every component is an *orthodox semigroup*, i.e. a semigroup in which the set of all idempotents is its subsemigroup.

We start with the following result.

Lemma 7.1 *The following conditions on a semigroup S are equivalent:*

(i) *$E(S)$ is a subsemigroup of S;*
(ii) *if $a, b \in S$ and $x \in V(a)$, $y \in V(b)$, then $yx \in V(ab)$;*
(iii) *for all $a, b, x, y \in S$, $a = axa$ and $b = byb$ implies $ab = abyxab$.*

If S is regular, then each of the previous conditions is equivalent to:

(iv) *every inverse of every idempotent from S is an idempotent.*

Proof. (i)$\Rightarrow$(ii) Assume $a, b \in S$, $x \in V(a)$, $y \in V(b)$. Then based on $xa, by \in E(S)$ and (i) we obtain that $xaby, byxa \in E(S)$, whence

$$\begin{aligned} abyxab &= axabyxabyb = a(xaby)^2b = axabyb = ab \\ yxabyx &= ybyxabyxax = y(byxa)^2x = ybyxax = yx. \end{aligned}$$

Therefore, $yx \in V(ab)$.

(ii)$\Rightarrow$(iii) Let $a = axa$, $b = byb$, $a, b, x, y \in S$. Then $xax \in V(a)$, $yby \in V(b)$, so by (ii), $ybyxax \in V(ab)$. Hence,

$$ab = ab(yby)(xax)ab = abyxab.$$

(iii)$\Rightarrow$(i) This follows immediately.

(i)$\Rightarrow$(iv) Let $e \in E(S)$ and let $x \in V(e)$. Then $xe, ex \in E(S)$, so based on (i) we obtain that

$$x = xex = (xe)(ex) = [(xe)(ex)]^2 = (xex)^2 = x^2.$$

Now, let S be a regular semigroup.

(iv)$\Rightarrow$(i) Assume $e, f \in E(S)$. Since S is regular, then there exists $x \in V(ef)$, whence

$$(ef)(fxe)(ef) = efxef = ef, \qquad (fxe)(ef)(fxe) = f(xefx)e = fxe,$$

so $ef \in V(fxe)$. On the other hand, $fxe = f(xefx)e = (fxe)^2$, i.e. $fxe \in E(S)$, so based on (iv) we obtain that $ef \in E(S)$. $\square$

According to the following lemma we describe some completely simple semigroups which are not orthodox, i.e. which are not rectangular groups.

Lemma 7.2 *Let R be the ring $\mathbf{Z}$ of all integers or the ring $\mathbf{Z}_p$ of all the rests of the integers by mod p, $p \in \mathbf{Z}^+$, $p \geq 2$, and let $I = \{0, 1\} \subseteq R$. The set $R \times I \times I$ under multiplication defined by*

$$(m; i, \lambda)(n; j, \mu) = (m + n - (i - j)(\lambda - \mu); i, \mu), \quad m, n \in R, i, j, \lambda, \mu \in I,$$

is a semigroup, in notation $\mathbf{E}(\infty) = \mathbf{Z} \times I \times I$, $\mathbf{E}(p) = \mathbf{Z}_p \times I \times I$. Also, $\mathbf{E}(\infty)$ and $\mathbf{E}(p)$, $p \in \mathbf{Z}^+$, $p \geq 2$, are completely simple semigroups and they are not rectangular groups.

Proof. It is evident that $\mathbf{E}(\infty)$ and $\mathbf{E}(p)$ are semigroups. Also, it is clear that $\mathbf{E}(\infty)$ ($\mathbf{E}(p)$) is a rectangular band of $I\times I$ groups $E_{i,\lambda} = \{(m;i,\lambda)\,|\,m\in R\}$, $i,\lambda\in I$, where $R=\mathbf{Z}$ ($R=\mathbf{Z}_p$), so based on Corollary 3.8, $\mathbf{E}(\infty)$ and $\mathbf{E}(p)$ are completely simple semigroups. The set of all idempotents from $\mathbf{E}(\infty)$ ($\mathbf{E}(p)$) is the set $\{(0;i,\lambda)\,|\,i,\lambda\in I\}$, and it is easy to prove that it is not a subsemigroup of $\mathbf{E}(\infty)$ ($\mathbf{E}(p)$). Thus, according to Theorem 3.6 $\mathbf{E}(\infty)$ and $\mathbf{E}(p)$, $p\in\mathbf{Z}^+$, $p\geq 2$ are not rectangular groups. □

A factor K of a semigroup S is a *completely π-regular factor* of S if each of its elements is completely π-regular.

The following theorem is the main result of this section.

Theorem 7.6 *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of nil-extensions of rectangular groups;*

(ii) *S is a semilattice of completely Archimedean semigroups and for all $e,f\in E(S)$ there exists $n\in\mathbf{Z}^+$ such that $(ef)^n=(ef)^{n+1}$;*

(iii) *S is completely π-regular and $(xy)^0=(xy)^0(yx)^0(xy)^0$;*

(iv) *S is π-regular and $a=axa$ implies $a=ax^2a^2$;*

(v) *S is a semilattice of completely Archimedean semigroups and the inverse of every idempotent from S is an idempotent;*

(vi) *S is a semilattice of completely Archimedean semigroups and between subsemigroups of S there are no $\mathbf{E}(\infty)$ and $\mathbf{E}(p)$, $p\in\mathbf{Z}^+$, $p\geq 2$ semigroups;*

(vii) *S is completely π-regular and between the completely π-regular factors of subsemigroups of S there are no $\mathbf{A}_2$, $\mathbf{B}_2$ and $\mathbf{E}(p)$, $p\in\mathbf{Z}^+$, $p\geq 2$ semigroups.*

Proof. (i)⇒(ii) Let S be a semilattice Y of semigroups S_α, $\alpha\in Y$, and for $\alpha\in Y$, let S_α be a nil-extension of a rectangular group K_α. Assume $e,f\in E(S)$. Then $ef,fe\in S_\alpha$, for some $\alpha\in Y$, so there exists $n\in\mathbf{Z}^+$ such that $(ef)^n,(fe)^n\in K_\alpha$. Furthermore, we have that $(ef)^n\in G_g$, $(fe)^n\in G_h$, for some $g,h\in E(K_\alpha)$, so $(ef)^nx=g$, $(fe)^ny=h$, for some $x\in G_g$, $y\in G_h$ and from Theorem 1.8 it follows that $(ef)^{n+1}\in G_g$. Since K_α is a rectangular group, then $ghg=g$. Now we have that

$$\begin{aligned}(ef)^n &= (ef)^ng=(ef)^n(ef)^nx=(ef)^ne(ef)^nx=(efe)^ng\\ &= e(fe)^ng=e(fe)^nhg=e(fe)^n(fe)^nyg=e(fe)^nf(fe)^nyg\\ &= (ef)^{n+1}hg=(ef)^{n+1}ghg=(ef)^{n+1}g=(ef)^{n+1}.\end{aligned}$$

(ii)$\Rightarrow$(i) Let S be a semilattice Y of completely Archimedean semigroups S_α, $\alpha \in Y$, and let for all $e, f \in E(S)$ there exists $n \in \mathbf{Z}^+$ such that $(ef)^n = (ef)^{n+1}$. For $\alpha \in Y$, let S_α be a nil-extension of a completely simple semigroup K_α. Assume $\alpha \in Y$, $e, f \in E(K_\alpha)$. Based on the hypothesis, $(ef)^n = (ef)^{n+1}$, for some $n \in \mathbf{Z}^+$, so $(ef)^n = (ef)^{n+1} \in E(S)$. On the other hand, $ef \in K_\alpha$, so $ef \in G_g$, for some $g \in E(K_\alpha)$. Since $\langle ef \rangle \subseteq G_g$, then $(ef)^n = (ef)^{n+1} = g$, whence $ef = efg = ef(ef)^n = (ef)^{n+1} = g \in E(S)$. Thus, $E(K_\alpha)$ is a subsemigroup of K_α, so based on Theorem 3.6 K_α is a rectangular group. Therefore, (i) holds.

(i)$\Rightarrow$(iii) Let S be a semilattice Y of semigroups S_α, $\alpha \in Y$, and for $\alpha \in Y$, let S_α be a nil-extension of a rectangular group. According to Theorem 7.4, S is completely π-regular. Assume $x, y \in S$. Then $xy, yx \in S_\alpha$, for some $\alpha \in Y$, whence $(xy)^0, (yx)^0 \in E(S_\alpha)$, so based on Corollary 3.12 $(xy)^0 = (xy)^0(yx)^0(xy)^0$.

(iii)$\Rightarrow$(iv) From (iii) it immediately follows that S is π-regular. Let $a = axa$, $a, x \in S$. Then $ax, xa \in E(S)$, whence $(ax)^0 = ax$, $(xa)^0 = xa$, and based on (iii) we obtain that $a = (ax)a = (ax)(xa)(ax)a = ax^2a^2xa = ax^2a^2$.

(iv)$\Rightarrow$(v) Let (iv) hold. Assume $a \in \mathrm{Reg}(S)$, $x \in V(a)$. Then from (iv) we obtain that $a = ax^2a^2 \in Sa^2$, and $x = xa^2x^2$, whence $a = axa = axa^2x^2a = a^2x^2a \in a^2S$. Thus, $a \in \mathrm{Gr}(S)$. Hence, $\mathrm{Reg}(S) = \mathrm{Gr}(S)$, so according to Theorem 7.4 S is a semilattice of completely Archimedean semigroups. Assume $e \in E(S)$, $y \in V(e)$. Then based on (iv) we have that $y = ye^2y^2 = yey^2 = y^2$. Therefore, (v) holds.

(v)$\Rightarrow$(vi) Let (v) hold. If S contains a subsemigroup isomorphic to $\mathbf{E}(\infty)$ or $\mathbf{E}(p)$, $p \in \mathbf{Z}^+$, $p \geq 2$, then there exists an idempotent from S and its inverse which is not an idempotent. Actually, the element $(1; 0, 0)$ is inverse of the idempotent $(0; 1, 1)$ in $\mathbf{E}(\infty)$, $\mathbf{E}(p)$ respectively, where $(1; 0, 0)$ is not an idempotent.

(vi)$\Rightarrow$(i) Let (vi) hold. If we want to prove (i), then it is enough to prove that every completely simple subsemigroup of S is a rectangular group. Let K be a completely simple subsemigroup of S. Assume that K is not a rectangular group. According to Theorem 3.6 there exist $e, f \in E(K)$ such that $ef \notin E(K)$. Hence, ef is a group element of the order $p \geq 2$ or of an infinite order in a semigroup K, and it is easy to prove that ef, efe, fef and fe are different elements of the same order (finite or infinite). Also, it is easy to prove that ef, efe, fef and fe are in the different $\mathcal{H}$-classes of K

and for K it holds:

$$ef\mathcal{L}fef, \quad ef\mathcal{R}efe, \quad fe\mathcal{L}efe, \quad fe\mathcal{R}fef. \tag{1}$$

According to Theorem 3.8, K is a rectangular band of $I \times \Lambda$ groups $H_{i\lambda}$, $i \in I$, $\lambda \in \Lambda$, which are $\mathcal{H}$-classes of K. For the sake of simplicity, we use the notation $ef \in H_{00}$, $fe \in H_{11}$, $0, 1 \in I$, $0, 1 \in \Lambda$. Based on (1), $efe \in H_{01}$, $fef \in H_{10}$. With G_{00}, G_{01}, G_{10}, G_{11} we respectively denote the monogenic subgroups of H_{00}, H_{01}, H_{10} and H_{11} generated by elements ef, efe, fef and fe, and let $T = G_{00} \cup G_{01} \cup G_{10} \cup G_{11}$. Now, there are two cases:

(A) The elements ef, efe, fef and fe are of an infinite order, i.e. the groups G_{00}, G_{01}, G_{10} and G_{11} are isomorphic to the additive group of integers. Then it is easy to prove that T is a subsemigroup of K isomorphic to $\mathbf{E}(\infty)$, where one isomorphism φ from $\mathbf{E}(\infty)$ to T is given by: for $n \in \mathbf{Z}$

$$(n; 0, 0)\varphi = (ef)^n, \qquad (n; 0, 1)\varphi = (efe)^n,$$

$$(n; 1, 0)\varphi = (fef)^n, \qquad (n; 1, 1)\varphi = (fe)^n.$$

(B) The elements ef, efe, fef and fe are of a finite order $p \geq 2$, i.e. the groups G_{00}, G_{01}, G_{10} and G_{11} are isomorphic to the additive group of the rest of the integers by mod p. Then it is easy to prove that T is a subsemigroup of K isomorphic to $\mathbf{E}(p)$, where one isomorphism φ from $\mathbf{E}(p)$ to T is given by: for $n \in \mathbf{Z}_p$

$$(n; 0, 0)\varphi = (ef)^n, (n; 0, 1)\varphi = (efe)^n, (n; 1, 0)\varphi = (fef)^n, (n; 1, 1)\varphi = (fe)^n.$$

Hence, in both cases we obtain a contradiction to the hypothesis in (vi). Therefore, K must be a rectangular group.

(vi)$\Leftrightarrow$(vii) This follows from Theorem 7.4 and from the fact that $\mathbf{E}(p)$ is a factor of $\mathbf{E}(\infty)$, for every $p \in \mathbf{Z}^+$, $p \geq 2$. $\square$

Lemma 7.3 *A semigroup S is a chain of rectangular bands if and only if for all $x, y \in S$ is $x = xyx$ or $y = yxy$.*

Proof. Let S be a chain Y of rectangular bands S_α, $\alpha \in Y$. Assume $x, y \in S$. Then $x \in S_\alpha$, $y \in S_\beta$, $\alpha, \beta \in Y$, and since T is a chain then $\alpha\beta = \alpha$ or $\alpha\beta = \beta$. If $\alpha\beta = \alpha$, then $x, xy \in S_\alpha$, so since S_α is a rectangular band, then $xyx = x(xy)x = x$. Similarly, from $\alpha\beta = \beta$ it follows that $yxy = y$.

Conversely, let $xyx = x$ and $yxy = y$ for all $x, y \in S$. Then, for $x \in S$ we have that $x = x^3$, and $x = xx^2x$ or $x^2 = x^2xx^2$, i.e. $x = x^4$ or $x = x^5$. Thus, $x = x^3$ or $x^2 = x^5$, whence $x = x^2$. Hence, S is a band, so based on Corollary 3.6, S is a semilattice Y of rectangular bands S_α, $\alpha \in Y$. It is easy to prove that Y is a chain. □

The chain of nil-extension of rectangular groups will be described by the following theorem.

Theorem 7.7 *The following conditions on a semigroup S are equivalent:*

(i) *S is a chain of nil-extensions of rectangular groups;*
(ii) *S is completely π-regular and* Reg(S) *is a chain of rectangular groups;*
(iii) *S is completely π-regular and $E(S)$ is a chain of rectangular bands.*

Proof. (i)⇒(ii) Let S be a chain Y of semigroups S_α, $\alpha \in Y$, and for $\alpha \in Y$ let S_α be a nil-extension of the rectangular group K_α. Based on Theorem 7.4, S is completely π-regular. Assume $e, f \in E(S)$. Then $e \in K_\alpha$, $f \in K_\beta$, $\alpha, \beta \in Y$. Since Y is a chain, then $\alpha\beta = \alpha$ or $\alpha\beta = \beta$. If $\alpha\beta = \alpha$, then $ef = e(ef) \in K_\alpha S_\alpha \subseteq K_\alpha$, while based on Theorem 7.6 we obtain that $(ef)^n = (ef)^{n+1}$, for some $n \in \mathbf{Z}^+$, whence $ef \in E(S_\alpha) = E(K_\alpha)$, so from Lemma 3.8 it follows that $e = e(ef)e = efe$. Similarly, from $\alpha\beta = \beta$ it follows that $ef \in E(S_\beta)$ and $f = fef$. Thus, $E(S)$ is a subsemigroup of S, and based on Lemma 7.3, $E(S)$ is a chain of rectangular bands.

(ii)⇒(iii) This is proved in a similar way as (i)⇒(iii).

(iii)⇒(i) and (iii)⇒(ii) This follows from Theorem 7.5. □

A semigroup S is a *singular band* if S is either a left zero band or a right zero band. A semigroup S is a *Rédei band* if for all $x, y \in S$, $xy = x$ or $xy = y$. The rectangular Rédei bands are described by the following lemma:

Lemma 7.4 *A semigroup S is a rectangular Rédei band if and only if S is a singular band.*

Proof. Let $S = I \times \Lambda$ be a rectangular band. Assume that is $|I| \geq 2$ and $|\Lambda| \geq 2$, i.e. assume $i, j \in I$, $i \neq j$, and $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$. Then $(i, \lambda)(j, \mu) = (i, \mu)$, so $(i, \lambda)(j, \mu) \neq (i, \lambda)$ and $(i, \lambda)(j, \mu) \neq (j, \mu)$, which is a contradiction of the hypothesis that S is a Rédei band. Thus, $|I| = 1$ or $|\Lambda| = 1$, so S is a singular band.

The converse, follows immediately. □

Now, we discuss a semilattice of semigroups in which an arbitrary component is a nil-extension or a nil-extension of a right group ("the mixed properties").

Theorem 7.8 *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of nil-extensions of left or right groups;*

(ii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n \in (ab)^n S(ba)^n \cup (ba)^n S(ab)^n$*;*

(iii) *S is π-regular and for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $(ab)^n \in Sa \cup bS$;*

(iv) *S is a semilattice of completely Archimedean semigroups and for all $e, f \in E(S)$ there exists $n \in \mathbf{Z}^+$ such that $(ef)^n = (efe)^n$ or $(ef)^n = (fef)^n$;*

(v) *S is completely π-regular and $(xy)^0 = (xy)^0(yx)^0$ or $(xy)^0 = (yx)^0(xy)^0$;*

(vi) *S is π-regular and $a = axa$ implies $ax = ax^2a$ or $ax = xa^2x$.*

Proof. (i)⇒(ii) Let S be a semilattice Y of semigroups S_α, $\alpha \in Y$, and for $\alpha \in Y$ let S_α be a nil-extension of a semigroup K_α, where K_α is a left or a right group. Assume $a, b \in S$. Then $ab, ba \in S_\alpha$, for some $\alpha \in Y$, whence there exists $n \in \mathbf{Z}^+$ such that $(ab)^n, (ba)^n \in K_\alpha$, so according to Theorem 3.7 and from its dual we obtain that

$$(ab)^n \in (ab)^n K_\alpha (ba)^n \subseteq (ab)^n S(ba)^n,$$

if K_α is a left group, whence

$$(ab)^n \in (ba)^n K_\alpha (ab)^n \subseteq (ba)^n S(ab)^n,$$

if K_α is a right group. Therefore, (ii) holds.

(ii)⇒(iii) This is evident.

(iii)⇒(iv) Let (iii) hold. Assume $a \in \mathrm{Reg}(S)$, $x \in V(a)$. Then, based on (iii) we obtain that $ax \in Sa \cup xS$ and $xa \in Sx \cup aS$. If $ax = ua$, for some $u \in S$, then $a = axa = ua^2 \in Sa^2$. If $ax = xv$, for some $v \in S$, then $a = axa = xva$, whence $a^2 = axva$ and $a = xva = xaxva = xa^2 \in Sa^2$. Thus, $ax \in Sa \cup xS$ implies that $a \in Sa^2$. Similarly, we prove that from $ax \in Sa \cup xS$ follows that $a \in a^2S$. Hence, $a \in \mathrm{Gr}(S)$, i.e. $\mathrm{Gr}(S) = \mathrm{Reg}(S)$, so based on Theorem 7.4, S is a semilattice of completely Archimedean semigroups.

For $e, f \in E(S)$, based on (iii), there exists $n \in \mathbf{Z}^+$ such that $(ef)^n \in Se \cup fS$. If $(ef)^n = ue$, for some $u \in S$, then $(ef)^n = ue = uee(ef)^n e = (efe)^n$. Similarly, from $(ef)^n \in fS$ it follows that $(ef)^n = (fef)^n$.

(iv)$\Rightarrow$(i) From (iv) it immediately follows that for all $e, f \in E(S)$ there exists $n \in \mathbf{Z}^+$ such that $(ef)^n = (ef)^{n+1}$, so based on Theorem 7.6 we obtain that S is a semilattice Y of semigroups S_α, $\alpha \in Y$, and for $\alpha \in Y$, S_α is a nil-extension of a rectangular group. Assume $\alpha \in Y$, $e, f \in E(S_\alpha)$. From (iv), using Corollary 3.12, it follows that $ef = efe = e$ or $ef = fef = f$, whence $E(S_\alpha)$ is a rectangular Rédei band, so based on Lemma 7.4 $E(S_\alpha)$ is a singular band. Thus, based on Theorem 3.17 S_α is a nil-extension of left or right groups.

(i)$\Rightarrow$(v) This proves similar as (i)$\Rightarrow$(iii) in Theorem 7.6.

(v)$\Rightarrow$(vi) This proves similar as (iii)$\Rightarrow$(iv) in Theorem 7.6.

(vi)$\Rightarrow$(i) From (vi) we obtain that from $a = axa$ it follows that $ax = ax^2a$ or $ax = xa^2x$, whence $a = (ax)a = ax^2a^2$ or $a = ax(ax)a = ax(xa^2x)a = ax^2a^2xa = ax^2a^2$. Thus, in both cases $a = ax^2a^2$, so based on Theorem 7.6, S is a semilattice Y of semigroups S_α, $\alpha \in Y$, and for $\alpha \in Y$, S_α is a nil-extension of a rectangular group. Assume $\alpha \in Y$, $e, f \in E(S_\alpha)$. Based on Corollary 3.12, $E(S_\alpha)$ is a rectangular band, so $e = efe$, and from (vi) we obtain that $ef = ef^2e = efe = e$ or $ef = fe^2f = fef = f$. Hence, $E(S_\alpha)$ is a rectangular Rédei band, so based on Lemma 7.4, $E(S_\alpha)$ is a singular band. Thus, according to Theorem 3.17, S_α is a nil-extension of a left or right group. $\square$

Using Theorem 7.8, the following result we prove in a similar way as Theorem 7.7.

Corollary 7.1 *The following conditions on a semigroup S are equivalent:*

(i) *S is a chain of nil-extensions of left or right groups;*

(ii) *S is completely π-regular and $\mathrm{Reg}(S)$ is a chain of left and right groups;*

(iii) *S is completely π-regular and $E(S)$ is a chain of singular bands;*

(iv) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in a^{2n}S(ab)^n \cup (ba)^n Sa^{2n} \ \vee\ b^n \in b^{2n}S(ba)^n \cup (ab)^n Sb^{2n}$.

Just like Theorem 7.8, we prove the following theorem:

Theorem 7.9 *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of nil-extensions of left groups;*
(ii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n \in (ab)^n S(ba)^n$*;*
(iii) *S is π-regular and a semilattice of left Archimedean semigroups;*
(iv) *S is a semilattice of completely Archimedean semigroups and for all $e, f \in E(S)$ there exists $n \in \mathbf{Z}^+$ such that $(ef)^n = (efe)^n$;*
(v) *S is completely π-regular and $(xy)^0 = (xy)^0(yx)^0$;*
(vi) *S is π-regular and $a = axa$ implies $ax = ax^2a$.*

Corollary 7.2 *The following conditions on a semigroup S are equivalent:*

(i) *S is a chain of nil-extensions of left groups;*
(ii) *S is completely π-regular and* $\mathrm{Reg}(S)$ *is a chain of left groups;*
(iii) *S is completely π-regular and $E(S)$ is a chain of left zero bands;*
(iv) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in a^{2n}S(ab)^n \cup (ba)^n Sa^{2n}$.

Exercises

1. The following conditions on a semigroup S are equivalent:

(a) S is a semilattice of nil-extensions of rectangular bands;
(b) S is π-regular and $E(S) = \mathrm{Reg}(S)$;
(c) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^{2n+1} = (ab)^n ba^2 (ab)^n$.

2. Prove that a semigroup S is a left (right) regular band if and only if S is a semilattice of left zero (right zero) bands.

3. The following conditions on a semigroup S are equivalent:

(a) S is a semilattice of nil-extensions of left groups;
(b) $(\forall x \in S)(\forall e \in E(S))\ x \mid e \Rightarrow ex = exe$;
(c) S is a semilattice of completely Archimedean semigroups and for all $e, f \in E(S)$ there exists $n \in \mathbf{Z}^+$ such that $(ef)^n \mathcal{L} (fe)^n$;
(d) S is a semilattice of completely Archimedean semigroups and $a = axa = aya$ implies $ax = ay$.

4. A completely simple semigroup S is not a rectangular group if and only if S contains some semigroup $\mathbf{E}(\infty)$ or $\mathbf{E}(p)$, $p \in \mathbf{Z}^+$, $p \geq 2$, as its own subsemigroup.

5. The following conditions on a completely π-regular semigroup S are equivalent:

(a) S is a band of left Archimedean semigroups;
(b) S satisfies the identity $(xy)^0 = (xy)^0(x^0y^0)^0$;
(c) there are no semigroups $\mathbf{A}_2$, $\mathbf{B}_2$, $\mathbf{R}_{3,1}$, $\mathbf{RZ}(n)$, for all $n > 1$, among the completely π-regular divisors of S.

6. The following conditions on a completely π-regular semigroup S are equivalent:

(a) S is a semilattice of left Archimedean semigroups;
(b) S satisfies the identity $(yx)^0 = (yx)^0(xy)^0$;
(c) there are no semigroups $\mathbf{A}_2$, $\mathbf{B}_2$, $\mathbf{R}_2$ among the completely π-regular divisors of S;
(d) each regular $\mathcal{D}$-class of S is a left group.

References

B. Biró, E. W. Kiss and P. P. Pálfy [1]; S. Bogdanović [16], [19]; S. Bogdanović and M. Ćirić [1], [2], [3], [6], [10], [16]; M. Ćirić and S. Bogdanović [1], [3], [6]; A. I. Evseev [1]; V. A. Fortunatov [1], [2]; J. I. Garcia [1]; X. J. Guo [1], [2], [3]; P. R. Jones [1], [2]; E. W. Kiss, L. Márki, P. Pröhle and W. Tholen [1]; M. S. Putcha [2], [8]; L. N. Shevrin [4]; K. P. Shum and X. M. Ren [1]; X. Tang [1], [2], [3], [4], [5]; Y. Wang, Y. Q. Jin and Z. Tian [1].

7.4 Locally Uniformly π-regular Semigroups

For any idempotent e of a semigroup S, the subsemigroup eSe is a maximal submonoid of S, and it is known under the name *local submonoid* of S. If $\mathcal{K}$ is some class or some property of semigroups, then S is said to be a *locally $\mathcal{K}$-semigroup* if any local submonoid of S belongs to $\mathcal{K}$ or has the property $\mathcal{K}$. The main purpose of this section is to characterize a more general kind of semigroups – π-regular semigroups whose any local submonoid is uniformly π-regular, and which are called *locally uniformly π-regular*.

We define the sets $Q(S)$ and $M(S)$ by

$$Q(S) = \bigcup_{e,f\in E(S)} eSf \qquad \text{and} \qquad M(S) = \bigcup_{e\in E(S)} eSe.$$

Let us note that $eSf = eS \cap Sf$, for all $e, f \in E(S)$.

If T is a subsemigroup of S then

$$\mathrm{Reg}(T) = \{a \in T \mid (\exists\, x \in T)\ a = axa\},$$
$$\mathrm{reg}(T) = \{a \in T \mid (\exists\, x \in S)\ a = axa\}.$$

Evidently, $\mathrm{Reg}(T) \subseteq \mathrm{reg}(T)$.

Recall that, a π-regular semigroup whose any regular element is completely regular is called *uniformly π-regular*.

Next we offer several results that describe some properties of the regular and group parts of quasi-ideals eSf, $e, f \in E(S)$, and bi-ideals eSe, $e \in E(S)$, of a semigroup S.

Lemma 7.5 *Let e, f be arbitrary idempotents of a semigroup S. Then the following conditions hold:*

(i) $\mathrm{Reg}(eSf) = \mathrm{Reg}(eS) \cap \mathrm{Reg}(Sf)$;

(ii) $\mathrm{Gr}(eSf) = eSf \cap \mathrm{Gr}(S)$.

Proof. (i) Let $a \in \mathrm{Reg}(eS) \cap \mathrm{Reg}(Sf)$. Then $a = ea = af$ and $a = axa = aya$, for some $x \in eS$ and $y \in Sf$, and from this it follows that $a \in eSf$ and

$$a = axaya \in aeSaSfa \subseteq a(eSf)a,$$

so $a \in \mathrm{Reg}(eSf)$. Thus, $\mathrm{Reg}(eS) \cap \mathrm{Reg}(Sf) \subseteq \mathrm{Reg}(eSf)$. The opposite inclusion is obvious.

(ii) Let $a \in eSf \cap \mathrm{Gr}(S)$. Then $a = ea = af$ and $a \in G_g$, for some $g \in E(S)$, and we have that $g = aa^{-1}a^{-1}a = eaa^{-1}a^{-1}af$, which yields $g = eg = gf$. Now

$$G_g = gG_gg = egG_ggf \subseteq eSf,$$

whence $a \in \mathrm{Gr}(eSf)$, so we have that $eSf \cap \mathrm{Gr}(S) \subseteq \mathrm{Gr}(eSf)$. The opposite inclusion is evident. □

Lemma 7.6 *Let e be an arbitrary idempotent of a semigroup S. Then the following conditions hold:*

(i) $\mathrm{Reg}(eSe) = \mathrm{reg}(eSe) = \mathrm{Reg}(Se) \cap \mathrm{Reg}(eS)$;

(ii) $\mathrm{Gr}(eSe) = eSe \cap \mathrm{Gr}(S)$;

(iii) $\mathrm{Gr}(Se) = Se \cap \mathrm{Gr}(S)$ *and* $\mathrm{Gr}(eS) = eS \cap \mathrm{Gr}(S)$.

Proof. (i) Based on Lemma 7.5 it follows that $\mathrm{Reg}(eSe) = \mathrm{Reg}(Se) \cap \mathrm{Reg}(eS)$. Let $a \in \mathrm{reg}(eSe)$. Then $a = ea = ae$ and $a = axa$ for some $x \in S$, and we have that $a = axa = aexea \in a(eSe)a$, so $a \in \mathrm{Reg}(eSe)$. Thus $\mathrm{reg}(eSe) \subseteq \mathrm{Reg}(eSe)$. It is clear that the opposite inclusion also holds.

(ii) This is also an immediate consequence of Lemma 7.5.

(iii) Evidently, $\mathrm{Gr}(Se) \subseteq Se \cap \mathrm{Gr}(S)$. Let $a \in Se \cap \mathrm{Gr}(S)$. Then $a = ae$ and $a \in G_f$, for some $f \in E(S)$, so by $f = a^{-1}a = a^{-1}ae \in Se$ it follows that $f = fe$. Therefore

$$G_f = G_f f = G_f fe \subseteq Se,$$

which implies $a \in \mathrm{Gr}(Se)$. Hence, $\mathrm{Gr}(Se) = Se \cap \mathrm{Gr}(S)$. In a similar way we prove that $\mathrm{Gr}(eS) = eS \cap \mathrm{Gr}(S)$. □

Lemma 7.7 *Let S be a semigroup with $E(S) \neq \varnothing$. Then*

$$\mathrm{Gr}(S) = \bigcup_{e\in E(S)} \mathrm{Gr}(Se) = \bigcup_{e\in E(S)} \mathrm{Gr}(eS) = \bigcup_{e\in E(S)} \mathrm{Gr}(eSe) = \bigcup_{e,f\in E(S)} \mathrm{Gr}(eSf).$$

Proof. From Lemma 7.5 it follows that

$$\bigcup_{e,f\in E(S)} \mathrm{Gr}(eSf) = \Big(\bigcup_{e,f\in E(S)} eSf\Big) \cap \mathrm{Gr}(S) = Q(S) \cap \mathrm{Gr}(S) = \mathrm{Gr}(S),$$

since $\mathrm{Gr}(S) \subseteq M(S) \subseteq Q(S)$. Similarly we prove the remaining equations. □

For a semigroup S, let the set $\mathrm{Reg}_M(S)$ be defined by

$$\mathrm{Reg}_M(S) = \bigcup_{e\in E(S)} \mathrm{Reg}(eSe).$$

Then the following equations hold:

Lemma 7.8 *Let S be a semigroup with $E(S) \neq \varnothing$. Then*

$$\mathrm{Reg}_M(S) = M(S) \cap \mathrm{Reg}(S) = \mathrm{Reg}(M(S)).$$

Proof. It is obvious that $\mathrm{Reg}_M(S) \subseteq M(S) \cap \mathrm{Reg}(S)$ and $\mathrm{Reg}_M(S) \subseteq \mathrm{Reg}(M(S))$. Let $a \in M(S) \cap \mathrm{Reg}(S)$. Then $a \in eSe$, for some $e \in E(S)$, so based on Lemma 7.6 we have that

$$a \in eSe \cap \mathrm{Reg}(S) = \mathrm{reg}(eSe) = \mathrm{Reg}(eSe) \subseteq \mathrm{Reg}_M(S).$$

Thus $M(S) \cap \mathrm{Reg}(S) \subseteq \mathrm{Reg}_M(S)$, whence $\mathrm{Reg}_M(S) = M(S) \cap \mathrm{Reg}(S)$. On the other hand

$$\mathrm{Reg}(M(S)) \subseteq M(S) \cap \mathrm{Reg}(S) = \mathrm{Reg}_M(S),$$

so we have proved $\mathrm{Reg}(M(S)) = \mathrm{Reg}_M(S)$. □

It is easy to verify that the following relationships between the sets $Gr(S)$, $Reg_M(S)$ and $Reg(S)$ hold on an arbitrary semigroup S:

$$Gr(S) \subseteq Reg_M(S) \subseteq Reg(S).$$

The conditions under which the first inclusion can be turned into an equality are determined by the following theorem.

Lemma 7.9 *Let S be a semigroup with $E(S) \neq \varnothing$. Then the following conditions are equivalent:*

(i) $\mathrm{Gr}(S) = \mathrm{Reg}_M(S)$;
(ii) $(\forall e \in E(S))\ \mathrm{Reg}(eSe) = \mathrm{Gr}(eSe)$;
(iii) $(\forall e \in E(S))\ \mathrm{reg}(eSe) = \mathrm{Gr}(eSe)$.

Proof. (i)$\Rightarrow$(ii) Let $\mathrm{Gr}(S) = \mathrm{Reg}_M(S)$ and let $e \in E(S)$. Then based on Lemma 7.6 we have that

$$\mathrm{Gr}(eSe) = eSe \cap \mathrm{Gr}(S) = eSe \cap \mathrm{Reg}_M(S) = \mathrm{Reg}(eSe).$$

(ii)$\Rightarrow$(i) Let $\mathrm{Reg}(eSe) = \mathrm{Gr}(eSe)$, for each $e \in E(S)$. Then Lemma 7.7 yields

$$\mathrm{Reg}_M(S) = \bigcup_{e \in E(S)} \mathrm{Reg}(eSe) = \bigcup_{e \in E(S)} \mathrm{Gr}(eSe) = \mathrm{Gr}(S).$$

(ii)$\Leftrightarrow$(iii) This follows immediately from Lemma 7.6. □

A bi-ideal of a π-regular semigroup is not necessarily π-regular. But, the principal bi-ideals generated by idempotents, that is to say, the local submonoids of a semigroup, have the following property:

Lemma 7.10 *Let S be a π-regular or a completely π-regular semigroup. Then for each $e \in E(S)$, the local submonoid eSe has the same property.*

Proof. Let S be a π-regular semigroup, and let $e \in E(S)$ and $a \in eSe$. Then there exists $n \in \mathbf{Z}^+$ such that $a^n \in \mathrm{Reg}(S)$, and based on Lemma 7.6 we have that $a^n \in eSe \cap \mathrm{Reg}(S) = \mathrm{Reg}(eSe)$. Thus eSe is π-regular, for every $e \in E(S)$.

Let S be a completely π-regular semigroup and let $a \in eSe$, for some $e \in E(S)$. Then there exists $n \in \mathbf{Z}^+$ such that $a^n \in \mathrm{Gr}(S)$, so again based on Lemma 7.6 it follows that $a^n \in eSe \cap \mathrm{Gr}(S) = \mathrm{Gr}(eSe)$. Hence, eSe is completely π-regular, for each $e \in E(S)$. □

A semigroup S is called *locally completely π-regular* if it is π-regular and eSe is completely π-regular, for every $e \in E(S)$, and it is called *locally uniformly π-regular* if S is π-regular and eSe is uniformly π-regular, for every $e \in E(S)$. The main result of this section is the following theorem that characterizes locally uniformly π-regular semigroups.

Theorem 7.10 *The following conditions on a semigroup S are equivalent:*

(i) *S is locally uniformly π-regular;*
(ii) *S is π-regular and if $a \in S$, $n \in \mathbf{Z}^+$ and $a' \in V(a^n)$, then $a'Sa^n$ (a^nSa') is uniformly π-regular;*
(iii) *S is π-regular and* $\mathrm{Reg}_M(S) = \mathrm{Gr}(S)$*;*
(iv) *S is π-regular and* $\mathrm{Reg}(eSe) = \mathrm{Gr}(eSe)$*, for each $e \in E(S)$;*
(v) *S is π-regular and* $\mathrm{reg}(eSe) = \mathrm{Gr}(eSe)$*, for each $e \in E(S)$;*
(vi) *S is locally completely π-regular, $\langle E(S)\rangle$ is locally uniformly π-regular and*

$$(\forall e, f, g \in E(S)) \quad e \geq f,\ e \geq g \ \&\ f|g \ \Rightarrow f|_{\langle E(eSe)\rangle} g.$$

Proof. (i)$\Leftrightarrow$(iv) This equivalence is an immediate consequence of the definition of a uniformly π-regular semigroup.

(i)$\Rightarrow$(ii) Let $a \in S$, $n \in \mathbf{Z}^+$ and $a' \in V(a^n)$. Set $e = a'a^n$ and $f = a^na'$. Then

$$eSe = a'a^nSa'a^n \subseteq a'Sa^n = a'a^na'Sa^na'a^n \subseteq a'a^nSa'a^n = eSe,$$

whence $eSe = a'Sa^n$, and from (i) it follows that $eSe = a'Sa^n$ is uniformly π-regular. In a similar way we prove that $a^nSa' = fSf$ is uniformly π-regular.

(ii)$\Rightarrow$(i) For each $e \in E(S)$, from $e \in V(e)$ and (ii) it follows that eSe is uniformly π-regular.

(iii)$\Leftrightarrow$(iv)$\Leftrightarrow$(v) These equivalences are immediate consequences of Lemma 7.9.

(i)$\Rightarrow$(vi) It is clear that S is locally completely π-regular. Since S is π-regular, then based on Lemma 2.11 we have that $\langle E(S)\rangle$ is π-regular, which implies that $e\langle E(S)\rangle e$, based on Lemma 7.10, is also π-regular, for every $e \in E(S)$. Based on (i)$\Leftrightarrow$(iv) we also have that $\mathrm{Reg}(eSe) = \mathrm{Gr}(eSe)$ for every $e \in E(S)$. Further, from

$$a \in \mathrm{Reg}(e\langle E(S)\rangle e) \subseteq \mathrm{Reg}(eSe) = \mathrm{Gr}(eSe)$$

it follows that for $a \in \mathrm{Reg}(e\langle E(S)\rangle e)$ there are $x \in eSe$ and $y \in e\langle E(S)\rangle e$ such that $a = axa = aya$ and $ax = xa \in E(eSe)$. Now we have that

$$a = axa = xa^2 \subseteq E(eSe)e\langle E(S)\rangle ea^2 \subseteq e\langle E(S)\rangle ea^2,$$

i.e. $a \in \mathrm{LReg}(e\langle E(S)\rangle e)$. Therefore $\mathrm{Reg}(e\langle E(S)\rangle e \subseteq \mathrm{LReg}(e\langle E(S)\rangle e)$ and $e\langle E(S)\rangle e$ is π-regular, which based on Theorem 7.4 means that $e\langle E(S)\rangle e$ is uniformly π-regular for every $e \in E(S)$. Thus $\langle E(S)\rangle$ is locally uniformly π-regular.

Let $e, f, g \in E(S)$, such that $e \geq f$, $e \geq g$ and $f|g$ in S. Then $f, g \in E(eSe)$ and $f|g$ in eSe and based on Theorem 7.4 we have that $f|g$ in $\langle E(eSe)\rangle$.

(vi)$\Rightarrow$(i) Let $e \in E(S)$. Based on Lemma 2.11 we have that $\langle E(eSe)\rangle$ is completely π-regular. On the other hand, from the hypothesis it follows that $e\langle E(S)\rangle e$ is uniformly π-regular. On the other hand $\langle E(eSe)\rangle \subseteq e\langle E(S)\rangle e$, so based on Theorem 7.4 and Lemma 2.5 we have that

$$\begin{aligned}\mathrm{Reg}(\langle E(eSe)\rangle) &= \langle E(eSe)\rangle \cap \mathrm{Reg}(e\langle E(S)\rangle e)\\ &= \langle E(eSe)\rangle \cap \mathrm{Gr}(e\langle E(S)\rangle e) = \mathrm{Gr}(\langle E(eSe)\rangle).\end{aligned}$$

Let $f, g \in E(eSe)$ such that $f|g$ in eSe. Then $e \geq f$, $e \geq g$ and $f|g$ in eSe, and based on the hypothesis we have that $f|g$ in $\langle E(eSe)\rangle$. Therefore, from Theorem 7.4 we obtain that eSe is uniformly π-regular for every $e \in E(S)$. Hence S is locally uniformly π-regular. $\square$

References

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7.5 Bands of π-groups

In this section we will discuss a band decomposition of semigroups whose components are π-groups, i.e. a nil-extension of groups.

First we prove the following theorem.

Theorem 7.11 *Let S be a π-regular semigroup and let for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that*

$$(ab)^n \in a^2Sb^2. \tag{2}$$

Then S is a semilattice of retractive nil-extensions of completely simple semigroups.

Proof. Assume $a \in \mathrm{Reg}(S)$, $x \in V(a)$. Based on (1), $(ax)^n \in a^2Sx^2$, for some $n \in \mathbf{Z}^+$, whence $a = axa = (ax)^n a \in a^2Sx^2a \subseteq a^2S$. Similarly we prove that $a \in Sa^2$. Based on this, $a \in \mathrm{Gr}(S)$, i.e. $\mathrm{Reg}(S) = \mathrm{Gr}(S)$, so according to Theorem 7.4, S is a semilattice Y of completely Archimedean semigroups S_α, $\alpha \in Y$. For $\alpha \in Y$, let S_α be a nil-extension of a completely simple semigroup K_α.

Assume $\alpha \in Y$, $e, f \in E(S_\alpha)$, $a \in T_e$. We will prove that

$$af = eaf \quad \text{and} \quad fa = fae. \tag{3}$$

First we will prove that for every $m \in \mathbf{Z}^+$ there exists $n \in \mathbf{Z}^+$ and $u \in S$ such that

$$(af)^n = a^m uf. \tag{4}$$

It is evident that (4) holds for $m = 1$. Assume that $(af)^n = a^m uf$ holds for some $m, n \in \mathbf{Z}^+$ and some $u \in S$. Then based on (2) we obtain that there exists $k \in \mathbf{Z}^+$ and $v \in S$ such that $(a^m uf)^k = a^{2m} v(uf)^2$, whence

$$(af)^{nk} = ((af)^n)^k = (a^m uf)^k = a^{2m} v(uf)^2 = a^{m+1} wf,$$

where $w = a^{m-1} vufu$. Now by induction for every $m \in \mathbf{Z}^+$ there exists $n \in \mathbf{Z}^+$ and $u \in S$ such that (4) holds.

Let $m \in \mathbf{Z}^+$ such that $a^m \in G_e$, and let $n \in \mathbf{Z}^+$, $u \in S$ such that (4) holds. Since $af \in K_\alpha = \mathrm{Gr}(S_\alpha)$, then $af = (af^2)y$, for some $y \in S$, whence

$$af = (af)^n y^{n-1} = a^m ufy^{n-1} = ea^m ufy^{n-1} = eaf.$$

By this we have proved the first part of statement (3). In a similar way we prove the second part of (3).

Now, we define the mapping $\varphi : S_\alpha \mapsto K_\alpha$ with

$$a\varphi = ae, \qquad \text{if } a \in T_e, e \in E(S_\alpha).$$

Assume $a \in T_e$, $b \in T_f$, $e, f \in E(S_\alpha)$, and assume that $ab \in T_g$, for some $g \in E(S_\alpha)$. Then based on (3) and based on Theorem 1.8 we obtain that

$$(ab)\varphi = abg = afbg = eafbg = eabg = eab = aeb = aebf = (a\varphi)(b\varphi).$$

Thus, φ is a homomorphism. Since $a\varphi = a$, then φ is a retraction, so S_α is a retractive nil-extension of K_α. □

From Theorem 7.11 we obtain the following corollary.

Corollary 7.3 *Let S be a π-regular semigroup and let for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $(ab)^n \in a^2Sa$. Then S is a semilattice of retractive nil-extensions of left groups.*

Proof. Assume $a, b \in S$. Then there exist $m, n \in \mathbf{Z}^+$ such that $(ab)^m \in a^2Sa$ and $(ba)^n \in b^2Sb$, whence $(ab)^{n+1} \in ab^2Sb^2$, so

$$(ab)^{m+n+1} \in a^2Saab^2Sb^2 \subseteq a^2Sb^2.$$

Thus, based on Theorem 7.11, S is a semilattice Y of semigroups S_α, $\alpha \in Y$, and for $\alpha \in Y$, S_α is a retractive nil-extension of a completely simple semigroup K_α. Just like in Theorem 7.8 we prove that K_α is a left group.□

By means of the following theorem we describe the relationship between a decomposition into a band of π-groups and the retraction of a semigroup on its regular part.

Theorem 7.12 *Let S be a band of π-groups and let* Reg(S) *be a subsemigroup of S. Then* Reg(S) *is a band of groups and a retract of S.*

Conversely, if S has a retract K which is a band of groups and if $\sqrt{K} = S$, then S is a band of π-groups.

Proof. Let S be a band B of π-groups S_i, $i \in B$, and let Reg(S) be a subsemigroup of S. For $i \in B$, let S_i be a nil-extension of a group G_i with the identity e_i. Then, $\mathrm{Reg}(S) = \mathrm{Gr}(S) = \cup\{G_i \,|\, i \in B\}$, so it is evident that Reg(S) is a band B of groups G_i, $i \in B$. Assume $i, j \in B$. From $e_ie_{ij} = (e_ie_{ij})e_{ij} \in S_{ij}G_{ij} = G_{ij}$ and $e_{ij}e_j = e_{ij}(e_{ij}e_j) \in G_{ij}S_{ij} = G_{ij}$ we obtain that

$$\begin{aligned}(e_ie_{ij})^2 &= e_i(e_{ij}(e_ie_{ij})) = e_i(e_ie_{ij}) = e_ie_{ij} \in S_{ij}\\ (e_{ij}e_j)^2 &= ((e_{ij}e_j)e_{ij})e_j = (e_{ij}e_j)e_j = e_{ij}e_j \in S_{ij},\end{aligned}$$

so since S_{ij} has an unique idempotent e_{ij}, then $e_ie_{ij} = e_{ij}e_j = e_{ij}$.

Now, we define the mapping $\varphi : S \mapsto \mathrm{Reg}(S)$ with:

$$x\varphi = xe_i, \quad \text{if} \quad x \in S_i, i \in B.$$

For $x_i \in S_i$, $x_j \in S_j$, $i, j \in B$ we have that:

$$\begin{aligned}
(x_i\varphi)(x_j\varphi) &= (x_ie_i)(x_je_j) \\
&= e_{ij}(x_ie_i)(x_je_j)e_{ij} && \text{(because } x_ie_ix_je_j \in G_iG_j \subseteq G_{ij}\text{)} \\
&= e_{ij}e_ix_ix_je_je_{ij} && \text{(from Theorem 1.8)} \\
&= e_{ij}e_ix_ix_je_{ij}e_je_{ij} && \text{(because } e_{ij}e_ix_ix_j \in G_{ij}S_{ij} \subseteq G_{ij}\text{)} \\
&= e_{ij}e_ix_ix_je_{ij} && \text{(because } e_{ij}e_j = e_{ij}\text{)} \\
&= e_{ij}e_ie_{ij}x_ix_je_{ij} && \text{(because } x_ix_je_{ij} \in S_{ij}G_{ij} \subseteq G_{ij}\text{)} \\
&= e_{ij}x_ix_je_{ij} && \text{(because } e_{ij}e_i = e_{ij}\text{)} \\
&= x_ix_je_{ij} && \text{(because } x_ix_je_{ij} \in G_{ij}\text{)} \\
&= (x_ix_j)\varphi.
\end{aligned}$$

Hence, φ is a homomorphism, so since $a\varphi = a$, for every $a \in \mathrm{Reg}(S)$, then φ is a retraction from S onto $\mathrm{Reg}(S)$.

Conversely, if S has a retract K which is a band B of groups G_i, $i \in B$, if $\sqrt{K} = S$, and if we assume that φ is a retraction from S onto K, then S is a band B of a semigroups $S_i = G_i\varphi^{-1}$, $i \in B$, since for every $i \in B$ it holds that $S_i \cap K = G_i$, $\sqrt{G_i} = S_i$, then S_i are π-groups. $\square$

From Theorem 7.12 it immediately follows:

Corollary 7.4 *A semigroup S is a retractive nil-extension of a completely simple semigroup if and only if S is a matrix of π-groups.*

Corollary 7.5 *A semigroup S is a retractive nil-extension of a left group if and only if S is a left zero band of π-groups.*

Let S be a semigroup. For $e \in E(S)$, by T_e we denote the set

$$T_e = \sqrt{G_e} = \{x \in S \mid (\exists n \in \mathbf{Z}^+)\ x^n \in G_e\}.$$

According to Theorem 1.8 and Theorem 1.7, for $e, f \in E(S)$, $e \neq f$, is $T_e \cap T_f = \emptyset$. On a semigroup S we define the relation $\mathcal{T}$ by:

$$a\mathcal{T}b \ \Leftrightarrow\ ((\exists e \in E(S))\ a, b \in T_e)\ \vee\ a = b, \qquad a, b \in S.$$

It is clear that $\mathcal{T}$ is an equivalence relation on S. If S is completely simple, then

$$a\mathcal{T}b \;\Leftrightarrow\; (\exists e \in E(S))\; a, b \in T_e.$$

Now, we prove the main result of this section.

Theorem 7.13 *The following conditions on a semigroups S are equivalent:*

(i) *S is a band of π-groups;*

(ii) *S is π-regular and for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $(ab)^n \in a^2bSab^2$;*

(iii) *S is completely π-regular and for all $a, b \in S$ is $ab\mathcal{T}a^2b\mathcal{T}ab^2$;*

(iv) *S is completely π-regular and $(xy)^0 = (x^2y)^0 = (xy^2)^0$.*

Proof. (i)$\Rightarrow$(ii) Let S be a band B of π-groups S_i, $i \in B$. Let $a \in S_i$, $b \in S_j$, $i, j \in B$. Then $ab, a^2b, ab^2 \in S_{ij}$, so (ii) holds.

(ii)$\Rightarrow$(iii) Let (ii) hold. Then based on Theorem 7.11 S is a semilattice Y of semigroups $S_\alpha, \alpha \in Y$, and for $\alpha \in Y$, S_α is a retractive nil-extension of a completely simple semigroup K_α, while based on Corollary 7.4, for every $\alpha \in Y$, S_α is a matrix of π-groups.

Assume $a, b \in S$. Then $ab, a^2b, ab^2 \in S_\alpha$, for some $\alpha \in Y$. Assume that S_α is a matrix $I \times \Lambda$ of π-groups $T_{i\lambda}$, $i \in I$, $\lambda \in \Lambda$. Assume that $ab \in T_{i\lambda}$, $a^2b \in T_{j\mu}$, $ab^2 \in T_{l\nu}$, for some $i, j, l \in I$, $\lambda, \mu, \nu \in \Lambda$. Let $e_{j\mu}$ be an idempotent from $T_{j\mu}$. Then $e_{j\mu}a^2b \in T_{j\mu}^2 \subseteq T_{j\mu}$ and

$$e_{j\mu}a^2b = e_{j\mu}e_{j\mu}aab \in T_{j\mu}S_{\alpha\beta}T_{i\lambda} \subseteq T_{j\lambda},$$

so $\mu = \lambda$. Similarly we prove that $l = i$. Also, from (ii) we obtain that there exists $n \in \mathbf{Z}^+$ and $u \in S$ such that $(ab)^n = a^2buab^2$, whence $uab^2a^2bu \in S_{\alpha\beta}$, so

$$(ab)^{2n} = a^2b(uab^2a^2bu)ab^2 \in T_{j\lambda}S_{\alpha\beta}T_{i\nu} \subseteq T_{j\nu}.$$

Since $(ab)^{2n} \in T_{i\lambda}$, then $j = i$ and $\nu = \lambda$. Therefore, $ab, a^2b, ab^2 \in T_{i\lambda}$, so (iii) holds.

(iii)$\Rightarrow$(i) Assume $a, b \in S$. Let $a \in T_e$, $b \in T_f$, for some $e, f \in E(S)$. Based on (iii), $ab\mathcal{T}a^kb$, for every $k \in \mathbf{Z}^+$. Let $k \in \mathbf{Z}^+$ such that $a^k \in G_e$. Then

$$eb = a^k(a^k)^{-1}b\mathcal{T}(a^k)^2(a^k)^{-1}b = a^keb = a^kb\mathcal{T}ab.$$

Thus, $ab\mathcal{T}eb$. Similarly we prove that $eb\mathcal{T}ef$. Hence, $ab\mathcal{T}ef$, so $\mathcal{T}$ is a congruence relation on S. It is evident that $\mathcal{T}$ is a band congruence and every $\mathcal{T}$-class is a π-group. Therefore, (i) holds.

(iii)⇔(iv) This follows immediately. □

Recall that a band S is a *normal* if for all $x, y, z \in S$ is $xyzx = xzyx$.

Based on Theorem 7.13 we gave the characterizations of a band of π-groups in general. Now, we will discuss some important types of bands of π-groups: normal bands, semilattices and Reédei bands of π-groups.

Theorem 7.14 *The following conditions on a semigroups S are equivalent:*

(i) *S is a normal band of π-groups;*
(ii) *S is π-regular and for all $a, b, c \in S$ there exists $n \in \mathbf{Z}^+$ such that $(abc)^n \in acSac$;*
(iii) *S is completely π-regular and for all $a, b, c, d \in S$ is $abcd\mathcal{T}acbd$;*
(iv) *S is completely π-regular and $(xyzu)^0 = (xzyu)^0$.*

Proof. (i)⇒(iii) This follows from Theorem 5.12.

(iii)⇒(ii) Let (iii) hold. It is evident that S is π-regular. Assume $a, b, c \in S$. From (iii) we have that

$$\begin{aligned} (abc)^2 &= ab(cab)c\mathcal{T}a(cab)bc = acab^2c \qquad \text{and} \\ (abc)^2 &= a(bca)bc\mathcal{T}ab(bca)c = ab^2cac, \end{aligned}$$

whence it follows that there exist $m, n \in \mathbf{Z}^+$ such that

$$(abc)^{2m} \in acS \quad \text{and} \quad (abc)^{2n} \in Sac,$$

so $(abc)^{2m+2n} \in acSac$. Hence, (ii) holds.

(ii)⇒(i) Let (ii) hold. Based on Corollary 5.7 S is a normal band B of t-Archimedean semigroups S_i, $i \in B$. Assume $a \in \mathrm{Reg}(S)$, $x \in V(a)$. Based on (ii), there exists $n \in \mathbf{Z}^+$ such that $ax = (axax)^n \in aaxSaax$, whence

$$a = axa \in a^2xSa^2xa \subseteq a^2Sa^2.$$

Thus, $a \in \mathrm{Gr}(S)$, so, S is a completely π-regular semigroup. According to Lemma 2.8, S_i are completely π-regular semigroups, and based on Theorem 3.18, S_i are π-groups.

(iii)⇔(iv). This follows immediately. □

Theorem 7.15 *The following conditions on a semigroups S are equivalent:*

(i) *S is a semilattice of π-groups;*

(ii) *S is π-regular and a semilattice of t-Archimedean semigroups;*

(iii) *S is a semilattice of completely Archimedean semigroups and for all $e, f \in E(S)$ there exists $n \in \mathbf{Z}^+$ such that $(ef)^n = (fe)^n$;*

(iv) *S is a semilattice of completely Archimedean semigroups and every regular element from S has a unique inverese element;*

(v) *S is completely π-regular and for all $a, b \in S$ is $ab\mathcal{T}ba$;*

(vi) *S is completely π-regular and $(xy)^0 = (yx)^0$;*

(vii) *S is π-regular and $a = axa$ implies $ax = xa$;*

(viii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n \in b^{2n}Sa^{2n}$.

Proof. (i)⇒(viii) Let S be a semilattice Y of π-groups S_α, $\alpha \in Y$, and for $\alpha \in Y$, let S_α be a nil-extension of a group G_α. Assume $a, b \in S$. Then $ab, b^m a^m \in S_\alpha$, for some $\alpha \in Y$ and for all $m \in \mathbf{Z}^+$. Then there exists $n \in \mathbf{Z}^+$ such that $(ab)^n \in G_\alpha$. Now for $m = 2n$ we have that $(b^{2n}a^{2n})^k \in G_\alpha$, for some $k \in \mathbf{Z}^+$. Therefore, $(ab)^n \in (b^{2n}a^{2n})^k G_\alpha (b^{2n}a^{2n})^k \subseteq b^{2n}Sa^{2n}$. Thus, (viii) holds.

(viii)⇒(ii) This follows from Corollary 5.3.

(ii)⇒(i) This follows from Lemma 2.7 and from Theorem 3.18.

(viii)⇒(iii) From (viii), and by Theorem 7.8, S is a semilattice of completely Archimedean semigroups. Assume $e, f \in E(S)$. By (viii), $(ef)^n = (fe)^n x (fe)^n$, for some $n \in \mathbf{Z}^+$, $x \in S$, so $(fe)^{n+1} = f(ef)^n e = f(fe)^n x (fe)^n e = (fe)^n x (fe)^n = (ef)^n$ and $(ef)^n = (fe)^n x (fe)^n = (fe)^n x (fe)^n e = (ef)^n e$, whence $(ef)^{n+1} = (ef)^n ef = (ef)^n f = (ef)^n$. Thus, $(ef)^{n+1} = (fe)^{n+1}$, so (iii) holds.

(iii)⇒(i) From (iii), for $e, f \in E(S)$ we obtain that $(ef)^n = (fe)^n$, for some $n \in \mathbf{Z}^+$, whence $(ef)^n = e(ef)^n f = e(fe)^n f = (ef)^{n+1}$, so based on Theorem 7.6, S is a semilattice Y of semigroups S_α, $\alpha \in Y$, and for $\alpha \in Y$, S_α is a nil-extension of a rectangular group K_α. Assume $\alpha \in Y$, $e, f \in E(K_\alpha)$. Since $E(K_\alpha)$ is a rectangular band, then from (iii) we obtain that $ef = fe$, so $|E(K_\alpha)| = 1$, i.e. K_α is a group.

(i)⇒(v) Let S be a semilattice Y of π-groups S_α, $\alpha \in Y$. Then $ab, ba \in S_\alpha$, for some $\alpha \in Y$, so for some $e \in E(S_\alpha)$ we have that $ab, ba \in S_\alpha = T_e$, whence $ab\mathcal{T}ba$.

(v)⇒(vi) and (vi)⇒(vii) This is evident.

(vii)$\Rightarrow$(iv) If (vii) hold, then $a = axa$ implies $ax = xa$, whence $a = axaxa = axxaa = ax^2a^2$, so based on Theorem 7.6, S is a semilattice of completely Archimedean semigroups. Assume $a \in \mathrm{Reg}(S)$, $x, y \in V(a)$. Based on (vii), $ax = xa$ and $ay = ya$, whence

$$\begin{aligned} x &= xax = x^2a = x^2aya = xay = axy \\ &= axyay = axay^2 = ay^2 = yay = y. \end{aligned}$$

Hence, (iv) holds.

(iv)$\Rightarrow$(i) From (iv) it follows that every inverse of every idempotent from S is also an idempotent, so based on Theorem 7.6, S is a semilattice Y of semigroups S_α, $\alpha \in Y$, and for $\alpha \in Y$, S_α is a nil-extension of a rectangular group K_α. Assume $\alpha \in Y$, $e, f \in E(K_\alpha)$. Then $E(K_\alpha)$ is a rectangular band, so $e, f \in V(e)$, whence, based on (iv), $e = f$. Thus, $|E(K_\alpha)| = 1$, so K_α is a group. Therefore, (i) holds. $\square$

A semigroup S is an *ordinal sum* Y of semigroups S_α, $\alpha \in Y$ if S is a chain Y of semigroups S_α, $\alpha \in Y$, and for $\alpha, \beta \in Y$, from $\alpha < \beta$, $a \in S_\alpha$, $b \in S_\beta$ it follows that $ab = ba = a$. Based on the following lemma we give the structural characterization of Rédei bands:

Lemma 7.11 *A semigroup S is Rédei band if and only if S is an ordinal sum of singular bands.*

Proof. Let S be a Rédei band. Based on Lemma 7.3, S is a chain Y of rectangular bands S_α, $\alpha \in Y$, while based on Lemma 7.4, S_α are singular bands. Assume that $\alpha, \beta \in Y$ are such that $\alpha < \beta$, and assume that $a \in S_\alpha$, $b \in S_\beta$. Then $a, ab, ba \in S_\alpha$ and $ab, ba \in \{a, b\}$, whence we obtain that $ab = ba = a$.

The converse follows immediately. $\square$

Theorem 7.16 *The following conditions on a semigroups S are equivalent:*

(i) *S is a Rédei band of π-groups;*
(ii) *S has a retract K which is a Rédei band and $\sqrt{K} = S$;*
(iii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in (ab)^nS(ab)^n \vee b^n \in (ab)^nS(ab)^n$.

Proof. (i)$\Rightarrow$(ii) Let S be a Rédei band B of π-groups S_i, $i \in B$. For $i \in B$, let S_i be a nil-extension of a group G_i with the identity e_i. It is evident that

$E(S) = \{e_i \mid i \in B\}$. Assume that $e_i, e_j \in E(S)$, $i, j \in B$. Then $e_ie_j \in S_{ij}$. If $ij = i$, then $e_ie_j \in S_i$, so $e_ie_j = e_i(e_ie_j) \in G_iS_i \subseteq G_i$ whence

$$(e_ie_j)^2 = ((e_ie_j)e_i)e_j = (e_ie_j)e_j = e_ie_j.$$

Similarly, from $ij = j$ it follows that $(e_ie_j)^2 = e_ie_j$. Thus, $E(S)$ is a subsemigroup of S, so based on Lemma 7.1 $\mathrm{Reg}(S)$ is a subsemigroup of S, whence based on Theorem 7.12 we obtain that (ii) holds.

(ii)$\Rightarrow$(i) This follows from Theorem 7.12.

(i)$\Rightarrow$(iii) Let S be a Rédei band B of π-groups S_i, $i \in B$. For $i \in B$, let S_i be a nil-extension of group G_i. Assume $a, b \in S$. Then $a \in S_i$, $b \in S_j$, for some $i, j \in B$. If $ij = i$, then $ab \in S_i$, so there exists $n \in \mathbf{Z}^+$ such that $(ab)^n, a^n \in G_i$, whence

$$a^n \in (ab)^nG_i(ab)^n \subseteq (ab)^nS(ab)^n.$$

Similarly, from $ij = j$ it follows that

$$b^n \in (ab)^nS(ab)^n,$$

for some $n \in \mathbf{Z}^+$. Thus, (iii) holds.

(iii)$\Rightarrow$(i) Let (iii) hold. It is evident that S is completely π-regular. Also, from (iii) it follows that $e \in Sf$ or $f \in eS$, for all $e, f \in E(S)$, so $E(S)$ is a Rédei band. Based on Lemma 7.11 and Corollary 7.1, S is a chain Y of semigroups S_α, $\alpha \in Y$, and for $\alpha \in Y$, S_α is a nil-extension of a semigroup K_α, where K_α is a left or right group.

Assume $\alpha \in Y$, $a, b \in S_\alpha$. Let K_α be a left group. Let $a \in T_e$, $b \in T_f$, $e, f \in E(S_\alpha)$, $e \neq f$. Based on (iii) we obtain that there exists $n \in \mathbf{Z}^+$ such that

$$a^n \in (af)^nS(af)^n \quad \text{or} \quad f \in (af)^nS(af)^n.$$

Assume that $f \in (af)^nS(af)^n \subseteq afSaf$, i.e. $f = afuaf$, for some $u \in S$. Since $af \in S_\alpha K_\alpha \subseteq K_\alpha$, then $af \in G_g$, for some $g \in E(S_\alpha)$. Now, based on Lemma 3.15 we obtain that

$$f = afuaf = g(afuaf)g = gfg \in gS_\alpha g = G_g,$$

whence $f = g$, i.e. $af \in G_f$. Also, $fa = f(fa) \in G_fK_\alpha \subseteq G_f$, because K_α is a left group, so $af = f(af) = (fa)f = fa$. Since $a^k \in G_e$, for some $k \in \mathbf{Z}^+$, and since K_α is a left group, then

$$a^k = a^ke = a^kef = a^kf = fa^k \in G_fG_e \subseteq G_f,$$

that is impossible. Thus, $a^n \in (af)^n S(af)^n$, whence $a^n \in afS_\alpha af \subseteq afK_\alpha af$, so based on Lemma 3.15 $a^n \mathcal{H} af$ in K_α. Thus, $af \in G_e$. In a similar way we prove that $be \in G_f$, so from Lemma 1.8 it follows that

$$be = fbe = bfe = bf = fb \text{ and } af = eaf = aef = ae = ea,$$

whence

$$abe = afb = eab.$$

Assume that $(ab)^m \in G_g$, for some $g \in E(S_\alpha)$, $m \in \mathbf{Z}^+$. Then

$$(ab)^m e \in G_g G_e \subseteq G_g \text{ and } (ab)^m e = e(ab)^m \in G_e G_g \subseteq G_g.$$

Hence, $g = e$, i.e. $(ab)^m \in G_e$, so $ab \in T_e = T_{ef}$. Thus, S_α is a left zero band $E(S_\alpha)$ of π-groups T_e, $e \in E(S_\alpha)$. If K_α is a right group, then in a similar way we prove that S_α is a right zero band $E(S_\alpha)$ of π-groups T_e, $e \in E(S_\alpha)$.

Assume $a \in T_e \subseteq S_\alpha$, $b \in T_f \subseteq S_\beta$, $\alpha, \beta \in Y$, $\alpha \neq \beta$. Let $\alpha < \beta$, i.e. $\alpha\beta = \beta\alpha = \alpha$ (a similar case is $\beta < \alpha$). Since $E(S)$ is a Rédei band and since $ef, fe, e \in S_\alpha$, $f \notin S_\alpha$, then $ef = fe = e$. Based on (iii), there exists $n \in \mathbf{Z}^+$ such that

$$b^n \in (be)^n S(be)^n \text{ or } e \in (be)^n S(be)^n.$$

If $b^n = (be)^n u(be)^n$, for some $u \in S$, then $u \in S_\gamma$, for some $\gamma \in Y$, so $\alpha\beta\gamma = \beta$, whence $\alpha\beta = \beta$, which is impossible. Hence, $e \in (be)^n S(be)^n$, whence

$$e \in beS_\alpha be.$$

Since $be = (be)e \in S_\alpha K_\alpha \subseteq K_\alpha$, from Lemma 3.15 it follows that $be \in G_e$. Similar we prove that $eb \in G_e$, so from Lemma 1.8 it follows that $eb = (eb)e = e(be) = be$ and $abe = aeb = eab$. Let $(ab)^m \in G_g$, for some $g \in E(S_\alpha)$, $m \in \mathbf{Z}^+$. Based on Lemma 3.15 we have that

$$\begin{aligned} (ab)^m &= (ab)^m g = (ab)^m geg = (ab)^m eg = e(ab)^m g = e(ab)^m \\ &= ee(ab)^m = e(ab)^m e \in eS_\alpha e = G_e. \end{aligned}$$

Hence, $(ab)^m \in G_e$, i.e. $ab \in T_e = T_{ef}$. Thus, S is a Rédei band $E(S)$ of π-group T_e, $e \in E(S)$. $\square$

From Theorem 7.16 it immediately follows that

Corollary 7.6 *A semigroup S is a Rédei band of periodic π-groups if and only if S is π-regular and for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $(ab)^n \in \langle a\rangle \cup \langle b\rangle$.*

Exercises

1. A semigroup S which satisfies the condition

$$x_1x_2\cdots x_{n+1}\in\langle x_1\rangle\cup\langle x_2\rangle\cup\cdots\cup\langle x_{n+1}\rangle,$$

for all $x_1,x_2,\cdots,x_{n+1}\in S$ we call a $\mathcal{U}_{n+1}$-*semigroup*. A semigroup $\mathcal{U}_2$ we call $\mathcal{U}$-*semigroup* for short. Prove that the following conditions hold:

(a) G is a $\mathcal{U}_{n+1}$-group if and only if G is a $\mathcal{U}$-group;
(b) G is a $\mathcal{U}$-group if and only if G is a cyclic group of the order p^k, $k\in\mathbf{Z}^+$, or quasi-cyclic $\mathbf{Z}_{p^\infty}$, for some prime p.

2. Let S be a monogenic semigroup. Then S is a $\mathcal{U}$- ($\mathcal{U}_{3k}$-, $\mathcal{U}_{3k+1}$-, $\mathcal{U}_{3k+2}$-) semigroup if and only if S is an ideal extension of a cyclic group by a 5- ((6k+1)-, (6k+3)-, (6k+5)-) nilpotent monogenic semigroup.

3. The following conditions on a semigroup S are equivalent:

(a) S is a regular $\mathcal{U}_{n+1}$-semigroup;
(b) S is a regular $\mathcal{U}$-semigroup;
(c) S is an ordinal sum of $\mathcal{U}$-groups and singular bands.

4. A band (chain) Y of semigroups S_α, $\alpha\in Y$, is a $\mathcal{U}_{n+1}$-*band* (*chain*) *of semigroups* S_α, $\alpha\in Y$, if

$$x_1x_2\cdots x_{n+1}\in\langle x_1\rangle\cup\langle x_2\rangle\cup\cdots\cup\langle x_{n+1}\rangle,$$

for all $x_1\in S_{\alpha_1},x_2\in S_{\alpha_2},\cdots,x_{n+1}\in S_{\alpha_{n+1}}$, where there are $i,j\in\{1,2,\ldots,n+1\}$ such that $S_{\alpha_i}\neq S_{\alpha_j}$. The $\mathcal{U}_2$-band (chain) of semigroups we call the $\mathcal{U}$-*band* (*chain*) *of semigroups*.

Prove that the following conditions on a semigroup S are equivalent:

(a) S is a $\mathcal{U}_{n+1}$-semigroup;
(b) S is a $\mathcal{U}_{n+1}$-chain of ideal extension of $\mathcal{U}$-groups by $\mathcal{U}_{n+1}$-nil-semigroups and a retractive extension of singular bands by $\mathcal{U}_{n+1}$-nil-semigroups;
(c) S is a $\mathcal{U}_{n+1}$-band of ideal extension of $\mathcal{U}$-groups by $\mathcal{U}_{n+1}$-nil-semigroups.

5. Let S be a $\mathcal{U}_{n+1}$-semigroup. Then $\mathrm{Reg}(S)$ is a retract of S.

6. A semigroup S is a $\mathcal{U}_{n+1}$-semigroup and $\mathrm{Reg}(S)$ is an ideal of S if and only if

$$x_1x_2\cdots x_{n+1}\in\cup_{i=1}^{n+1}\{x_i^k\mid k\in\mathbf{Z}^+,k\geq 2\},$$

for all $x_1,x_2,\cdots,x_{n+1}\in S$.

7. A semigroup S is an n-inflation of Rédei's band if and only if

$$x_1x_2\cdots x_{n+1}\in\{x_1^{n+2},x_2^{n+2},\ldots,x_{n+1}^{n+2}\},$$

for all $x_1,x_2,\cdots,x_{n+1}\in S$.

8. A semigroup S in which for all $x_1, x_2, \cdots, x_{n+1} \in S$ there exists $m \in \mathbf{Z}^+$ such that
$$(x_1x_2\cdots x_{n+1})^m \in \langle x_1\rangle \cup \langle x_2\rangle \cup \cdots \cup \langle x_{n+1}\rangle,$$
we call a $\mathcal{GU}_{n+1}$-*semigroup*. The $\mathcal{GU}_2$-semigroup is the $\mathcal{GU}$-*semigroup*.

Prove that S is a π-regular $\mathcal{GU}_{n+1}$-semigroup if and only if S is a π-regular $\mathcal{GU}$-semigroup.

9. A chain Y of semigroups S_α, $\alpha \in Y$, is a $\mathcal{GU}$-*chain* of semigroups S_α, $\alpha \in Y$, if for all $\alpha, \beta \in Y$, $\alpha \neq \beta$, and for all $a \in S_\alpha$, $b \in S_\beta$ there exists $m \in \mathbf{Z}^+$ such that $(ab)^m \in \langle a\rangle \cup \langle b\rangle$.

Prove that the following conditions on a semigroup S are equivalent:

(a) S is a Rédei's band of a periodic π-groups;
(b) S is a π-regular $\mathcal{GU}$-semigroup;
(c) S is a periodic $\mathcal{GU}$-semigroup;
(d) S is a $\mathcal{GU}$-chain of retractive nil-extensions of periodic left and right groups;
(e) S has a retract T which is a regular $\mathcal{GU}$-semigroup and $\sqrt{T} = S$.

10. Let $\mathfrak{C}$ be a class of semigroups with a modular lattice of subsemigroups, or a class of semigroups with a distributive lattice of subsemigroups or a class of $\mathcal{U}$-semigroups. Then the following conditions on a semigroup S are equivalent:

(a) $S \in \mathfrak{C}$;
(b) S is a $\mathcal{U}$-band of ideal extensions of groups from the class $\mathfrak{C}$ by $\mathcal{U}$-nil-semigroups;
(c) S is a $\mathcal{U}$-chain of ideal extensions of groups from the class $\mathfrak{C}$ by $\mathcal{U}$-nil-semigroups and retractive extensions of singular bands by $\mathcal{U}$-nil-semigroups.

11. Let S be a completely π-regular semigroup and $\overline{xy} = \overline{x}\,\overline{y}$. Then S is a semilattice of retractive nil-extensions of completely simple semigroups by commutative maximal subgroups and $x = x^3$, for every $x \in \langle E(S)\rangle$.

12. Let S be a completely π-regular semigroup and $\overline{\overline{xy}} = \overline{\overline{x}}\,\overline{\overline{y}}$. Then S is a semilattice of retractive nil-extensions of completely simple semigroups.

13. Let S be a completely π-regular semigroup and $\mathcal{J} \subseteq \mathcal{T}$, then S is a semilattice of π-groups.

14. Let S be a semilattice of π-groups. Then a relation $\xi = \{(x, y) \in S \times S \mid (\exists e \in E(S))\ ex = ey\}$ is the smallest congruence on S such that S/ξ is a group.

15. The following conditions on a semigroup S are equivalent:

(a) $T(\mathcal{H})$ is a band congruence;
(b) S is a band of π-groups;
(c) $(\forall a, b \in S)\ abT(\mathcal{H})a^2bT(\mathcal{H})ab^2$.

16. The following conditions on a semigroup S are equivalent:

(a) S is a semilattice of π-groups;
(b) S is completely π-regular and each regular $\mathcal{D}$-class of S is a group;

(c) S is completely π-regular and there are no semigroups $\mathbf{A}_2$, $\mathbf{B}_2$, $\mathbf{L}_2$, $\mathbf{R}_2$ among the completely π-regular divisors of S;
(d) S is a semilattice of completely Archimedean semigroups and does not contain $\mathbf{L}_2$ and $\mathbf{R}_2$ as subsemigroups.

17. Let $\mathbf{V} = \langle e, f \,|\, e^2 = e, f^2 = f, fe = 0\rangle = \{e, f, ef, 0\}$. The following conditions on a semigroup S are equivalent:

(a) S is completely π-regular and $\overline{xy} = \overline{y}\,\overline{x}$;
(b) S is completely π-regular and $(xy)^0 = (yx)^0$, $x^0y^0 = (x^0y^0)^0$;
(c) S is a semilattice of π-groups and $ef = fe$, for all $e, f \in E(S)$;
(d) S is π-regular and $\mathrm{Reg}(S)$ is a semilattice of groups;
(e) S is completely π-regular and there are no semigroups $\mathbf{B}_2$, $\mathbf{L}_2$, $\mathbf{R}_2$ and $\mathbf{V}$ among the completely π-regular divisors of S.

18. The following conditions on a π-regular semigroup S are equivalent:

(a) S is a band of t-Archimedean semigroups;
(b) S satisfies the identity $(xy)^0 = (x^0y^0)^0$;
(c) there are no semigroups $\mathbf{A}_2$, $\mathbf{B}_2$, $\mathbf{L}_{3,1}$, $\mathbf{R}_{3,1}$, $\mathbf{LZ}(n)$, $\mathbf{RZ}(n)$ among the completely π-regular divisors of S.

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