Proximity and uniform spaces

Radoslav Dimitrijević

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Preface

The basic aim of this book is to provide the reader with the complete insight into the basics of the theory of proximity spaces, uniform spaces and their generalizations. Although they are created in different time periods and independent of each other, proximity spaces and uniform spaces are essential connected not only with topological spaces, but also with each other. Nowadays, being the parts of general topology, these spaces are examined in almost all the books related to the study of general topology. However, not many books are devoted to these two areas or to one of them. This book is precisely of that kind. The author has made considerable effort to achieve the balance while presenting these two areas and preserve the richness of the materials resulting from their interconnection, if possible.

Over seventy years have passed since Dj. Kurepa implicitly and A. Weil explicitly introduced uniform spaces, and almost sixty years since V. A. Effremovich formulated the axioms of proximity spaces. Since then, a huge number of papers have been devoted to the study of these spaces. Another basic aim of this book is to collect at one place, the most significant results obtained through the study of these spaces, which are spread in various journals all over the world. This is the reason why in the end of the book the author gives huge bibliography which should direct the reader towards further study of the subject matter presented here.

The book is, first of all, dedicated to the students of Ph.D. studies, but also to the students at higher courses who acquired knowledge in general topology and want to expand their knowledge about uniform spaces and proximity spaces. Each chapter of the book can be presented within elective courses for the students of graduate studies. Chapter 2 and 3 can be read independently and used for the lectures at the elective courses. However, for better understanding of these chapters we need to know the subject matter presented in Chapter 1.

Chapter 1 presents the results related to proximity and uniform spaces, which were, historically, first introduced as axiomatic. Both of them represent symmetric structures. In Chapter 2 the results related to symmetric generalizations of these spaces is being exposed, while the results related to non-symmetric generalizations was presented in Chapter 3.

Each chapter is divided into section, and these sections are further divided into subsections. Each section ends with historical and bibliographical notes. The proofs of assertions end with the symbol \clubsuit . The same symbol can be found in the formulation of the assertions whose proofs are obvious. For easier reading and orientation in the text, there are subject index and index of symbols at the end of the book. The text of the book is formatted in the programme package IAT_EX , and the numeration of chapters, sections, subsections, definitions, theorems, as well as all the citations of the quotations in the book are predefined by this programme.

Anyone who has written the book of this kind is aware that, since the beginning of writing, until the promotion of the book, the debts have been accumulated. These debts must be acknowledged.

Therefore, I would like to thank professor G. Di Maio and professor Lj. Kochinac who were reviewers of my book and who gave me useful suggestions and contributed to the quality of this text. I am grateful to my students who showed great interest in the subject matter while attending the seminars about proximity spaces and uniform spaces held at the Faculty of Science and Mathematics, University of Nish, and who motivated me to transform the materials into the text of the book in front of you. I would like to express my thanks to Vojislava Ignjatovic, an English teacher, for the review of the English version of the book. Thanks to Miroslav Dimitrijevic for the book cover design. The printing house SVEN gave its contribution to the technical aspect of the book. In the end, I should thank my wife, Zlatica, for her support and forbearance while I have been working on this book.

Such a voluminous material cannot be flawless, regardless of the multiple reading by author, the reviewers and the lector. For all oversights and flaws in the text I, being the author, am the only responsible, and the one who read the manuscript for the last time before its printing. I am deeply grateful and open for all the comments and suggestions.

Radoslav Dimitriyevic

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Chapter 1

Proximity spaces and uniform spaces

1.1 Proximity spaces

1.1.1 Definition and basic properties of proximity relation

Definition 1.1.1.1 A relation δ on the family P(X) of all subsets of a set X is called a **proximity** on X if δ satisfies the following conditions:

- (B_1) if $A\delta B$, then $B\delta A$;
- (B_2) $A\delta(B \cup C)$ if and only if either $A\delta B$ or $A\delta C$;

 $(B_3) X \delta \emptyset;$

 (B_4) {x} δ {x} for each $x \in X$;

(B₅) if $A\overline{\delta}B$, then there exists $E \in P(X)$ such that $A\overline{\delta}E$ and $X - E\overline{\delta}B$. The pair (X, δ) is called a **proximity space**. If (B₄) is replaced by

 (B'_4) {x} δ {y} if and only if x = y,

then δ is called a separated or Hausdorff proximity relation and (X, δ) is called a separated or Hausdorff proximity space.

Strictly speaking, one should use the notation $(A, B) \in \delta$ or $(A, B) \notin \delta$ when the sets A and B are either near each other or not, but we shall simply write $A\delta B$ or $A\overline{\delta}B$.

Just as the class of all topologies on a given set can be partially ordered by inclusion, one can impose a partial order on the class \mathcal{P} of all proximities defined on a set X in the following manner:

Definition 1.1.1.2 If δ_1 and δ_2 are two elements of \mathcal{P} , we define

 $\delta_1 > \delta_2$ if and only if $A\delta_1 B$ implies $A\delta_2 B$.

In this case we say that δ_1 is **finer** then δ_2 , or δ_2 is **coarser** than δ_1 .

According to the above definition we have the following:

Proposition 1.1.1.1 Let δ_1 , δ_2 , δ_3 be proximities on X. Then

- (a) $\delta_1 < \delta_1$;
- (b) if $\delta_1 < \delta_2$ and $\delta_2 < \delta_1$, then it implies that $\delta_1 = \delta_2$;
- (c) if $\delta_1 < \delta_2$ and $\delta_2 < \delta_3$, then it implies that $\delta_1 < \delta_3$.

In other words, the set \mathcal{P} of all proximities on the set X is partially ordered by the relation <.

Example 1.1.1.1 Just as discrete and indiscrete topology can be defined on any set, we have discrete and indiscrete proximity.

(a) If we define $A\delta_0 B$ if and only if $A \cap B \neq \emptyset$, then δ_0 is the **discrete proximity** on X.

(b) On the other hand, if $A\delta_1 B$ for every pair of non-empty subsets A and B of X, then we obtain the **indiscrete proximity** on X.

It is obvious that $\delta_0 > \delta > \delta_1$ for any proximity δ on X.

We shall often need the following simple proposition:

Proposition 1.1.1.2 Let (X, δ) be a proximity space. Then

- (a) if $A\delta B$ and $B \subseteq C$, then $A\delta C$;
- (b) if $A\overline{\delta}B$ and $C \subseteq B$, then $A\overline{\delta}C$;
- (c) if there exists a point $x \in X$ such that $A\delta\{x\}$ and $\{x\}\delta B$, then $A\delta B$;
- (d) if $A \cap B \neq \emptyset$, then $A\delta B$;
- (e) $A\delta\emptyset$ for every $A \subseteq X$;
- (f) if $A\delta B$, then $A \neq \emptyset$ and $B \neq \emptyset$.

Proof: Statement (a) immediately follows from (B_2) and the fact that $C = B \cup C$. (b) This statement follows from (a). (c) Suppose that $A\delta\{x\}$ and $\{x\}\delta B$ for some $x \in X$. If $A\overline{\delta}B$, then from (B_5) there exists a set $E \subset X$ such that $A\overline{\delta}E$ and $X - E\overline{\delta}B$. If $x \in E$, then, taken from (b), there follows $A\overline{\delta}\{x\}$, which is a contradiction. If $x \in X - E$, then also taken from (b), there follows that $\{x\}\delta B$, which is a contradiction, too. (d) Let $x \in A \cap B$. It follows that $\{x\}\delta\{x\}$ from (B_4) . Since $\{x\} \subset A$, then $\{x\}\delta A$ according to the statement (a). In an analogous manner we can conclude that $B\delta\{x\}$, and therefore according to the statement (c) and (B_1) it follows that $A\delta B$. (e) This statement follows from (B_3) and the statement (b). (f) It immediately follows from (e).

Proposition 1.1.1.3 If δ is a proximity relation on a set X, then the axiom (B_5) is equivalent to each of the following statements:

 (B'_5) if $A\delta B$, then there are sets C and D such that $A\delta C$, $B\delta D$ and $C \cup D = X$;

 (B_5'') if $A\overline{\delta}B$, then there are sets C and D such that $A\overline{\delta}X - C$, $X - D\overline{\delta}B$ and $C\overline{\delta}D$;

 (B_5''') if $A\overline{\delta}B$, then there are sets C and D such that $C \cap D = \emptyset$, $A\overline{\delta}X - C$ and $B\overline{\delta}X - D$.

Proof: Let us prove that the axiom (B_5) is equivalent to the statement (B'_5) . The fact that the other two statements are equivalent to the axiom (B_5) , can be proved immediately.

Let us suppose that the axiom (B'_5) is true and let $A\delta B$. Then there exists a set $E \subset X$ such that $A\overline{\delta}E$ and $X - E\overline{\delta}B$. If E = C and X - E = D, then it is obvious that $A\overline{\delta}C$, $D\overline{\delta}B$ and $D = E \cup (X - E) = X$.

Conversely, let us suppose that (B'_5) holds and let $A\delta B$. Then there exist sets C and D for which we have $A\overline{\delta}C$, $B\overline{\delta}D$ and $C \cup D = X$. Let E = C. Then $A\overline{\delta}E$. Since $C \cup D = X$, we have $X - E = X - C \subset D$, and since $B\overline{\delta}D$, according to Proposition 1.1.1.2 and the axiom (B_1) , $X - E\overline{\delta}B$ follows.

Definition 1.1.1.3 Let (X, δ) be a proximity space. We say that the sets $A, B \subset X$ are in the relation \ll and write $A \ll B$ if $A\overline{\delta}X - B$. When $A \ll B$, we call B a **proximity** or δ -neighborhood of A.

Theorem 1.1.1.1 Let (X, δ) be a proximity space. Then the relation \ll satisfies the following properties:

 $(O_1) X \ll X;$

 (O_2) if $A \ll B$, then $A \subset B$;

 $(O_3) A \subset B \ll C \subset D \text{ implies } A \ll D;$

 $(O_4) A \ll B \text{ implies } X - B \ll X - A;$

(O₅) $A \ll B_k$ is true for k = 1, 2, ..., n if and only if $A \ll \bigcap_{k=1}^n B_k$;

 (O_6) if $A \ll B$, then there exists a set $C \subset X$ such that $A \ll C \ll B$.

If δ is a separated proximity, then

 (O_7) {x} $\ll X - \{y\}$ if and only if $x \neq y$.

Proof: (O_1) According to the axiom (B_3) there follows that $X\delta\emptyset$, and therefore $X \ll X$ holds from the definition of the relation \ll .

 (O_2) If $A \ll B$, then $A\overline{\delta}X - B$ holds. By Proposition 1.1.1.2 (d) there follows that $A \cap (X - B) = \emptyset$, hence $A \subseteq B$.

 (O_3) Let $A \subset B \ll C \subset D$. Then $B\overline{\delta}X - C$, and since $A \subset B$ and $X - D \subset X - C$, by Proposition 1.1.1.2 (b) it follows that $A\overline{\delta}X - D$, i.e. $A \ll D$.

 (O_4) If $A \ll B$, then $A\overline{\delta}X - B$. Therefore, by (B_1) it follows that $X - B\overline{\delta}A$, i.e. $X - B \ll X - A$.

 (O_5) It is sufficient to prove the statement for the case n = 2. Let $A \ll B_1$ and $A \ll B_2$. Then $A\overline{\delta}X - B_1$ and $A\overline{\delta}X - B_2$; thus by (B_2) we have that $A\overline{\delta}(X - B_1) \cup (X - B_2)$, i.e. $A\overline{\delta}X - (B_1 \bigcap B_2)$. But then $A \ll (B_1 \bigcap B_2)$ is true. It is obvious that the converse holds as well.

 $(O_6) A \ll B$ implies $A\overline{\delta}X - B$. By (B_5) , there exists a set X - C such that $A\overline{\delta}X - C$ and $C\overline{\delta}X - B$. But then $A \ll C \ll B$.

 (O_7) By axiom (B'_3) , $x \neq y$ is true if and only if $\{x\}\overline{\delta}\{y\}$. This is equivalent with $\{x\} \ll X - \{y\}$.

Corollary 1.1.1.1 If $A_k \ll B_k$ for k = 1, 2, ..., n, then

$$\bigcap_{k=1}^{n} A_k \ll \bigcap_{k=1}^{n} B_k \quad and \quad \bigcup_{k=1}^{n} A_k \ll \bigcup_{k=1}^{n} B_k. \quad \clubsuit$$

All of the separated proximity axioms are used in the above proofs. In particular, we note that (B_5) is equivalent to the property (O_6) of the relation \ll in the above theorem, and (B'_4) is equivalent to the property (O_7) . The following theorem is the converse of Theorem 1.1.1.1.

Theorem 1.1.1.2 If \ll is a binary relation on the power set of X satisfying the properties $(O_1) - (O_6)$ of Theorem 1.1.1.1, then the binary relation δ defined on P(X) with

 $A\overline{\delta}B$ if and only if $A \ll X - B$,

is a proximity relation on X. Moreover, if \ll also satisfies the axiom (O₇) of Theorem 1.1.1.1, then δ is a separated proximity on X. A set B is a δ -neighborhood of a set A if and only if $A \ll B$.

Proof: (B_1) If $A\overline{\delta}B$, then $A \ll X - B$. By the axiom (O_4) , $B \ll X - A$, so $B\overline{\delta}A$.

 (B_2) Let us suppose that $(A \cup B)\overline{\delta}C$. Then $(A \cup B) \ll X - C$. Thus, by the axiom (O_3) it follows that $A \ll X - C$ and $B \ll X - C$, i.e. $A\overline{\delta}C$ and $B\overline{\delta}C$. To prove the converse, let us suppose that $(A \cup B)\delta C$, i.e. $C\delta(A \cup B)$. Then $C \ll X - (A \cup B)$, i.e. $C \ll (X - A) \cap (X - B)$; hence by the axiom (O_5) , it follows that $C \not\ll X - A$ or $C \not\ll X - B$. But then $C\delta A$ or $C\delta B$, and it follows, since δ is symmetric, that $A\delta C$ or $B\delta C$.

 (B_3) This axiom is a direct consequence of the axiom (O_1) .

 (B_4) Let us suppose that $\{x\}\overline{\delta}\{y\}$, i.e. $\{x\}\overline{\delta}X - (X - \{y\})$. Then $\{x\} \ll X - \{y\}$ holds, and therefore $\{x\} \ll X - \{y\}$. Now, by the axiom (O_2) , there follows the inclusion $\{x\} \subset X - \{y\}$. Hence $x \neq y$.

 (B_5) Let us suppose $A\overline{\delta}B$, i.e. $A \ll X - B$. Then by the axiom (O_6) , there exists a set $C \subset X$ such that $A \ll X - C \ll X - B$. Thus, there exists a $C \subset X$ such that $A\overline{\delta}X - C$ and $X - C\overline{\delta}B$.

 (B'_4) According to the axiom (O_7) , $x \neq y$ is true if and only if $\{x\} \ll X - \{y\}$, i.e. if and only if $\{x\}\overline{\delta}\{y\}$.

Let us consider the family $\mathcal{F}(A)$ of all δ -neighborhoods of a set A in a proximity space (X, δ) . If $A = \emptyset$, then, by the axiom (B_3) , $\mathcal{F}(A)$ consists of all the subsets of X. On the other hand, the following proposition holds:

Proposition 1.1.1.4 If (X, δ) is a proximity space, $A \subset X$, $A \neq \emptyset$, then $\mathcal{F}(A)$ is a filter on X.

Proof: First, let as note that each element of the family $\mathcal{F}(A)$ is a nonempty set. Indeed, if $B \in \mathcal{F}(A)$, then $A\overline{\delta}X - B$, and thus, by Proposition 1.1.1.2 (d), $A \subset B$, which proves that $B \neq \emptyset$. Let $B \in \mathcal{F}(A)$ and $B \subset C$. Since $B \in \mathcal{F}(A)$, we have that $A\overline{\delta}X - B$. But then, by Proposition 1.1.1.2 (b), $A\overline{\delta}X - C$ holds, and therefore, $C \in \mathcal{F}(A)$. Finally, if $B, C \in \mathcal{F}(A)$, then $A\overline{\delta}X - B$ and $A\overline{\delta}X - C$, thus, by the axiom $(B_2) A\overline{\delta}(X - B) \cup (X - C)$, i.e. $A\overline{\delta}X - (B \cap C)$. But then we have that $B \cap C \in \mathcal{F}(A)$. Hence, $\mathcal{F}(A)$ is a filter on X.

The family $\mathcal{F}(A)$ of all δ -neighborhoods of a set A, where A is a nonempty subset of the proximity space (X, δ) , is called the **proximity filter** or δ -filter of A. Let us give some properties of δ -filters.

Proposition 1.1.1.5 Let (X, δ) be a proximity space. Then

(a) $B \in \mathcal{F}(A)$ implies $A \subset B$; (b) $B \in \mathcal{F}(A)$ implies $X - A \in \mathcal{F}(X - B)$; (c) if $A \subset B$, then $\mathcal{F}(A) \subset \mathcal{F}(B)$; (d) $\mathcal{F}(A \cup B) = \mathcal{F}(A) \cap \mathcal{F}(B)$; (e) if $B \in \mathcal{F}(A)$, then there exists a $C \in \mathcal{F}(A)$ such that $B \in \mathcal{F}(C)$; (f) $\mathcal{F}(A) \cap \mathcal{F}(B) \subset \mathcal{F}(A \cap B)$, where $\mathcal{F}(A) \cap \mathcal{F}(B) = \{C \cap D : C \in \mathcal{F}(A), D \in \mathcal{F}(B)\}$.

The proof of this proposition is left to the reader.

1.1.2 Topology generated by a proximity

In this subsection we shall be consider the topology on X induced by a proximity on X, and study its elementary properties.

Definition 1.1.2.1 Let (X, δ) be a proximity space. A subset $F \subset X$ is defined to be **closed** if and only if $x\delta F$ implies $x \in F$. By τ_{δ} denote the family of complements of all the sets defined in such a way.

Theorem 1.1.2.1 If (X, δ) is a proximity space, then the family τ_{δ} is a topology on the set X.

Proof: Obviously X and \emptyset are closed sets. Let $\{F_i\}_{i\in I}$ be an arbitrary collection of the closed subsets of X. If $\{x\}\delta\bigcap_{i\in I}F_i$, then, by Proposition 1.1.1.2 (a), $\{x\}\delta F_i$ for each $i \in I$. Since the sets F_i are closed, $x \in F_i$ for each $i \in I$. Thus $x \in \bigcap_{i\in I}F_i$, which means that by the definition of a closed set, $\bigcap_{i\in I}F_i$ is a closed set. Finally, if F_1 and F_2 are closed sets and $x\delta(F_1 \cup F_2)$, then by the axiom (B_2) , either $x\delta F_1$ or $x\delta F_2$ holds. Since the sets F_1 and F_2 are closed, there follows that $x \in F_1$ or $x \in F_2$. Therefore, $x \in F_1 \cup F_2$. Thus, $F_1 \cup F_2$ is a closed set.

Proposition 1.1.2.1 Let (X, δ) be a proximity space and $\tau = \tau_{\delta}$. Then the τ -closure \overline{A} of a set A is given by $\overline{A} = \{x : x \delta A\}$.

Proof: If \overline{A} denotes the intersection of all closed sets containing A and $A^{\delta} = \{x : x\delta A\}$, then it should be proved that $\overline{A} = A^{\delta}$. If $x \in A^{\delta}$, then $\{x\}\delta A$. By Proposition 1.1.1.2 (a) this implies $x\delta \overline{A}$ and, since \overline{A} is closed, $x \in \overline{A}$. Thus $A^{\delta} \subseteq \overline{A}$. To prove the reverse inclusion it suffices to prove that A^{δ} is closed, i.e. $x\delta A^{\delta}$ implies $x \in A^{\delta}$. Assuming that $x \notin A^{\delta}$, then $x\overline{\delta}A$ so that, by the axiom (B_5) , there exists a set E such that $x\overline{\delta}E$ and $X - E\overline{\delta}A$. Thus, no point of the set X - E is near A, i.e. $A^{\delta} \subseteq E$, which, together with $x\overline{\delta}E$, implies that $x\overline{\delta}A^{\delta}$.

An alternative method of introducing the same topology on a proximity space (X, δ) would be to define the subset A^{δ} of X for each subset A of X and to show that it is a Kuratowski closure operator.

Theorem 1.1.2.2 Let (X, δ) be a proximity space. Then $A \to A^{\delta}$, where $A^{\delta} = \{x \in X : x \delta A\}$, is a Kuratowski closure operator.

Proof: (K_1) By Proposition 1.1.1.2 (e), for each $x \in X$ it follows that $\{x\}\overline{\delta}\emptyset$, from which $\emptyset^{\delta} = \emptyset$ follows.

 (K_2) If $x \in A \subset X$, then $\{x\}\delta A$ according to Proposition 1.1.1.2 (d). Hence, $x \in A^{\delta}$, which proves that $A \subseteq A^{\delta}$.

 (K_3) By (B_2) , $x \in (A \cup B)^{\delta}$ if and only if $x \delta A \cup B$ if and only if $\{x\} \delta A$ or $\{x\} \delta B$ if and only if $x \in A^{\delta}$ or $x \in B^{\delta}$ if and only if $x \in A^{\delta} \cup B^{\delta}$. Thus, $(A \cup B)^{\delta} = A^{\delta} \cup B^{\delta}$.

 (K_4) To prove that $(A^{\delta})^{\delta} \subseteq A^{\delta}$ is true, let us suppose that $x \notin A^{\delta}$, i.e. $x\overline{\delta}A$. Then, by the (B_5) there exists a set E such that $x\overline{\delta}E$ and $X - E\overline{\delta}A$. Now $A^{\delta} \subseteq E$ and $x\overline{\delta}E$, so that $x\overline{\delta}A^{\delta}$ and $x \notin (A^{\delta})^{\delta}$.

It is known that, with the help of the proximity introduced in the pseudometric space, the neighborhoods of each point can be characterized: the set V is a neighborhood of the point x if and only if $\{x\}$ and X - V are far from each other. In other words, this means that the neighborhood filter of the point x is identical with the proximity filter of the set $\{x\}$. According to this, a topology can be introduced in any proximity space as follows:

Theorem 1.1.2.3 Let (X, δ) be a proximity space and let us call the neighborhood filter of the point $x \in X$ the proximity filter $\mathcal{F}(\{x\})$ of the set $\{x\}$. Then we obtain a topology on X, called the topology of the proximity space (X, δ) , or the topology induced by the proximity δ and also denoted by τ_{δ} or $\tau(\delta)$.

Proof: As a consequence of Proposition 1.1.1.4 $\mathcal{F}(\{x\})$ is a filter of every set which contains $\{x\}$ (by Proposition 1.1.1.5). Thus, τ_{δ} is in any case a neighborhood structure. Moreover, by Proposition 1.1.1.5 (e), if $V \in \mathcal{F}(\{x\})$, then there exists an $U \in \mathcal{F}(\{x\})$ such that $V \in \mathcal{F}(U)$. Then $y \in U$ is implied, by Proposition 1.1.1.5 (c), $V \in \mathcal{F}(\{y\})$; thus, the system of the filters $(\mathcal{F}(\{x\}))_{x \in X}$ is a neighborhood structure which is, according to the above facts, a topology on X.

Let us give some properties of the sets which are open or closed in topology τ_{δ} .

Proposition 1.1.2.2 If G is a subset of a proximity space (X, τ) , then G is open in topology τ_{δ} if and only if $\{x\}\overline{\delta}X - G$ for every $x \in G$.

Proof: Let G be an open set in the topology τ_{δ} and let $x \in G$. The set X - G is closed, so that from $\{y\}\delta X - G$ it follows that $y \in X - G$. Since $x \notin X - G$, then $\{x\}\overline{\delta}X - G$. Conversely, let us suppose that $\{x\}\overline{\delta}X - G$ for each $x \in G$. This means that $x \notin (X - G)^{\delta} = \overline{X - G}^{\tau_{\delta}}$. Hence, no point of

the set G is in closure of the set X - G. But then all closure points of the set X - G are in the set X - G, which proves that it is closed. Therefore, the set G is open as its complement.

Proposition 1.1.2.3 If A and B are subsets of a proximity space (X, δ) , then $A\overline{\delta}B$ implies:

(a) $\overline{B} \subset X - A$, and (b) $B \subset int(X - A)$,

where the closure and the interior are taken with respect to the topology τ_{δ} .

Proof: (a) Let us suppose that $x \notin X - A$. Since $B\overline{\delta}X - (X - A)$, by Proposition 1.1.1.2 (b), we have that $x\overline{\delta}B$, from which follows that $x \notin \overline{B}$. (b) Since $A\overline{\delta}B$, then by the axiom (B_1) and previously proved inclusion we have that $\overline{A} \subset X - B$. Therefore $B \subset X - \overline{A} = int(X - A)$.

Proposition 1.1.2.4 For the subsets A and B of the proximity space (X, δ) we have that

 $A\delta B$ if and only if $\overline{A}\delta \overline{B}$,

where the closure is taken with respect to the topology τ_{δ} .

Proof: If $A\delta B$, then according to Proposition 1.1.1.2 (a) we have that $\overline{A}\delta\overline{B}$. To prove the converse, let us suppose that $A\overline{\delta}B$. Then by axiom (B_5) there exists a set E such that $A\overline{\delta}E$ and $X - E\overline{\delta}B$. Now by means of Proposition 2.3.1.1 from $B\overline{\delta}X - E$ we conclude that $\overline{B} \subset E$. Since $A\overline{\delta}E$ and $\overline{B} \subset E$, by Proposition 1.1.1.2 (b) it follows that $A\overline{\delta}\overline{B}$. In an analogous manner from $A\overline{\delta}\overline{B}$, by means of axiom (B_1) , it follows that $\overline{A}\overline{\delta}\overline{B}$.

Proposition 1.1.2.5 Let (X, δ) be a proximity space. If \overline{A} and Int A denote, respectively, the closure and the interior of the set A with respect to the topology τ_{δ} , then

(a) $A \ll B$ implies $\overline{A} \ll B$, and (b) $A \ll B$ implies $A \ll Int B$.

Proof: (a) Let $A \ll B$. Then by Definition 1.1.1.3 we have that $A\overline{\delta}X - B$, from which, by the above proposition, it follows $\overline{A}\overline{\delta}X - B$. But then $\overline{A} \ll B$.

(b) From $A\overline{\delta}X - B$, by the previous proposition, it follows that $A\overline{\delta}\overline{X} - \overline{B}$, so that $A\overline{\delta}X - \operatorname{Int}B$, i.e. $A \ll \operatorname{Int}B$.

From the second assertion of this proposition it follows that any δ -neighborhood of some set is also a topological neighborhood of this set,

of course, with respect to the topology generated by the proximity relation δ . However, a δ -neighborhood in general is not an open set with respect to this topology.

Proposition 1.1.2.6 The intersection of all δ -neighborhoods of a set A is equal to the closure of the set A.

Proof: If $A \ll B$, then by the previous proposition it follows that $\overline{A} \ll B$. But then, the set \overline{A} is contained in the intersection of all δ -neighborhoods of A. To prove that intersection of all δ -neighborhoods of the set A is equal to the set \overline{A} , it is sufficient to prove that for every point $x \notin \overline{A}$ there exists a δ -neighborhood of the set A which does not contain the point x. If $x \notin \overline{A}$, then $x\overline{\delta}A$, so that by Proposition 1.1.1.3 there are disjoint δ -neighborhoods of the point x and the set A.

Proposition 1.1.2.7 The topology τ_{δ} generated by a proximity relation δ on a space X is regular.

Proof: Let U be any neighborhood of a point $x \in X$. Then $U \in \mathcal{F}(\{x\})$, so by Proposition 1.1.1.5 (e) there exists a set $V \in \mathcal{F}(\{x\})$ such that $U \in \mathcal{F}(V)$, i.e. $V\overline{\delta}X - U$. Now \overline{V} is a neighborhood of the point x for which, according to Proposition 1.1.2.4, $\overline{V}\overline{\delta}X - U$, from which, by means of Proposition 1.1.1.2 (d), it follows that $\overline{V} \subset U$. In this way we proved that the topology τ_{δ} on the space X is regular.

Proposition 1.1.2.8 A proximity space (X, δ) is separated if and only if the topology τ_{δ} generated by the proximity relation δ is a T_0 -topology.

Proof: If the proximity relation δ is separated and $x \neq y$, then $X - \{y\} \in \mathcal{F}(\{x\})$ is a neighborhood of the point x not containing the point y. Conversely, if U is a neighborhood of the point x not containing the point y, then $U \in \mathcal{F}(\{x\})$. Since $U \subset X - \{y\}$, then $X - \{y\} \in \mathcal{F}(\{x\})$, so that $\{x\}\overline{\delta}\{y\}$.

The following proposition gives the connection between the comparison of topologies and proximity relations:

Proposition 1.1.2.9 Let δ_1 and δ_2 be two proximity relations defined on the set X. If $\delta_1 < \delta_2$, then $\tau(\delta_1) \subset \tau(\delta_2)$.

Proof: Let us suppose $G \in \tau(\delta_1)$. Then by Proposition 1.1.2.2 $\{x\}\overline{\delta}_1X - G$ for each $x \in G$. Since $\delta_1 < \delta_2$, then $\{x\}\overline{\delta}_2X - G$ for each $x \in G$, so that $G \in \tau(\delta_2)$. Hence $\tau(\delta_1) \subset \tau(\delta_2)$.

Example 1.1.2.1 If δ is an arbitrary proximity on a set X, then $\delta_0 < \delta < \delta_1$. Thus, by the above proposition, it follows that $\tau(\delta_1) \subset \tau_{\delta} \subset \tau(\delta_0)$.

On the other hand, the converse in general is not true.

1.1.3 Compatibility of topology with a proximity relation

Definition 1.1.3.1 Let τ and δ be a topology and a proximity relation respectively, both defined on a set X. If $\tau = \tau_{\delta}$, then τ and δ are said to be compatible.

Theorem 1.1.3.1 Let (X, τ) be a completely regular space. Then the relation δ , which is defined on the power set PX of the set X by

(1)
$$A\delta B \quad if and only if A \neq \emptyset \neq B and there is not a continuous function f: X \to I, such that f(x) = 0 for x \in A, and f(x) = 1 for x \in B,$$

is a proximity relation compatible with the topology τ . If (X, τ) is a Tychonoff space, then the proximity δ is separated.

Proof: From the definition of the proximity relation δ immediately follows that it satisfies axioms (B_1) , (B_3) and (B_4) . To prove that the axiom (B_2) holds, it suffices to show that from $A\overline{\delta}B$ and $A\overline{\delta}C$, $A\overline{\delta}(B \cup C)$ is true. Since $A\overline{\delta}B$, there exists a continuous function $f: X \to I$ such that f(x) = 0 for $x \in A$ and f(x) = 1 for $x \in B$. There also exists a continuous function $g: X \to I$ such that g(x) = 0 for $x \in A$ and g(x) = 1 for $x \in C$. Function $h(x) = \max\{f(x), g(x)\}$ is continuous on the set X, h(x) = 0 on the set Aand h(x) = 1 on the set $B \cup C$, so that $A\overline{\delta}(B \cup C)$. Finally, let us prove that axiom (B_5) is satisfied. Let us suppose that $A\overline{\delta}B$ and let $f: X \to I$ be a continuous function for which f(A) = 0 and f(B) = 1 holds. Let us prove that $A\overline{\delta}E$ and $X - E\overline{\delta}B$ holds for the set $E = \{x \in X : 1/2 \leq f(x) \leq 1\}$. Let us examine the function defined in the following way:

$$g(y) = \begin{cases} 2y, \ 0 \le y \le 1/2, \\ 1, \ 1/2 \le y \le 1. \end{cases}$$

It is obvious that $g: I \to I$ is a continuous function, so that $g \circ f: X \to I$ is also a continuous function for which $(g \circ f)(A) = 0$ and $(g \circ f)(E) = 1$ holds. Therefore $A\overline{\delta}E$. In a similar way it can be proved that $X - E\overline{\delta}B$. Let us prove now that $\tau = \tau_{\delta}$. Let us suppose that $G \in \tau$ and let $x \in G$. Then x is not in the closed set X - G, so there exists a continuous function $f: X \to I$ such that f(x) = 0 and f(X - G) = 1. Therefore $\{x\}\overline{\delta}X - G$, so that $G \in \tau_{\delta}$. Conversely, if $G \in \tau_{\delta}$ and $x \in G$, then $\{x\}\overline{\delta}X - G$. But then, according to the definition of the proximity relation δ , there exists a continuous function $f: X \to I$ such that f(x) = 0 and f(X - G) = 1. Now there follows that $f^{-1}([0, 1/2))$ is a τ -open neighborhood of the point x which is contained in the set G. In this manner we have proved that $G \in \tau$.

To prove that δ is separated if (X, τ) is Tychonoff, let us note that if $x \neq y$ then $x \notin \overline{y}$ since (X, τ) is T_1 . From the definition of completely regular space, we are assured that x and \overline{y} are functionally distinguishable, implying that $x\overline{\delta y}$.

The proximity space (X, δ) in the above theorem will be called a **fine** proximity space.

Theorem 1.1.3.2 If (X, τ) is a normal topological space and

(2)
$$A\overline{\delta}B \text{ if and only if } \overline{A} \cap \overline{B} = \emptyset,$$

then δ is a proximity relation on the set X.

Proof: It is obvious that the axioms (B_1) , (B_3) and (B_4) hold. Let us prove that axiom (B_2) holds. Let $A\delta(B \cup C)$. Then $\overline{A} \cap (\overline{B} \cup \overline{C}) \neq \emptyset$, so that $\overline{A} \cap (\overline{B} \cup \overline{C}) \neq \emptyset$. But then $(\overline{A} \cap \overline{B}) \cup (\overline{A} \cap \overline{C}) \neq \emptyset$, so that $\overline{A} \cap \overline{B} \neq \emptyset$ or $\overline{A} \cap \overline{C}$. Therefore $A\delta B$ or $A\delta C$. To prove that the axiom (B_5) is true, let us suppose that $A\overline{\delta}B$. Then $\overline{A} \cap \overline{B} = \emptyset$, and since X is a normal space, there exist open sets C and D such that $\overline{A} \subset C$, $\overline{B} \subset D$ and $C \cap D = \emptyset$. The set X - C is closed and $\overline{A} \cap (X - C) = \emptyset$, so that $\overline{A}\overline{\delta}X - C$. But then, according to Proposition 1.1.2.4, it follows that $A\overline{\delta}X - C$. It can be proved in an analogous manner that $B\overline{\delta}X - D$, so that by Proposition 1.1.1.3 axiom (B_5) is satisfied. \clubsuit

In a normal space (X, τ) , the proximities defined by (1) and (2) are equivalent.

Definition 1.1.3.2 The non-empty sets A and B are said to be **discon**nected in a topological space if they have disjoint neighborhoods. A and B are said to be **separated** in a topological space if A has a neighborhood disjoint from B and B has a neighborhood disjoint from A; A and B are **weakly separated** if at least one of them possesses a neighborhood not intersecting the other one. **Definition 1.1.3.3** A topological space X is said to be

-an S_1 -space if any two weakly separated points are separated;

-an S_2 -space if any two weakly separated points are disconnected;

-an S_3 -space if any point $x \in X$ and any closed set not containing x are disconnected.

A normal S_1 -space is said to be an S_4 -space.

Definition 1.1.3.4 The proximity relation defined in the previous theorem on an S_4 -space is called the elementary proximity relation.

Theorem 1.1.3.3 Let (X, τ) be a normal topological space, and let δ^* be the proximity on the set X defined in the previous theorem. Then the topology τ_{δ_*} generated by the proximity relation δ^* is coarser than the topology τ . This topologies are identical if and only if the space (X, τ) is an S_4 -space.

Proof: The set U is a τ_{δ^*} -neighborhood of the point $x \in X$ if and only if $\{x\}\overline{\delta}^*X - U$, which is by Proposition 1.1.2.4 equivalent to $\overline{\{x\}}\overline{\delta}^*\overline{X} - \overline{U}$. From this, according to Proposition 1.1.1.2, we have that $\overline{\{x\}} \cap \overline{X} - \overline{U} = \emptyset$, so that $x \in X - \overline{X} - \overline{U} \subset U$. Therefore U is a τ -neighborhood of the point x. This proves that the topology τ_{δ^*} is coarser than the topology τ .

Let us suppose now that $\tau_{\delta^*} = \tau$. Then for every τ -neighborhood Uof the point x the inclusion $\overline{\{x\}} \subset X - \overline{X - U} \subset U$ holds, so that X is an S_1 -space, and also an S_4 -space. To prove converse, let us suppose that (X, τ) is an S_4 -space and let U be a τ -neighborhood of the point x. Then there exists a τ -open set V such that $x \in V \subset U$. Since X is an S_1 -space, the set V is a τ -neighborhood of the set $\overline{\{x\}}$. Therefore $\overline{\{x\}} \subset V$ and $\overline{\{x\}} \cap (X - V) = \overline{\{x\}} \cap \overline{X - V} = \emptyset$, so that $\overline{\{x\}} \cap \overline{X - U} = \emptyset$, which proves that the set U is a τ_{δ^*} -neighborhood of the point x.

Lemma 1.1.3.1 Let (X, δ) be a proximity space. If K is a compact, and F is a closed set in the topology τ_{δ} and if $K \cap F = \emptyset$, then $K\overline{\delta}F$.

Proof: If $x \in K$, then X - F is a neighborhood of the point x in topology τ_{δ} , so that $\{x\}\overline{\delta}F$. Then, by virtue of Proposition 1.1.1.3, for each point $x \in K$ there exist the sets C_x and D_x such that $C_x \cap D_x = \emptyset$, $\{x\}\overline{\delta}X - C_x$ and $F\overline{\delta}X - D_x$. Since C_x is a neighborhood of the point x, then, because of compactness of the set K, there exists a finite set of the points x_1, x_2, \ldots, x_n such that $K \subset \bigcup_{i=1}^{n} C_{x_i}$. For the set $D = \bigcap_{i=1}^{n} D_{x_i}$ it holds $(\bigcup_{i=1}^{n} C_{x_i}) \cap D = \emptyset$, so that $K \cap D = \emptyset$. Since $X - D = \bigcup_{i=1}^{n} (X - D_{x_i})\overline{\delta}F$, by Proposition 1.1.1.2 (b), it follows that $K\overline{\delta}F$.

Theorem 1.1.3.4 On a compact T_2 -space the elementary proximity relation is the unique compatible proximity.

Proof: Every compact T_2 -space is a T_4 -space, so the elementary proximity δ , by virtue of Theorem 1.1.3.3, is a proximity relation compatible with the topology of this space. Let δ^* be any proximity relation on X compatible with the topology of this space. Let us prove that $\delta = \delta^*$.

Let $A\overline{\delta}^*B$. Then, by Proposition 1.1.2.4, $\overline{A} \overline{\delta}^*\overline{B}$, so that according to Proposition 1.1.1.2 (d) it follows that $\overline{A} \cap \overline{B} = \emptyset$. Therefore, $A\overline{\delta}B$, which proves that $\delta^* < \delta$.

To prove the converse, let us suppose that $A\overline{\delta}B$, which is, by Proposition 1.1.2.4, equivalent to the fact that $\overline{A}\,\overline{\delta}\,\overline{B}$, i.e. $\overline{A}\cap\overline{B}=\emptyset$. Then by virtue of the previous lemma it follows that $\overline{A}\,\overline{\delta}^*\overline{B}$, so that, again by Proposition 1.1.2.4, it follows that $A\overline{\delta}^*B$. In this way we have proved that $\delta = \delta^*$; therefore, δ is the unique proximity on X compatible with topology on the space X.

Theorem 1.1.3.5 On a compact S_2 -space the elementary proximity is the unique proximity compatible with the topology of that space.

Proof: Let (X, τ) be a compact S_2 -space and let δ be a proximity relation on X for which $\tau = \tau_{\delta}$. Let us prove that $\delta = \delta^*$, where δ^* is the elementary proximity relation defined in Theorem 1.1.3.3. The compact S_2 -space X is normal, so that the relation δ^* defined in Theorem 1.1.3.3 is a proximity relation. Therefore, the proximity δ , by the above proposition, is coarser than the proximity relation defined in Theorem 1.1.3.3. Let us prove the converse, i.e. that from $A\delta B$ it follows $\overline{A} \cap \overline{B} \neq \emptyset$. Indeed, if $\overline{A} \cap \overline{B} = \emptyset$, then by Lemma 1.1.3.1 we have that $\overline{A} \delta \overline{B}$, so that by Proposition 1.1.2.4 $A\overline{\delta}B$, contrary to the assumption. Now from the facts that $\overline{A} \cap \overline{B} = \emptyset$, and that δ^* is a proximity relation compatible with the topology τ , it follows that $\overline{A} \delta^* \overline{B}$. But then, according to Proposition 1.1.2.4, $A\overline{\delta}^* B$ holds.

1.1.4 Comparison of proximity relations

In the first subsection of this section we have introduced the order defined on the set \mathcal{P} of all proximity relations on a set X. We have seen that every proximity lies between the discrete and indiscrete proximities. We have also proved that from $\delta_1 < \delta_2$ follows $\tau(\delta_1) \subset \tau(\delta_2)$. The following example proves that the converse in general is not true. **Example 1.1.4.1** If $X = \mathbb{R}$, d(x, y) = |x - y| and $\delta_1 = \delta_d$, while δ_2 is the proximity defined in Theorem 1.1.3.2 with the help of the topology τ_d , which is a T_5 - and therefore an S_4 -space, then $\tau(\delta_1) = \tau(\delta_2)$, although $\delta_1 \neq \delta_2$. To prove this fact, let us take the sets $A = \mathbb{N}$ and $B = \{n+1/2n : n \in \mathbb{N}\}$. Then d(A, B) = 0 implies $A\delta_1 B$. On the other hand, $A = \overline{A}, B = \overline{B}, \overline{A} \cap \overline{B} = \emptyset$, so that $A\overline{\delta}_2 B$. It should be observed that in this example $\delta_1 < \delta_2$. This can be seen from the following proposition:

Proposition 1.1.4.1 Let (X, τ) be a normal space, δ being the proximity defined in Theorem 1.1.3.2 and δ^* an arbitrary proximity on X for which $\tau_{\delta^*} < \tau$. Then $\delta^* < \delta$.

Proof: Let $A\delta B$, i.e. $\overline{A}^{\tau} \cap \overline{B}^{\tau} \neq \emptyset$. Then $A\delta^*B$, because otherwise, by Proposition 1.1.2.4, $\overline{A}^{\tau_{\delta^*}} \overline{\delta}^* \overline{B}^{\tau_{\delta^*}}$ would follows. But then, according to the supposition that $\tau_{\delta^*} < \tau$, it will be $\overline{A}^{\tau} \cap \overline{B}^{\tau} = \emptyset$, which is in contradiction with the supposition.

Theorem 1.1.4.1 Let $\{\delta_i : i \in I \neq \emptyset\}$ be any family of proximity relations on X. For the sets $A, B \subset X$ let us define the relation

(1)
$$A\delta B$$
 if for any finite decompositions $A = \bigcup_{j=1}^{m} A_j$, $B = \bigcup_{k=1}^{n} B_k$
there exist indecis j and k such that $A_j \delta_i B_k$ for each $i \in B_k$

Then the proximity relation δ is the coarsest of the proximities finer than all proximities δ_i and is denoted by $\delta = \sup\{\delta_i : i \in I\}$. For the corresponding topologies we have:

$$\tau_{\delta} = \sup\{\tau(\delta_i) : i \in I\}.$$

Proof: Let us first prove that δ is a proximity relation on the set X. It is obvious that the axioms (B_1) , (B_3) and (B_4) are satisfied. Let us denote $J_k = \{1, 2, \ldots, k\}$. Let us suppose that $A\overline{\delta}B$ and $A\overline{\delta}C$. Then there exist decompositions of the sets

$$A = \bigcup_{j=1}^{p} A_j, \quad B = \bigcup_{k=1}^{q} B_k,$$

such that for each $j \in J_p$ and each $k \in J_q$ there exists an index i = i(j, k) for which $A_j \overline{\delta}_i B_k$ holds. There also exist decompositions of the sets

$$A = \bigcup_{m=1}^{r} A'_m, \quad C = \bigcup_{n=1}^{s} C_n$$

such that for each $m \in J_r$ and for each $n \in J_s$ there exists some i' = i'(m, n) for which $A'_m \overline{\delta}_{i'} C_n$ holds. Let us consider decompositions of the sets

$$A = \bigcup_{j=1}^{p} \bigcup_{m=1}^{r} (A_j \bigcap A'_m), \quad B \cup C = \bigcup_{k=1}^{q} B_j \cup \bigcup_{n=1}^{s} C_n.$$

Since $A_j\overline{\delta}_iB_k$ for i = i(j,k) and every $(j,k) \in J_p \times J_q$ and $A'_m\overline{\delta}_{i'}C_n$ for i' = i'(m,n) and every $(m,n) \in J_q \times J_s$, so that $A_j \cap A'_m\overline{\delta}_iB_k$ for i = i(j,k), every $(j,k) \in J_p \times J_q$ and every $m \in J_r$ and $A_j \cap A'_m\overline{\delta}_{i'}C_n$ for i' = i'(m,n), every $(m,n) \in J_r \times J_s$ and every $j \in J_m$. In both cases it follows that $A\overline{\delta}(B \cup C)$, so the axiom (B_2) holds.

To prove the axiom (B_5) , let us suppose that $A\overline{\delta}B$. Then there exist decompositions of the sets

$$A = \bigcup_{j=1}^{m} A_j, \quad B = \bigcup_{k=1}^{n} B_k$$

such that $A_j\delta_i B_k$ for some i = i(j,k) and each $(j,k) \in J_m \times J_n$. Therefore for each pair $(j,k) \in J_m \times J_n$ by virtue of Proposition 1.1.1.3 there exist sets P_{jk} and Q_{jk} for which $A_j\overline{\delta}_{i(j,k)}X - P_{jk}$, $B_k\overline{\delta}_{i(j,k)}X - Q_{jk}$ and $P_{jk} \cap Q_{jk} = \emptyset$ hold. Let us consider the sets

$$P_j = \bigcap_{k=1}^n P_{jk}, \ P = \bigcup_{j=1}^m P_j, \ Q_j = \bigcup_{k=1}^n Q_{jk}, \ Q = \bigcap_{j=1}^m Q_j.$$

First let us notice that $P_{jk} \cap Q_{jk} = \emptyset$ for each $(j,k) \in J_m \times J_n$, so that $P_j \cap Q_j = \emptyset$ for each $j \in J_m$, from which it follows that $P \cap Q = \emptyset$. Moreover it is evident that if $C\overline{\delta}_i D$ at least for one $i \in I$, then $C\delta D$. According to this fact we can conclude that $A_j\overline{\delta}X - P_{jk}$ and $B_k\overline{\delta}X - Q_{jk}$ for each $(j,k) \in J_m \times J_n$. But then $A\overline{\delta}X - P$ and $B\overline{\delta}X - Q$. Let us prove now that $A\overline{\delta}X - P$. Since $A_j\overline{\delta}X - P_{jk}$ for each $(j,k) \in J_m \times J_n$, i.e. $A_j \ll B_{jk}$ for each $(j,k) \in J_m \times J_n$, it follows by Corollary 1.1.4.1 that $\cup A_j \ll B_{jk}$ holds for each $k \in J_n$. According to the same corollary we also have that $\cup A_j \ll \bigcap P_{jk} = P_j$ holds for each $j \in J_m$, so that $\cup A_j \ll \cup P_j = P$. Hence from Theorem 1.1.1.1 $(O_3) A \ll P$, i.e. $A\overline{\delta}X - P$. In a similar way it can be proved that $B\overline{\delta}X - Q$.

Let us prove that the proximity relation δ is finer than all proximities δ_i . Indeed, we have already concluded that, if $A\delta_i B$ for some $i \in I$, then $A\delta B$. Therefore $\delta_i < \delta$. Let now δ^* be a proximity relation which is finer than all proximity relations δ_i and let $A\overline{\delta}B$. Then there exist decompositions (1) of the sets A and B so that $A_j \overline{\delta}_{i(j,k)} B_k$ for each $(j,k) \in J_m \times J_n$. Therefore $A_j \overline{\delta}^* B_k$ for each $(j,k) \in J_m \times J_n$, so that

$$A = \bigcup_{j=1}^{m} A_j \overline{\delta}^* B_k , \quad A \overline{\delta}^* \bigcup_{k=1}^{n} B_k = B ,$$

which proves that $\delta < \delta^*$. In this manner we have proved that $\delta = \sup\{\delta_i : i \in I\}$.

Since $\delta_i < \delta$, according to Proposition 1.1.2.9 it follows that $\tau(\delta_i) < \tau_{\delta}$ for every $i \in I$, so that $\tau < \tau_{\delta}$, where $\tau = \sup\{\tau(\delta_i) : i \in I\}$. To prove the converse, let us take any τ_{δ} -neighborhood G of the point x. Then $\{x\}\overline{\delta}X - G$, so there exists a decomposition of the set $X - G = \bigcup_{k=1}^{n} B_k$ such that $\{x\}\overline{\delta}_{i(k)}B_k$ for some $i = i(k) \in I$ and every $k \in J_n$. Thus $X - B_k$ is a $\tau(\delta_{i(k)})$ -neighborhood of the point x, and also a τ -neighborhood of that point. Therefore $G = \bigcap_{k=1}^{n} (X - B_k)$ is also a τ -neighborhood of the point x. In this way we have proved that $\tau_{\delta} < \tau$ and the theorem is proved.

Corollary 1.1.4.1 For any non-empty family of proximity relations δ_i on the set X there exists a proximity relation δ which is the finest of all proximities coarser than all δ_i . It is denoted by $\delta = \inf{\{\delta_i : i \in I\}}$.

Proof: Since the indiscrete proximity of the set X is coarser than all proximities δ_i , we can speak of the supremum of the proximities coarser than all proximities δ_i . Denoting it by δ , we clearly obtain a relation with the required property.

Corollary 1.1.4.2 If there exists a compatible proximity for a topology τ , then there exists the finest one among the proximities compatible with the topology τ .

Proof: The proof immediately follows from Theorem 1.1.4.1.

Definition 1.1.4.1 The proximity relation in the above corollary is called the *Czech-Stone proximity* of the topology τ .

1.1.5 The subspace of the proximity space

Theorem 1.1.5.1 Let (X, δ) be a proximity space and $\emptyset \neq Y \subset X$. For sets $A, B \subset Y$ let

(1)
$$A\delta_Y B$$
 if and only if $A\delta B$.

Then (Y, δ_Y) is a proximity space.

Proof: It is obvious that the relation δ_Y satisfies the axioms (B_1) - (B_4) . Let us prove that it satisfies also the axiom (B_5) . Let us suppose $A\overline{\delta}_Y B$, $A, B \subset Y$. Then $A\overline{\delta}B$, so there exists a set $E^* \subset X$ such that $A\overline{\delta}E^*$ and $X - E^*\overline{\delta}B$. Let $E = Y \bigcap E^*$. Then $Y - E = Y - E^* \subset X - E^*$, $E \subset E^*$, and therefore, according to Proposition 1.1.1.2 (b), it follows that $A\overline{\delta}E$ and $Y - E\overline{\delta}B$. Hence, $A\overline{\delta}_Y E$ and $Y - E\overline{\delta}_Y B$.

Definition 1.1.5.1 Let (X, δ) be a proximity space, and let $\emptyset \neq Y \subset X$. The proximity relation δ_Y defined in the above proposition on the subset Y of the set X is called the **restriction on Y of the proximity** δ and is denoted by $\delta|Y$. The ordered pair $(Y, \delta|Y)$ is called the proximity subspace of the proximity space (X, δ) .

Proposition 1.1.5.1 If (X, δ) is a proximity space and $\emptyset \neq Y \subset X$, then a $\delta | Y$ -proximity filter of any subset $A \subset Y$, $A \neq \emptyset$, is $\mathcal{F}(A) \bigcap \{Y\}$. Moreover, the equality $\tau_{\delta|Y} = \tau_{\delta} | Y$ holds.

Proof: Let us denote by $\mathcal{F}_Y(A)$ a δ_Y -proximity filter of the set A in the subspace $(Y, \delta|Y)$ and let $F \in \mathcal{F}_Y(A)$. Then $F \subset Y$ and $A\overline{\delta}_Y Y - F$. Let us denote by $H = F \cup (X - Y)$. Then X - H = Y - F, so that $A\overline{\delta}X - H$, $H \in \mathcal{F}(A)$ and $F = H \cap Y$. From $A\overline{\delta}X - H$ it follows that $H \in \mathcal{F}(A)$, and since $F = H \cap Y$, we have that $F \in \mathcal{F}(A) \cap \{Y\}$. This proves that $\mathcal{F}_Y(A) \subset \mathcal{F}(A) \cap \{Y\}$.

To prove the converse, let us suppose that $F \in \mathcal{F}(A)$, i.e. $A\delta X - F$. Since $Y - (F \cap Y) \subset X - F$, then $A\overline{\delta}_Y Y - (F \cap Y)$, from which it follows that $F \cap Y \in \mathcal{F}_Y(A)$, which had to be proved. The second part of the assertion immediately follows if we put $A = \{x\}$.

Finally, let us give some obvious consequences of the above considerations.

Corollary 1.1.5.1 If δ_1 and δ_2 are proximities on the set X, $\emptyset \neq Y \subset X$, and if $\delta_1 < \delta_2$, then $\delta_1 | Y < \delta_2 | Y$.

Corollary 1.1.5.2 If δ is a proximity relation on the set X and if $\emptyset \neq Z \subset Y \subset X$, then $(\delta|Y)|Z = \delta|Z$.

Corollary 1.1.5.3 Let $\{\delta_i : i \in I\}$ be a non-empty family of the proximities on the set X and let $\delta = \sup\{\delta_i : i \in I\}$. If $\emptyset \neq Y \subset X$, then

$$\sup\{\delta_i|Y:i\in I\}=\delta|Y.\clubsuit$$

The restriction $\delta | Y$ of the proximity relation δ can be considered as the special case of a more general concept.

Let us consider a mapping $f : X \to Y$, where (Y, δ) is a proximity space and let us define a relation on the power set P(X) of the set X in the following way:

(2)
$$A\delta^*B$$
 if and only if $f(A)\delta f(B)$.

Let us prove that δ^* is a proximity relation on the set X. For this purpose it is enough to check the axiom (B_5) , because the other axioms obviously hold. Let $A\overline{\delta}^*B$. Then $f(A)\delta f(B)$, so there exist sets P and Q such that $f(A)\overline{\delta}Y - P$, $f(B)\overline{\delta}Y - Q$ and $P \cap Q = \emptyset$. Since $f(X - f^{-1}(P)) =$ $f(f^{-1}(Y - P)) \subset Y - P$, then by virtue of Proposition 1.1.1.2 we have that $f(A)\overline{\delta}f(X - f^{-1}(P))$, so that $A\overline{\delta}^*X - f^{-1}(P)$. In a similar way it can be proved that $B\overline{\delta}^*X - f^{-1}(Q)$. Finally, from $P \cap Q = \emptyset$ it follows that $f^{-1}(P) \cap f^{-1}(Q) = \emptyset$. Thus the axiom (B_5) is true.

The proximity relation δ^* defined in such a way is called the **inverse image of the proximity** δ and denoted by $f^{-1}(\delta)$. According to Theorem 1.1.5.1 and the above consideration the following corollary holds.

Corollary 1.1.5.4 Let (X, δ) be a proximity space and let $\emptyset \neq Y \subset X$. If $f: Y \to X$ is the canonical injection, then $f^{-1}(\delta) = \delta | Y$.

Proposition 1.1.5.2 If $f : X \to Y$ and δ is a proximity on the set Y, then $f^{-1}(\tau_{\delta}) = \tau(f^{-1}(\delta)).$

Proof: Let $F \in \mathcal{F}(\{f(x)\})$, i.e. $\{f(x)\}\overline{\delta}Y - F$. Then $f^{-1}(F)$ is an element of the neighborhood base of the point x in the topology $f^{-1}(\tau_{\delta})$ and $\{x\}\overline{f^{-1}(\delta)}f^{-1}(Y-F)$ holds. Since $f(f^{-1}(Y-F)) \subset Y - F$, then $\{f(x)\}\overline{\delta}f(f^{-1}(Y-F))$, so that $X - f^{-1}(Y-F) = f^{-1}(F)$ is a neighborhood of the point x with respect to the topology $\tau(f^{-1}(\delta))$. To prove the converse, let us suppose that F is a neighborhood of the point x in the topology $\tau(f^{-1}(\delta))$, i.e. $f(x)\overline{\delta}f(X-F)$. But then $F = Y - f(X-F) \in \mathcal{F}(\{f(x)\})$, so that $F \in f^{-1}(\tau_{\delta})$.

Corollary 1.1.5.5 If $f : X \to Y$ and if δ_1 and δ_2 are the proximities on Y for which $\delta_1 < \delta_2$, then $f^{-1}(\delta_1) < f^{-1}(\delta_2)$ holds.

Corollary 1.1.5.6 Let $f: X \to Y$, $g: Y \to Z$, $h = g \circ f$ and let δ be a proximity relation on the set Z. Then $h^{-1}(\delta) = f^{-1}(g^{-1}(\delta))$.

Theorem 1.1.5.2 If $f : X \to Y$ and if $\{\delta_i : i \in I\}$ is a non-empty family of the proximities on the set Y, and $\delta = \sup\{\delta_i : i \in I\}$, then

$$\sup\{f^{-1}(\delta_i) : i \in I\} = f^{-1}(\delta)$$

Proof: Let $A\delta^*B$, where $\delta^* = \sup\{f^{-1}(\delta_i) : i \in I\}$. Then for every decompositions $\{A_j : j \in J_m\}$ and $\{B_k : k \in J_n\}$ of the sets A and B respectively, there exist some indices $j \in J_m$ and $k \in J_n$ such that $A_j f^{-1}(\delta_i)B_k$, i.e. $f(A_j)\delta_i f(B_k)$ for every $i \in I$. But then $Af^{-1}(\delta^*)B$, i.e. $f(A)\delta^*f(B)$. Indeed, let $\{A'_j : j \in J'_r\}$ and $\{B'_k : k \in J'_s\}$ be the decompositions of the sets f(A) and f(B) respectively. Then $\{A \cap f^{-1}(A'_j) : j \in J'_r\}$ and $\{B \cap f^{-1}(B'_k) : k \in J'_s\}$ are the decompositions of the sets A and B, so that $f(A \cap f^{-1}(A'_j))\delta_i f(B \cap f^{-1}(B'_k))$ for every $i \in I$. Since $f(A \cap f^{-1}(A'_j)) \subset A'_j$ and $f(B \cap f^{-1}(B'_k)) \subset B'_k$, then by Proposition 1.1.1.2 (a) it follows that $A'_j\delta_i B'_k$ for every $i \in I$.

Conversely, if $Af^{-1}(\delta)B$, i.e. $f(A)\delta f(B)$, then for every two decompositions $\{f(A_j) : j \in J_m\}$ and $\{f(B_k) : k \in J_n\}$ of the sets f(A) and f(B)respectively, there exist indices $j \in J_m$ and $k \in J_n$ such that $f(A_j)\delta_i f(B_k)$ for every $i \in I$. Therefore $A_j f^{-1}(\delta_i)B_k$ for every $i \in I$, so that $A\delta^*B$.

1.1.6 Proximally continuous mapping

Definition 1.1.6.1 Let (X, δ_X) and (Y, δ_Y) be two proximity spaces. The mapping $f : X \to Y$ is said to be **proximally** or δ -continuous if $A\delta_X B$ implies $f(A)\delta_Y f(B)$ for every two sets $A, B \subset X$.

Proposition 1.1.6.1 A mapping $f : X \to Y$ of a proximity space (X, δ_X) into a proximity space (Y, δ_Y) is δ -continuous if and only if for every two sets $P, Q \subset Y, P\overline{\delta}_Y Q$ implies $f^{-1}(P)\overline{\delta}_X f^{-1}(Q)$.

Proof: If f is δ -continuous and $P\overline{\delta}_Y Q$, then $f^{-1}(P)\delta_X f^{-1}(Q)$ cannot hold as this would imply $f(f^{-1}(P))\delta_Y f(f^{-1}(Q))$ although $f(f^{-1}(P)) \subset P$ and $f(f^{-1}(Q)) \subset Q$. To prove the converse, let us suppose that $A\delta_X B$. Since $f^{-1}(f(A)) \supset A$ and $f^{-1}(f(B)) \supset B$, then according to Proposition 1.1.1.2, it follows that $f^{-1}(f(A))\delta_X f^{-1}(f(B))$. But then $f(A)\delta_Y f(B)$, because contrary to this case there follows a contradiction to the fact that $f^{-1}(f(A))\delta_X f^{-1}(f(B))$.

Corollary 1.1.6.1 A mapping $f : X \to Y$ of a proximity space (X, δ_X) into a proximity space (Y, δ_Y) is δ -continuous if and only if $P \ll_Y Q$ implies that $f^{-1}(P) \ll_X f^{-1}(Q)$ for every two sets $P, Q \subset Y$.

Corollary 1.1.6.2 Let $f : X \to Y$ be a mapping from a set X on a proximity space (Y, δ_Y) . Then $\delta_X = f^{-1}(\delta_Y)$ is the coarsest proximity on X for which f is a δ -continuous mapping.

Proof: We have already proved that δ_X is a proximity on the set X. Let δ'_X be an arbitrary proximity on X and suppose that f is a δ -continuous mapping with respect to this proximity. Then for any two sets $A, B \subset X$ from $A\delta'_X B$ follows $f(A)\delta_Y f(B)$. Since $f(A)\delta_Y f(B)$ is equivalent to the fact that $Af^{-1}(\delta_Y)B$, then $\delta'_X > f^{-1}(\delta_Y) = \delta_X$.

The following proposition gives an interesting characterization of proximity $f^{-1}(\delta)$ which considered in the previous corollary.

Proposition 1.1.6.2 Let f be a mapping from a set X into a proximity space (Y, δ_Y) . The coarsest proximity δ_X which may be assigned to X in order that f be δ -continuous is defined by

$$A\overline{\delta}_X B$$
 if and only if there exists a set $C \subset Y$
such that $f(A)\overline{\delta}_Y(Y-C)$ and $f^{-1}(C) \subset X-B$.

Proof: Let us first prove that δ_X is a proximity on the set X. Let us suppose that $A\overline{\delta}_X B$ and let $C \subset Y$ be the set for which $f(A)\overline{\delta}_Y(Y-C)$ and $f^{-1}(C) \subset X - B$ hold. Let us consider the set D = Y - f(A). Since $f(B) \subset Y - C$ and $f(A)\overline{\delta}_Y Y - C$, we have that $f(B)\overline{\delta}_Y Y - D$. Moreover, $f^{-1}(D) = X - f^{-1}(f(A)) \subset X - A$. Hence $B\overline{\delta}_X A$, which proves the axiom (B_1) .

To prove the axiom (B_2) , let us suppose that $(A \cup B)\overline{\delta}_X C$. Then there exists a set $D \subset Y$ such that $[f(A) \cup f(B)]\overline{\delta}_Y Y - D$ and $f^{-1}(D) \subset X - C$, from which $A\overline{\delta}_X C$ and $B\overline{\delta}_X C$ follow. Conversely, if $A\overline{\delta}_X C$ and $B\overline{\delta}_X C$, then there exist D_1 and D_2 such that $f(A)\overline{\delta}_Y Y - D_1$, $f(B)\overline{\delta}_Y Y - D_2$, $f^{-1}(D_1) \subset$ X - C and $f^{-1}(D_2) \subset X - C$. Therefore $[f(A) \cup f(B)]\overline{\delta}_Y [Y - (D_1 \cup D_2)]$ and $f^{-1}(D_1 \cup D_2) \subset X - C$, i.e. $(A \cup B)\overline{\delta}_X C$. If $A = \emptyset$, then for $C = \emptyset$ we have that $f(A)\overline{\delta}_Y Y$ and $f^{-1}(\emptyset) \subset X - X$. Hence $X\overline{\delta}_X \emptyset$; thus the axiom (B_3) is true.

Let us now prove the axiom (B_4) . For this we shall prove that $A\overline{\delta}_X B$ implies $A \cap B = \emptyset$, because then it is obvious that $\{x\}\delta_X\{x\}$ for every $x \in X$. Since $A\overline{\delta}_X B$, then there exists a set $C \subset Y$ such that $f(A)\overline{\delta}_Y Y - C$ and $f^{-1}(C) \subset X - B$. Therefore $f(A) \cap (Y - C) = \emptyset$ and $f^{-1}(f(A)) \cap f^{-1}(Y - C) = \emptyset$. Since $A \subset f^{-1}(f(A))$ and $B \subset f^{-1}(Y - C)$, $A \cap B = \emptyset$ follows.

If $A\overline{\delta}_X B$, then there exists a set $C \subset Y$ such that $f^{-1}(C) \subset X - B$ and $f(A)\overline{\delta}_Y Y - C$. Then according to the axiom (B_5) there exists a set $D \subset Y$ such that $f(A)\overline{\delta}_Y D$ and $Y - D\overline{\delta}_Y Y - C$. Let $E = f^{-1}(D)$. Since $f(A)\overline{\delta}_Y D$, so that $A\overline{\delta}_X E$. But now from $f(X - E) \subset (Y - D)$, $Y - D\overline{\delta}_Y Y - C$ and $f^{-1}(C) \subset X - B$, it follows that $X - E\overline{\delta}_X B$. So, we have proved that δ_X is a proximity on the set X.

To prove that $f: (X, \delta_X) \to (Y, \delta_Y)$ is a δ -continuous mapping, let us suppose that $f(A)\overline{\delta}_Y f(B)$. Since $f(A) \ll Y - f(B)$, there exists a set Csuch that $f(A) \ll C \ll Y - f(B)$ by Corollary 1.1.6.1. Thus $f(A)\overline{\delta}_Y Y - C$ and $f^{-1}(C) \subset X - f^{-1}(f(B)) \subset X - B$, i.e. $A\overline{\delta}_X B$.

It remains to show that if δ_1 is any proximity on X such that $f : (X, \delta_1) \to (Y, \delta_Y)$ is δ -continuous, then δ_1 is finer than δ_X . If $A\overline{\delta}_X B$, then there exists a set $C \subset Y$ such that $f(A)\overline{\delta}_Y Y - C$ and $f^{-1}(C) \subset X - B$. Since f is δ -continuous, we have that $A\overline{\delta}_1 X - f^{-1}(C)$. But then $B \subset X - f^{-1}(C)$ implies $A\overline{\delta}_1 B$. Thus $\delta_1 > \delta_X$.

Corollary 1.1.6.3 The composition of δ -continuous mappings is a δ -continuous mapping.

Corollary 1.1.6.4 Let δ_1 and δ_2 be two proximities on the set X. The identity mapping $i : (X, \delta_1) \to (X, \delta_2)$ of the set X is a δ -continuous mapping if and only if $\delta_1 > \delta_2$.

Corollary 1.1.6.5 Let $f : (X, \delta_X) \to (Y, \delta_Y)$ be a δ -continuous mapping. If δ'_X is a proximity on X finer than proximity δ_X , and δ'_Y a proximity on Y coarser than proximity δ_Y , then the mapping $f : (X, \delta'_X) \to (Y, \delta'_Y)$ is δ -continuous.

Proposition 1.1.6.3 Let δ_X be a proximity relation on X, $\{\delta_Y^i : i \in I\}$ be a non-empty family of proximities on Y and $\delta_Y = \sup\{\delta_Y^i : i \in I\}$. A mapping $f : (X, \delta_X) \to (Y, \delta_Y)$ is δ -continuous if and only if $f : (X, \delta_X) \to (Y, \delta_Y^i)$ is a δ -continuous mapping for each $i \in I$.

Proof: By Corollary 1.1.6.2 the mapping $f : (X, \delta_X) \to (Y, \delta_Y)$ is δ continuous if and only if $f^{-1}(\delta_Y) < \delta_X$, while the mapping $f : (X, \delta_X) \to (Y, \delta_Y^i)$ is δ -continuous if and only if $f^{-1}(\delta_Y^i) < \delta_X$. Now the assertion follows from Theorem 1.1.5.2.

Proposition 1.1.6.4 Let $\{\delta_X^i : i \in I\}$ be a non-empty family of proximities on X, $\delta_X = \inf\{\delta_X^i : i \in I\}$ and let δ_Y be a proximity on Y. The mapping $f : (X, \delta_X) \to (Y, \delta_Y)$ is δ -continuous if and only if $f : (X, \delta_X^i) \to (Y, \delta_Y)$ is a δ -continuous mapping for each $i \in I$.

Proof: Let $\delta'_X = f^{-1}(\delta_Y)$. If $f : (X, \delta^i_X) \to (Y, \delta_Y)$ is a δ -continuous mapping for each $i \in I$, then by Corollary 1.1.6.2 $\delta'_X < \delta^i_X$ for each $i \in I$ holds. Therefore $\delta'_X < \delta_X$, so that $f : (X, \delta_X) \to (Y, \delta_Y)$ is a δ -continuous mapping. Conversely, if $f : (X, \delta_X) \to (Y, \delta_Y)$ is a δ -continuous mapping, then by virtue of Corollary 1.1.6.5 the mapping $f : (X, \delta^i_X) \to (Y, \delta_Y)$ is δ -continuous for each $i \in I$.

The proofs of the following three propositions are easy and left to the reader.

Proposition 1.1.6.5 Let (X, δ) be a proximity space and $\emptyset \neq Y \subset X$. The canonical injection $f : (Y, \delta|Y) \to (X, \delta)$ is δ -continuous.

Proposition 1.1.6.6 Let (X, δ_X) and (Y, δ_Y) be the proximity spaces, $f : X \to Y$, $f(X) \subset Y_0 \subset Y$. The mapping $f : (X, \delta_X) \to (Y, \delta_Y)$ is δ -continuous if and only if the mapping $f|_X^{Y_0} : (X, \delta_X) \to (Y_0, \delta_Y|Y_0)$ is δ -continuous.

Proposition 1.1.6.7 Let $f : X \to Y$ and $\emptyset \neq X_0 \subset X$. If the mapping $f : (X, \delta_X) \to (Y, \delta_Y)$ is δ -continuous, then $f|X_0 : (X_0, \delta_X|X_0) \to (Y, \delta_Y)$ is also a δ -continuous mapping.

Proposition 1.1.6.8 If f is a δ -continuous mapping of the proximity space (X, δ_X) into the proximity space (Y, δ_Y) , then it is continuous with respect to the topologies $\tau(\delta_X)$ and $\tau(\delta_Y)$.

Proof: Let $x \in \overline{A}^{\tau(\delta_X)}$. Then $\{x\}\delta_X A$, and since f is a δ -continuous mapping, then $\{f(x)\}\delta_Y f(A)$, i.e. $f(x) \in \overline{f(A)}^{\tau(\delta_Y)}$. This proves that $f(\overline{A}^{\tau(\delta_X)}) \subset \overline{f(A)}^{\tau(\delta_Y)}$, thus the mapping is continuous with respect to the topologies $\tau(\delta_X)$ and $\tau(\delta_Y)$.

The converse in general case is not true. Indeed, if in Example 1.1.4.1 we take the identical mapping, then it is continuous with respect to the topologies $\tau(\delta_1)$ and $\tau(\delta_2)$, but it is not a δ -continuous mapping of the proximity space (X, δ_1) onto the proximity space (X, δ_2) . The following two propositions give us the conditions when the converse is true.

Proposition 1.1.6.9 Let (X, δ_X) and (Y, δ_Y) be the proximity spaces and δ_X the Czech-Stone proximity of the topology $\tau = \tau(\delta_X)$. If $f : (X, \tau(\delta_X)) \to (Y, \tau(\delta_Y))$ is a continuous mapping, then $f : (X, \delta_X) \to (Y, \delta_Y)$ is a δ -continuous mapping.

Proof: From the general topology it is known that $f^{-1}(\tau(\delta_Y)) < \tau(\delta_X)$ holds. Let $\delta'_X = \sup\{\delta_X, f^{-1}(\delta_Y)\}$. Then $\tau(\delta'_X) = \sup\{\tau(\delta_X), f^{-1}(\tau(\delta_Y))\}$ holds on account of Theorem 1.1.4.1, since by Proposition 1.1.5.2 it follows that $\tau(f^{-1}(\delta_Y)) = f^{-1}(\tau(\delta_Y))$. According to the definition of the Czech-Stone proximity there follows that $\delta'_X < \delta_X$, so that $\delta'_X = \delta_X$, hence $f^{-1}(\delta_Y) < \delta_X$. Now by Corollary 1.1.6.2 and Corollary 1.1.6.5 it follows that the mapping $f: (X, \delta_X) \to (Y, \delta_Y)$ is δ -continuous.

Proposition 1.1.6.10 If (X, δ_X) and (Y, δ_Y) are the proximity spaces and X is a compact space with respect to the topology $\tau(\delta_X)$, then every mapping $f : X \to Y$ which is continuous with respect to the topologies $\tau(\delta_X)$ and $\tau(\delta_Y)$ is also δ -continuous with respect to the proximities δ_X and δ_Y .

Proof: Let $A\delta_X B$. Then by Proposition 1.1.2.4 $\overline{A} \cap \overline{B} \neq \emptyset$, so that $f(\overline{A}) \cap f(\overline{B}) \neq \emptyset$. But then according to Proposition 1.1.1.2 (d) it follows that $f(\overline{A})\delta_Y f(\overline{B})$. Since f is a continuous mapping, then $f(\overline{A}) \subset \overline{f(A)}$ and $f(\overline{B}) \subset \overline{f(B)}$, so that by Proposition 1.1.1.2 (a) $\overline{f(A)}\delta_Y \overline{f(B)}$. Now by Proposition 1.1.2.4 it follows that $f(A)\delta_Y f(B)$, so that f is a δ -continuous mapping.

Definition 1.1.6.2 If $f : (X, \delta_X) \to (Y, \delta_Y)$ is a bijective δ -continuous mapping and $f^{-1} : (Y, \delta_Y) \to (X, \delta_X)$ is a δ -continuous mapping, then fis said to be a proximally equimorphism, proximally isomorphism or δ -homeomorphism from X onto Y. The proximity spaces (X, δ_X) and (Y, δ_Y) are proximally equimorphic, proximally isomorphic or δ -homeomorphic if there exists a δ -homeomorphism between them.

This relation is reflexive, symmetric and transitive. Since an δ -equimorphism is a homeomorphism between (X, τ_{δ_X}) and (Y, τ_{δ_Y}) , δ -equimorphic spaces are also homeomorphic.

Proposition 1.1.6.11 Let f be a given mapping from the proximity space (X, δ_X) into the proximity space (Y, δ_Y) . Then

(a) f is δ -continuous if and only if $\delta_X > f^{-1}(\delta_Y)$;

(b) if f is an injective mapping, then $\delta_X = f^{-1}(\delta_Y)$ if and only if the mapping $h = f|_X^{f(X)}$ is a δ -homeomorphism from the proximity space (X, δ_X) onto the proximity space $(f(X), \delta_Y | f(X))$;

(c) if f is bijective mapping, then $\delta_X = f^{-1}(\delta_Y)$ holds if and only if f is a δ -homeomorphism from (X, δ_X) onto (Y, δ_Y) .

Proof: (a) Let f be a δ -continuous mapping. If $A\delta_X B$, then $f(A)\delta_Y f(B)$, so that $Af^{-1}(\delta_Y)B$ by the definition of the proximity $f^{-1}(\delta_Y)$. This proves that $\delta_X > f^{-1}(\delta_Y)$. Conversely, let us suppose that $\delta_X > f^{-1}(\delta_Y)$ and let $A\delta_X B$. Then $Af^{-1}(\delta_Y)B$, which is equivalent with $f(A)\delta_X f(B)$. But then the mapping f is δ -continuous.

(b) Since $\delta_X = f^{-1}(\delta_Y)$, then by Corollary 1.1.6.2 $f: (X, f^{-1}(\delta_Y)) \to (Y, \delta_Y)$ is a δ -continuous mapping. But then by Proposition 1.1.6.6 $h|_X^{f(X)}: (X, f^{-1}(\delta_Y)) \to (f(X), \delta_Y | f(X))$ is a δ -continuous mapping. Let $g = h^{-1}$. It is obvious that $f \circ g: f(X) \to Y$ is a canonical injection. Thus, by Corollary 1.1.5.4 and Corollary 1.1.5.6, it follows that $\delta_Y | f(X) = (f \circ g)^{-1}(\delta_Y) = g^{-1}(f^{-1}(\delta_Y))$. But then, by Proposition 1.1.6.2, $g: (f(X), \delta_Y | f(X)) \to (X, f^{-1}(\delta_Y))$ is a δ -continuous mapping, so that h is a δ -homeomorphism. To prove the converse, let $h = f|_X^{f(X)}$ be a δ -homeomorphism from the proximity space (X, δ_X) onto the proximity space $(f(X), \delta_Y | f(X))$. Then the identical mapping $g \circ h$ is a δ -homeomorphism from (X, δ_X) onto space $(X, f^{-1}(\delta_Y))$. Now by Corollary 1.1.6.4 we have that $\delta_X = f^{-1}(\delta_Y)$.

(c) The assertion is a special case of (b). \clubsuit

1.1.7 Product of proximity spaces

Proposition 1.1.7.1 Let $\{(X_i, \delta_i) : i \in I\}$ be a non-empty family of the proximity spaces, $X \neq \emptyset$ and $f_i : X \to X_i$ being a given mapping for each $i \in I$. Then there exists the coarsest proximity δ^* on the set X for which $f_i : (X, \delta^*) \to (X_i, \delta_i)$ is δ -continuous mapping for every $i \in I$. In this case $\delta^* = \sup\{f_i^{-1}(\delta_i) : i \in I\}.$

Proof: Let δ be an arbitrary proximity on the set X for which $f_i : (X, \delta) \to (X_i, \delta_i)$ is a δ -continuous mapping for every $i \in I$. In this case by Proposition 1.1.6.11 (a) there follows that $\delta > f_i^{-1}(\delta_i)$ for every $i \in I$. However,

according to Theorem 1.1.4.1, there is the coarsest of the proximities on X, say δ^* , which is finer than all the proximities $f_i^{-1}(\delta_i)$. Therefore it is coarsest than the proximity δ . Since δ is an arbitrary proximity on X, the proof of the proposition is finished.

Definition 1.1.7.1 The proximity δ^* in the above proposition is called the proximity projectively generated by the system $\{f_i, \delta_i : i \in I\}$.

The inverse image $f^{-1}(\delta)$ of a proximity δ is nothing other than the proximity projectively generated by the system $\{f, \delta\}$ of one element, where $f: X \to (Y, \delta)$ is a given mapping. On the other hand, if $X_i = X$ for every $i \in I$, while f_i is the identical mapping on X, then the system $\{f_i, \delta_i : i \in I\}$ generates projectively precisely the proximity $\sup\{\delta_i : i \in I\}$.

In the following we shall always use the notation introduced in Proposition 1.1.7.1.

Corollary 1.1.7.1 For the sets in the proximity space (X, δ^*) it holds that

 $A\delta^*B$ if and only if for any two finite decompositions $\{A_j : j \in J_m\}$ and $\{B_k : k \in J_n\}$ of the sets A and B respectively, there are indices j and k such that $f_i(A_j)\delta_i f_i(B_k)$ for every $i \in I$.

Proof: The proof immediately follows from Theorem 1.1.4.1 and definition of the inverse image of proximity.

Corollary 1.1.7.2 If (Y, δ_Y) is a proximity space and $g : Y \to X$, then $g : (Y, \delta_Y) \to (X, \delta^*)$ is δ -continuous if and only if $f_i \circ g : (Y, \delta_Y) \to (X_i, \delta_i)$ is a δ -continuous mapping for every $i \in I$.

Proof: Let $g: Y \to X$ be a δ -continuous mapping. Since $f_i: (X, \delta^*) \to (X_i, \delta_i), i \in I$, is a δ -continuous mapping according to Proposition 1.1.7.1, then $f_i \circ g: (X, \delta) \to (X_i, \delta_i), i \in I$, is a δ -continuous mapping by virtue of Corollary 1.1.6.3. Conversely, let us suppose that $f_i \circ g: (X, \delta^*) \to (X_i, \delta_i)$ is a δ -continuous mapping for every $i \in I$. If $P\overline{\delta}^*Q$, where $\delta^* = \sup\{f^{-1}(\delta_i): i \in I\}$, then there are decompositions $\{P_j: j \in J_m\}$ and $\{Q_k: k \in J_n\}$ of the sets P and Q respectively, so that for every $(j,k) \in J_m \times J_n$ there exists some i = i(j,k) for which $f_i(P_j)\overline{\delta}_i f_i(Q_k)$ holds. Since $f_i \circ g$ is a δ -continuous mapping for every $i \in I$, then for every $(j,k) \in J_m \times J_n$ we have that $(f_i \circ g)^{-1}(f_i(P_j))\overline{\delta}_Y(f_i \circ g)^{-1}(f_i(Q_k))$, i.e. $g^{-1}(f_i^{-1}(f_i(P_j)))\overline{\delta}_Y g^{-1}(f_i^{-1}(f_i(Q_k)))$ where i = i(j,k). Since $P_j \subset f_i^{-1}(f_i(P_j))$ and $Q_k \subset f_i^{-1}(f_i(Q_k))$, then $g^{-1}(P_j)\overline{\delta}_Y g^{-1}(Q_k)$ for every $(j,k) \in J_m \times J_n$. But then according to the axiom (B_2) it follows that $g^{-1}(P)\overline{\delta}_Y g^{-1}(Q)$, so that g is a δ -continuous mapping.

Corollary 1.1.7.3 Let $J_i \neq \emptyset$ be a set of indices for every $i \in I$, (X_{ij}, δ_{ij}) a proximity space for $i \in I$ and $j \in J_i$, $f_{ij} : X_i \to X_{ij}$ a mapping such that δ_i is the proximity on X_i projectively generated by the system $\{f_{ij}, \delta_{ij} : j \in J_i\}$. Then δ^* is identical with the proximity projectively generated by the system $\{f_{ij} \circ f_i, \delta_{ij} : i \in I, j \in J_i\}$.

Proof: This immediately follows from the previous corollary.

Corollary 1.1.7.4 If $g: Y \to X$, $Y \neq \emptyset$, then proximity $g^{-1}(\delta^*)$ coincides with the proximity projectively generated by the system $\{f_i \circ g, \delta_i : i \in I\}$.

Proof: This assertion is a special case of the previous corollary.

Corollary 1.1.7.5 Let $\emptyset \neq Y \subset X$ and $f_i(Y) \subset Y_i \subset X_i$ for every $i \in I$. Then $\delta^*|Y$ is a proximity projectively generated by system $\{f_i|_Y^{Y_i}, \delta_i|Y_i : i \in I\}$.

Proof: The assertion immediately follows from Corollary 1.1.7.3, Corollary 1.1.5.4 and Proposition 1.1.6.11.

Corollary 1.1.7.6 Let (Y_i, δ'_i) be a proximity space, $g_i : X_i \to Y_i$ a mapping for which $\delta_i = g_i^{-1}(\delta'_i)$ holds. Then δ^* is a proximity identical with the proximity projectively generated by the system $\{g_i \circ f'_i, \delta'_i : i \in I\}$.

Proof: This follows from Corollary 1.1.7.3.

Corollary 1.1.7.7 If $g_i : X_i \to Y_i$ is a δ -homeomorphism for every $i \in I$, then the proximity δ^* is identical with the proximity projectively generated by the system $\{g_i \circ f_i : i \in I\}$.

Proof: This immediately follows from Proposition 1.1.6.11 (c).

Corollary 1.1.7.8 If δ'_i is a proximity on X_i such that $\delta_i < \delta'_i$ for every $i \in I$ and δ^{**} is the proximity projectively generated by the system $\{f_i, \delta'_i : i \in I\}$, then $\delta^* < \delta^{**}$.

Proof: This follows from Corollary 1.1.5.5.

Corollary 1.1.7.9 The topology projectively generated by the system $\{f_i, \tau(\delta_i) : i \in I\}$ coincides with the topology $\tau(\delta^*)$.

Proof: The topology projectively generated by the system $\{f_i, \tau(\delta_i) : i \in I\}$ is defined as $\sup\{f_i^{-1}(\tau(\delta_i)) : i \in I\}$. According to Theorem 1.1.4.1 we have the equality $\sup\{f_i^{-1}(\tau(\delta_i)) : i \in I\} = \sup\{\tau(f_i^{-1}(\delta_i)) : i \in I\}$. Since by Proposition 1.1.5.2 $\sup\{\tau(f_i^{-1}(\delta_i)) : i \in I\} = \tau(\sup\{f_i^{-1}(\delta_i) : i \in I\}) =$ $\tau(\delta^*)$, then $\sup\{f_i^{-1}(\tau(\delta_i)) : i \in I\} = \tau(\delta^*)$.

Definition 1.1.7.2 Let (X_i, δ_i) be a proximity space for every $i \in I \neq \emptyset$, $X = \prod_{i \in I} X_i$ and $p_i : X \to X_i$ the *i*-th projection. The proximity relation δ which is projectively generated on X by the system $\{p_i, \delta_i : i \in I\}$ is called the **product of the proximities** δ_i and is denoted by $\prod_{i \in I} \delta_i$. The proximity space $(\prod_{i \in I} X_i, \prod_{i \in I} \delta_i)$ is the **product of the proximity spaces** (X_i, δ_i) .

The above notations will be used henceforth.

Now by virtue of definition of the product of the proximities and Proposition 1.1.7.1 we have the following

Corollary 1.1.7.10 For the sets $A, B \subset X$ $A\delta B$ holds if and only if for any decompositions $\{A_j : j \in J_m\}$ and $\{B_k : k \in J_n\}$ of the sets A and Brespectively there exist indices $j \in J_m$ and $k \in J_n$ such that $p_i(A_j)\delta_i p_i(B_k)$ for every $i \in I$.

Corollary 1.1.7.11 Let (Y, δ_Y) be a proximity space and $g : Y \to X$ a mapping. The mapping $g : (Y, \delta_Y) \to (X, \delta)$ is a δ -continuous if and only if the composition $p_i \circ g : (Y, \delta_Y) \to (X_i, \delta_i)$ is δ -continuous for every $i \in I$.

Proof: This immediately follows by virtue of Corollary 1.1.6.3 and Corollary 1.1.7.8.

Corollary 1.1.7.12 If δ'_i is a proximity relation on a set X_i , so that $\delta_i < \delta'_i$ for every $i \in I$, then $\prod_{i \in I} \delta_i < \prod_{i \in I} \delta'_i$.

Proof: This follows from Corollary 1.1.7.8.

Corollary 1.1.7.13 If $\emptyset \neq Y_i \subset X_i$ and $Y = \prod_{i \in I} Y_i$, then $\prod_{i \in I} (\delta_i | Y_i) = \delta | Y$.

Corollary 1.1.7.14 If $\emptyset \neq Y_i \subset X_i$ and $Y = \prod_{i \in I} Y_i$, then $\prod_{i \in I} (\delta_i | Y_i) = \delta | Y$. If $Y_i = X_i$ for some $i \in I$, while for the other indices $Y_i = \{y_i\}$, where $y_i \in X_i$, then $p_j | Y : (Y, \delta | Y) \to (X_i, \delta_i)$ is a δ -homeomorphism.

Proof: The first part of the assertion follows from Corollary 1.1.7.4. The second part follows from Proposition 1.1.6.11 and the fact that any mapping from a proximity space onto one element set equipped with indiscrete proximity is δ -continuous.

Corollary 1.1.7.15 Let (Y_i, δ_Y^i) be a proximity space for every $i \in I$, $f_i : (X_i, \delta_i) \to (Y_i, \delta_Y^i)$ a δ -continuous mapping, $Y = \prod_{i \in I} Y_i$, $\delta_Y = \prod_{i \in I} \delta_Y^i$, $p'_i : Y \to Y_i$ the *i*-th projection and $f : X \to Y$ the mapping for which $f_i \circ p_i = p'_i \circ f$. Then $f : (X, \delta) \to (Y, \delta_Y)$ is a δ -continuous mapping. If f_i is a δ -homeomorphism, then f is also a δ -homeomorphism.

Proof: Since f_i is a δ -continuous mapping, then by Corollary 1.1.6.3 $f_i \circ p_i$ is a δ -continuous mapping. But then $p'_i \circ f$ is also a δ -continuous mapping, from which, by Corollary 1.1.7.11, it follows that the mapping f is δ -continuous. The last part of the assertion follows from the assertion previously proved and the fact that $f_i^{-1} \circ p'_i = p_i \circ f^{-1}$.

Let $f: I \to J$ be a bijection and $Y_i = X_{f(i)}$. Then the mapping $g: \prod_{i \in I} Y_i \to \prod_{j \in J} X_j$ defined by: if g(b) = a, where $b = (b_i)$, then $a_{f(i)} = b_i$, is a bijection from the set $Y = \prod_{i \in I} Y_i$ onto the set $X = \prod_{j \in J} X_j$. For the mapping defining in such a way there follows:

Corollary 1.1.7.16 Let δ_i be a proximity space on the set Y_i , $\delta'_{f(i)} = \delta_i$, $\delta = \prod_{i \in I} \delta_i$ and $\delta' = \prod_{j \in J} \delta'_j$. Then $g: (Y, \delta) \to (X, \delta')$ is a δ -homeomorphism.

Let us consider a bijection $f: I \to J$. Let $I = \bigcup_{j \in J} I_j$, $I_{j_1} \cap I_{j_2} = \emptyset$ for $j_1 \neq j_2$, $Y_j = \prod_{i \in I_j} X_i$, $X = \prod_{i \in I} X_i$, $Y = \prod_{j \in J} Y_j$, and let $p_i: X \to X_i$, $q_j: Y \to Y_j$ and $r_{j_i}: Y_j \to X_i$ be the projections. If $r_{j_i}(q_j(f(x))) = p_i(x)$ for every $x \in X$, then $f: X \to Y$ is a bijection for which the following assertion holds:

Corollary 1.1.7.17 If δ_i is a proximity on the set X_i , $\delta = \prod_{i \in I} \delta_i$, $\delta'_j = \prod_{i \in I_i} \delta_i$, $\delta' = \prod_{i \in J} \delta'_j$, then $f : (X, \delta) \to (Y, \delta')$ is a δ -homeomorphism.

The last two assertions immediately follow from Corollary 1.1.7.11 while from Corollary 1.1.7.9 the following assertion holds.

Corollary 1.1.7.18 If $\delta = \prod_{i \in I} \delta_i$, then $\tau_{\delta} = \prod_{i \in I} \tau(\delta_i)$. However, the product of separated proximities is also a separated proximity.

Finally, let us give a theorem which is analogous to the embedding theorem in topological spaces. **Theorem 1.1.7.1** Let (X_i, δ_i) be a proximity space for every $i \in I \neq \emptyset$, Y a given set, $f_i : Y \to X_i$ a given mapping, δ^* a proximity on the set Y projectively generated by the system $\{f_i, \delta_i : i \in I\}$, $X = \prod_{i \in I} X_i$, $\delta = \prod_{i \in I} \delta_i$, $p_i : X \to X_i$ the *i*-th projection, $f : Y \to X$ the mapping for which $p_i \circ f = f_i$, and $h = f|_Y^{f(Y)}$. If for $x, y \in X$ from $x \neq y$ follows $f_i(x) \neq f_i(y)$ for at least one $i \in I$, then $h : (Y, \delta^*) \to (f(X), \delta | f(X))$ is a δ -homeomorphism. This assertion is certainly fulfilled if δ^* is a separated proximity.

Proof: According to Corollary 1.1.7.4 there follows that $\delta^* = f^{-1}(\delta)$. Since *h* is a bijective mapping, then, by Proposition 1.1.6.11, it is a δ -homeomorphism. If the proximity δ^* is separated and if for elements $x, y \in X \ x \neq y$ holds, then $\{x\}\overline{\delta}^*\{y\}$, so that by Corollary 1.1.7.1 there exists some index $i \in I$ for which $\{f_i(x)\}\overline{\delta}_i\{f_i(y)\}$ holds. Therefore $f_i(x) \neq f_i(y)$.

1.1.8 Quotient space of proximity spaces

Let now (X_i, δ_i) be a proximity space for every $i \in I \neq \emptyset$, $f_i : X_i \to X$ a given mapping and let us consider the finest proximity δ^* on the set X for which each one of the mappings f_i is δ -continuous. This proximity exists since every f_i is δ -continuous with respect to the indiscrete proximity on X and then we have to take only all the proximities δ for which every f_i is δ -continuous and denote their supremum by δ^* :

Proposition 1.1.8.1 Let (X_i, δ_i) be a proximity space for every $i \in I \neq \emptyset$, $f_i : X_i \to X$ a given mapping, and δ^* the supremum of those proximities δ on the set X for which every $f_i : (X_i, \delta_i) \to (X, \delta)$ is δ -continuous. Then δ^* is the finest among the proximities considered.

Proof: It needs only to be checked that every $f_i : (X_i, \delta_i) \to (X, \delta^*)$, $i \in I$, is δ -continuous. By Proposition 1.1.6.11 this will hold if and only if $f_i^{-1}(\delta^*) < \delta_i$ for every $i \in I$. The last inequality, by Theorem 1.1.5.2, is equivalent to $\sup_{\delta} f_i^{-1}(\delta) < \delta_i$, $i \in I$, where supremum is considered with respect to all the proximities δ on X for which f_i is δ -continuous. Since $f_i : (X_i, \delta_i) \to (X, \delta)$ is δ -continuous by Proposition 1.1.6.11 if and only if $\delta_i > f_i^{-1}(\delta)$, the inequality $\sup_{\delta} f_i^{-1}(\delta) < \delta_i$ obviously holds.

Definition 1.1.8.1 For the proximity δ^* in the above proposition it is said to be inductively generated by the system $\{f_i, \delta_i : i \in I\}$.
There is an essential difference with respect to the inductive generation of topologies, namely that δ^* cannot be constructed, in general, in a simple way by means of the given f_i and δ_i , not even in the special case of a single proximity space (X', δ') and a single mapping $f : X' \to X$. In this case the notation $\delta^* = f(\delta')$ is used and we speak of the **quotient proximity** belonging to δ' and f.

In fact, if there is a given proximity on X for which f is δ -continuous, then for the sets $A, B \subset X$, $f^{-1}(A)\delta'f^{-1}(B)$ implies that $f(f^{-1}(A)) \subset A$ and $f(f^{-1}(B)) \subset B$ are near to each other. Therefore if a relation δ is defined in such a way that $A\delta B$ holds if and only if $f^{-1}(A)\delta'f^{-1}(B)$ and δ defined in this manner is a proximity on X, then the fewest possible pairs of sets will be near to each other with respect to δ , and thus δ will be identical with $f(\delta')$. Now it can be easily proved that, if f is surjective, then δ defined in this way will certainly fulfil the axioms (B_1) to (B_4) . The axiom (B_5) is fulfilled if and only if $f^{-1}(A)\overline{\delta}'f^{-1}(B)$ implies that there exist the sets Pand Q such that $P \cap Q = \emptyset$, $f^{-1}(A)\overline{\delta}'f^{-1}(X-P)$ and $f^{-1}(B)\overline{\delta}'f^{-1}(X-Q)$.

The latter condition is not always fulfilled. For that reason let us consider the following:

Example 1.1.8.1 Let for example $X' = \mathbb{R}$, $\delta' = \delta_{d_1}$ (where d_1 is the Euclidean metric), X being the set of integers, $f : X' \to X$ defined by f(x) = [x]. $A = \{0\}$, $B = \{2\}$ imply $f^{-1}(A) = [0, 1)$, $f^{-1}(B) = [2, 3)$, so that $f^{-1}(A)\overline{\delta'}f^{-1}(B)$. However, for the arbitrary sets $P, Q \subset X$, the conditions $[0, 1)\overline{\delta'}f^{-1}(X - P)$, $[2, 3)\overline{\delta'}f^{-1}(X - Q)$ imply that both $f^{-1}(P)$ and $f^{-1}(Q)$ intersect the interval [1, 2), hence $1 \in P \cap Q$ is fulfilled.

As a result of the previous consideration the following proposition holds:

Proposition 1.1.8.2 Let (Y, δ_Y) be a proximity space, $f: Y \to X$ a given mapping and δ be a relation on the set X defined by: $A\delta B$ if and only if $f^{-1}(A)\delta_Y f^{-1}(B)$. Whenever δ , defined in this way, is a proximity on X, then $\delta = f(\delta_Y)$. This is the case if f is surjective and if for $A, B \subset X$, $f^{-1}(A)\overline{\delta}_Y f^{-1}(B)$, there exist $C, D \subset X$ such that $C \cap D = \emptyset$, $f^{-1}(A)\overline{\delta}_Y f^{-1}$ (X - C) and $f^{-1}(B)\overline{\delta}_Y f^{-1}(X - D)$.

Corollary 1.1.8.1 If $g: X \to Y$ is a bijection, $f = g^{-1}$, δ_Y is a proximity on the set Y, then $f(\delta_Y) = g^{-1}(\delta_Y)$ (with the notations introduced in the previous proposition).

Proof: Let us suppose that for the sets $A, B \subset X$ holds $f^{-1}(A)\overline{\delta}_Y f^{-1}(B)$. Then there exist sets $C', D' \subset Y$ for which $f^{-1}(A)\overline{\delta}_Y Y - C', f^{-1}(B)\overline{\delta}_Y Y - D'$ and $C' \cap D' = \emptyset$ holds. Then for the sets C = f(C') and D = f(D') it follows that $f^{-1}(A)\overline{\delta}_Y f^{-1}(Y-C), f^{-1}(B)\overline{\delta}_Y f^{-1}(Y-D)$ and $C \cap D = \emptyset$.

Starting from the definition, we have the following propositions.

Proposition 1.1.8.3 $\delta^* = \inf\{f_i(\delta_i) : i \in I\}.$

Proposition 1.1.8.4 If δ'_i is a proximity on X_i for which $\delta_i < \delta'_i$, then the proximity inductively generated by the system $\{f_i, \delta'_i : i \in I\}$ is finer than the proximity δ^* .

Proposition 1.1.8.5 Let (Y, δ_Y) be a proximity space and $g : X \to Y$ a given mapping. The mapping $g : (X, \delta^*) \to (Y, \delta_Y)$ is δ -continuous if and only if $g \circ f_i : (X_i, \delta_i) \to (Y, \delta_Y)$ is a δ -continuous mapping for every $i \in I$.

Proposition 1.1.8.6 Let (X_{ij}, δ_{ij}) be a proximity space for every $i \in I \neq \emptyset$ and $j \in J_i \neq \emptyset$, $f_{ij} : X_{ij} \to X_i$ a mapping such that δ_i is the proximity inductively generated by the system $\{f_{ij}, \delta_{ij} : j \in J_i\}$. Then δ^* is the proximity inductively generated by the system $\{f_i \circ f_{ij}, \delta_{ij}\}$.

Proposition 1.1.8.7 If (Y_i, δ'_i) is a proximity space for every $i \in I$ and $g_i : Y_i \to X_i$ a mapping for which $\delta_i = g_i(\delta'_i)$ holds, then the proximity δ^* is identical with the proximity inductively generated by the system $\{f_i \circ g_i, \delta'_i : i \in I\}$. The statement also holds in the case when g_i is a δ -homeomorphism for every $i \in I$.

Proposition 1.1.8.8 Let (X, δ_X) and (Y, δ_Y) be two proximity spaces, $f : X \to Y$ a given mapping. Then the following statements are equivalent:

- (a) f is a δ -continuous mapping;
- (b) $\delta_Y < f(\delta_X);$
- (c) $f^{-1}(\delta_Y) < \delta_X$.

Proposition 1.1.8.9 Let (X, δ_X) , (Y, δ_Y) and (Z, δ_Z) be proximity spaces, $f : X \to Y$ and $g : Y \to Z$ given mappings, and let $\delta_Y = f(\delta_X)$. The mapping g is δ -continuous if and only if $g \circ f$ is a δ -continuous mapping.

Proposition 1.1.8.10 If (X_i, δ_i) is a proximity space for every $i \in I \neq \emptyset$, $X = \prod_{i \in I} X_i, \ \delta = \prod_{i \in I} \delta_i \text{ and } p_i : X \to X_i \text{ is the i-th projection, then}$ $\delta_i = p_i(\delta)$ for every $i \in I$. **Proof:** According to Proposition 1.1.8.2 it must be shown that if $A_j, B_j \subset X_j$, then $A_j \delta_j B_j$ holds if and only if $p_j^{-1}(A_j) \delta p_j^{-1}(B_j)$. Now as p_j is proximally continuous, it follows that the latter relation implies the former one. Let us suppose therefore that $A_j \delta_j B_j$ and let $A = p_j^{-1}(A_j), B = p_j^{-1}(B_j)$ and let $\{C_r : r \in I_a\}$ and $\{D_s : s \in I_b\}$ be decompositions of the sets A and B respectively. Let us consider the sets $p_j(C_r) = C'_r, r \in I_a$, and let us construct all the sets of the form $\bigcap_{r=1}^a E_r$, where $E_r = C'_r$ or $E_r = A_j - C'_r$ for every $r \in I_a$. Let us denote these intersections by P_1, P_2, \ldots, P_p . It is obvious that $A_j = \bigcup_{m=1}^p P_m$, the sets P_m are disjoint, and every set C'_r is the union of those P_m which are contained in it. Disjoint sets Q_1, Q_2, \ldots, Q_q can be similarly constructed such that $B_j = \bigcup_{n=1}^q Q_n$ and every set $D'_s = p_j(D_s)$ is the union of the sets Q_n contained in it.

Now it is evident that there exist indices m and n such that $P_m \delta_j Q_n$. For every index $i \in I - \{j\}$ let $x_i = y_j \in X_i$ be arbitrarily chosen element so that $x_i \in P_m$ and $y_j \in Q_n$. Then $x = (x_i) \in \bigcup_{r=1}^a C_r$, $y = (y_j) \in \bigcup_{s=1}^b D_s$, so that $x \in C_r$, $y \in D_s$ for suitable indices r and s, thus $x_j \in C'_r$, $y_j \in D'_s$. Therefore $P_m \subset C'_r$, $Q_n \subset D'_s$, and thus $p_j(C_r)\delta_j p_j(D_s)$. Furthermore, if $i \neq j$ then $x_i \in p_i(C_r)$, $y_i \in p_i(D_s)$ implies that $p_i(C_r)\delta_i p_i(D_i)$, from which, by Corollary 1.1.7.10, it follows that $A\delta B$.

In the following by **partition** of a set X we understand a system S of sets whose elements are pairwise disjoint, non-empty and their union is X. The elements of the partition S are called the **cells** of the partition. Often a partition on X is given by defining an equivalence relation on X and identifying the cells of the partition with the equivalence classes. In this case the quotient space obtained is called the **quotient space belonging to the equivalence relation**

An important example of this is the following: let $\mathcal{N}(x)$ be the neighborhood filter of the point x in the topological space (X, τ) . Let x be said to be equivalent to y if and only if $\mathcal{N}(x) = \mathcal{N}(y)$. Then we obtain evidently an equivalence relation on X. The partition \mathcal{S} belonging to it is called the **separative partition** belonging to the topology τ .

In the same way as in the case of topological spaces we can also speak of the **quotient space of a proximity** space belonging to a partition or an equivalence relation. It is worth to study in particular the quotient space with respect to the separative partition of the topology τ_{δ} :

Proposition 1.1.8.11 Let (X, δ) be a proximity space, S the separative partition corresponding to the topology τ_{δ} , $p: X \to S$ the canonical surjection. Then

(a) x and y belong to the same cell $Z \in S$ if and only if $\{x\}\delta\{y\}$;

- (b) $Ap(\delta)B$ if and only if $p^{-1}(A)\delta p^{-1}(B)$;
- (c) $p^{-1}(p(\delta)) = \delta;$
- (d) $\tau(p(\delta)) = p(\tau_{\delta});$
- (e) $p(\delta)$ is a separated proximity.

Proof: (a) If $\{x\}\overline{\delta}\{y\}$, then $X - \{y\} \in \mathcal{F}(\{x\}) = \mathcal{F}(x)$, so that $\mathcal{F}(y) \neq \mathcal{F}(x)$. Conversely, if $\mathcal{F}(y) \neq \mathcal{F}(x)$, then there exists a τ_{δ} -open set G such that $x \in G, y \notin G$. Therefore $\{x\}\overline{\delta}X - G$, and since $\{y\} \subset X - G$, then $\{x\}\overline{\delta}\{y\}$.

(b) Let us notice first that the condition formulated in Proposition 1.1.8.2 is fulfilled by p. Indeed, if $A, B \subset S$, $p^{-1}(A)\overline{\delta}p^{-1}(B)$, then let $C, D \subset X$ be such that $p^{-1}(A)\overline{\delta}X - C$, $p^{-1}(B)\overline{\delta}X - D$ and $C \bigcap B = \emptyset$. Then by Proposition 1.1.2.4 $p^{-1}(A)\overline{\delta}\overline{X-C}$, $p^{-1}(B)\overline{\delta}\overline{X-D}$ holds. Since every closed set in the topology τ_{δ} is a union of the classes of equivalence, then $p^{-1}(p(X-C)) \subset \overline{X-C}$ and $p^{-1}(p(X-D)) \subset \overline{X-D}$. But then the sets $P = \mathcal{S}-p(X-C)$ and $Q = \mathcal{S}-p(X-D)$ are disjoint, and $p^{-1}(\mathcal{S}-P) = p^{-1}(p(X-C))\overline{\delta}p^{-1}(A)$, $p^{-1}(\mathcal{S}-Q) = p^{-1}(p(X-D))\overline{\delta}p^{-1}(B)$ holds. Accordingly, Proposition 1.1.8.2 can be applied and shows that the statement is true.

(c) Since $p: (X, \delta) \to (S, p(\delta))$ is δ -continuous mapping, then by Proposition 1.1.6.11 we have that $p^{-1}(p(\delta)) < \delta$. On the other hand, if $A\overline{\delta}B$ is true, then by Proposition 1.1.2.4 $\overline{A}\overline{\delta}\overline{B}$ holds, from where it follows that $p^{-1}(p(A))\overline{\delta}p^{-1}(p(B))$ (because it is closed in the topology τ_{δ} as the union of the classes of equivalence). But then, according to the assertion (b), $p(A)\overline{p(\delta)}p(B)$, from which it follows that $A\overline{p^{-1}(p(\delta))}B$.

(d) Since the projection $p: (X, \delta) \to (\mathcal{S}, p(\delta))$ is δ -continuous mapping, then by Proposition 1.1.6.8 $p: (X, \tau_{\delta}) \to (\mathcal{S}, \tau(p(\delta)))$ is continuous mapping, so that $\tau(p(\delta)) < p(\tau_{\delta})$. To prove the converse, let us suppose that G is a $p(\tau_{\delta})$ -open set, i.e. that $p^{-1}(G)$ is a τ_{δ} -open set and let $x \in p^{-1}(G)$. Then $\{x\}\overline{\delta}X - p^{-1}(G)$, so that according to Proposition 1.1.2.4 $\overline{\{x\}}\overline{\delta}X - p^{-1}(G)$ holds. But then $p^{-1}(p(x))\overline{\delta}X - p^{-1}(G)$, so by (b) $\{p(x)\}\overline{p(\delta)}\mathcal{S} - G$. Therefore G is a $\tau(p(\delta))$ -open set.

(e) This assertion follows from (d) and the fact that $p(\tau_{\delta})$ is a T_0 -topology.

Historical and bibliographic notes

Although it had been suggested as early as 1908 by F. Riesz [273] and the idea was revived in 1941 by Wallace [329], the theory of proximity had its real beginning with V. A. Effermovich in 1952 [92], and was developed by several authors (largely in the Soviet Union), notably Yu. M. Smirnoff. The axioms for a proximity space were originally given by V. A. Efremovich, although they appeared in a slightly different but equivalent form to those presented in subsection 1.1. The theorems in this subsection are mainly due to V. A. Efremovich, just as the concept of δ -neighborhood. They have been collectively presented by Smirnoff in his early survey of proximity spaces [294]. The results concerning proximity mappings were first established by Smirnoff [294]. For an account of a proximity on the product of proximity spaces, see Leader [186].

1.2 Uniform spaces

The concept of a uniform space can be considered either as an axiomatization of some geometric notions, close to yet quite independent of the concept of a topological space, or as convenient tools for an investigation of topological spaces. Uniformities, when introduced by Weil, were considered as such tools, suitable, in contrast to metrics, for studying topological spaces with no countability assumptions. Burbaki, who pays a great attention to the theory of uniform spaces in their book, emphasizes its character as an independent theory which is, however, strongly related to the theory of topological spaces. The relation between the two theories consists in the fact that to uniform spaces and uniformly continuous function one can assign, in a standard way, topological space and continuous mappings.

1.2.1 Definition and basic properties of uniform spaces

In a pseudo-metric space (X, d) (x_n) is a Cauchy sequence if, for every $\varepsilon > 0$, there exists an index $n_{\varepsilon} \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ whenever $m, n > n_{\varepsilon}$. If $U_{d,\varepsilon} = \{(x, y) \in X^2 : d(x, y) < \varepsilon\}$, then we can say that (x_n) is a Cauchy sequence, if, beginning with an index $n_{\varepsilon} \in \mathbb{N}$, all its elements are in the set $U_{d,\varepsilon}$.

The uniform continuity of a function $f : \mathbb{R}^n \to \mathbb{R}$ is well known and plays an important role in mathematical analysis. A function f of this type is said to be uniformly continuous if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $x, y \in \mathbb{R}^n$, $d_n(x, y) < \delta$ implies $|f(x) - f(y)| < \varepsilon$. The definition can be extended word for word to real functions defined on a pseudo-metric space (X, d). The difficulty in extending it to an arbitrary topological space is the fact that, in order to do this, we should have to extend the expression "the points x and y are nearer to each other than δ for some $\delta > 0$ ", i.e. we should need a suitable generalization of "the system of pairs of points (x, y)nearer to each other than δ ".

On account of this, it can be expected that the notion of a uniform continuity can be extended to functions defined on a set X where some sets of pairs - the elements of which belong to X - are distinguished. The set of these pairs will then take the role played in the case of a pseudo-metric spaces by pairs of points which are nearer to each other than δ . In order to obtain a suitable generalization, let us look at some simple properties of sets of such pairs of a pseudo-metric spaces.

Every set $U_{d,\varepsilon}$ is non-empty, because $\Delta \subset U_{d,\varepsilon}$ for every $\varepsilon > 0$, where $\Delta = \{(x,x) : x \in X\}$ is the diagonal of the set X. Furthermore, $U_{d,\varepsilon} \subset U_{d,\varepsilon_1} \cap U_{d,\varepsilon_2}$, where $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$, so that the system of the sets $\{U_{d,\varepsilon} : \varepsilon > 0\}$ is a filter base in $X \times X$.

The sets $U_{d,\varepsilon}$ are symmetric, because the pseudo-metric d is a symmetric function, i.e. $U_{d,\varepsilon} = U_{d,\varepsilon}^{-1}$. Furthermore, for every $U_{d,\varepsilon}$, there exists an $U_{d,\rho}$ such that $U_{d,\rho} \circ U_{d,\rho} \subset U_{d,\varepsilon}$. This assertion immediately follows from the triangle inequality.

The sets of the form $U_{d,\varepsilon}$ are said to be an ε -surrounding of the pseudometric space (X, d).

Definition 1.2.1.1 A non-void family \mathcal{U} of subsets (called the entourages of the diagonal) of the set $X \times X$ is a uniformity or a uniform structure on the set X, if the following conditions are satisfied:

 $(U_1) \Delta \subset U$ for every element $U \in \mathcal{U}$;

 (U_2) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;

 (U_3) if $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$;

 (U_4) if $U \in \mathcal{U}$ and $U \subset V$, then $V \in \mathcal{U}$;

 (U_5) for every $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ such that $V \circ V \subset U$.

The pair (X, \mathcal{U}) is called a **uniform space**. A uniformity \mathcal{U} is called **separated** or **Hausdorff**, if for $x, y \in X$, $x \neq y$ there exists an entourage $U \in \mathcal{U}$ such that $(x, y) \notin U$. A uniform space (X, \mathcal{U}) is **Hausdorff**, if the uniformity \mathcal{U} is Hausdorff.

Definition 1.2.1.2 A subfamily $\mathcal{B} \subset \mathcal{U}$ is called a **base for the unifor**mity \mathcal{U} if for every $U \in \mathcal{U}$ there exists a $B \in \mathcal{B}$ such that $B \subset U$.

Obviously, a uniformity \mathcal{U} can have many bases. However, every uniformity \mathcal{U} is completely determined by any of its bases in the following way.

A subset U of the product $X \times X$ is an element of the uniformity \mathcal{U} if and only if there exists an element $B \in \mathcal{B}$ such that $B \subset U$.

Definition 1.2.1.3 Let \mathcal{A} and \mathcal{B} be systems of sets. We say that \mathcal{A} is coarser than \mathcal{B} , or \mathcal{B} is finer than \mathcal{A} , denoted by $\mathcal{A} < \mathcal{B}$, if for for each set $A \in \mathcal{A}$ there is a set $B \in \mathcal{B}$ such that $B \subset A$. If $\mathcal{A} < \mathcal{B}$ and $\mathcal{B} < \mathcal{A}$ hold simultaneously then we say that the systems of sets \mathcal{A} and \mathcal{B} are equivalent.

In spite of all this, the families \mathcal{U} and \mathcal{B} are equivalent. By definition of the base, $\mathcal{B} > \mathcal{U}$ holds. However, $\mathcal{B} < \mathcal{U}$ also holds. Indeed, let \mathcal{B} be any element of \mathcal{B} . Then $\mathcal{B} \in \mathcal{U}$, because $\mathcal{B} \subset \mathcal{U}$, so there exists an element $U \in \mathcal{U}$ such that $U \circ U \subset \mathcal{B}$. But then $U = U \circ \Delta \subset U \circ U \subset \mathcal{B}$.

Proposition 1.2.1.1 A family \mathcal{B} of the subsets of the product $X \times X$ is a base of a uniformity on the set X if and only if the following conditions are fulfilled:

- (a) $\Delta \subset B$ for every element $B \in \mathcal{B}$;
- (b) if $U \in \mathcal{B}$, then U^{-1} contains an element of \mathcal{B} ;
- (c) for every $U \in \mathcal{B}$ there exists a $V \in \mathcal{B}$ such that $V \circ V \subset U$;
- (d) the intersection of every two elements of $\mathcal B$ contains an element of $\mathcal B$.

According to the above proposition we have the following simple corollary.

Corollary 1.2.1.1 A family $\mathcal{B} = \{U_{d,\varepsilon} : \varepsilon > 0\}$ of ε -entourages $U_{d,\varepsilon}$ defined in the pseudo-metric space (X, d) is a base of some uniformity \mathcal{U}_d on the set X. The filter \mathcal{B} generated by it is the **uniformity of the pseudo-metric** space (X, d).

Proposition 1.2.1.2 If \mathcal{B} is a base of some uniformity on the set X, and \mathcal{B}' is a family of symmetric subsets of the product $X \times X$ which is equivalent to the family \mathcal{B} ($\mathcal{B} < \mathcal{B}'$ and $\mathcal{B}' < \mathcal{B}$), then the family \mathcal{B}' is a base of the uniformity on the set X.

Proof: Let $B' \in \mathcal{B}'$. Since the families \mathcal{B} and \mathcal{B}' are equivalent, then $\mathcal{B} > \mathcal{B}'$, so there exists a $B \in \mathcal{B}$ such that $B \subset B'$. Therefore $\Delta \subset B'$, so that \mathcal{B}' is a filter base on X. Let us choose an element $B' \in \mathcal{B}'$ and let $B \in \mathcal{B}$ be a set for which $B \subset B'$. Since \mathcal{B} is a base of the uniformity on X, then by Proposition 1.2.1.1 (c) there exists a $B_1 \in \mathcal{B}$ such that $B_1 \circ B_1 \subset B$. But then, since $\mathcal{B}' > \mathcal{B}$, for the set B_1 there exists a set $B'_1 \in \mathcal{B}'$ for which $B'_1 \subset B_1$ holds, from where it follows the inclusion $B'_1 \circ B'_1 \subset B_1 \circ B_1 \subset B \subset B'$. This proves that \mathcal{B}' is a base of the uniformity on X.

Furthermore we can suppose that entourages are symmetric elements of uniformity which contains diagonal.

It is easy to see that all entourages of some uniformity \mathcal{U} constitute a uniform base generating \mathcal{U} . This is at the same time the largest uniform base generating \mathcal{U} .

Definition 1.2.1.4 A subfamily S of a uniform structure U is a subbase for U if the family of all finite intersections of elements of S form a base for U.

Proposition 1.2.1.3 A family S of the subsets of the product $X \times X$ is a subbase of a uniformity on the set X if and only if the following conditions are fulfilled:

(a) $\Delta \subset U$ for every element U of S;

- (b) for every $U \in S$ the set U^{-1} contains an element V of S;
- (c) for every $U \in S$ there exists an element $V \in S$ such that $V \circ V \subset U$.

Proof: Let us prove that the family \mathcal{B} of all finite intersections of the elements of \mathcal{S} fulfills the conditions of Proposition 1.2.1.1. It is therefore sufficient to notice the following facts. If U_1, U_2, \ldots, U_n and V_1, V_2, \ldots, V_n are any subsets of the product $X \times X$, $U = \bigcap_{i=1}^n U_i$, $V = \bigcap_{i=1}^n V_i$, then $V \subset U^{-1}$ (that is $V \circ V \subset U$), whenever $V_i \subset U_i^{-1}$ (that is $V_i \circ V_i \subset U_i$) for every $i = 1, 2, \ldots, n$.

Another important method to obtain uniformities is the following. Let Σ be an arbitrary non-empty family of the pseudo-metrics defined on the set X. For every $\sigma \in \Sigma$ and every $\varepsilon > 0$ the set

$$U_{\sigma,\varepsilon} = \{(x,y) : \sigma(x,y) < \varepsilon\} \subset X \times X$$

is evidently an entourage in X, because $\Delta \subset U_{\sigma,\varepsilon}$ and $U_{\sigma,\varepsilon} = U_{\sigma,\varepsilon}^{-1}$; moreover, $U_{\sigma,\varepsilon/2} \circ U_{\sigma,\varepsilon/2} \subset U_{\sigma,\varepsilon}$, so that, by the above proposition, the family $\{U_{\sigma,\varepsilon} : \sigma \in \Sigma, \varepsilon > 0\}$ is a subbase of some uniform structure on X. Let us assign now to every finite subset $\emptyset \neq \Sigma' \subset \Sigma$ and every $\varepsilon > 0$ the set

$$U_{\Sigma',\varepsilon} = \{(x,y) : \sigma(x,y) < \varepsilon, \ \sigma \in \Sigma'\} \subset X \times X.$$

Let us denote by \mathcal{B}_{Σ} the family of all sets $U_{\Sigma',\varepsilon}$, where Σ' runs over all finite non-empty subsets of Σ , and ε runs over all positive real numbers. Then the following proposition holds: **Proposition 1.2.1.4** If Σ is a non-empty family of pseudo-metrics on the set X, then the system of sets

$$\mathcal{B}_{\Sigma} = \{ U_{\Sigma',\varepsilon} : \emptyset \neq \Sigma' \subset \Sigma \text{ is a finite set }, \varepsilon > 0 \}$$

is a uniform base on X.

Proof: According to the definition of a subbase, the finite intersections of the sets $U_{\sigma,\varepsilon}$ constitute a uniform base on X. On the other hand, \mathcal{B}_{Σ} is equivalent to this base since, for $\Sigma' = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$,

$$U_{\Sigma',\varepsilon} = \bigcap_{i=1}^n U_{\sigma_i,\varepsilon};$$

moreover, if $0 < \varepsilon \leq \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$, then

n

$$\bigcap_{i=1}^{n} U_{\sigma_{i},\varepsilon_{i}} \subset U_{\Sigma',\varepsilon} \cdot \clubsuit$$

The uniform structure \mathcal{U} defined in this proposition is called the **uni**formity induced by the family of pseudo-metrics Σ .

The notion of a uniform structure can be introduced on the set X with the help of a family of covers of the set X.

The following definition was given by Smirnoff (see [294]) and it presents an insignificantly modification of the definition which was given by Tukey (see [323]).

Definition 1.2.1.5 A family Σ of the coverings of the set X is a uniform structure or a uniformity on the set X, if the following conditions are fulfilled:

(P₁) if the covering α is inscribed in the covering β and if $\alpha \in \Sigma$, then $\beta \in \Sigma$;

(P₂) for any two coverings α and β of the family Σ the intersection $\alpha \cap \beta$ also belongs to the family Σ ;

(P₃) For any covering $\alpha \in \Sigma$ there exists a covering $\beta \in \Sigma$ such that the covering $\{st(x,\beta): x \in X\}$ is inscribed in the covering α .

The pair (X, Σ) is called the **uniform space**. If it is satisfied the following additional condition:

(P₄) for every pair of distinct points $x \in X$ and $y \in X$ there exists a covering $\gamma \in \Sigma$ such that $y \notin st(x, \gamma)$,

then the uniform structure Σ on X is said to be separated or Hausdorff.

Let us prove that Definition 1.2.1.5 is equivalent to Definition 1.2.1.1. Let us denote with \mathbf{W} the set of all uniform structures on the set X given by A. Weil, and with \mathbf{T} the set of all uniform structures given by J. Tukey. We will show that the order in \mathbf{T} and the topology defined on X by a structure $\Sigma \in \mathbf{T}$ coincides with those of A. Weil.

First, to each set $U \in \mathcal{U}$ we associate the corresponding covering $\gamma_U = \{U[x] : x \in X\}$ consisting of neighborhoods U[x] of the points $x \in X$. The mapping $\xi : \mathbf{W} \to \mathbf{T}$ is constructed as follows: the image of the Weil structure $\mathcal{U} \in \mathbf{W}$ is the system $\Sigma = \xi(\mathcal{U})$ consisting of all coverings γ of the set X in each of which is inscribed the covering of the form $\gamma_U, U \in \mathcal{U}$.

It is easy to see that the condition (P_1) of Definition 1.2.1.5 is fulfilled in $\Sigma = \xi(\mathcal{U})$. Furthermore, we see that if $W = U \cap V$, $U, V \in \mathcal{U}$, then $W[x] = U[x] \cap V[x]$ for any $x \in X$. This means that for any $U \in \mathcal{U}$ and $V \in \mathcal{U}$ the covering $\gamma_{U \cap V}$ is inscribed in the intersection of the coverings γ_U and γ_V , whence the condition (P_2) of Definition 1.2.1.5 follows. To prove the condition (P_3) , for any element $U \in \mathcal{U}$ we choose a $W \in \mathcal{U}$ such that $W \circ W \subseteq U$. It is easy to prove that for the set $V = W \cap W^{-1} \in \mathcal{U}$ the star $st(x, \gamma_V)$ of each point $x \in X$ is contained in the neighborhood U[x] of the point x. Thus for any $\mathcal{U} \in \mathbf{W}$ the system $\Sigma = \xi(\mathcal{U})$ is a uniform structure from \mathbf{T} .

To prove that ξ is a bijection, let us construct a one-one mapping η from the set **T** into the set **W** which is inverse to ξ , i.e. such that $\xi(\eta(\Sigma)) = \Sigma$ and $\eta(\xi(\mathcal{U})) = \mathcal{U}$ for any $\Sigma \in \mathbf{T}$ and any $\mathcal{U} \in \mathbf{W}$. It is constructed as follows: each covering $\gamma \in \Sigma$ is made to correspond to the set $V_{\gamma} \supseteq \Delta$ which is the union of the sets of the form $\Gamma \times \Gamma$ of each $\Gamma \in \gamma$ with itself: $V_{\gamma} = \bigcup_{\Gamma \in \gamma} (\Gamma \times \Gamma)$; after this the image of the structure $\Sigma \in \mathbf{T}$ under the mapping η is the system $\mathcal{U} = \eta(\Sigma)$ which consists of all sets $V \subseteq X \times X$ each of which containing a set of the form $V_{\gamma}, \gamma \in \Sigma$.

It is easy to see that the conditions (U_1) and (U_4) of Definition 1.2.1.1 are fulfilled in $\mathcal{U} = \eta(\Sigma)$. For any $\gamma \in \Sigma$ the set $V_{\gamma}^{-1} = V_{\gamma}$. Hence condition (U_2) of Definition 1.2.1.1 is fulfilled. Since for any two sets $A, B \subseteq X$ we always have $(A \cap B) \times (A \cap B) = (A \times A) \cap (B \times B)$, then for any two coverings $\alpha, \beta \in \Sigma$ the equality $V_{\alpha \cap \beta} = V_{\alpha} \cap V_{\beta}$ holds. The condition (U_3) of Definition 1.2.1.1 now follows from this equality. To prove the condition (U_5) we note that if the covering $\alpha \in \Sigma$ is star-inscribed in the covering $\beta \in \Sigma$, then $V_{\alpha}^2 \subseteq V_{\beta}$. Thus the system $\mathcal{U} = \eta(\Sigma) \in \mathbf{W}$.

Let now $\alpha \in \Sigma$. It follows that the covering α is inscribed in the covering $\gamma_{V_{\alpha}} \in \xi(\eta(\Sigma))$. This means that every covering $\gamma \in \xi(\eta(\Sigma))$ belongs to Σ . Conversely, for any covering $\alpha \in \Sigma$ we choose a covering $\beta \in \Sigma$ which is starinscribed in the covering α . Then the covering $\gamma_{V_{\beta}} \in \xi(\eta(\Sigma))$ is inscribed in the covering α . Consequently, every covering $\alpha \in \Sigma$ belongs to $\xi(\eta(\Sigma))$. Thus the equality $\xi(\eta(\Sigma)) = \Sigma$ has been proved. The inequality $\eta(\xi(\mathcal{U})) = \mathcal{U}$ can be proved analogously.

Let us suppose now that $\Sigma \in \mathbf{T}$ is a separated uniformity. If $y \notin st(x, \gamma)$, then no element of the covering γ simultaneously contains the points x and y, and so, $(x, y) \notin V_{\gamma}$. Hence we get that for the Weil structure $\mathcal{U} = \eta(\Sigma) \in \mathbf{W}$ the equality $\bigcap_{V \in \mathcal{U}} V = \Delta$ is true. Conversely, if for the Weil structure $\mathcal{U} = \eta(\Sigma) \in \mathbf{W}$ the equality $\bigcap_{V \in \mathcal{U}} V = \Delta$ holds, then for any pair of distinct points $x, y \in X$ there exists a $V \in \mathcal{U}$ such that $(x, y) \notin V$. This means that for the covering $\gamma_V \in \Sigma$, the point $y \notin st(x, \gamma_V)$, which was to be proved.

It follows immediately from the definition of the mappings ξ and η that if $\Sigma > \Sigma'$, then the Weil structure $\mathcal{U} = \eta(\Sigma)$ is finer than the uniform structure $\mathcal{U}' = \eta(\Sigma')$, and conversely, if the the Weil structure \mathcal{U} is finer than \mathcal{U}' , then $\Sigma > \Sigma'$.

In the following subsection we shall prove that the topologies generated by the uniform structures Σ and \mathcal{U} coincide. The proof about equivalence of uniform structures introduced with the help of Definitions 1.2.1.5 and 1.2.1.1 were proved by A. Kochetkov, and this proof is given in the paper of Ju. Smirnoff (see [294], p. 573-574).

1.2.2 Proximity and topology of uniform spaces

The sets A and B in a pseudo-metric space (X, d) are said to be near if d(A, B) = 0. This means that for every $\varepsilon > 0$ there are $x \in A$ and $y \in B$ such that $d(x, y) < \varepsilon$, i.e. $(x, y) \in U_{\varepsilon}$. Accordingly, let us agree to say that, in a uniform space (X, \mathcal{U}) , the sets A and B are near if there are, for every entourage $U \in \mathcal{U}, x \in A$ and $y \in B$ such that $(x, y) \in U$. Therefore the following holds:

Proposition 1.2.2.1 If (X, \mathcal{U}) is a uniform space, then the relation $\delta_{\mathcal{U}}$ defined by

 $\begin{array}{ll} A\delta_{\mathcal{U}}B & \text{if and only if } (A \times B) \bigcap U \neq \emptyset \\ & \text{for every entourage } U \in \mathcal{U} \end{array}$

is a proximity on the set X which is called the **proximity induced by the** uniformity \mathcal{U} . **Proof:** (B_1) follows from (U_2) . If $A\delta_{\mathcal{U}}(B \cup C)$, then $[A \times (B \cup C)] \cap U \neq \emptyset$ for every $U \in \mathcal{U}$, i.e. $[(A \times B) \cup (A \times C)] \cap U \neq \emptyset$ for every $U \in \mathcal{U}$. This is equivalent to the fact that $[(A \times B) \cap U] \cup [(A \times C) \cap U] \neq \emptyset$ for every $U \in \mathcal{U}$, so that $(A \times B) \cap U \neq \emptyset$ for every $U \in \mathcal{U}$ or $(A \times C) \cap U \neq \emptyset$ for every $U \in \mathcal{U}$. Hence $A\delta_{\mathcal{U}}B$ or $A\delta_{\mathcal{U}}C$, so (B_2) holds. (B_3) obviously holds, and (B_4) is true according to (U_1) . It remains to prove the axiom (B_5) . In order to do this, let us suppose that $A\delta_{\mathcal{U}}B$. Then there exists an element $U \in \mathcal{U}$ for which $(A \times B) \cap U = \emptyset$ holds. But then by (U_5) there exists a $V \in \mathcal{U}$ such that $V \circ V \subset U$. Let C = V[A], D = V[B]. Then $C \cap D = \emptyset$. Indeed, if there exists $y \in C \cap D$, then there exists $x \in A$ and $z \in B$ such that $(x, y) \in V$ and $(z, y) \in V$. Since the entourage V is symmetric, $(y, z) \in V$, so that $(x,z) \in V \circ V \subset U$. Then $(x,z) \in (A \times B) \cap U$, which contradicts the fact that $(A \times B) \cap U = \emptyset$. Let us prove that $A\overline{\delta}_{\mathcal{U}}X - C$ and $B\overline{\delta}_{\mathcal{U}}X - D$. It is suffices to prove that $(A \times (X - C)) \cap V = \emptyset$ and $(B \times (X - D)) \cap V = \emptyset$. Let $(x, y) \in A \times (X - V[A])$. Then $x \in A$ and $y \notin V[A]$, so that $(x, y) \notin V$, from which the first equality follows. The second equality can be proved in an analogous manner. 🌲

Corollary 1.2.2.1 Let \mathcal{B} be a base of the uniformity \mathcal{U} on the set X. Then $A\delta_{\mathcal{U}}B$ if and only if $(A \times B) \cap U \neq \emptyset$ for every entourage $U \in \mathcal{B}$.

Proof: It is obvious that $A\delta_{\mathcal{U}}B$ is equivalent to the relation $\emptyset \notin \{A \times B\} \cap \mathcal{U}$, so that \mathcal{U} can be replaced here by any system of subsets of the product $X \times X$ equivalent to it. \clubsuit

Corollary 1.2.2.2 The proximity δ_d of a pseudo-metric space (X, d) coincides with the proximity $\delta_{\mathcal{U}_d}$ generated by the pseudo-metric uniformity \mathcal{U}_d .

Proof: Applying the previous corollary to the system of ε -entourages of the pseudo-metric space (X, d), we come to the proof of the assertion.

Proposition 1.2.2.2 Let (X, U) be a uniform space, \mathcal{B} a base of the uniformity $\mathcal{U}, \emptyset \neq A \subset X, \mathcal{F}(A)$ a $\delta_{\mathcal{U}}$ -filter of the set A. Then the filter base $\{U[A] : U \in \mathcal{B}\}$ constitutes a base of the filter $\mathcal{F}(A)$.

Proof: Since $(A \times (X - U[A])) \cap U = \emptyset$, then $A\overline{\delta}_{\mathcal{U}}X - U[A]$, so that $U[A] \in \mathcal{F}(A)$. On the other hand, if $P \in \mathcal{F}(A)$, then by Corollary 1.2.2.1 there exists a $U \in \mathcal{B}$ such that $(A \times (X - P)) \cap U = \emptyset$. But then $U[A] \subset P$. Indeed, if $x \in U[A]$ then $(a, x) \in U$ for some $a \in A$. Since $(A \times (X - P)) \cap U = \emptyset$, then $(a, x) \notin A \times (X - P)$. Since $a \in A, x \notin X - P$ holds, so that $x \in P$.

By the **topology induced** or **generated by the uniformity** \mathcal{U} , denoted by $\tau_{\mathcal{U}}$, we mean the topology $\tau_{\delta_{\mathcal{U}}}$ induced by $\delta_{\mathcal{U}}$. Applying the previous corollary for the case $A = \{x\}$, the following assertion holds:

Corollary 1.2.2.3 Let (X, U) be a uniform space, \mathcal{B} a base of the uniformity \mathcal{U} . Then the filter base $\{U[x] : U \in \mathcal{B}\}$ is a $\tau_{\mathcal{U}}$ -neighborhood base of the point x.

Proposition 1.2.2.3 Let \mathcal{U} be a uniformity on a set X. For every entourage $U \in \mathcal{U}$ let us define a relation \ll_U for the subsets of X in the following way:

 $A \ll_U B$ if and only if $U[A] \subset B$.

Then the relation \ll_U fulfills the conditions $(O_1) - (O_5)$ of Theorem 1.1.1.1. Moreover,

 (O'_6) if for entourages $U, U_1 \in \mathcal{U} \ U_1 \circ U_1 \subset U$ holds, then $A \ll_U B$ implies that there exists a set C such that $A \ll_{U_1} C \ll_{U_1} B$.

Proof: Since $U[\emptyset] = \emptyset$, the condition (O_1) obviously holds. The condition (O_2) holds, since $A \subset U[A]$. Since $A \subset B$ implies $U[A] \subset U[B]$, the condition (O_3) is fulfilled. (O_4) is fulfilled as well, since $U[A] \subset B$ and $U[X - B] \subset X - A$ both mean, because $U = U^{-1}$, that $x \in A$, $(x, y) \in U$ implies $y \in B$. (O_5) follows from the definition of the relation \ll_U . Finally, let us prove (O'_6) . Let us suppose that for entourage U and U_1 holds $U_1 \circ U_1 \subset U$, and let $A \ll_U B$, i.e. $U[A] \subset B$. Let $C = U_1[A]$. Then it is obvious that $A \ll_{U_1} C$. On the other hand, we have that $U_1[C] = U_1[U_1[A]] = (U_1 \circ U_1)[A] \subset U[A] \subset B$, so that $C \ll_{U_1} B$.

According to Proposition 1.1.2.8 we can conclude:

Corollary 1.2.2.4 A uniformity \mathcal{U} is separated if and only if the proximity $\delta_{\mathcal{U}}$ is separated, i.e. if and only if $\tau_{\mathcal{U}}$ is a T_0 -topology.

The proximity $\delta_{\mathcal{U}_{\Sigma}}$ and topology $\tau_{\mathcal{U}_{\Sigma}}$ of the uniformity \mathcal{U}_{Σ} generated by the family Σ of pseudo-metrics are called the **proximity** and **topology induced by the family** Σ of **pseudo-metrics** and denoted by δ_{Σ} and τ_{Σ} .

Let us consider the uniform structure on some set X introduced by Definition 1.2.1.5.

Proposition 1.2.2.4 If the family of coverings Σ is a uniform structure on the set X, then by

 $\begin{array}{ll} A\delta B & \mbox{if and only if every covering } \gamma \in \Sigma \mbox{ contains some element} \\ \Gamma \in \gamma \mbox{ which has the non-empty intersection with both} \\ sets \ A \ and \ B \end{array}$

the proximity on the set X is defined.

Proof: The conditions (B_1) , (B_3) and (B_4) of Definition 1.1.1.1 are obviously fulfilled. Let us verify the condition (B_2) . If the union of the sets A and B is far from the set C, then obviously each of the sets A and B is far from the set C. Conversely, let each of the sets A and B be far from the set C. Then there exist coverings $\alpha \in \Sigma$ and $\beta \in \Sigma$ such that $st(C, \alpha) \cap A = \emptyset$ and $st(C, \beta) \cap B = \emptyset$. The covering $\alpha \cap \beta$ is by the condition (P_2) of Definition 1.2.1.5 an element of the uniform structure Σ , and since $st(C, \alpha \cap \beta) \subseteq st(C, \alpha) \cap st(C, \beta)$, it follows that $st(C, \alpha \cap \beta) \cap (A \cup B) = \emptyset$, which proves that the set C is far from the union of the sets A and B.

Finally we will show that the condition (B_5) of Definition 1.1.1.1 is fulfilled. According to Proposition 1.1.1.3 it is sufficient to prove that the sets A and B, which are far in X, have disjoint δ -neighborhoods. In order to do this, let us choose a covering $\gamma \in \Sigma$ such that $A \cap st(B, \gamma) = \emptyset$. According to the condition (P_3) of Definition 1.2.1.5 we can choose a covering α which is the star inscribed in γ . We will now show that the δ -neighborhoods $st(A, \alpha)$ and $st(B, \alpha)$ of the sets A and B are disjoint. Indeed, if there were a point $x \in st(A, \alpha) \cap st(B, \alpha)$, then the set $st(x, \alpha)$ would meet both sets A and B. Since the set $st(x, \alpha)$ is contained in some set $\Gamma \in \gamma$, then the set Γ would meet both sets A and B, which is impossible, because the sets A and B are far. \clubsuit

Proposition 1.2.2.5 Let a family of coverings Σ be a uniform structure on a set X. If δ is a proximity relation on X generated by this uniform structure, then the topology τ_{δ} can be obtained directly from the uniform structure Σ in the following way: for any point $x \in X$, the family int $st(x, \gamma)$, $\gamma \in \Sigma$, is a base for the space X at x.

Proof: Indeed, since for every point $x \in X$ the set $st(x, \gamma)$ is a neighborhood of the point x, then $x \in int st(x, \gamma)$. Conversely, if O_x is any neighborhood of the point x, since $X - O_x$ is a closed set, it follows that $x\overline{\delta}(X - O_x)$. Therefore there exists a covering $\gamma \in \Sigma$ such that $st(x, \gamma) \subseteq O_x$, which was to be proved.

Proposition 1.2.2.6 The topology generated on the set X by the uniform structure $\Sigma \in \mathbf{T}$ coincides with the topology generated by the uniform structure $\mathcal{U} = \eta(\Sigma) \in \mathbf{W}$.

Proof: Indeed, for every covering $\alpha \in \Sigma$ and every point $x \in X$ we have that $V_{\alpha}[x] = st(x, \alpha)$, where $V_{\alpha} = \bigcup_{A \in \alpha} (A \times A)$, so that $intV_{\alpha}[x] \subseteq int st(x, \alpha)$.

Conversely, let $U \in \mathcal{U}$ and $x \in X$. Let us choose a set $W \in \mathcal{U}$ for which $W \circ W \subseteq U$ holds and let $V = W \cap W^{-1}$. Then $st(x, \gamma_V) \subseteq U[x]$, where $\gamma_V = \{V[x] : x \in X\}$. Thus $int st(x, \gamma_V) \subseteq int U[x]$, which was to be proved.

1.2.3 Comparison of uniformities

The comparison of uniformities is based on the comparison of the filters. We shall say that the uniformity \mathcal{U}_1 is **coarser** than the uniformity \mathcal{U}_2 , i.e. that the uniformity \mathcal{U}_2 is **finer** than the uniformity \mathcal{U}_1 and denoted by $\mathcal{U}_1 < \mathcal{U}_2$, whenever this relation holds for the filters \mathcal{U}_1 and \mathcal{U}_2 .

It is obvious that the relation < is a partial order on the set of all uniform structures defined on the set X. Let us notice that, on any set $X \neq \emptyset$, the uniform base consisting only of the diagonal of $X \times X$ generates a uniformity inducing the discrete proximity; this is evidently the finest uniformity on X and is called the **discrete uniformity**. On the other hand, the uniformity consisting only of $X \times X$ itself is the coarsest uniformity on X. It is called the **indiscrete uniformity** and induces the indiscrete proximity.

Proposition 1.2.3.1 If \mathcal{U}_1 and \mathcal{U}_2 are uniformities on the set X for which $\mathcal{U}_1 < \mathcal{U}_2$ holds, then $\delta_{\mathcal{U}_1} < \delta_{\mathcal{U}_2}$ and $\tau_{\mathcal{U}_1} < \tau_{\mathcal{U}_2}$.

Proof: If $A\overline{\delta}_{\mathcal{U}_1}B$, then there exists a $U \in \mathcal{U}_1$ such that $(A \times B) \cap U = \emptyset$. Since $\mathcal{U}_1 < \mathcal{U}_2$, then $U \in \mathcal{U}_2$, so that $A\overline{\delta}_{\mathcal{U}_2}B$. In this way we have proved that $\delta_{\mathcal{U}_1} < \delta_{\mathcal{U}_2}$. The second statement results from this by Proposition 1.1.2.9.

It is to be noticed here that, on the other hand, $\delta_{\mathcal{U}_1} < \delta_{\mathcal{U}_2}$ does not imply $\mathcal{U}_1 < \mathcal{U}_2$. Moreover, it may happen that $\delta_{\mathcal{U}_1} = \delta_{\mathcal{U}_2}$, but $\mathcal{U}_1 \neq \mathcal{U}_2$. To prove this fact, let us consider the following

Example 1.2.3.1 Let d be the metric defined on an infinite set X by the formula

$$d(x,y) = \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y, \end{cases}$$

and let $\mathcal{U}_1 = \mathcal{U}_d$. Then $\delta_{\mathcal{U}_1}$ is a discrete proximity on the set X. On the other hand, let \mathcal{D} be an arbitrary decomposition $X = \bigcup_{i=1}^{n} X_i$ of the set X into a finite number of pairwise disjoint subsets and let $U_{\mathcal{D}} = \bigcup_{i=1}^{n} (X_i \times X_i)$. The sets $U_{\mathcal{D}}$ corresponding to all decompositions \mathcal{D} of this type constitute a uniform base \mathcal{B} since $U_{\mathcal{D}} \subset U_{\mathcal{D}_1} \cap U_{\mathcal{D}_2}$, where $\mathcal{D}_1, \mathcal{D}_2$ and \mathcal{D} denote decompositions $X = \bigcup_1^m A_i, X = \bigcup_1^n B_j$ and $X = \bigcup_1^m \bigcup_1^n (A_i \cap B_j)$ respectively, and it is clearly that $U_{\mathcal{D}} \circ U_{\mathcal{D}} = U_{\mathcal{D}}$ for every \mathcal{D} . Let \mathcal{U}_2 be the uniformity generated by the above base \mathcal{B} . Then $\delta_{\mathcal{U}_2}$ is the discrete proximity on the set X, since $A \cap B = \emptyset$ implies $(A \times B) \cap U_{\mathcal{D}} = \emptyset$ for the decomposition $\mathcal{D} : X = \{A, X - A\}$. Finally, $\mathcal{U}_1 \neq \mathcal{U}_2$ as in the uniformity \mathcal{U}_1 , for $0 < \varepsilon < 1$, U_{ε} is identical with the diagonal of $X \times X$ which cannot contain $U_{\mathcal{D}}$ for any finite decomposition \mathcal{D} since at least one member of the decomposition consists of several points.

Proposition 1.2.3.2 Let \mathcal{U}_i be a uniform structure on a set X, $i \in I \neq \emptyset$, \mathcal{B}_i a uniform base generating \mathcal{U}_i . Then $\bigcup_{i \in I} \mathcal{B}_i$ is a uniform subbase and the filter in $X \times X$ generated by it is identical with the coarsest of all uniformities finer than each uniformity \mathcal{U}_i . It is denoted by $\sup \{\mathcal{U}_i : i \in I\}$.

Proof: It is easy to prove that the family $\mathcal{P} = \bigcup_{i \in I} \mathcal{B}_i$ fulfills all the conditions of Proposition 1.2.3.2, so that it is a subbase of some uniformity \mathcal{U} on the set X. Since $\mathcal{B}_i \subset \mathcal{P} \subset \mathcal{U}$, then $\mathcal{U}_i < \mathcal{U}$ for every $i \in I$. If \mathcal{U}' is any uniformity on the set X which is finer than every uniformity \mathcal{U}_i , then $\mathcal{U}_i \subset \mathcal{U}'$ for every $i \in I$, so that $\mathcal{P} \subset \mathcal{U}'$. But then $\mathcal{U} \subset \mathcal{U}'$, i.e. $\mathcal{U} < \mathcal{U}'$, which was to be proved.

Let us observe that, with the help of the operation introduced above, the uniformity induced by the family Σ of pseudo-metrics can be constructed from the uniformities induced by the single pseudo-metrics $d \in \Sigma$:

Proposition 1.2.3.3 Let Σ be a family of pseudo-metrics on the set X. Then

$$\mathcal{U}_{\Sigma} = \sup\{\mathcal{U}_d : d \in \Sigma\}$$

Proof: In the proof of Proposition 1.2.1.4 we have seen that \mathcal{U}_{Σ} is generated by the uniform subbase

$$\bigcup_{d\in\Sigma}\bigcup_{\varepsilon>0}U_{d,\varepsilon}=\bigcup_{d\in\Sigma}\mathcal{B}_d\,,$$

where \mathcal{B}_d is a uniform base generating uniformity \mathcal{U}_{σ} .

Proposition 1.2.3.4 Let U_i be a uniform structure on the set X for every $i \in I$, $U = \sup\{U_i : i \in I\}$. Then

$$\tau_{\mathcal{U}} = \sup\{\tau_{\mathcal{U}_i} : i \in I\}.$$

Proof: On account of Propositions 1.2.3.4 and 1.2.1.4, $\tau_{\mathcal{U}_i} < \tau_{\mathcal{U}}$ holds for each $i \in I$, so that, with the notation $\tau = \sup\{\tau_{\mathcal{U}_i} : i \in I\}, \tau < \tau_{\mathcal{U}}$. To prove the converse, let us suppose that V is a $\tau_{\mathcal{U}}$ -neighborhood of the point $x \in X$. Then by Proposition 1.2.1.4 and Corollary 1.2.2.3 it follows that $\bigcap_{k=1}^n U_{i_k}[x] \subset V$, where $U_{i_k} \in \mathcal{U}_{i_k}, i_k \in I, k = 1, 2, \ldots, n$. If $U = \bigcap_{k=1}^n U_{i_k}$, then $U[x] = \bigcap_{k=1}^n U_{i_k}[x]$. Since $U_{i_k}[x]$ is a $\tau_{\mathcal{U}_{i_k}}$ -neighborhood of the point x, it is also a τ -neighborhood of the point x. Therefore V is a τ -neighborhood of the point x, i.e. $\tau_{\mathcal{U}} < \tau$.

Proposition 1.2.3.5 Let \mathcal{U}_i , $i \in I \neq \emptyset$, be a uniformity on the set X. Then there exists a uniformity \mathcal{U} on X which is the finest among all uniformities coarser than all uniformity \mathcal{U}_i , denoted by $\mathcal{U} = \inf{\{\mathcal{U}_i : i \in I\}}$.

Proof: Since the indiscrete uniformity on X is coarser than all the uniformities \mathcal{U}_i , we can speak of the supremum of the uniformities coarser than all \mathcal{U}_i . This coincides with \mathcal{U} .

Corollary 1.2.3.1 If a topology τ can be induced by a uniformity, then there exists the finest among all uniformities inducing τ .

Proof: According to Proposition 1.2.3.4, this is the supremum of all uniformities inducing the topology τ .

1.2.4 Subspaces of uniform spaces

Proposition 1.2.4.1 Let (X, U) be a uniform space, $\emptyset \neq Y \subset X$, and \mathcal{B} a base of the uniformity \mathcal{U} . Then $\mathcal{U} \cap \{Y \times Y\}$ is a uniformity on Y denoted by $\mathcal{U}|Y$, while $\mathcal{B} \cap \{Y \times Y\}$ is a base of the uniformity $\mathcal{U}|Y$ denoted by $\mathcal{B}|Y$.

Proof: It is suffices to prove that $\mathcal{U} \cap \{Y \times Y\}$ is a filter on the set $Y \times Y$, i.e. that $\emptyset \notin \mathcal{U} \cap \{Y \times Y\}$. It immediately follows from the fact that $(x, x) \in U \cap (Y \times Y)$ for every $x \in Y$ and all $U \in \mathcal{U}$. If \mathcal{B} is a base of the uniformity \mathcal{U} and $\mathcal{B}|Y = \mathcal{B} \cap (Y \times Y)$, then it is easy to check that the conditions (a), (b) and (d) of Proposition 1.2.1.1 are fulfilled. That the condition (c) is true follows from the fact that $V \circ V \subset U$ evidently implies the inclusion

 $(V \cap (Y \times Y)) \circ (V \cap (Y \times Y)) \subset U \cap (Y \times Y).$

Thus $\mathcal{B}|Y$ is a uniform base on Y. \clubsuit

Definition 1.2.4.1 The uniform structure $\mathcal{U}|Y$ defined in the previous proposition is called the **restriction of the uniformity** \mathcal{U} on Y, and $(Y, \mathcal{U}|Y)$ is said to be the **uniform subspace** of the space (X, \mathcal{U}) .

Proposition 1.2.4.2 Let Σ be a family of pseudo-metrics on the set X, $\emptyset \neq Y \subset X$ and $\sigma^* = \sigma | Y$ for $\sigma \in \Sigma$. Then $\Sigma^* = \{\sigma^* : \sigma \in \Sigma\}$ is a family of pseudo-metrics on the set Y and $\mathcal{U}_{\Sigma^*} = \mathcal{U}_{\Sigma} | Y$.

Proof: For any finite set of pseudo-metrics $\emptyset \neq \Sigma_1 \subset \Sigma$, let us denote by Σ_1^* the set of the restrictions σ^* of the pseudo-metrics $\sigma \in \Sigma_1$. Then it is evident that

$$U_{\Sigma_1^*,\varepsilon} = \{(x,y) : x, y \in Y, \ \sigma^*(x,y) < \varepsilon, \sigma^* \in \Sigma_1^*\} = \\ = \{(x,y) : x, y \in Y, \ \sigma(x,y) < \varepsilon, \sigma \in \Sigma_1\} = \\ = U_{\Sigma_1,\varepsilon} \cap (Y \times Y).$$

The assertion now follows from the fact that the sets on the left-hand side of the above equality generate the uniformity \mathcal{U}_{Σ^*} , while the sets on the right-hand side of the same equality generate the uniformity $\mathcal{U}_{\Sigma}|Y$.

Proposition 1.2.4.3 If (X, U) is a uniform space, and $\emptyset \neq Y \subset X$, then $\delta_{\mathcal{U}|Y} = \delta_{\mathcal{U}}|Y$ and $\tau_{\mathcal{U}|Y} = \tau_{\mathcal{U}}|Y$.

Proof: The first equality follows from the fact that $(A \times B) \cap U = (A \times B) \cap U \cap (Y \times Y)$ for every entourage $U \in \mathcal{U}$ and every two sets $A, B \subset Y$. The second equality follows from this according to Proposition 1.1.5.1.

Corollary 1.2.4.1 If \mathcal{U}_1 and \mathcal{U}_2 are the uniform structures on the set X with $\mathcal{U}_1 < \mathcal{U}_2$ and if $\emptyset \neq Y \subset X$, then $\mathcal{U}_1 | Y < \mathcal{U}_2 | Y$.

Proposition 1.2.4.4 Let \mathcal{U}_i be the uniformity on the set X, $i \in I \neq \emptyset$, $\emptyset \neq Y \subset X$ and $\mathcal{U} = \sup\{\mathcal{U}_i : i \in I\}$. Then $\sup\{\mathcal{U}_i | Y : i \in I\} = \mathcal{U} | Y$.

Proof: For an arbitrary family of the entourages $U_{i_k} \in \mathcal{U}_{i_k}$, $i_k \in I$, $k = 1, 2, \ldots, n$, the following equality is true:

$$\left(\bigcap_{k=1}^{n} U_{i_k}\right) \cap (Y \times Y) = \bigcap_{k=1}^{n} (U_{i_k} \cap (Y \times Y)).$$

The assertion follows from the fact that the uniformity $\mathcal{U}|Y$ is generated by the filter base composed of the sets on the left-hand side of the above equality, while the uniformity $\sup \{\mathcal{U}_i | Y : i \in I\}$ is composed of the sets on the right-hand of this equality. **Corollary 1.2.4.2** If \mathcal{U} is a uniformity on the set X, $\emptyset \neq X_0 \subset X_1 \subset X$, then $(\mathcal{U}|X_1)|X_0 = \mathcal{U}|X_0$.

The operation of the restriction can also be considered as a special case of a more general operation.

Proposition 1.2.4.5 Let \mathcal{U} be a uniform structure on the set $Y, f: X \to Y$ and let $g: X \times X \to Y \times Y$ be the mapping which carries (x, y) into (f(x), f(y)). Then the filter in $X \times X$ generated by the filter base $g^{-1}(\mathcal{U})$ is a uniformity on X.

Proof: Let $\mathcal{B} = \{g^{-1}(U) : U \text{ is a entourage from } \mathcal{U}\}$. Then \mathcal{B} is a uniform base on X. Indeed, the conditions (a), (b) and (d) of Proposition 1.2.1.1 obviously hold. It remains to prove that the condition (c) is fulfilled. Indeed, if $V \circ V \subset U$, then

$$g^{-1}(V) \circ g^{-1}(V) \subset g^{-1}(U)$$
.

To prove this inclusion, let us suppose that $(x, y) \in g^{-1}(V)$ and $(y, z) \in g^{-1}(V)$. Then $(f(x), f(y)) \in V$, $(f(y), f(z)) \in V$, so that $(f(x), f(z)) \in U$, i.e. $(x, z) \in g^{-1}(U)$. The family of all entourages of the diagonal of Y form the base of the uniformity \mathcal{U} which is equivalent to the family of subsets of the product $X \times X$. But then the family \mathcal{B} is equivalent to $g^{-1}(\mathcal{U})$, so that the filter on $X \times X$ generated by the family $g^{-1}(\mathcal{U})$ is equivalent to the filter generated by the family \mathcal{B} , for which we have already proved that it is the base of the uniformity on the set X.

Definition 1.2.4.2 The uniformity on the set X described in the previous proposition will be called the **inverse image of the uniformity** \mathcal{U} and denoted by $f^{-1}(\mathcal{U})$.

Corollary 1.2.4.3 Let (X, \mathcal{U}) be a uniform space, Y a non-empty subset of the set X and $f: Y \to X$ a canonical injection. Then $f^{-1}(\mathcal{U}) = \mathcal{U}|Y$.

Proposition 1.2.4.6 Let $f : X \to Y$, Σ be a family of pseudo-metrics on Y. For every $\sigma \in \Sigma$ let σ^* be the pseudo-metric on X defined by

$$\sigma^*(x, y) = \sigma(f(x), f(y)).$$

Then $\mathcal{U}_{\Sigma^*} = f^{-1}(\mathcal{U}_{\Sigma})$, where $\Sigma^* = \{\sigma^* : \sigma \in \Sigma\}$.

Proof: It is obvious that σ^* is a pseudo-metric. If $\emptyset \neq \Sigma_1 \subset \Sigma$ is a finite set and $\Sigma_1^* = \{\sigma^* : \sigma \in \Sigma_1\}$, then

$$U_{\Sigma_1^*,\varepsilon} = \{(x,y) : \sigma^*(x,y) < \varepsilon \text{ if } \sigma^* \in \Sigma_1^*\} = \\ = \{(x,y) : \sigma(f(x), f(y)) < \varepsilon \text{ if } \sigma \in \Sigma_1\} = \\ = g^{-1}(U_{\Sigma_1,\varepsilon}),$$

where g is the mapping defined in Proposition 1.2.4.5. Since the filter base composed of the entourages $U_{\Sigma_1,\varepsilon}$ generates \mathcal{U}_{Σ} , while the one composed of $U_{\Sigma_1^*,\varepsilon}$ does the same for \mathcal{U}_{Σ^*} , then $\mathcal{U}_{\Sigma^*} = f^{-1}(\mathcal{U}_{\Sigma})$.

Proposition 1.2.4.7 If $f : X \to Y$ and \mathcal{U} is a uniformity on the set Y, then

$$f^{-1}(\delta_{\mathcal{U}}) = \delta_{f^{-1}(\mathcal{U})}, \quad f^{-1}(\tau_{\mathcal{U}}) = \tau_{f^{-1}(\mathcal{U})}.$$

Proof: If $A, B \subset X$ and $U \in \mathcal{U}$ is an entourage, then $(A \times B) \cap g^{-1}(U) \neq \emptyset$ is equivalent to $(f(A) \times f(B)) \cap U \neq \emptyset$.

Corollary 1.2.4.4 If $f : X \to Y$, \mathcal{U}_1 and \mathcal{U}_2 are the uniformities on Y for which $\mathcal{U}_1 < \mathcal{U}_2$ holds, then $f^{-1}(\mathcal{U}_1) < f^{-1}(\mathcal{U}_2)$.

Proposition 1.2.4.8 Let \mathcal{U}_i , $i \in I \neq \emptyset$, be a uniformity on the set Y, $\mathcal{U} = \sup\{\mathcal{U}_i : i \in I\}$ and $f : X \to Y$. Then $\sup\{f^{-1}(\mathcal{U}_i) : i \in I\} = f^{-1}(\mathcal{U})$.

Proof: The uniformity \mathcal{U} is generated by the sets of the form $\bigcap_{k=1}^{n} U_{i_k}$, where $U_{i_k} \in \mathcal{U}_{i_k}$, $i_k \in I$, k = 1, 2, ..., n, is an arbitrary entourage in Y. Thus the uniformity $f^{-1}(\mathcal{U})$ is generated by the sets of the form

$$g^{-1}\left(\bigcap_{k=1}^{n} U_{i_k}\right) = \bigcap_{k=1}^{n} g^{-1}(U_{i_k}).$$

However, the same sets on the right-hand side of the last equality generate the supremum of the uniformities $f^{-1}(\mathcal{U}_i)$.

Proposition 1.2.4.9 If $f_1 : X \to Y$ and $f_2 : Y \to Z$ are the given mappings, $f_3 = f_2 \circ f_1$, and if \mathcal{U} is a uniformity on the set Z, then $f_3^{-1}(\mathcal{U}) = f_1^{-1}(f_2^{-1}(\mathcal{U}))$.

Proof: As in Proposition 1.2.4.5, let us define the mappings g_1 , g_2 and $g_3 = g_2 \circ g_1$. Then $g_3^{-1}(\mathcal{U}) = g_1^{-1}(g_2^{-1}(\mathcal{U}))$. Since the families $g_2^{-1}(\mathcal{U})$ and $f_2^{-1}(\mathcal{U})$ are equivalent, the families $g_1^{-1}(g_2^{-1}(\mathcal{U}))$ and $g_1^{-1}(f_2^{-1}(\mathcal{U}))$ are also equivalent. However, the last family is equivalent to the family $f_1^{-1}(f_2^{-1}(\mathcal{U}))$.

1.2.5 Uniformly continuous mappings

A mapping $f: X \to Y$ from a pseudo-metric space (X, d_X) into a pseudometric space (Y, d_Y) is uniformly continuous if, for every $\varepsilon > 0$, there exists a $\delta_{\varepsilon} > 0$, such that, for every $x, y \in X$, $d_X(x, y) < \delta_{\varepsilon}$ implies $d_Y(f(x), f(y)) < \varepsilon$. On the other hand, a mapping f is uniformly continuous on the set X if, for every set $U_{d_Y,\varepsilon}$, there exists a set $U_{d_X,\delta}$ such that for every $(x, y) \in U_{d_X,\delta}$ we have $(f(x), f(y)) \in U_{d_Y,\varepsilon}$. This formulation can be extended to arbitrary uniform spaces.

Definition 1.2.5.1 The mapping $f : X \to Y$ from the uniform space (X, U)into the uniform space (Y, V) is said to be **uniformly continuous** if, for every entourage $V \in V$, there exists an entourage $U \in U$ such that $(x, y) \in U$ implies $(f(x), f(y)) \in V$.

This can be also formulated by means of the mapping g introduced in Proposition 1.2.4.5.

Proposition 1.2.5.1 Let (X, U) and (Y, V) be uniform spaces, $f : X \to Y$ a given mapping, and $g : X \times X \to Y \times Y$ the mapping which carries $(x, y) \in X \times X$ into $(f(x), f(y)) \in Y \times Y$. The mapping f is uniformly continuous if and only if g(U) > V, or equivalently, if and only if $U > g^{-1}(V)$.

Proof: If \mathcal{A} and \mathcal{B} denote bases consisting of all entourages in \mathcal{U} and \mathcal{V} respectively, then the condition in the definition can be written in the form $g(\mathcal{A}) > \mathcal{B}$ or $\mathcal{U} > g^{-1}(\mathcal{V})$. The statement follows from this by virtue of the properties of the relation >. \clubsuit

Corollary 1.2.5.1 If (X, U) is an arbitrary uniform space, (Y, V) a discrete uniform space (especially if Y consists of a single point), then any mapping $f: X \to Y$ is uniformly continuous.

Corollary 1.2.5.2 Let $f : X \to Y$ and \mathcal{V} be a uniformity on Y. Then $\mathcal{U} = f^{-1}(\mathcal{V})$ is the coarsest uniformity on X for which f is uniformly continuous.

Corollary 1.2.5.3 The composition of uniformly continuous mappings is a uniformly continuous mapping.

Proposition 1.2.5.2 If $f : (X, U) \to (Y, V)$ is a uniformly continuous mapping, then it is also δ -continuous with respect to the proximities $\delta_{\mathcal{U}}$ and $\delta_{\mathcal{V}}$, hence it is continuous with respect to the topologies $\tau_{\mathcal{U}}$ and $\tau_{\mathcal{V}}$.

Proof: Suppose $A\overline{\delta}_{\mathcal{V}}B$, $A, B \subset Y$. Then there exists a $V \in \mathcal{V}$ such that $(A \times B) \cap V = \emptyset$. The mapping f is uniformly continuous, so there exists an $U \in \mathcal{U}$ such that $g(U) \subset V$. But then $(f^{-1}(A) \times f^{-1}(B)) \cap U = \emptyset$, i.e. $f^{-1}(A)\overline{\delta}_{\mathcal{U}}f^{-1}(B)$, so that the mapping f, by Proposition 1.1.6.1, is δ -continuous. But then it is by Proposition 1.1.6.8 also continuous with respect to the topologies $\tau_{\mathcal{U}}$ and $\tau_{\mathcal{V}}$.

Let us notice that the converse of the above proposition in general is not true.

Example 1.2.5.1 Let $X = \mathbb{R}$, and let \mathcal{U} be the usual metric uniformity on X, and let \mathcal{V} be a subspace uniformity on X induced by the uniformity of its Smirnoff compactification corresponding to the usual metric proximity. Clearly, \mathcal{U} and \mathcal{V} induce the same (metric) proximity on X. However, since \mathcal{U} is not totally bounded whereas \mathcal{V} is totally bounded, \mathcal{U} and \mathcal{V} are different uniformities. Identical mapping $i : (X, \mathcal{V}) \to (X, \mathcal{U})$ is δ -continuous, but it is not uniformly continuous.

Proposition 1.2.5.3 Let \mathcal{U} and \mathcal{V} be two uniform structures on the set X. The identity mapping $i : (X, \mathcal{U}) \to (X, \mathcal{V})$ is uniformly continuous if and only if $\mathcal{U} > \mathcal{V}$.

Proof: Follows from Proposition 1.2.5.1, where g is the identical mapping on $X \times X$.

Corollary 1.2.5.4 Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping. If U' is a uniformity on X finer than U, and V' a uniformity on Y coarser than V, then the mapping $f : (X, U') \to (Y, V')$ is uniformly continuous.

Proof: Follows from Corollary 1.2.5.3 and the above proposition.

Proposition 1.2.5.4 Let \mathcal{U} be a uniformity on X, \mathcal{U}_i a uniformity on Y for every $i \in I \neq \emptyset$, $\mathcal{U}' = \sup\{\mathcal{U}_i : i \in I\}$ and $f : X \to Y$. The mapping $f : (X, \mathcal{U}) \to (Y, \mathcal{U}')$ is uniformly continuous if and only if the mapping $f : (X, \mathcal{U}) \to (Y, \mathcal{U}_i)$ is uniformly continuous for every $i \in I$.

Proof: According to Corollary 1.2.5.2, Corollary 1.2.5.4 and Proposition 1.2.5.1 the uniform continuity of the mapping $f : (X, \mathcal{U}) \to (Y, \mathcal{U}')$ is equivalent to the relation $f^{-1}(\mathcal{U}') < \mathcal{U}$, and the uniform continuity of the mapping $f : (X, \mathcal{U}) \to (Y, \mathcal{U}_i)$ to the relation $f^{-1}(\mathcal{U}_i) < \mathcal{U}$. Thus, the statement follows from Proposition 1.2.4.8.

Proposition 1.2.5.5 Let \mathcal{U}_i be a uniformity on X for every $i \in I \neq \emptyset$, $\mathcal{U}' = \inf{\{\mathcal{U}_i : i \in I\}}, \mathcal{U}$ a uniformity on Y and $f : X \to Y$ a given mapping. The mapping $f : (X, \mathcal{U}') \to (Y, \mathcal{U})$ is uniformly continuous if and only if the mapping $f : (X, \mathcal{U}_i) \to (Y, \mathcal{U})$ is uniformly continuous for every $i \in I$.

Proof: As in the proof of the above proposition, we have to deal with the fact that $f^{-1}(\mathcal{U}) < \mathcal{U}'$ holds if and only if $f^{-1}(\mathcal{U}) < \mathcal{U}_i$ for every $i \in I$.

Proposition 1.2.5.6 Let \mathcal{U} and \mathcal{V} be the uniformities on X and Y respectively, $f : X \to Y$, $f(X) \subset Y_0 \subset Y$ and $g = f|_X^{Y_0}$. The mapping $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is uniformly continuous if and only if the mapping $g : (X, \mathcal{U}) \to (Y_0, \mathcal{V}|Y_0)$ is uniformly continuous.

Proof: Let $h : Y_0 \to Y$ denote the canonical injection. The mapping $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is uniformly continuous if and only if $f^{-1}(\mathcal{V}) < \mathcal{U}$. Since $f = h \circ g$, the last relation is equivalent to the relation $g^{-1}(h^{-1}(\mathcal{V})) < \mathcal{U}$. Since $h^{-1}(\mathcal{V}) = \mathcal{V}|Y_0$ by Proposition 1.2.4.3, then $g^{-1}(\mathcal{V}|Y_0) < \mathcal{U}$, so that the mapping $g : (X, \mathcal{U}) \to (Y_0, \mathcal{V}|Y_0)$ is uniformly continuous.

Proposition 1.2.5.7 If $f : (X, U) \to (Y, V)$ is a uniformly continuous mapping and $\emptyset \neq X_0 \subset X$, then $f|X_0 : (X_0, U|X_0) \to (Y, V)$ is a uniformly continuous mapping.

Proof: If $g: X_0 \to X$ is the canonical injection, then by Proposition 1.2.4.5 $\mathcal{U}|X_0 = g^{-1}(\mathcal{U})$ holds. Therefore $g: (X_0, \mathcal{U}|X_0) \to (X, \mathcal{U})$ is a uniformly continuous mapping, so that $f|X_0 = f \circ g$ is a uniformly continuous mapping as the composition of the uniformly continuous mappings.

Proposition 1.2.5.8 Let \mathcal{U}_1 and \mathcal{U}_2 be uniformities on X and Y respectively, and τ_1 and τ_2 topologies generated by them, and, let \mathcal{U}_1 , in particular, be the finest uniformity generating the topology τ_1 . If $f : (X, \tau_1) \to (Y, \tau_2)$ is a continuous mapping, then $f : (X, \mathcal{U}_1) \to (Y, \mathcal{U}_2)$ is uniformly continuous.

Proof: Let $\mathcal{U} = \sup\{\mathcal{U}_1, f^{-1}(\mathcal{U}_2)\}$. Then, by virtue of Proposition 1.2.3.4, Proposition 1.2.4.7 and the fact that $\tau_{\mathcal{U}_2}$ is the coarsest among the topologies with respect to which f is a continuous mapping, we have that $\tau_{\mathcal{U}} = \sup\{\tau_{\mathcal{U}_1}, \tau_{f^{-1}(\mathcal{U}_2)}\} = \sup\{\tau_1, f^{-1}(\tau_{\mathcal{U}_2})\} = \tau_1$. By hypothesis $\mathcal{U} < \mathcal{U}_1$, i.e. $\mathcal{U} = \mathcal{U}_1$ and then $f^{-1}(\mathcal{U}_2) < \mathcal{U}_1$, so that the statement follows from Proposition 1.2.3.4 and Corollary 1.2.5.4.

An important statement of the same type can be made in case of pseudometric spaces. **Proposition 1.2.5.9** Let (X, d_X) and (Y, d_Y) be any pseudo-metric spaces, and $f: X \to Y$ a given mapping. If $f: (X, \delta_{d_X}) \to (Y, \delta_{d_Y})$ is δ -continuous mapping, then $f: (X, \mathcal{U}_{d_X}) \to (Y, \mathcal{U}_{d_Y})$ is uniformly continuous mapping.

Proof: Otherwise, there would be $\varepsilon > 0$ such that, for every $n \in \mathbb{N}$, we could find points $x_n, y_n \in X$ such that

(1)
$$d_X(x_n, y_n) < \frac{1}{n} \text{ and } d_Y(f(x_n), f(x_n)) \ge \varepsilon.$$

Let us suppose first that there exist an index n_0 and an infinite sequence of natural numbers (n_i) for which

(2)
$$d_Y(f(y_{n_0}), f(x_{n_i})) < \frac{\varepsilon}{4}$$
 for every $i \in \mathbb{N}$.

In this case let $A = \{x_{n_i} : i \in \mathbb{N}\}$ and $B = \{y_{n_i} : i \in \mathbb{N}\}$. Then $d_Y(f(A), f(B)) \ge \varepsilon/2$ as otherwise, for suitable *i* and *j*, $d_Y(f(x_{n_i}), f(y_{n_j})) < \varepsilon/2$ would be valid and hence by (2) we should get

$$d_Y(f(x_{n_j}), f(y_{n_j})) \leq d_Y(f(x_{n_j}), f(y_{n_0})) + d_Y(f(y_{n_0}), f(x_{n_i})) + d_Y(f(x_{n_i}), f(y_{n_j})) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon,$$

which contradicts (1). On the other hand, the first inequality of (1) would imply $d_X(A, B) = 0$ and the δ -continuity of f would nevertheless follow $d_Y(f(A), f(B)) = 0$.

From this we can see that there cannot exist any index n_0 and any sequence (n_i) fulfilling (2). In the same way, there does not exist any n_0 and any infinite sequence (n_i) for which $d_Y(f(x_{n_0}), f(y_{n_i})) < \varepsilon/4$ for every $i \in \mathbb{N}$. In other words, for every $n \in \mathbb{N}$, there exists an index k_n such that $i \ge k_n$ implies

$$d_Y(f(y_n), f(x_i)) \ge \frac{\varepsilon}{4}, \quad d_Y(f(x_n), f(y_i)) \ge \frac{\varepsilon}{4}.$$

Therefore, starting from the value $n_1 = 1$, an increasing sequence $n_1 < n_2 < < \ldots$ can be constructed such that

(3)
$$d_Y(f(y_{n_j}), f(x_{n_i})) \ge \frac{\varepsilon}{4} \text{ and } d_Y(f(x_{n_j}), f(y_{n_i})) \ge \frac{\varepsilon}{4}$$

for every $i \in \mathbb{N}$ and every j = 1, 2, ..., i - 1. For this purpose, it is enough to choose n_i larger than all the indices $k_{n_1}, k_{n_2}, ..., k_{n_{i-1}}$. Constructing the sets $A = \{x_{n_i} : i \in \mathbb{N}\}$ and $B = \{y_{n_i} : i \in \mathbb{N}\}$ by means of this sequence (n_i) , again $d_X(A, B) = 0$, but $d_Y(f(A), f(B)) \ge \varepsilon/4$ for, in addition to (3), $d_Y(f(x_{n_i}), f(y_{n_i})) \ge \varepsilon/4$ also holds. \clubsuit **Definition 1.2.5.2** Let (X, \mathcal{U}) and (Y, \mathcal{V}) be uniform spaces, and $f : X \to Y$ a given bijection. If $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ and $f^{-1} : (Y, \mathcal{V}) \to (X, \mathcal{U})$ are the uniformly continuous mappings, then the mapping f is said to be the unimorphism or the uniform isomorphism. The uniform spaces X and Y are called unimorphic or uniformly isomorphic if there exists a unimorphism $f : X \to Y$.

This relation is reflexive, symmetrical and transitive. As a uniform isomorphism is at the same time a δ -isomorphism and a homeomorphism, therefore, if (X_1, \mathcal{U}_1) and (X_2, \mathcal{U}_2) are uniformly isomorphic, then $(X_1, \delta_{\mathcal{U}_1})$ and $(X_2, \delta_{\mathcal{U}_2})$ are δ -isomorphic while $(X_1, \tau_{\mathcal{U}_1})$ and $(X_2, \tau_{\mathcal{U}_2})$ are homeomorphic. Similar to Proposition 1.1.6.11, the following can be proved:

Proposition 1.2.5.10 Let (X, \mathcal{U}) and (Y, \mathcal{V}) be two uniform spaces, $f : X \to Y$ a mapping. Then

(a) f is a uniformly continuous mapping if and only if $\mathcal{U} > f^{-1}(\mathcal{V})$;

(b) if f is an injective, then $\mathcal{U} = f^{-1}(\mathcal{V})$ holds if and only if $h = f|_X^{f(X)}$: $(X,\mathcal{U}) \to (f(X),\mathcal{V}|f(X))$ is a uniform isomorphism;

c) if f is a bijective mapping, then $\mathcal{U} = f^{-1}(\mathcal{V})$ holds if and only if $f: (X, \mathcal{U}) \to (Y, \mathcal{V})$ is a uniform isomorphism.

1.2.6 Totally bounded uniform spaces

First let (X, d) be a pseudo-metric space. Let us call it **totally bounded** if, for every $\varepsilon > 0$, there exists a finite covering of the space X with sets having diameters less than ε . The terminology is motivated by the fact that a totally bounded pseudo-metric space is bounded. Conversely, however, an infinite discrete metric space is clearly bounded without being totally bounded.

It is an important fact that these notions coincide in the case of the subsets of the space \mathbb{R}^n .

The totally bounded uniform spaces are a natural generalization of totally bounded pseudo-metric spaces.

Definition 1.2.6.1 Let (X, U) be a uniform space. For a set $A \subset X$ let us call to be **small of order** U, where $U \in U$, if $A \times A \subset U$. In this case we shall write d(A) < U.

Definition 1.2.6.2 The uniform space (X, U) (or the uniformity U) is said to be **totally bounded** if, for every entourage $U \in U$, there exists a finite covering of X consisting of the sets which are small of order U.

According to the fact that, if $U_1 \subset U$, a set small of order U_1 is small of order U, it would be enough to speak here of the entourages U belonging to a uniform base generating \mathcal{U} . From these circumstances and from the fact, that, in a pseudo-metric space, the diameter of a set small of order U_{ε} is $\leq \varepsilon$ and, on the other hand, if $d(A) < \varepsilon$, then A is evidently small of order U_{ε} , can formulate the following:

Proposition 1.2.6.1 A pseudo-metric space (X, d) is totally bounded if and only if (X, \mathcal{U}_d) is a totaly bounded uniform space.

Definition 1.2.6.3 The subset $Y \subset X$ of the uniform space (X, U) is **to**tally bounded if $Y = \emptyset$ or $Y \neq \emptyset$ and the uniformity $\mathcal{U}|Y$ is totaly bounded.

It is obvious that the set $Y \subset X$ is totaly bounded if and only if, for every entourage $U \in \mathcal{U}$ there exists a finite covering of the set Y consisting of the sets small of order U. It clearly follows from this and the fact that all the subsets of the set small of order U have the same property:

Corollary 1.2.6.1 Every subspace of a totally bounded uniform space is totally bounded.

Corollary 1.2.6.2 If A_i , i = 1, 2, ..., n, are totally bounded subsets of the uniform space (X, U), then $\bigcup_{i=1}^{n} A_i$ is a totally bounded set.

Proposition 1.2.6.2 If $f : X \to Y$ and if \mathcal{U} is a totally bounded uniformity on Y, then $f^{-1}(\mathcal{U})$ is a totally bounded uniformity on X.

Proof: For the given entourage $U \in \mathcal{U}$, let $Y = \bigcup_{i=1}^{n} Y_i$, where Y_i is small of order U. Then $X = \bigcup_{i=1}^{n} f^{-1}(Y_i)$ and $f^{-1}(Y_i)$ is clearly small of order $g^{-1}(U)$, where g is the mapping introduced in Proposition 1.2.4.5.

Proposition 1.2.6.3 Let $f : (X, U) \to (Y, V)$ be a uniformly continuous surjective mapping. If U is totally bounded uniformity, then V is also totally bounded uniformity.

Proof: Let $V \in \mathcal{V}$. Since f is uniformly continuous, there exists an $U \in \mathcal{U}$ such that $g(U) \subset V$. The space X is totally bounded, so that there

exists a finite covering $\{A_1, A_2, \ldots, A_n\}$ of the space X, where A_i is small of order U, for which $A_i \times A_i \subset U$ holds. The mapping f is surjective, so that $\{f(A_1), f(A_2), \ldots, f(A_n)\}$ is a finite cover of the space Y, where $f(A_i) \times f(A_i) = g(A_i) \subset g(U) \subset V$, i.e. $f(A_i)$ is small of order V for every $i = 1, 2, \ldots, n$.

Corollary 1.2.6.3 If \mathcal{U} and \mathcal{V} are uniformities on X, $\mathcal{U} < \mathcal{V}$, and if \mathcal{V} is a totally bounded uniformity, then \mathcal{U} is also a totally bounded uniformity.

Proposition 1.2.6.4 Let U_i , $i \in I \neq \emptyset$, be a totally bounded uniformities on X. Then $U = \sup\{U_i : i \in I\}$ is a totally bounded uniformity on X.

Proof: Let $U = \bigcap_{k=1}^{n} U_{i_k}$ be any entourage of the uniform base which generates the uniformity \mathcal{U} , where $U_{i_k} \in \mathcal{U}_{i_k}$, $i_k \in I$, k = 1, 2, ..., n. If $X = \bigcup_{j=1}^{n_k} A_{kj}$ is a finite covering, where A_{kj} is small of order U_{i_k} , then all sets of the form

 $\bigcap_{k=1}^{n} A_{kj_k}, \quad 1 \le j_k \le n_k$

are small of order U and form a finite covering of the set X. \clubsuit

Proposition 1.2.6.5 Let (X, U) be a uniform space. If Y is a τ_U -dense and totally bounded in X, then the space X is totally bounded.

Proof: Let $U \in \mathcal{U}$ be an arbitrary entourage, $V \in \mathcal{U}$ an entourage for which $V \circ V \circ V \subset U$ holds. Let $\{S_i : i = 1, 2, ..., n\}$ be a finite covering of Y consisting of the sets small of order V. Since $X = \overline{Y} = \bigcup_{i=1}^{n} \overline{S}_i$, it is suffices to show that \overline{S}_i is small of order U.

Now if $x, y \in \overline{S}_i$, then, by Corollary 1.2.2.3, there are $u, v \in S_i$ such that $u \in V[x]$ and $v \in V[y]$, i.e. such that $(x, u) \in V$ and $(v, y) \in V$. Since $(u, v) \in V$, then $(x, y) \in V \circ V \circ V \subset U$. Furthermore $\bigcup_{i=1}^n \overline{S}_i = \overline{Y} = X$, where $\{\overline{S}_i : i = 1, 2, ..., n\}$ is a finite covering of X consisting of the sets small of order U.

In Example 3.1.1.1 we have proved that a δ -continuous mapping does not have to be uniformly continuous. The following theorem gives the conditions under which this statement holds.

Theorem 1.2.6.1 Let (X, \mathcal{U}) and (Y, \mathcal{V}) be uniform spaces, where \mathcal{V} is a totally bounded uniformity. If $f : (X, \delta_{\mathcal{U}}) \to (Y, \delta_{\mathcal{V}})$ is a δ -continuous mapping, then $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is uniformly continuous as well.

Proof: Let $V \in \mathcal{V}$. Then there exists a $V_1 \in \mathcal{V}$ such that $V_1 \circ V_1 \circ V_1 \subset V$. Since Y is totally bounded, there exists a finite covering $\{Y_1, Y_2, \ldots, Y_n\}$ of Y consisting of the sets small of order V_1 . Let us consider those pairs of indices (i, j) for which $Y_i \overline{\delta}_{\mathcal{V}} Y_j$ holds. The mapping f is δ -continuous, so that for these pairs we have that $f^{-1}(Y_i)\overline{\delta}_{\mathcal{U}}f^{-1}(Y_j)$. But then there exists $U_{ij} \in \mathcal{U}$ such that $(f^{-1}(Y_i) \times f^{-1}(Y_j)) \cap U_{ij} = \emptyset$. Let $U \in \mathcal{U}$ be an entourage which is a subset of every U_{ij} .

If now $(x, y) \in U$ and $x \in f^{-1}(Y_i)$, $y \in f^{-1}(Y_j)$, then the pair (i, j)cannot belong to those considered above, so that $Y_i \delta_{\mathcal{V}} Y_j$ and therefore there can be found a pair of points (u, v) such that $(u, v) \in (Y_i \times Y_j) \cap V_1$. Since $f(x), u \in Y_i, f(y), v \in Y_j$, then $(f(x), u) \in V_1$ and $(v, f(y)) \in V_1$ because Y_i and Y_j are the sets small of order V_1 . Furthermore $(u, v) \in V_1$, so that finally we have $(f(x), f(y)) \in V$.

Corollary 1.2.6.4 If \mathcal{U} is a totally bounded uniformity on X, then \mathcal{U} is the coarsest uniformity on X inducing the proximity $\delta = \delta_{\mathcal{U}}$ on X.

Proposition 1.2.6.6 Let \mathcal{U}_1 and \mathcal{U}_2 be the two uniformities on X, where \mathcal{U}_1 is totally bounded. If $\mathcal{U} = \sup\{\mathcal{U}_1, \mathcal{U}_2\}$, then $\delta_{\mathcal{U}} = \sup\{\delta_{\mathcal{U}_1}, \delta_{\mathcal{U}_2}\}$.

Proof: Let us denote $\delta = \sup\{\delta_{\mathcal{U}_1}, \delta_{\mathcal{U}_2}\}$. Since $\mathcal{U}_i < \mathcal{U}, i = 1, 2$, by Proposition 1.2.3.4 it follows that $\delta_{\mathcal{U}_i} < \delta_{\mathcal{U}}, i = 1, 2$, so that $\delta < \delta_{\mathcal{U}}$.

To prove the converse, let us suppose that $A\delta B$ and prove that $A\delta_{\mathcal{U}}B$. Let $U_1 \in \mathcal{U}_1$ and $U_2 \in \mathcal{U}_2$ be an arbitrary entourages. According to Corollary 1.2.2.1 and Proposition 1.2.3.2, we have to prove that $(A \times B) \cap U_1 \cap U_2 \neq \emptyset$. Let now $U'_1 \in \mathcal{U}_1$ be an entourage for which $U'_1 \circ U'_1 \circ U'_1 \subset U_1$ holds, and $\{G_1, G_2, \ldots, G_n\}$ a covering of X, where the sets G_i are small of order U'_1 . This covering exists because \mathcal{U}_1 is a totally bounded uniformity. Let $A_j = A \cap G_j, B_j = B \cap G_j$. Then $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ are the coverings of the sets A and B respectively, so by Proposition 1.2.3.2 there exist indices j and k such that $A_j\delta_{\mathcal{U}_1}B_k$ and $A_j\delta_{\mathcal{U}_2}B_k$. Therefore $(A_j \times B_k) \cap U'_1 \neq \emptyset$, hence there exist an $a \in A_j$ and a $b \in B_k$ such that $(a, b) \in U'_1$. Then for every $x \in A_j$ and every $y \in B_k$, $(x, a) \in U'_1$ and $(y, b) \in U'_1$, because the sets A_j and B_k are small of order U'_1 . But then $(x, y) \in U_1$. Moreover, there are $x \in A_j$ and $y \in B_k$ such that $(x, y) \in U_2$, and then, for this pair, $(x, y) \in (A_j \times B_k) \cap U_1 \cap U_2 \subset (A \times B) \cap U_1 \cap U_2$. In this way we have proved that $A\delta_{\mathcal{U}}B$.

Proposition 1.2.6.7 Let U_i , $i \in I \neq \emptyset$, be totally bounded uniformities on X. If $U = \sup\{U_i : i \in I\}$, then

$$\delta_{\mathcal{U}} = \sup\{\delta_{\mathcal{U}_i} : i \in I\}.$$

Proof: Let $\delta = \sup\{\delta_{\mathcal{U}_i} : i \in I\}$. As in the previous proposition it can be proved that $\delta < \delta_{\mathcal{U}}$. Thus we have to see again that $A\delta B$ implies $A\delta_{\mathcal{U}}B$, which means that, for any finite family of entourages $U_{i_k} \in \mathcal{U}_{i_k}, i_k \in I, k = 1, 2, \ldots, n$,

$$(A \times B) \cap \bigcap_{k=1}^{n} U_{i_k} \neq \emptyset.$$

Let $U'_{i_k} \in \mathcal{U}_{i_k}$ be entourage such that $U'_{i_k} \circ U'_{i_k} \circ U'_{i_k} \subset U_{i_k}, k = 1, 2, ..., n$, and

$$X = \bigcup_{j=1}^{n_k} G_{kj}, \quad k = 1, 2, \dots, n,$$

where the sets G_{kj} , $j = 1, 2, ..., n_k$, are small of order U'_{i_k} . All the sets of the form

$$\bigcap_{k=1}^{n} G_{kj_k}, \quad 1 \leqslant j_k \leqslant n_k,$$

form a covering

$$X = \bigcup_{r=1}^{s} H_r \,,$$

and every set H_r is small of order U'_{i_k} for every index k = 1, 2, ..., n. Let

$$A = \bigcup_{r=1}^{s} (A \cap H_r), \quad B = \bigcup_{r=1}^{s} (B \cap H_r).$$

According to Theorem 1.1.4.1, there exist indices p and q such that the sets $A \cap H_p$ and $B \cap H_q$ are $\delta_{\mathcal{U}_i}$ -near for every $i \in I$. Therefore there exist, for every $k, x_k \in A \cap H_p$ and $y_k \in B \bigcap H_q$ such that $(x_k, y_k) \in U'_{i_k}$. But then $x \in A \cap H_p$ and $y \in B \cap H_q$ imply $(x, y) \in U_{i_k}$, according to the fact that $(x, x_k) \in U'_{i_k}$ and $(y_k, y) \in U'_{i_k}$. Thus, for every $x \in A \cap H_p$ and $y \in B \cap H_q$ it follows that

$$(x,y) \in (A \times B) \cap \bigcap_{k=1}^{n} U_{i_k}$$
.

Proposition 1.2.6.8 Let \mathcal{U}_i , $i \in I \neq \emptyset$, be the uniformities on X, where \mathcal{U}_i are totally bounded with the exception of one of them at the most. If $\mathcal{U} = \sup\{\mathcal{U}_i : i \in I\}$, then $\delta_{\mathcal{U}} = \sup\{\delta_{\mathcal{U}_i} : i \in I\}$.

Proof: Let the uniformity \mathcal{U}_i be totally bounded for every $i \in I - \{i_0\}$. Then the uniformity $\mathcal{U}' = \sup\{\mathcal{U}_i : i \neq i_0\}$ is totally bounded by Proposition 1.2.6.4. It is obvious that $\mathcal{U} = \sup\{\mathcal{U}_{i_0}, \mathcal{U}'\}$. Then by Proposition 1.2.6.6 $\delta_{\mathcal{U}} = \sup\{\delta_{\mathcal{U}_{i_0}}, \delta_{\mathcal{U}'}\}$, and on account of the previous proposition it follows that $\delta_{\mathcal{U}'} = \sup\{\delta_{\mathcal{U}_i} : i \neq i_0\}$. Therefore $\delta_{\mathcal{U}} = \sup\{\delta_{\mathcal{U}_i} : i \in I\}$.

1.2.7 The product of uniform spaces

The definitions and theorems in this subsection can be established in the way analogous to the generation of proximities.

First of all, let us notice that from Proposition 1.2.3.2 and Proposition 1.2.4.5 we come to the following result:

Proposition 1.2.7.1 Let (X_i, \mathcal{U}_i) be a uniform space for every $i \in I \neq \emptyset$, $X \neq \emptyset$ and $f_i : X \to X_i$. Then there exists the coarsest uniformity \mathcal{U}^* on X for which every mapping f_i is uniformly continuous, namely

(1)
$$\mathcal{U}^* = \sup\{f_i^{-1}(\mathcal{U}_i) : i \in I\}. \clubsuit$$

Definition 1.2.7.1 \mathcal{U}^* in (1) is called the uniformity projectively generated by the system $\{f_i, \mathcal{U}_i : i \in I\}$.

In the following we shall always use the notations from the of previous proposition. The following proposition gives a more precise description of the uniformity \mathcal{U}^* , and its proof is established according to Proposition 1.2.3.2 and Proposition 1.2.4.5.

Proposition 1.2.7.2 Let \mathcal{B}_i be a base of uniformity \mathcal{U}_i , and $g_i : X \times X \to X_i \times X_i$ the mapping given by the formula $g_i(x, y) = (f_i(x), f_i(y))$. Then the sets of the form $g_i^{-1}(U_i)$, where $i \in I$, $U_i \in \mathcal{B}_i$, constitute a uniform subbase for the uniformity \mathcal{U}^* , while entourages of the form $\bigcap_{j=1}^n g_{i_j}^{-1}(U_{i_j})$, where $i \in I$, $U_i \in \mathcal{B}_{i_j}$, $j = 1, 2, \ldots, n$, constitute a base of uniformity \mathcal{U}^* .

Proposition 1.2.7.3 Let (Y, U) be a uniform space, $f : Y \to X$ a given mapping. $f : (Y, U) \to (X, U^*)$ is uniformly continuous if and only if $f_i \circ f :$ $(Y, U) \to (X_i, U_i)$ is uniformly continuous for every $i \in I$.

Proof: It follows by using Corollary 1.2.5.3, Proposition 1.2.5.10 (a), Proposition 1.2.4.9 and Proposition 1.2.4.8 and the proof is analogous to the proof of the suitable proposition for proximity spaces. \clubsuit

Corollary 1.2.7.1 For every $i \in I$, let $J_i \neq \emptyset$, $(X_{ij}, \mathcal{U}_{ij})$ and $f_{ij} : X_i \rightarrow X_{ij}, j \in J_i$, be such that \mathcal{U}_i is the uniformity projectively generated by the system $\{f_{ij}, \mathcal{U}_{ij} : j \in J_i\}$. Then \mathcal{U}^* is identical with the uniformity projectively generated by the system $\{f_{ij} \circ f_i, \mathcal{U}_{ij} : i \in I, j \in J_i\}$

Corollary 1.2.7.2 If $f : Y \to X$, then $f^{-1}(\mathcal{U}^*)$ is the uniformity projectively generated by the system $\{f_i \circ f, \mathcal{U}_i : i \in I\}$.

Corollary 1.2.7.3 Let $\emptyset \neq Y \subset X$ and $f_i(Y) \subset Y_i \subset X_i$. Then $\mathcal{U}^*|Y$ coincides with the uniformity projectively generated by the system $\{f_i|_Y^{Y_i}, \mathcal{U}_i|Y_i : i \in I\}$.

Corollary 1.2.7.4 Let (Y_i, \mathcal{U}'_i) be a uniform space and $h_i : X_i \to Y_i$, $i \in I$, a mapping for which $\mathcal{U}_i = h_i^{-1}(\mathcal{U}'_i)$ holds. Then \mathcal{U}^* is identical with the uniformity projectively generated by the system $\{h_i \circ f_i, \mathcal{U}'_i : i \in I\}$. Especially, if h_i is a uniform isomorphism, then \mathcal{U}^* is identical with the uniformity projectively generated by the system $\{h_i \circ f_i, \mathcal{U}'_i : i \in I\}$.

Corollary 1.2.7.5 Let $I = \bigcup_{j \in J} I_j$ and \mathcal{U}_j^* be the uniformity projectively generated by the system $\{f_i, \mathcal{U}_i : i \in I_j\}$. Then $\mathcal{U}^* = \sup\{\mathcal{U}_i^* : j \in J\}$.

Corollary 1.2.7.6 If \mathcal{U}'_i is a uniformity on X_i for which $\mathcal{U}_i < \mathcal{U}'_i$ holds and if \mathcal{U}^{**} is a uniformity projectively generated by the system $\{f_i, \mathcal{U}'_i : i \in I\}$, then $\mathcal{U}^* < \mathcal{U}^{**}$.

Proposition 1.2.7.4 The topology $\tau_{\mathcal{U}^*}$ coincides with the topology projectively generated by the system $\{f_i, \tau_{\mathcal{U}_i} : i \in I\}$

Proof: According to Proposition 1.2.3.4 we have that $\tau_{\mathcal{U}^*} = \sup\{\tau_{f_i^{-1}(\mathcal{U}_i)}: i \in I\}$. Since by Proposition 1.2.4.7 $\tau_{f_i^{-1}(\mathcal{U}_i)} = f_i^{-1}(\tau_{\mathcal{U}_i})$, then $\tau_{\mathcal{U}^*} = \sup\{f_i^{-1}(\tau_{\mathcal{U}_i}): i \in I\}$, so that, by definition, the topology $\tau_{\mathcal{U}^*}$ is projectively generated by the system $\{f_i, \tau_{\mathcal{U}_i}: i \in I\}$.

Definition 1.2.7.2 Let (X_i, \mathcal{U}_i) be a uniform space for every $i \in I \neq \emptyset$, $X = \prod_{i \in I} X_i, p_i : X \to X_i$ the *i*-th projection. The uniformity \mathcal{U} on Xprojectively generated by the system $\{p_i, \mathcal{U}_i : i \in I\}$ is called the **product** of the uniformities \mathcal{U}_i and denoted by the symbol $\prod_{i \in I} \mathcal{U}_i$. The uniform space $(\prod_{i \in I} X_i, \prod_{i \in I} \mathcal{U}_i)$ is the **product of the uniform spaces** (X_i, \mathcal{U}_i) .

In the following we shall always use the above notations. According to Proposition 1.2.7.2 the following holds:

Proposition 1.2.7.5 Let \mathcal{B}_i be a base for uniformity \mathcal{U}_i and $g_i : X \times X \to X_i \times X_i$ a mapping defined with $g_i(x, y) = (p_i(x), p_i(y))$. Then the sets of the form $g_i^{-1}(U_i)$, $i \in I$, $U_i \in \mathcal{B}_i$, constitute a subbase for uniformity \mathcal{U} , while the sets of the form $\bigcap_{j=1}^n g_{i_j}^{-1}(U_{i_j})$, where $i_j \in I$, $U_{i_j} \in \mathcal{B}_{i_j}$, j = 1, 2, ..., n constitute a base for \mathcal{U} .

Corollary 1.2.7.7 Let (Y, U') be a uniform space, $g : Y \to X$. The mapping $g : (Y, U') \to (X, U)$ is uniformly continuous if and only if $p_i \circ g : (Y, U') \to (X_i, U_i)$ is uniformly continuous for every $i \in I$.

Corollary 1.2.7.8 Let $\emptyset \neq Y_i \subset X_i$, $Y = \prod_{i \in I} Y_i$. Then $\prod_{i \in I} \mathcal{U}_i | Y_i = \mathcal{U} | A$. If $Y_j = X_j$ for an index $j \in I$, while for the other indices $Y_i = \{x_i\}, x_i \in X_i$, then $p_j | Y : (Y, \mathcal{U} | Y) \to (X_j, \mathcal{U}_j)$ is a uniform isomorphism.

Corollary 1.2.7.9 If \mathcal{U}'_i is a uniformity on X_i and $\mathcal{U}_i < \mathcal{U}'_i$ for every $i \in I$, then $\prod_{i \in I} \mathcal{U}_i < \prod_{i \in I} \mathcal{U}'_i$.

Corollary 1.2.7.10 Let (Y_i, \mathcal{U}'_i) be a uniform space, $f_i : (X_i, \mathcal{U}_i) \to (Y_i, \mathcal{U}'_i)$ a uniformly continuous for every $i \in I$, $Y = \prod_{i \in I} Y_i$, $\mathcal{U}' = \prod_{i \in I} \mathcal{U}'_i$, $p'_i : Y \to Y_i$ the *i*-th projection, $f : X \to Y$ the mapping for which $f_i \circ p_i = p'_i \circ f$ for every $i \in I$. Then $f : (X, \mathcal{U}) \to (Y, \mathcal{U}')$ is uniformly continuous. If every $f_i : (X_i, \mathcal{U}_i) \to (Y_i, \mathcal{U}'_i)$ is a uniform isomorphism, then f is again a uniform isomorphism.

With the notations of Corollary 1.1.7.16 and Corollary 1.1.7.17, according to Corollary 1.2.7.7, we can formulate the following statements:

Corollary 1.2.7.11 Let \mathcal{U}_i be a uniformity on Y_i , $\mathcal{U}'_{f(i)} = \mathcal{U}_i$, $\mathcal{U} = \prod_{i \in I} \mathcal{U}_i$, $\mathcal{U}' = \prod_{i \in J} \mathcal{U}'_i$. Then g is a uniform isomorphism.

Corollary 1.2.7.12 Let \mathcal{U}_i be a uniformity on X_i , $\mathcal{U} = \prod_{i \in I} \mathcal{U}_i$, $\mathcal{U}'_j = \prod_{i \in I_i} \mathcal{U}_i$ i $\mathcal{U}' = \prod_{j \in J} \mathcal{U}'_j$. Then f is a uniform isomorphism.

From Proposition 1.2.7.4 it follows that:

Corollary 1.2.7.13 If $\mathcal{U} = \prod_{i \in I} \mathcal{U}_i$, then $\tau_{\mathcal{U}} = \prod_{i \in I} \tau_{\mathcal{U}_i}$.

Of course, an embedding theorem is true for uniform spaces, too.

Theorem 1.2.7.1 Let (X_i, \mathcal{U}_i) be a uniform space for every $i \in I \neq \emptyset$, $Y \neq \emptyset$ a given set, $f_i : Y \to X_i$, \mathcal{U}^* the uniformity projectively generated by the system $\{f_i, \mathcal{U}_i : i \in I\}, X = \prod_{i \in I} X_i, \mathcal{U} = \prod_{i \in I} \mathcal{U}_i, p_i : X \to X_i$ the *i*-th projection and $f: Y \to X$ the mapping for which $p_i \circ f = f_i$, and finally $h = f|_Y^{f(Y)}$. If for the elements $x, y \in Y, x \neq y$ implies $f_i(x) \neq f_i(y)$ for at least one $i \in I$, then $h: (X, \mathcal{U}^*) \to (f(X), \mathcal{U}|f(X))$ is a uniform isomorphism. This condition certainly holds if \mathcal{U}^* is separated.

Proof: From Corollary 1.2.7.2 it follows that $\mathcal{U}^* = f^{-1}(\mathcal{U})$, so that by Proposition 1.2.5.10 (b) the mapping h is a uniform isomorphism. If \mathcal{U}^* is a separated uniformity, then $x \neq y$ implies $(f_i(x), f_i(y)) \notin U_i$ for at least one $i \in I$ and a suitable entourage $U_i \in \mathcal{U}_i$, with $f_i(x) \neq f_i(y)$ in the end.

1.2.8 Quotient spaces of uniform spaces

Let $(X_i, \mathcal{U}_i), i \in I$, be a non-empty family of uniform spaces, $f_i : X_i \to X$ given mappings from the non-empty set X_i into X for every $i \in I$. The uniformity \mathcal{U}^* inductively generated by the system $\{f_i, \mathcal{U}_i : i \in I\}$ on the set X if \mathcal{U}^* is the finest uniformity for which every mapping f_i : $(X_i, \mathcal{U}_i) \to (X, \mathcal{U}^*)$ is uniformly continuous. The existence of the uniformity \mathcal{U}^* with this property can be proved in the same way as the analogous statement according to the inductive generation of the proximities. Here again the case, when a single mapping $\{f : Y \to X\}$ and a single uniform space $\{(Y, \mathcal{V})\}$ are given, is the most important one: in this case \mathcal{U}^* is called the **quotient uniformity** belonging to f and \mathcal{V} and is denoted by $f(\mathcal{V})$.

Defining the mapping $g: Y \times Y \to X \times X$ by means of the formula g(x, y) = (f(x), f(y)) as usual, it is clear that only entourages U such that $g^{-1}(U) \in \mathcal{V}$ can belong to a uniformity for which f is uniformly continuous. Thus, if all these U constitute a base of the uniformity on X, then $f(\mathcal{V})$ is certainly generated by them. In fact, these U fulfil conditions (a), (b) and (d) of Proposition 1.2.1.1, whereas it fulfils the condition (c) only if, for every U of this kind, there is a U_1 of the same property for which $U_1 \circ U_1 \subset U$ holds. Hence we obtain the following statement:

Proposition 1.2.8.1 Let (Y, \mathcal{V}) be a uniform space, $f : Y \to X$, $g : Y \times Y \to X \times X$ defined by the formula g(x, y) = (f(x), f(y)) and let as suppose that there are those entourages U in X for which $f^{-1}(U) \in \mathcal{V}$ forms a uniform base \mathcal{B} . Then $f(\mathcal{V})$ is generated by \mathcal{B} . This condition is fulfilled if and only if, for every entourage U in X for which $g^{-1}(U) \in \mathcal{V}$ holds, there can be found an entourage U_1 in X such that $g^{-1}(U_1) \in \mathcal{V}$ and $U_1 \circ U_1 \subset U$.

Proposition 1.2.8.2 If (X_i, \mathcal{U}_i) is a uniform space for every $i \in I \neq \emptyset$, $X = \prod_{i \in I} X_i, \ \mathcal{U} = \prod_{i \in I} \mathcal{U}_i \text{ and } p_i : X \to X_i \text{ is the i-th projection, then}$ $\mathcal{U}_j = p_j(\mathcal{U}) \text{ for every } j \in I.$

Proof: By previous proposition it is enough to show that $g_j^{-1}(U) \in \mathcal{U}$ holds for an entourage U in X_j if and only if $U \in \mathcal{U}_j$, where $g_j : X \times X \to X_j \times X_j$ is defined here by means of the formula $g_j(x, y) = (p_j(x), p_j(y))$. However, if $U \in \mathcal{U}_j$, then $g_j^{-1}(U) \in \mathcal{U}$ in view of Proposition 1.2.5.10. Conversely, if U is an entourage in X and $g_j^{-1}(U) \in \mathcal{U}$, then there exist indices $i_1, i_2, \ldots, i_n \in I$ and for $k = 1, \ldots, n$ entourages $U_{i_k} \in \mathcal{U}_{i_k}$ such that

$$U' = \bigcap_{k=1}^{n} g_{i_k}^{-1}(U_{i_k}) \subset g_j^{-1}(U) \,,$$

where $g_{i_k}(x, y) = (p_{i_k}(x), p_{i_k}(y))$. Let $x_j, y_j \in X_j$, while for $i \neq j, x_i = y_i \in X_i$. If j does not occur among the indices i_1, i_2, \ldots, i_n , then, for arbitrary $x_j, y_j \in X_j$, with the notation $x = (x_i), y = (y_i)$, in view of $(x, y) \in U'$, we have $(x_j, y_j) \in U$, i.e. $U = E_j \times E_j$, so that $U \in \mathcal{U}_j$. On the other hand, if j occurs among the i_k , say $i_1, \ldots, i_p = j, i_{p+1}, \ldots, i_n \neq j$, then $(x, y) \in U'$ for $(x_j, y_j) \in \bigcap_{k=1}^p U_{i_k}$, so that $(x_j, y_j) \in U$. Thus $\bigcap_{k=1}^p U_{i_k} \subset U$ and since $\bigcap_{k=1}^p U_{i_k} \in \mathcal{U}_j$, then $U \in \mathcal{U}_j$.

As a particularly important example, the case of the separative partition may again be mentioned.

Proposition 1.2.8.3 Let (X, U) be a uniform space, S the separative partition belonging to the topology τ_{U} , $p : X \to S$ the canonical surjection. Then

(a) x and y belong to the same cell $Z \in S$ if and only if $(x, y) \in U$ for every entourage $U \in \mathcal{U}$;

(b) if $U \in \mathcal{U}$ is an arbitrary entourage, then the sets

(*)
$$U' = \{(Z_1, Z_2) : Z_1, Z_2 \in \mathcal{S}, Z_1 \times Z_2 \subset U\}$$

form a base of the uniformity $p(\mathcal{U})$;

(c) $p^{-1}(p(\mathcal{U})) = \mathcal{U};$ (d) $\delta_{p(\mathcal{U})} = p(\delta_{\mathcal{U}});$ (e) $\tau_{p(\mathcal{U})} = p(\tau_{\mathcal{U}});$ (f) $p(\mathcal{U})$ is a separated uniformity.

Proof: (a) It can be obtained by applying Proposition 1.1.8.11 (a) to the proximity $\delta_{\mathcal{U}}$.

(b) Introducing the notation q(x, y) = (p(x), p(y)), we see that $q^{-1}(U') = U \in \mathcal{U}$ holds for entourage $U' \in \mathcal{S}$ if and only if $U \in \mathcal{U}$ and U' is given by means of the formula (*). Accordingly the condition occurring in Proposition 1.2.8.1 is fulfilled by p: if $q^{-1}(U') = U \in \mathcal{U}$, let $U_1 \in \mathcal{U}$ be an entourage such that $U_1 \circ U_1 \subset U$, and let U'_1 be defined by means of (*) putting U_1 instead of U. By (a) U'_1 is an entourage. Furthermore, $U'_1 \circ U'_1 \subset U'$ holds. Indeed, if $(Z_1, Z_2) \in U'_1$ and $(Z_2, Z_3) \in U'_1$, then choosing a point $z_2 \in Z_2$ for the arbitrary points $z_1 \in Z_1$ and $z_3 \in Z_3$, then $(z_1, z_2) \in U_1$ and $(z_2, z_3) \in U_1$ implies $(z_1, z_3) \in U$, i.e. $Z_1 \times Z_3 \subset U$ and $(Z_1, Z_3) \in U'$. Hence the statement follows from Proposition 1.2.8.1.

(c) We have seen that $q^{-1}(U') = U$ for U' defined by (*), hence the statement follows from Proposition 1.2.4.5.

(d) As $p: (X, \mathcal{U}) \to (\mathcal{S}, p(\mathcal{U}))$ is uniformly continuous, then $p: (X, \delta_{\mathcal{U}}) \to (\mathcal{S}, \delta_{p(\mathcal{U})})$ is δ -continuous, so that $\delta_{p(\mathcal{U})} < p(\delta_{\mathcal{U}})$. On the other hand, if $A\overline{p(\delta_{\mathcal{U}})}B$, i.e. if $p^{-1}(A)\overline{\delta}_{\mathcal{U}}p^{-1}(B)$, then there exists an entourage $U \in \mathcal{U}$ such that $(p^{-1}(A) \times p^{-1}(B)) \cap U = \emptyset$. Hence $Z_1 \in A$ and $Z_2 \in B$ imply $(Z_1, Z_2) \notin U'$, where U' is again given by (*). Indeed, for any points $z_1 \in Z_1$ and $z_2 \in Z_2, z_1 \in p^{-1}(A)$ and $z_2 \in p^{-1}(B)$ implies $(z_1, z_2) \notin U$. Therefore $(A \times B) \cap U' = \emptyset$, and $A\overline{\delta}_{p(\mathcal{U})}B$.

(e) and (f) follow from (d) by means of Proposition 1.1.8.11. \clubsuit

1.2.9 Complete uniform spaces

We have seen that some properties of the metric or pseudo-metric space can be extended to the proximity and uniform spaces as more general spaces. An important property of pseudo-metric spaces is the fact that for them there is a necessary condition for the convergence of a sequence without need of knowing its limit point. Let us examine the situation in the case of uniform and proximity spaces from this point of view; more generally, we shall consider the filter bases instead of sequences, since it is known that for the convergence in the general topological spaces the role of the sequences is played by the filter base.

First, let us notice that, in a pseudo-metric space (X, d), the sequence of points (x_n) is a Cauchy sequence if and only if, for every $\varepsilon > 0$, there exists an $n_{\varepsilon} \in \mathbb{N}$ such that $m, n \ge n_{\varepsilon}$ implies $d(x_m, x_n) < \varepsilon$. The last inequality can be noted in the form $(x_m, x_n) \in U_{\varepsilon}$, where $U_{\varepsilon} = U_{d,\varepsilon}$ is the entourage of the pseudo-metrizable uniformity. In other words, (x_n) is a Cauchy sequence if, for every $\varepsilon > 0$, there exists an element of the filter base of the sequence (x_n) small of order U_{ε} . By virtue of this idea, the following definition is now plausible:

Definition 1.2.9.1 Let (X, U) be a uniform space and \mathcal{F} a filter base on X. \mathcal{F} is said to be a **Cauchy filter base** or a \mathcal{U} -**Cauchy filter base** if for every entourage $U \in \mathcal{U}$ there exists an element $F \in \mathcal{F}$ small of order U. The sequence (x_n) is called **Cauchy sequence** if the corresponding sequential filter base is a Cauchy filter base.

Obviously, it would be sufficient to speak of the entourages belonging to a uniform base \mathcal{B} generating the uniformity \mathcal{U} instead of all the entourages $U \in \mathcal{U}$.

Definition 1.2.9.2 Let us agree that the convergence of the filter base in a uniform space (X, \mathcal{U}) (or in a proximity space) means the convergence with respect to the topology $\tau_{\mathcal{U}}$ (respectively τ_{δ}).

Then it can be seen that the definition introduced above will suit the purpose required:

Proposition 1.2.9.1 If \mathcal{F} is a convergent filter base in the uniform space (X, \mathcal{U}) , then it is a Cauchy filer base.

Proof: Let us suppose $\mathcal{F} \to x$. For a given entourage $U \in \mathcal{U}$, let $U_1 \in \mathcal{U}$ be an entourage such that $U_1 \circ U_1 \subset U$. Then there exists an $F \in \mathcal{F}$ such that $F \subset U_1[x]$. But then the set F is small of order U. Indeed, if $y, z \in F$, then $(y, x) \in U_1$ and $(x, z) \in U_1$, so that $(y, z) \in U_1 \circ U_1 \subset U$.

Now we shall study some properties of the Cauchy filter bases.

Proposition 1.2.9.2 Let (X, U) and (Y, V) be uniform spaces and let $f : X \to Y$ be a uniformly continuous mapping. If \mathcal{F} is a Cauchy filter base in X, then $f(\mathcal{F})$ is a Cauchy filter base in Y.

Proof: If $U \in \mathcal{U}$ is an entourage choose for the set $V \in \mathcal{V}$ such that $(x,y) \in U$ implies $(f(x), f(y)) \in V$, and $F \in \mathcal{F}$ is small of order U, then f(F) is the set small of order V.

Corollary 1.2.9.1 If U_1 and U_2 are the uniformities on X and $U_1 < U_2$, then every U_2 -Cauchy filter base is also a U_1 -Cauchy filter base.
Proposition 1.2.9.3 Let \mathcal{U}_i , $i \in I \neq \emptyset$, be the uniformities on X, and $\mathcal{U} = \sup{\mathcal{U}_i : i \in I}$. If \mathcal{F} is a \mathcal{U}_i -Cauchy filter base for every $i \in I$, then it is also a \mathcal{U} -Cauchy filter base.

Proof: It is sufficient to consider the entourages of the form $U = \bigcap_{1}^{n} U_{i_k}$ of the uniformity \mathcal{U} , where $U_{i_k} \in \mathcal{U}_{i_k}$, $i_k \in I$, k = 1, 2, ..., n. If the set $F_k \in \mathcal{F}$ is small of order U_{i_k} , and $F \in \mathcal{F}$ a set for which $F \subset \bigcap_{1}^{n} F_k$, then F is the set small of order U.

Proposition 1.2.9.4 Let $f : X \to Y$ and \mathcal{V} be a uniformity on Y. A filter base \mathcal{F} in X is an $f^{-1}(\mathcal{V})$ -Cauchy filter base if and only if $f(\mathcal{F})$ is a \mathcal{V} -Cauchy filter base.

Proof: For any entourage $U \in \mathcal{V}$ a set F is small of order $g^{-1}(U)$ if and only if $g^{-1}(U)$ is small of order U, where $g = f \times f : X \times X \to Y \times Y$ is the mapping with the usual notation.

Corollary 1.2.9.2 Let (X, U) be a uniform space, $\emptyset \neq Y \subset X$. A filter base in Y is a (U|Y)-Cauchy filter base if and only if it is U-Cauchy.

Corollary 1.2.9.3 Every filter base finer than a Cauchy filter base is itself a Cauchy filter base. Equivalent filter bases are simultaneously Cauchy.

Proposition 1.2.9.5 Let (X, d) be a pseudo-metric space. A filter base \mathcal{F} in X is Cauchy (i.e. \mathcal{U}_d -Cauchy) if and only if for every $\varepsilon > 0$ there exists a set $F \in \mathcal{F}$ with the diameter $< \varepsilon$.

Proof: If $d(F) < \varepsilon$, then the set F is small of order U_{ε} . On the other hand, if $0 < \varepsilon_1 < \varepsilon$ and F is the set small of order U_{ε_1} , then $d(F) \leq \varepsilon_1 < \varepsilon$.

Definition 1.2.9.3 A uniform space (X, U) (or a uniformity U) is complete, if every U-Cauchy filter base is τ_U -convergent.

It is not evident that, by applying this definition for pseudo-metric spaces, it will be equivalent to the previous one, for the completeness of a pseudo-metric space was defined with the help of the Cauchy sequence instead of the Cauchy filter bases. Nevertheless, the following holds:

Proposition 1.2.9.6 A pseudo-metric space (X, d) is complete if and only if the uniformity \mathcal{U}_d is complete.

Proof: Let us suppose that \mathcal{U}_d is a complete uniformity on X and let (x_n) be a Cauchy sequence with respect to the pseudo-metric d. Then the filter base of the sequence (x_n) is a \mathcal{U}_d -Cauchy filter base, it is convergent in the topology $\tau_{\mathcal{U}_d}$, and with it the sequence (x_n) is convergent as well.

Let us suppose now that the pseudo-metric space (X, d) is complete and let \mathcal{F} be a \mathcal{U}_d -Cauchy filter base. Then by Proposition 1.2.9.5 for every $n \in \mathbb{N}$ there exists an element $F_n \in \mathcal{F}$ of diameter smaller than 1/n. Let $x_n \in F_n$ be an arbitrary point for every $n \in \mathbb{N}$. The sequence (x_n) is a Cauchy sequence. To prove this assertion, let $\varepsilon >$ be an arbitrary real number, and $n_{\varepsilon} > 2/\varepsilon$. Then, for $m, n \ge n_{\varepsilon}, F_m \bigcap F_n \neq \emptyset$ implies $d(x_m, x_n) \le d(F_m) + d(F_n) < 1/n + 1/m < \varepsilon$. Hence the sequence (x_n) is convergent. If $x_n \to x$, then $\mathcal{F} \to x$. Indeed, if for given $\varepsilon > 0$, n is so large that $d(x_n, x) < \varepsilon/2$ and $n > 2/\varepsilon$, then clearly $F_n \subset S(x, \varepsilon)$.

Proposition 1.2.9.7 Let (X, U) and (Y, V) be uniform spaces, $f : X \to Y$ a bijection, $f : (X, U) \to (Y, V)$ uniformly continuous and $f^{-1} : (Y, \tau_V) \to (X, \tau_V)$ a continuous mapping. If the uniformity V is complete, then the uniformity U is complete as well.

Proof: If \mathcal{F} is a \mathcal{U} -Cauchy filter base, then, by Proposition 1.2.9.2, $f(\mathcal{F})$ is a \mathcal{V} -Cauchy filter base, so that $f(\mathcal{F}) \to y \in Y$ with respect to the topology $\tau_{\mathcal{V}}$. Then by the well known fact from the general topology $f^{-1}(f(\mathcal{F})) = \mathcal{F} \to f^{-1}(y) \in X$ with respect to the topology $\tau_{\mathcal{U}}$.

Corollary 1.2.9.4 Let (X, U) and (Y, V) be uniformly isomorphic spaces. If one of them is complete, then the other one is also complete.

Corollary 1.2.9.5 Let \mathcal{U} and \mathcal{V} be uniformities on X for which $\mathcal{U} < \mathcal{V}$. If $\tau_{\mathcal{U}} = \tau_{\mathcal{V}}$ and \mathcal{U} is a complete uniformity, then \mathcal{V} is also a complete uniformity.

Proposition 1.2.9.8 Let $f : X \to Y$ be a surjection. If \mathcal{V} is a complete uniformity on Y, then $f^{-1}(\mathcal{V})$ is a complete uniformity on X.

Proof: If \mathcal{F} is a $f^{-1}(\mathcal{V})$ -Cauchy filter base, then by Proposition 1.2.9.4 $f(\mathcal{F})$ is a \mathcal{V} -Cauchy filter base, so that $f(\mathcal{F}) \to y \in Y$ with respect to the topology $\tau_{\mathcal{V}}$. Let $x \in X$ be such that f(x) = y. Then by the known fact from the general topology $\mathcal{F} \to x$ with respect to topology $f^{-1}(\tau_{\mathcal{V}})$. Since by Proposition 1.2.4.7 $f^{-1}(\tau_{\mathcal{V}}) = \tau_{f^{-1}(\mathcal{V})}$, then $\mathcal{F} \to x$ with respect to the topology $\tau_{f^{-1}(\mathcal{V})}$.

Proposition 1.2.9.9 Let (X, U) be a complete uniform space and $\emptyset \neq Y \subset X$ a τ_U -closed set. Then U|Y is a complete uniformity on Y.

Proof: If \mathcal{F} is a $\mathcal{U}|Y$ -Cauchy filter base in Y, then by Corollary 1.2.9.2 \mathcal{F} is a \mathcal{U} -Cauchy filter base, so that $\mathcal{F} \to x$ with respect to the topology $\tau_{\mathcal{U}}$. According to the well known proposition of the general topology, $x \in Y$, and then $\mathcal{F} \to x$ with respect to the topology $\tau_{\mathcal{U}}|Y$. Now, on account of Proposition 1.2.4.3, $\tau_{\mathcal{U}}|Y = \tau_{\mathcal{U}|Y}$, so that $\mathcal{F} \to x$ with respect to the topology $\tau_{\mathcal{U}}|Y$.

Proposition 1.2.9.10 Let (X, U) be a separated uniform space, $\emptyset \neq Y \subset X$. If U|Y is a complete uniformity, then Y is a τ_U -closed set.

Proof: It is sufficient to show that if \mathcal{F} is a filter base in Y and $\mathcal{F} \to x \in X$ with respect to the topology $\tau_{\mathcal{U}}$, then $x \in Y$. However, in this case by Proposition 1.2.9.1 \mathcal{F} is a \mathcal{U} -Cauchy filter base and by Corollary 1.2.9.2 it is also a $\mathcal{U}|Y$ -Cauchy filter base. Therefore $\mathcal{F} \to y \in Y$ with respect to the topology $\tau_{\mathcal{U}|Y}$, and by Proposition 1.2.4.3 with respect to the topology $\tau_{\mathcal{U}}|Y$, and hence with respect to the topology $\tau_{\mathcal{U}}$ as well. Since by Proposition 1.1.2.7 and Corollary 1.2.2.4 $\tau_{\mathcal{U}}$ is a T_3 -topology and a fortiory a T_2 -topology, therefore $x = y \in Y$.

1.2.10 Completely regular spaces

The notions of the proximity and uniformity previously introduced, raise many further problems. First of all, it may be asked which topologies can be induced by a proximity. It is known that a topology of this kind has to be regular and that, on the other hand, all S_4 -topologies have this property. Accordingly the condition looked for, has to be somewhere between axioms S_3 and S_4 .

Then we have seen that every uniformity induces a proximity. It may be asked, on the other hand, which proximities can be induced by uniformities. Every family of the pseudo-metrics induces a uniformity. On the other hand, it may be asked which uniformities can be induced by a family of pseudometrics, and by a single pseudo-metric in particular.

The answer to these questions, and many others as well, is based on an important theorem, called Urysohn's lemma. In order to prove this in a sufficiently general form, it is useful to introduce suitable notations first and study the basic relations connected with them. In order to do this, let us consider a sequence $\{\ll_n : n = 0, 1, 2...\}$ of the relations defined for the subsets of a set X. Let us suppose that, for each of the relations $\ll = \ll_n$, the statements $(O_1) - (O_5)$ of Theorem 1.1.1.1 are fulfilled as well as the additional condition:

if $A \ll_n B$, then there exists a C such that $A \ll_{n+1} C \ll_{n+1} B$.

A function $f: X \to \mathbb{R}$ is said to be **associated** with the sequence (\ll_n) , if $f(X) \subset I$, where I = [0, 1], and if for the sets $P, Q \subset I$, $d(P, Q) > 1/2^n$ implies $f^{-1}(P) \ll_{n+2} f^{-1}(I-Q)$ for every $n \in \mathbb{N}$. d(x, y) = |x-y| denotes here the Euclidean metric on \mathbb{R} , and $d(P, Q) = \inf\{d(x, y) : x \in P, y \in Q\}$.

Lemma 1.2.10.1 (Urysohn's lemma) Let us suppose that, for every relation \ll_n , n = 0, 1, 2, ..., defined for the subsets of a set X, $(O_1) - (O_5)$ are valid (with \ll_n instead of \ll), and if $A \ll_n B$, then there exists a set Csuch that $A \ll_{n+1} C \ll_{n+1} B$. If now $M \ll_0 N$, then there exists a function f associated with the sequence (\ll_n) for which f(M) = 0 and f(X - N) = 1holds.

Proof: Let us first define, for each fraction between 0 and 1 and of the form $p/2^n \in [0, 1]$, $p = 0, 1, ..., 2^n$, a set $A_{p/2^n}$ such that $A_0 = M$, $A_1 = N$ and $A_{p/2^n} \ll_n A_{(p+1)/2^n}$ holds for each $n = 0, 1, 2, ...; p = 0, 1, ..., 2^n - 1$. We do this by recursion with respect to n starting from the definition

$$(1) A_0 = M, \quad A_1 = N;$$

then $A_0 \ll_0 A_1$ is indeed true.

Let us suppose that the sets $A_{p/2^n}$ have already been defined for some integer $n \ge 0$ and all the values of $p = 0, 1, 2..., 2^n$ in such a way that

(2)
$$A_{p/2^n} \ll_n A_{(p+1)/2^n}$$
 for $p = 0, 1, \dots, 2^n - 1$.

Then, by assumption, for every p of this kind, there exists a set C_p such that

$$A_{p/2^n} \ll_{n+1} C_p \ll_{n+1} A_{(p+1)/2^n};$$

let us define the set $A_{(2p+1)/2^{n+1}}$ by the formula

$$A_{(2p+1)/2^{n+1}} = C_p$$
 for $p = 0, 1, 2, \dots, 2^n - 1$.

Then the definition of the sets $A_{q/2^{n+1}}$, for every $q = 0, 1, \ldots, 2^{n+1}$, is clear and similarly to (2),

$$A_{q/2^{n+1}} \ll_{n+1} A_{(q+1)/2^{n+1}}$$
.

Hence we can continue the recursion for all numbers n = 0, 1, 2, ..., and (1) and (2) are fulfilled, the latter for every n.

Let us also define A_r if r > 1 denotes a dyadic rational number (i.e. having the form $p/2^n$ where $n = 0, 1, ..., p = 2^n + 1, 2^n + 2, ...$) by the formula

$$(3) A_r = X, \ r > 1.$$

It can be seen from (O_1) and (O_3) that now

(4)
$$A_{p/2^n} \ll_n A_{(p+1)/2^n}$$
 for $n = 0, 1, 2, \dots; p = 0, 1, 2, \dots$

But then from (O_2) it can be seen immediately that

$$A_{p/2^n} \subset A_{(p+1)/2^n}$$
 for $n = 0, 1, 2, \dots; p = 0, 1, 2, \dots$

hence, in general, denoting the set of the non-negative dyadic rational numbers by \mathbf{R} ,

(5)
$$A_r \subset A_s \text{ if } r < s, r, s \in \mathbf{R}.$$

Now the required function f can be defined as:

(6)
$$f(x) = \inf\{r : x \in A_r, r \in \mathbf{R}\}, x \in X.$$

It is immediately clear that $x \in X$ implies $f(x) \ge 0$, while, from (3) and by the definition of $f, f(x) \le 1$ for every $x \in X$ holds, so that $f(X) \subset I$. From (1) we have that f(M) = 0, while f(X - N) = 1. Indeed, if $x \in X - N$, then $x \notin N = A_1$, and hence for every r < 1 $x \notin A_r$ holds by (5). But then according to (3) it follows that f(x) = 1.

It remains to show that the function f is associated with the sequence (\ll_n) . For this purpose, let $P, Q \subset I$, $d(P,Q) > 1/2^n$, and

$$I_p = \left[\frac{p}{2^{n+1}}, \frac{p+1}{2^{n+1}}\right], \quad J_p = \left[\frac{p-1}{2^{n+1}}, \frac{p+2}{2^{n+1}}\right], \quad p = 0, 1, \dots, 2^{n+1} - 1.$$

Clearly, if $P \cap I_p \neq \emptyset$, then $Q \cap J_p = \emptyset$, as the distance of an arbitrary point of I_p and any point of J_p is $\leq 1/2^n$. If P' denotes the union of the intervals I_p intersecting P, and Q' the union of the corresponding intervals J_p , then

$$P \subset P' \subset Q' \cap I \subset I - Q.$$

Thus

$$f^{-1}(P) \subset f^{-1}(P') \subset f^{-1}(Q' \cap I) \subset f^{-1}(I-Q)$$
.

By (O_3) and (O_5) , it suffices to show that

$$f^{-1}(I_p) \ll_{n+2} f^{-1}(J_p \cap I) = f^{-1}(J_p)$$

for every $p = 0, 1, ..., 2^{n+1} - 1$. However,

$$f^{-1}(I_p) = f^{-1}\left(\left[0, \frac{p+1}{2^{n+1}}\right]\right) \cap f^{-1}\left(\left[\frac{p}{2^{n+1}}, 1\right]\right),$$

$$f^{-1}(J_p) = f^{-1}\left(\left[0, \frac{p+2}{2^{n+1}}\right]\right) \cap f^{-1}\left(\left[\frac{p-1}{2^{n+1}}, 1\right]\right),$$

therefore, again by (O_5) , it is sufficient to show that

(7)
$$f^{-1}\left(\left[0, \frac{p+1}{2^{n+1}}\right]\right) \ll_{n+2} f^{-1}\left(\left[0, \frac{p+2}{2^{n+1}}\right]\right),$$

(8)
$$f^{-1}\left(\left[\frac{p}{2^{n+1}},1\right]\right) \ll_{n+2} f^{-1}\left(\left[\frac{p-1}{2^{n+1}},1\right]\right).$$

If $p = 2^{n+1} - 1$, then (7) goes over into the formula $X \ll_{n+2} X$, which holds on account of (O_1) . On the other hand, if $p < 2^{n+1} - 1$, then $f(x) \leq (p + 1)/2^{n+1}$ implies by (5) and (6) that $x \in A_{(2p+3)/2^{n+2}}$, and $x \in A_{(2p+4)/2^{n+2}} = A_{(p+2)/2^{n+1}}$ implies $f(x) \leq (p+2)/2^{n+1}$ by (6). In other words, from (4), we obtain now

$$f^{-1}\left(\left[0,\frac{p+1}{2^{n+1}}\right]\right) \subset A_{(2p+3)/2^{n+2}} \ll_{n+2} A_{(2p+4)/2^{n+2}} \subset f^{-1}\left(\left[0,\frac{p+2}{2^{n+1}}\right]\right),$$

hence by (O_3) (7) is also valid.

In the case p = 0, (8) goes over into the relation $X \ll_{n+2} X$, and if p = 1, into

$$f^{-1}\left(\left[\frac{1}{2^{n+1}},1\right]\right) \ll_{n+2} X\,,$$

which holds by (O_1) and (O_3) . In the case $p = 2, \ldots, 2^{n+1} - 1$ we obtain from (O_4) that it is sufficient to show that

$$f^{-1}\left(\left[0, \frac{p-1}{2^{n+1}}\right]\right) \ll_{n+2} f^{-1}\left(\left[0, \frac{p}{2^{n+1}}\right]\right).$$

However, if $f(x) < (p-1)/2^{n+1}$, then by (5) and (6) $x \in A_{(p-1)/2^{n+1}}$. On the other hand, $x \in A_{(2p-1)/2^{n+2}}$ implies $f(x) \leq (2p-1)/2^{n+2} < p/2^{n+1}$, so that

$$f^{-1}\left(\left[0,\frac{p-1}{2^{n+1}}\right]\right) \subset A_{(2p-2)/2^{n+1}} \ll_{n+2} A_{(2p-1)/2^{n+2}} \subset f^{-1}\left(\left[0,\frac{p}{2^{n+1}}\right]\right),$$

and again by (O_3) , we get the statement.

The Urysohn's lemma can be formulated more briefly by introducing the following terminology. A function $f: X \to \mathbb{R}$, defined on the set X, is said to **separate** the sets $A, B \subset X$, if $f(X) \subset I$, f(A) = 0 and f(B) = 1. Moreover if Φ is a family of functions defined on the set X, then the sets $A, B \subset X$ are called Φ -separable or Φ -separated if there exists a function $f \in \Phi$ separating them. Now the conclusion of Lemma 1.2.10.1 states that there is a function f associated with the sequence (\ll_n) which separates M and X - N. It is also worth mentioning the following:

Proposition 1.2.10.1 Under the hypotheses of the Urysohn's lemma, let f be a function associated with the sequence (\ll_n) . If $|f(x) - f(y)| > 1/2^n$, $n = 1, 2, ..., then \{x\} \ll_{n+2} X - \{y\}$.

Proof: By hypothesis, $d(\lbrace f(x) \rbrace, \lbrace f(y) \rbrace) > 1/2^n$, so that

$$f^{-1}(f(x)) \ll_{n+2} f^{-1}(I - \{f(y)\}) = X - f^{-1}(f(y))$$

But then from $\{x\} \subset f^{-1}(f(x)), \{y\} \subset f^{-1}(f(y))$ and (O_3) we have that $\{x\} \ll_{n+2} X - \{y\}$.

As the first application of the Urysohn's lemma, let us prove the following proposition:

Proposition 1.2.10.2 Let (X, δ) be a proximity space, Φ the family of those δ -continuous functions on X for which $x \in X$ implies $0 \leq f(x) \leq 1$. If $A\overline{\delta}B$, then the sets A and B are Φ -separable.

Proof: The conditions of the Urysohn's lemma are fulfilled whenever for all n, \ll_n is identical with the relation \ll defined in Theorem 1.1.1.1. $A\overline{\delta}B$ then implies $A \ll X - B$; hence, according to Urysohn's lemma, there exists a function f associated with this sequence separates the sets A and B. For this $f, f(X) \subset I = [0,1]$ and f is δ -continuous. Indeed, if $U, V \subset \mathbb{R}$ and d(U,V) > 0, then, with the notations $P = U \cap I, Q = V \cap I, d(P,Q) > 1/2^n$ holds for suitable $n, f^{-1}(U) = f^{-1}(P), f^{-1}(V) = f^{-1}(Q)$ and $f^{-1}(P) \ll f^{-1}(I-Q) = X - f^{-1}(Q)$ implies $f^{-1}(P)\overline{\delta}f^{-1}(Q)$, thus Proposition 1.1.6.1 can be applied.

From this, we obtain the following proposition (strictly speaking, this is what Urysohn proved):

Proposition 1.2.10.3 Let (X, τ) be a normal topological space, and Φ the family of continuous functions on X. If A and B are closed and disjoint sets, then A and B are Φ -separable.

Proof: Let us introduce on X the proximity δ defined in Theorem 1.1.3.2. Then $A\overline{\delta}B$, and thus, by the previous proposition, the sets A and B can be separated by a δ -continuous function $f \in \Phi$. By Proposition 1.1.6.8 δ -continuous function f is τ_{δ} -continuous. By Theorem 1.1.3.3 $\tau_{\delta} < \tau$, therefore, f is τ -continuous.

Proposition 1.2.10.4 Let (X, δ) be a proximity space, F a τ_{δ} -closed set and $x \notin F$. Then the sets $\{x\}$ and F can be separated by a τ_{δ} -continuous function.

Proof: Since X - F is a τ_{δ} -neighborhood of the point x, then $\{x\}\overline{\delta}F$. On account of Proposition 1.2.10.2, the sets $\{x\}$ and F are separated by a δ -continuous function f which is τ_{δ} -continuous again by Proposition 1.1.6.8.

Our next task will be to show that a topology can be induced by a proximity if and only if it is completely regular. In order to do this, and for other purposes as well, the concept of a function family will be a suitable tool.

Let $X \neq \emptyset$ be an arbitrary set. By the **function family** on X we understand an arbitrary non-empty set Φ of real functions defined on X. The function family Φ is said to be **bounded** if all the functions $f \in \Phi$ are bounded, i.e. $f(X) \subset \mathbb{R}$ is a bounded set.

Every function family on X induces a family of pseudo-metrics on X. This is a consequence of the following remark: if $f: X \to \mathbb{R}$, then

$$d_f(x,y) = |f(x) - f(y)|$$

is a pseudo-metric on X.

The family of pseudo-metrics induced by the function family Φ is the family of pseudo-metrics

$$\Sigma_{\Phi} = \{ d_f : f \in \Phi \} \,.$$

The uniformity $\mathcal{U}_{\Sigma_{\Phi}}$, the proximity $\delta_{\Sigma_{\Phi}}$ and the topology $\tau_{\Sigma_{\Phi}}$ induced by Σ_{Φ} are briefly called the **uniformity**, the **proximity** and the **topology induced by the function family** Φ and denoted by \mathcal{U}_{Φ} , δ_{Φ} and τ_{Φ} respectively.

Proposition 1.2.10.5 Let Φ be a function family on X, $\emptyset \neq Y \subset X$, and $\Phi_Y = \{f | Y : f \in \Phi\}$. Then Φ_Y is a function family on Y for which $\mathcal{U}_{\Phi_Y} = \mathcal{U}_{\Phi} | Y$, $\delta_{\Phi_Y} = \delta_{\Phi} | Y$ and $\tau_{\Phi_Y} = \tau_{\Phi} | Y$ hold. **Proof:** Immediately follows from Proposition 1.2.4.2.

Proposition 1.2.10.6 Let Φ be a function family on X, d_1 Euclidean metric on \mathbb{R} . If $f \in \Phi$, then $\mathcal{U}_{d_f} = f^{-1}(\mathcal{U}_{d_1})$, where d_f is a pseudo-metric generated by f, while $\mathcal{U}_{\Phi} = \sup\{f^{-1}(\mathcal{U}_{d_1}) : f \in \Phi\}$.

Proof: Let $(x, y) \in U_{d_f,\varepsilon} \in \mathcal{U}_{d_f}$. This holds if and only if $|f(x) - f(y)| < \varepsilon$, i.e. if and only if $(f(x), f(y)) \in U_{d_1,\varepsilon} \in \mathcal{U}_{d_1}$, which is, by Proposition 1.2.4.5, equivalent to the fact that $(x, y) \in g^{-1}(U_{d_1,\varepsilon}) \in f^{-1}(\mathcal{U}_{d_1})$, where g(x, y) = (f(x), f(y)). Now, according to the proved equality and Proposition 1.2.3.3 we have $\mathcal{U}_{\Phi} = \mathcal{U}_{\Sigma_{\Phi}} = \sup\{\mathcal{U}_{d_f} : f \in \Phi\} = \sup\{f^{-1}(\mathcal{U}_{d_1}) : f \in \Phi\}$.

Corollary 1.2.10.1 If Φ is an arbitrary function family on X, then \mathcal{U}_{Φ} is the coarsest uniformity for which all functions $f \in \Phi$ are uniformly continuous.

Proof: Follows from the previous proposition and Corollary 1.2.5.2.

Proposition 1.2.10.7 If Φ is a bounded function family on X, then \mathcal{U}_{Φ} and \mathcal{U}_{d_f} , $f \in \Phi$, are totally bounded uniformities.

Proof: If $f \in \Phi$ and $g: f(X) \to \mathbb{R}$ is the canonical injection, $h = f|_X^{f(X)}$, then $f = g \circ h$, thus $f^{-1}(\mathcal{U}_{d_1}) = h^{-1}(g^{-1}(\mathcal{U}_{d_1})) = h^{-1}(\mathcal{U}_{d_1}|f(X))$ according to Proposition 1.2.4.9 and Corollary 1.2.4.3. However, by Proposition 1.2.6.1, $\mathcal{U}_{d_1}|f(X)$ is totally bounded so that, according to Proposition 1.2.6.2 and Proposition 1.2.10.6, $\mathcal{U}_{d_f} = f^{-1}(\mathcal{U}_{d_1})$ and, according to Proposition 1.2.6.4, uniformity \mathcal{U}_{Φ} is also bounded.

Proposition 1.2.10.8 Let Φ be a function family on X, d_1 the Euclidean metric on \mathbb{R} . Then for every function $f \in \Phi$ there holds $\delta_{d_f} = f^{-1}(\delta_{d_1})$, and if Φ is bounded as well, then

$$\delta_{\Phi} = \sup\{\delta_{d_f} : f \in \Phi\}.$$

Proof: First, let us notice that by Corollary 1.2.2.2 $\delta_{d_f} = \delta_{\mathcal{U}_{d_f}}$. By Proposition 1.2.10.6 $\delta_{\mathcal{U}_{d_f}} = \delta_{f^{-1}(\mathcal{U}_{d_1})}$ holds, and by Proposition 1.2.4.7 there holds $\delta_{f^{-1}(\mathcal{U}_{d_1})} = f^{-1}(\delta_{\mathcal{U}_{d_1}})$. Since by Corollary 1.2.2.2 $f^{-1}(\delta_{\mathcal{U}_{d_1}}) = f^{-1}(\delta_{d_1})$, then $\delta_{d_f} = f^{-1}(\delta_{d_1})$. The second part follows from the first part and Proposition 1.2.6.7, as \mathcal{U}_{d_f} , $f \in \Phi$, is totally bounded on account of Proposition 1.2.10.7.

Notice that the second statement of the previous proposition holds even if all but one of the functions $f \in \Phi$ are bounded. Without this condition the statement is not necessarily true. This is shown by the following example: **Example 1.2.10.1** Let $X = \mathbb{R}^2$ and let Φ consist of two functions: f(x, y) = x and g(x, y) = y. Let us choose $A = \{(x, y) : x + y \leq 0\}, B = \{(x, y) : x + y \geq 1\}$. Then $A\overline{\delta}_{\Phi}B$, since $(A \times B) \cap U_{\Phi,1/2} = \emptyset$ as $(x_1, y_1) \in A$, $|x_2 - x_1| < 1/2, |y_2 - y_1| < 1/2$ implies $(x_2, y_2) \notin B$. On the other hand, with the notation $\delta = \sup\{\delta_{d_f}, \delta_{d_g}\}$, we have $A\delta B$. In fact, if $\{A_i : i \in J_p\}$ and $\{B_j : j \in J_q\}$ are two arbitrary decompositions of the sets A and B respectively, then there exists an *i* for which A_i contains a point (u, -u) and a point (v, -v) such that v > u + 1, since the sets A_i , finite in number, cover the line x + y = 0; thus, at least one of them has to contain an unbounded part of this line. But in this case $(v, -u) \in B$, therefore $(v, -u) \in B_j$ for some *j* and then $A_i\delta_{d_f}B_j, A_i\delta_{d_g}B_j$, as $d_f(A_i, B_j) = d_g(A_i, B_j) = 0$, since

$$|f(v,-v) - f(v,-u)| = |v - v| = 0,$$

$$|g(u,-u) - g(v,-u)| = |-u - (-u)| = 0.$$

This example also shows that the condition of total boundedness cannot be omitted in Proposition 1.2.6.8.

As an immediate consequence of the previous Proposition and Corollary 1.1.6.2 we have the following:

Corollary 1.2.10.2 If Φ is a bounded function family, then δ_{Φ} is the coarsest proximity for which every $f \in \Phi$ is δ -continuous.

Proposition 1.2.10.9 For any function family Φ , $f \in \Phi$ implies $\tau_{d_f} = f^{-1}(\mathcal{E})$, where \mathcal{E} is the Euclidean topology of the real line and

$$au_{\Phi} = \sup\{f^{-1}(\mathcal{E}) : f \in \Phi\}.$$

Proof: The first part follows from Proposition 1.2.10.8 and Proposition 1.1.5.2, the second from this on account of Proposition 1.2.3.3 and Proposition 1.2.3.4.

Now we can show that every completely regular topology can be induced by a function family. First, the following proposition will be proved:

Proposition 1.2.10.10 Let (X, τ) be a topological space, Φ a function family consisting of τ -continuous functions such that, if $x \in X$, $\overline{F} = F \subset X$, $x \notin F$, then $\{x\}$ and F are Φ -separated. Then $\tau_{\Phi} = \tau$.

Proof: According to the previous proposition we have that $\tau_{\Phi} < \tau$. To prove the converse, let us suppose that G is an open set in the topology τ and let $x \in G$. Then there exists a function $f \in \Phi$ which separates $\{x\}$ and X - G. Then $U_{d_f,1}[x] \subset G$, so that G is open in the topology τ_{Φ} . Hence $\tau < \tau_{\Phi}$.

Corollary 1.2.10.3 If (X, τ) is a completely regular topological space and Φ is the function family consisting of all τ -continuous functions, then $\tau_{\Phi} = \tau$.

From the comparison of the above corollary and Proposition 1.2.10.4, by applying the equality $\tau_{\Phi} = \tau_{\delta_{\Phi}}$, we have the following:

Corollary 1.2.10.4 A topology τ can be induced by a proximity if and only if it is completely regular.

This is the answer to one of basic questions put at the beginning of this subsection. It should also be noticed that the proximity δ_{Φ} induced by Φ and occurring in Corollary 1.2.10.3 is a distinguished proximity of the completely regular topology τ :

Proposition 1.2.10.11 Let τ be a completely regular topology, Φ the function family consisting of all τ -continuous functions, Φ^* that of all the bounded τ -continuous functions. Then $\delta_{\Phi} = \delta_{\Phi^*} = \delta$ coincides with the Czech-Stone proximity of τ . $A\overline{\delta}B$ holds if and only if A and B can be separated by a τ -continuous function.

Proof: For the sake of brevity let $\delta_1 = \delta_{\Phi}$, $\delta_2 = \delta_{\Phi^*}$, and let δ be the Czech-Stone proximity compatible with the topology τ . Since $\Phi^* \subset \Phi$, then $\Sigma_{\Phi^*} \subset \Sigma_{\Phi}$, so that $\delta_2 < \delta_1 < \delta$, since, by Corollary 1.2.10.3, $\tau_{\delta_1} = \tau$ and δ is the finest proximity for which $\tau_{\delta} = \tau$. If $A\bar{\delta}B$, then by Proposition 1.2.10.2 there exists a δ -continuous function f which separates A and B. By Proposition 1.1.6.8 the function f is τ -continuous. On the other hand, if some τ -continuous function f separates A and B, then $0 \leq f \leq 1$ implies $f \in \Phi^*$ and $(A \times B) \cap U_{d_{f,1}} = \emptyset$, so that $A\bar{\delta}_2 B$. Hence $\delta < \delta_2$, thus $\delta_2 = \delta_1 = \delta$, and $A\bar{\delta}B$ holds if and only if A and B can be separated by τ -continuous function.

With the help of the following proposition, an answer can be given to our second basic question about the inducement of proximities by uniformities:

Proposition 1.2.10.12 Let (X, δ) be a proximity space and Φ a bounded function family consisting of δ -continuous functions such that, if $A\overline{\delta}B$, then A and B are Φ -separable. Then $\delta = \delta_{\Phi}$.

Proof: First, let us notice that $\delta_{\Phi} < \delta$ holds by Corollary 1.2.10.2. To prove the converse, let us suppose that $A\overline{\delta}B$. Then there exists an $f \in \Phi$

which separates A and B, hence for which $(A \times B) \cap U_{d_f,1} = \emptyset$. Therefore $A\overline{\delta}_{\Phi}B$, so that $\delta < \delta_{\Phi}$.

As a direct consequence of this proposition and Proposition 1.2.10.2 we have the following:

Corollary 1.2.10.5 If (X, δ) is an arbitrary proximity space, Φ is the family of the bounded δ -continuous functions and Φ^* the family of the functions for which $0 \leq f(x) \leq 1$ for every $x \in X$, then $\delta_{\Phi} = \delta_{\Phi^*} = \delta$

From the equality $\delta_{\Phi} = \delta_{\mathcal{U}_{\Phi}}$, we can now obtain the following corollary:

Corollary 1.2.10.6 Every proximity can be induced by a uniformity.

Moreover, according to Proposition 1.2.10.7, we can add to this:

Corollary 1.2.10.7 Every proximity can be induced by a totally bounded uniformity.

Summarizing results of Corollary 1.2.6.4 and Corollary 1.2.10.5, the following can be concluded:

Corollary 1.2.10.8 The set of all proximities δ on the set X and the set of all totally bounded uniformities \mathcal{U} on X are in a one-to-one correspondence with each other. To the totally bounded uniformity \mathcal{U} , there corresponds the proximity $\delta = \delta_{\mathcal{U}}$; on the other hand the totaly bounded uniformity $\mathcal{U} = \mathcal{U}_{\Phi}$ corresponds to the proximity δ , where Φ denotes the family of all the bounded δ -proximally continuous functions.

Let us consider the question when a uniformity can be induced by a family of pseudo-metrics. In order to clear up this basic question, let us notice the following two lemmas.

Lemma 1.2.10.2 Let σ_i , $i \in I \neq \emptyset$, be an arbitrary set of the pseudometrics defined on the set X and let us suppose that, for any $x, y \in X$, the set of numbers $\{\sigma_i(x, y) : i \in I\}$ is bounded from above. Then

$$\sigma(x,y) = \sup\{\sigma_i(x,y) : i \in I\}$$

is a pseudo-metric on X.

Proof: Let us prove only the triangle inequality. If $x, y, z \in X$, then

 $\sigma_i(x,z) \leqslant \sigma_i(x,y) + \sigma_i(y,z) \leqslant \sigma(x,y) + \sigma(y,z)$

for every $i \in I$, hence $\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z)$.

Lemma 1.2.10.3 Let σ be a pseudo-metric on X, c > 0. If $\sigma' = \min(\sigma, c)$, then σ' is a pseudo-metric on X.

Proof: Again the triangle inequality should be proved. But $\sigma'(x, z) \leq \sigma'(x, y) + \sigma'(y, z)$ is obvious if any member of the right-hand side is equal to c, while otherwise we can refer to the inequality concerning σ .

Theorem 1.2.10.1 Let us suppose that $U = U_0, U_1, U_2, ...$ is a sequence of entourages in X such that $U_{n+1} \circ U_{n+1} \subset U_n$ for every n = 0, 1, 2, ... Then $\mathcal{B} = \{U_n : n = 0, 1, 2, ...\}$ is a base of some uniformity \mathcal{U} on X. Let \ll_n denote the relation for the subsets of X for which $A \ll_n B$ holds if and only if $U_n[A] \subset B$. Let Φ be the family of all the functions f associated with the sequence (\ll_n) and

(*)
$$\sigma_U(x,y) = \sup\{\sigma_f(x,y) : f \in \Phi\}, \ x, y \in X.$$

 σ_U is then a pseudo-metric on X and $\mathcal{U} = \mathcal{U}_{\sigma_U}$.

Proof: To prove that \mathcal{B} is a uniform base, we have to show that \mathcal{B} is a filter base. This follows from the fact that, by $U_{n+1} \subset U_{n+1} \circ U_{n+1} \subset U_n$, if n < m, then $U_m \subset U_n$. On account of Proposition 1.2.10.7, the sequence (\ll_n) fulfils the conditions of Urysohn's lemma, so that we can speak of the functions associated with this sequence. As the values of these functions lie in [0, 1], by Lemma 1.2.10.2, σ_U defined by (*) is a pseudo-metric on X.

It must be shown that $1/2^n < \varepsilon$ implies $U_{\sigma_U,\varepsilon} \supset U_{n+2}$. In fact, if $(x,y) \notin U_{\sigma_U,\varepsilon}$, i.e. if $\sigma_U(x,y) \ge \varepsilon$, then, by $\sigma_U(x,y) > 1/2^n$, there exists a function $f \in \Phi$ such that $\sigma_f(x,y) > 1/2^n$, i.e. $|f(x) - f(y)| > 1/2^n$. However, f is associated with the sequence (\ll_n) , so that by Proposition 1.2.10.1, $\{x\} \ll_{n+2} X - \{y\}$, hence $(x,y) \notin U_{n+2}$ holds.

On the other hand, it will be shown that $U_{\sigma_U,2^{-m}} \subset U_m$ for every m = 0, 1, 2, ... In fact, if $(x, y) \notin U_m$, then Urysohn's lemma can be applied for the system $\{\ll_n: n = m, m + 1, ...\}$, since $\{x\} \ll_m X - \{y\}$ and hence there exists a function f associated with the sequence $\ll_m, \ll_{m+1}, ...$ for which f(x) = 0, f(y) = 1. Let us consider the function $g(x) = f(x)/2^m$. It is associated with the sequence $\ll_0, \ll_1, \ll_2, ...$ Indeed, on the one hand $g(X) \subset [0, 1/2^m] \subset [0, 1] = \mathbf{I}$, on the other hand, $P, Q \subset \mathbf{I}, d(P, Q) > 1/2^n$, $n \in \mathbb{N}$, imply either n > m and then $|g(x') - g(y')| > 1/2^n$ for any points $x' \in g^{-1}(P)$ and $y' \in g^{-1}(Q)$, thus $|f(x') - f(y')| > 1/2^{n-m}$ so that, on account of the fact that f is associated with the sequence $\ll_m, \ll_{m+1}, ...,$ by 1.2.10.1 $(x', y') \notin U_{m+n-m+2} = U_{n+2}$, i.e. $g^{-1}(P) \ll_{n+2} X - g^{-1}(Q) =$ $g^{-1}(\mathbf{I} - Q)$, or $n \leqslant m$ and then one of the sets $g^{-1}(P)$ and $g^{-1}(Q)$ is empty so that according to (O_1) and $(O_3) g^{-1}(P) \ll_{n+2} X - g^{-1}(Q) = g^{-1}(\mathbf{I} - Q)$ is again true. Now from the fact that g is associated with the sequence (\ll_n) , i.e. $g \in \Phi$, it follows that $\sigma_U(x,y) \ge |g(x) - g(y)| = 1/2^m$, hence $(x,y) \not\in U_{\sigma_U,2^{-m}}$.

Theorem 1.2.10.2 Let us suppose that (X, U) is a uniform space, \mathcal{B} a base of the uniformity \mathcal{U} . For every entourage $U \in \mathcal{B}$, let us select entourages $U_n \in \mathcal{B}$, n = 0, 1, 2, ..., such that $U_0 = U$, $U_{n+1} \circ U_{n+1} \subset U_n$, n = 0, 1, 2, ..., and let σ_U be the pseudo-metric constructed from the sequence (U_n) according to the formula (*). If $\Sigma = \{\sigma_U : U \in \mathcal{B}\}$, then $\mathcal{U} = \mathcal{U}_{\Sigma}$.

Proof: First, let us notice that, by Proposition 1.2.1.1 (c), for every entourage $U \in \mathcal{B}$, a sequence (U_n) with the above properties can be constructed. By the previous theorem for $U \in \mathcal{U}$ there exists an $\varepsilon > 0$ such that $U_{\sigma_U,\varepsilon} \subset U$ (moreover, according to the proof, $\varepsilon = 1$ will do). On the other hand, let us consider a finite subset $\emptyset \neq \Sigma' \subset \Sigma$, and let us say that $\Sigma' = \{\sigma_i : i = 1, 2, ..., m\}, \sigma_i = \sigma_{vi}, U^i \in \mathcal{B}$. Then for any $\varepsilon > 0$

$$U_{\Sigma',\varepsilon} = \bigcap_{i=1}^m U_{\sigma_i,\varepsilon}$$

and by the previous theorem, for every *i*, there is an n_i such that $U_{n_i}^i \subset U_{\sigma_i,\varepsilon}$ (moreover, according to the proof, $n_i = n + 2$ will do whenever $1/2^n < \varepsilon$). Hence, let us choosing $U \in \mathcal{B}$ such that $U \subset \bigcap_{i=1}^m U_{n_i}^i$, which is possible by $U_{n_i}^i \in \mathcal{B}$. But then $U \subset U_{\Sigma',\varepsilon}$ holds.

Now the answer to our basic question can be given:

Corollary 1.2.10.9 Every uniformity can by induced by a family of pseudo-metrics.

A further important statement can be obtained from Theorem 1.2.10.1.

Theorem 1.2.10.3 A uniformity \mathcal{U} is pseudo-metrizable if and only if there exists a countable uniform base generating \mathcal{U} and it is metrizable if and only if it admits a countable base and is separated.

Proof: If $\mathcal{U} = \mathcal{U}_d$, then $\{U_{d,1/n} : n \in \mathbb{N}\}$ is obviously a countable uniform base which generates \mathcal{U} . If \mathcal{U} is a metrizable uniformity, then its topology is a T_0 -topology, thus it is separated by Corollary 1.2.2.4.

Conversely, let us suppose that $\{U'_n : n = 0, 1, 2, ...\}$ is a base of the uniformity \mathcal{U} . Let $U = U_0 = U'_0$ and, if the entourage U_n is already chosen,

let $U_{n+1} \in \mathcal{U}$ be a entourage such that $U_{n+1} \circ U_{n+1} \subset U_n$ and $U_{n+1} \subset U'_{n+1}$. An entourage with this property can evidently be found by using properties of the base of uniformity. Then $\mathcal{B} = \{U_n : n = 0, 1, 2, ...\}$ generates \mathcal{U} and therefore, according to Theorem 1.2.10.1, $\mathcal{U} = \mathcal{U}_{\sigma_U}$ holds. Moreover, if uniformity \mathcal{U} is separated, then its topology is T_0 , thus σ_U is a metric according to the known result from general topology.

Proposition 1.2.10.13 If $\{\mathcal{U}_i : i \in I\}$ is a countable family of a (pseudo-)metrizable uniformities on X, then $\mathcal{U} = \sup\{\mathcal{U}_i : i \in I\}$ is a (pseudo-)metrizable uniformity on X.

Proof: Let \mathcal{B}_i be a countable base for the uniformity \mathcal{U}_i . Then $\bigcup_{i \in I} \mathcal{B}_i$ is countable and the finite intersections of its elements constitute a countable uniform base for \mathcal{U} according to Proposition 1.2.3.2. Therefore the part of the statement concerning the pseudo-metrizability follows from the previous proposition. If the uniformities \mathcal{U}_i are metrizable, and even if at least one of them is metrizable, then the topology of this one will be a T_0 -topology, and $\tau_{\mathcal{U}}$ has the same property, thus \mathcal{U} is metrizable.

Theorem 1.2.10.4 Every uniform space can be embedded with the help of a uniform isomorphism into the product of pseudo-metrizable uniform spaces.

Proof: If $\Sigma = \{\sigma_i : i \in I\}$ is a family of pseudo-metrics such that $\mathcal{U}_{\Sigma} = \mathcal{U}^*$, $X_i = X$ for every $i, \mathcal{U}_i = \mathcal{U}_{\sigma_i}$, and f_i is the identity mapping of the set X, then the system $\{f_i, \mathcal{U}_i : i \in I\}$ generates projectively the uniformity \mathcal{U}^* by Proposition 1.2.3.3 and, for a given \mathcal{U}^* , there can always be found a Σ of this kind by Corollary 1.2.10.9. The hypotheses of Theorem 1.2.7.1 are clearly fulfilled, hence the statement follows.

Completely regular spaces can be characterized by the property that they admit bases or subbases with certain special properties. For this purpose, let us first introduce some notations and a suitable terminology.

Let (X, τ) be a topology space, $f: X \to \mathbb{R}$ a function defined on X. Let

$$Z_f = \{x \in X : f(x) = 0\}, \quad N_f = \{x \in X : f(x) \neq 0\} = X - Z_f$$

The elements of the systems

$$\mathcal{Z}(\tau) = \{Z_f : f \text{ is } \tau - \text{continuous function}\},\\ \mathcal{N}(\tau) = \{N_f : f \text{ is } \tau - \text{continuous function}\}\$$

are said to be the **zero-sets** and **cozero-sets** of the space (X, τ) respectively.

Proposition 1.2.10.14 Let (X, τ) be a topological space. Then:

(a) $A, B \in \mathcal{N}(\tau)$ implies $A \cup B, A \cap B \in \mathcal{N}(\tau)$;

(b) if $x \in N \in \mathcal{N}(\tau)$, then there exists a $Z \in \mathcal{Z}(\tau)$ such that $x \in Z \subset N$; (c) if $A, B \in \mathcal{Z}(\tau), A \cap B = \emptyset$, then there exist $C, D \in \mathcal{Z}(\tau)$ such that $C \cup D = X, A \cap C = B \cap D = \emptyset$; (d) $\emptyset, X \in \mathcal{N}(\tau)$.

Proof: (a) If $A = N_f$, $B = N_g$, then $A \cup B = N_h$, $A \cap B = N_k$, where $h = f^2 + g^2$, k = fg. Together with f and g, h and k are continuous.

(b) If $N = N_f$, $f(x) = c \neq 0$, g = f - c, then $x \in Z_g \subset N$, and, together with f, g is continuous as well.

(c) If $A = Z_f$, $B = Z_g$, then let

$$h = \max\left(\frac{g^2}{f^2 + g^2}, \frac{1}{2}\right) - \frac{1}{2}, \quad k = \max\left(\frac{f^2}{f^2 + g^2}, \frac{1}{2}\right) - \frac{1}{2},$$

 $C = Z_h, D = Z_k$. For $x \in X$, either $f(x) \ge g(x)$ or $f(x) \le g(x)$, accordingly either h(x) = 0 or k(x) = 0 and $x \in C$ or $x \in D$. If $x \in A$, then f(x) = 0, h(x) = 1/2, hence $x \notin C$; similarly, if $x \in B$, then $x \notin D$.

(d) $\emptyset = N_f$ for $f \equiv 0, X = N_g$ for $g \equiv 1$.

Proposition 1.2.10.15 A topological space (X, τ) is completely regular if and only if $\mathcal{N}(\tau)$ is a base for the topology τ .

Proof: If τ is a completely regular topology on $X, x \in X$ and G is an open neighborhood of x, then there exists a continuous function $f: X \to \mathbf{I}$ such that f(x) = 0 and f(X - G) = 1. Then g = 1 - f is τ -continuous, and $x \in N_g \subset G$.

Conversely, let us suppose that $\mathcal{N}(\tau)$ is a base for τ , and let x and G be as above. If f is a τ -continuous function and $x \in N_f \subset G$, then let

$$c = f(x), \ g = \frac{1}{c}f, \ h = \min\{1, \max\{g, 0\}\}, \ k = 1 - h$$

Then h is a τ -continuous function and clearly separates $\{x\}$ and X - G.

Theorem 1.2.10.5 Let (X, τ) be a topological space, S a subbase for the topology τ and $\mathcal{T} = \{X - S : S \in S\}$ the family of sets with the following properties:

(a) $\emptyset, X \in \mathcal{S};$

(b) if $x \in S \in S$, then there exists a $T \in T$ such that $x \in T \subset S$;

(c) if $A, B \in \mathcal{T}$, $A \cap B = \emptyset$, then there exists $T_i \in \mathcal{T}$, i = 1, 2, ..., n, such that $\bigcup_{i=1}^{n} T_i = X$ and, for each *i*, either $A \cap T_i = \emptyset$ or $B \cap T_i = \emptyset$.

Then (X, τ) is a completely regular space.

Proof: According to Corollary 1.2.10.4, it suffices to show that τ is induced by a suitable proximity δ .

In order to obtain this proximity, let

 $A\overline{\delta}B$ if and only if there exist decompositions $\{A_i : i \in J_m\}$ and $\{B_j : j \in J_n\}$ of A and B respectively such that for every i and every j there exist $C_{ij} \in \mathcal{T}$ and $D_{ij} \in \mathcal{T}$ for which $A_i \subset C_{ij}$, $B_j \subset D_{ij}$, $C_{ij} \cap D_{ij} = \emptyset$ hold,

and let us first prove that δ is a proximity on X. Condition (B_1) is obviously fulfilled. Let us suppose $A\overline{\delta}B$ and $A\overline{\delta}C$. Then there exist $A_i, B_j, C_{ij}, D_{ij} \in \mathcal{T}$ satisfying conditions of definition with respect to the sets A and B, and also sets $A'_k, B'_l, C'_{kl}, D'_{kl} \in \mathcal{T}$ such that $\{A'_k : k \in J_p\}$ and $\{B'_l : l \in J_q\}$ are decompositions of A and C respectively, where $A'_k \subset C'_{kl} \in \mathcal{T}, B'_l \subset D'_{kl} \in \mathcal{T}$ and $C'_{kl} \cap D'_{kl} = \emptyset$ for every $k \in J_p$ and every $l \in J_q$. Now we have

$$A = \bigcup_{i=1}^{m} \bigcup_{k=1}^{p} (A_i \cap A'_k), \quad B \cup C = \bigcup_{j=1}^{n} B_j \cup \bigcup_{l=1}^{q} B'_l,$$

$$A_i \cap A'_k \subset C_{ij} \in \mathcal{T}, \quad B_j \subset D_{ij} \in \mathcal{T}, \quad C_{ij} \cap D_{ij} = \emptyset,$$

$$A_i \cap A'_k \subset C'_{kl} \in \mathcal{T}, \quad B'_l \subset D'_{kl} \in \mathcal{T}, \quad C'_{kl} \cap D'_{kl} = \emptyset$$

so that $A\overline{\delta}(B \cup C)$ by definition of relation δ . The converse can be easily proved, so (B_2) holds. Property (B_3) holds because $\emptyset, X \in \mathcal{T}$, and (B_4) obviously holds. Let us prove (B_5) . Let $A\overline{\delta}B$ and let $A_i, B_j, C_{ij}, D_{ij} \in \mathcal{T}$ be the sets described in the definition of the relation δ . Then by the condition (c) of the theorem there exist sets $T_{ijk} \in \mathcal{T}, k = 1, 2, \ldots, n_{ij}$ such that

$$\bigcup_{k=1}^{n_{ij}} T_{ijk} = X, \ i = 1, 2, \dots, m, \ j = 1, 2, \dots, n,$$

and, for $k = 1, 2, ..., n_{ij}$, either $C_{ij} \cap T_{ijk} = \emptyset$ or $D_{ij} \bigcap T_{ijk} = \emptyset$. Let P_{ij} be the union of those T_{ijk} for which $C_{ij} \cap T_{ijk} = \emptyset$, and similarly, let Q_{ij} be the union of those T_{ijk} for which $D_{ij} \cap T_{ijk} = \emptyset$. Then $P_{ij} \cup Q_{ij} = X$ for every $i \in J_m$ and every $j \in J_n$. It is obvious that

$$C_{ij}\overline{\delta}P_{ij}, \quad D_{ij}\overline{\delta}Q_{ij}.$$

By properties $(B_1) - (B_4)$ established already

$$A_i \overline{\delta} \bigcup_{j=1}^n P_{ij} = P'_i, \quad i = 1, 2, \dots, m,$$
$$A = \bigcup_{i=1}^m A_i \overline{\delta} \bigcap_{i=1}^m P'_i = P',$$
$$B_j \overline{\delta} \bigcup_{i=1}^m Q_{ij} = Q'_j, \quad j = 1, 2, \dots, n,$$
$$B = \bigcup_{j=1}^n B_j \overline{\delta} \bigcap_{j=1}^n Q'_j = Q'$$

and moreover $P' \cup Q' = X$. Indeed, if $x \notin P'$, then there exists an index i_0 such that $x \notin P'_{i_0}$, so that $x \notin P_{i_0j}$ for every j = 1, 2, ..., n. Hence $x \in Q_{i_0j}$ for every j = 1, 2, ..., n, thus $x \in Q'_j$ for every j, from which follows that $x \in Q'$. Therefore P = X - P' and Q = X - Q' fulfil

$$P \cap Q = \emptyset, \ A\overline{\delta}X - P, \ B\overline{\delta}X - Q.$$

According to this, δ is a proximity indeed. We show that δ generates τ . If G is a τ -neighborhood of x, then there exist $S_1, S_2, \ldots, S_n \in \mathcal{S}$ such that

$$x \in \bigcap_{i=1}^n S_i \subset G.$$

By (b), there exist $T_i \in \mathcal{T}$ such that $x \in T_i \subset S_i$, i = 1, 2, ..., n. Hence $T_i \overline{\delta} X - S_i \in \mathcal{T}$ and $\{x\} \overline{\delta} X - G$ since

$$X - G \subset \bigcup_{i=1}^{n} (X - S_i).$$

To prove the converse, let us suppose that $\{x\}\overline{\delta}X - G$ for some $x \in X$ and $G \subset X$. Then

$$\{x\} = \bigcup_{i=1}^{m} A_i, \quad X - G = \bigcup_{j=1}^{n} B_j,$$
$$A_i \subset C_{ij} \in \mathcal{T}, \quad B_j \subset D_{ij} \in \mathcal{T}, \quad C_{ij} \cap D_{ij} = \emptyset.$$

Hence $x \in A_i$ for some *i*, so that

$$X - G \subset \bigcup_{j=1}^{n} D_{ij}, \ x \in \bigcap_{j=1}^{n} C_{ij} \subset \bigcap_{j=1}^{n} (X - D_{ij}) = B,$$

so that $x \in B \subset G$ and

$$B = \bigcap_{j=1}^{n} (X - D_{ij})$$

is the intersection of a finite number of the elements of the subbase S. Therefore G is a τ -neighborhood of x.

The following theorem summarizes the preceding results:

Theorem 1.2.10.6 A topological space (X, τ) is completely regular if and only if there exists a subbase S satisfying (with notation $\tau = \{X - S : S \in S\}$) conditions (a) - (c) of the previous theorem.

Proof: If τ is completely regular, then by Proposition 1.2.10.14 and Proposition 1.2.10.15 it follows that $\mathcal{N}(\tau)$ is a base with the required properties.

1.2.11 Compact proximity spaces

As in uniform spaces, in proximity spaces can also be given necessary conditions for convergence of a filter base. For this purpose, the following terminology is used.

Definition 1.2.11.1 In a proximity space (X, δ) the filter base \mathcal{F} is said to be **compressed** (or δ -compressed) if, for any two sets A and B δ -far from each other, there exists an $F \in \mathcal{F}$ which intersects at the most one of them, or equivalently, if the fact that $A \cap F \neq \emptyset \neq B \cap F$ for every set $F \in \mathcal{F}$ implies $A\delta B$.

Proposition 1.2.11.1 Every convergent filter base in a proximity space (X, δ) is compressed.

Proof: If, in the proximity space $(X, \delta), \mathcal{F} \to x$ and $A\overline{\delta}B$, then according to Proposition 1.1.1.3 there exist C and D such that $C \cap D = \emptyset, A\overline{\delta}X - C$ and $B\overline{\delta}X - D$. For the element $x \in X$ at least one of the relations $x \notin C$, $x \notin D$ holds, let us say $x \notin C$. Then by Proposition 1.1.1.2 (b) $\{x\}\overline{\delta}A$, so that $X - A \in \mathcal{F}(\{x\})$. Therefore, by Theorem 1.1.2.3, X - A is a τ_{δ} -neighborhood of x. Since $\mathcal{F} \to x$, there exists an $F \in \mathcal{F}$ for which $F \subset X - A$, i.e. $A \cap F = \emptyset$. Similarly, if $x \notin D$, then there is an $F \in \mathcal{F}$ such that $F \subset X - B$. **Proposition 1.2.11.2** Let (X, δ_X) and (Y, δ_Y) be proximity spaces, $f : X \to Y$ a δ -continuous mapping. If \mathcal{F} is a compressed filter base in X, then $f(\mathcal{F})$ is compressed in Y.

Proof: If for $A, B \subset Y A\overline{\delta}_Y B$ holds, then $f^{-1}(A)\overline{\delta}_X f^{-1}(B)$ holds by Proposition 1.1.6.1. Since \mathcal{F} is a compressed filter base in X, there exists an $F \in \mathcal{F}$ such that $F \cap f^{-1}(A) = \emptyset$. But then $f(F) \cap A = \emptyset$.

Corollary 1.2.11.1 If δ_1 and δ_2 are the proximities on X for which $\delta_1 < \delta_2$, and the filter base \mathcal{F} is δ_2 -compressed, then it is δ_1 -compressed as well.

Proposition 1.2.11.3 Let δ_i , $i \in I \neq \emptyset$, be a proximities on X and $\delta = \sup{\delta_i : i \in I}$. If \mathcal{F} is δ_i -compressed for every $i \in I$, then it is δ -compressed as well.

Proof: If $A\overline{\delta}B$, then for any two decompositions $\{A_j : j \in J_p\}$ and $\{B_k : k \in J_q\}$ of the sets A and B respectively, there exists an index $i(j,k) \in I$ such that for every $j \in J_p$ and $k \in J_q A_j \overline{\delta}_{i(j,k)} B_k$ holds. Let $F_{jk} \in \mathcal{F}$ be a set which intersects at most one of the sets A_j and B_k , and $F \in \mathcal{F}$ such that $F \subset \bigcap_{j=1}^p \bigcap_{k=1}^q F_{jk}$. If $A \cap F \neq \emptyset$, then, for an index $j, A_j \cap F \neq \emptyset$, and then for this j and all k there follows $A_j \cap F_{jk} \neq \emptyset$. But then $B_k \cap F_{jk} = \emptyset$ and $B_k \cap F = \emptyset$. Hence $B \cap F = \emptyset$.

Proposition 1.2.11.4 Let $f : X \to Y$, δ be a proximity on Y. A filter base \mathcal{F} in X is $f^{-1}(\delta)$ -compressed if and only if $f(\mathcal{F})$ is δ -compressed.

Proof: Let $f(\mathcal{F})$ be a δ -compressed filter base. If $A\overline{f^{-1}(\delta)}B$, then by $f(A)\overline{\delta}f(B)$ there exists an $F \in \mathcal{F}$ such that e.g. $f(F) \cap f(A) = \emptyset$. In this case $F \cap A = \emptyset$. On the other hand, if \mathcal{F} is a $f^{-1}(\delta)$ -compressed and $C\overline{\delta}D$, then by $f^{-1}(C)\overline{f^{-1}(\delta)}f^{-1}(D)$, there exists an $F \in \mathcal{F}$ such that e.g. $F \cap f^{-1}(C) = \emptyset$. But then $f(F) \cap C = \emptyset$.

Corollary 1.2.11.2 Let (X, δ) be a proximity space, $\emptyset \neq Y \subset X$. A filter base \mathcal{F} in Y is $\delta | Y$ -compressed if and only if it is δ -compressed.

Corollary 1.2.11.3 A filter base finer than a compressed filter base is itself compressed. Equivalent filter bases are simultaneously compressed.

The following propositions establish a connection between compressed filter bases and Cauchy filter base. **Proposition 1.2.11.5** If (X, U) is a uniform space and \mathcal{F} a \mathcal{U} -Cauchy filter base, then \mathcal{F} is $\delta_{\mathcal{U}}$ -compressed.

Proof: If $A\overline{\delta}_{\mathcal{U}}B$, then there exists an $U \in \mathcal{U}$ such that $(A \times B) \cap U = \emptyset$. Let $F \in \mathcal{F}$ be a set small of order U. Then $F \times F \subset U$, so that $(A \times B) \cap (F \times F) = (A \cap F) \times (B \cap F) = \emptyset$. But then at least one of the sets $A \cap F$, $B \cap F$ is empty.

Proposition 1.2.11.6 If (X, U) is a totaly bounded uniform space, \mathcal{F} a $\delta_{\mathcal{U}}$ -compressed filter base, then \mathcal{F} is a \mathcal{U} -Cauchy filter base.

Proof: Let \mathcal{F} be a $\delta_{\mathcal{U}}$ -compressed filter base. For a given entourage $U \in \mathcal{U}$, let the entourage $V \in \mathcal{U}$ be chosen in such a way that $V \circ V \circ V \subset U$, and let $X = \bigcup_{i=1}^{n} G_i$, where G_i is small of order V. For all pairs of indices (i, j) for which $G_i \delta G_j$ holds, let $F_{ij} \in \mathcal{F}$ be a set which intersects one of the sets G_i and G_j at the most. Finally, let $F \in \mathcal{F}$ be a set for which $F \subset \cap F_{ij}$. Let us prove that $F \times F \subset U$. If $(x, y) \in F \times F$, then $x \in G_i$ and $y \in G_j$ for some indices i and j. For these indices $G_i \delta_{\mathcal{U}} G_j$ holds, because contrary to this the set F will intersect at least one of the sets G_i and G_j . Therefore $(G_i \times G_j) \cap V \neq \emptyset$, so there exist $u \in G_i$ and $v \in G_j$ such that $(u, v) \in V$. Since $(x, u) \in G_i \times G_i \subset V, (v, y) \in G_j \times G_j \subset V$, then $(x, y) \in V \circ V \circ V \subset U$. Hence F is small of order U, which was to be proved.

Definition 1.2.11.2 Let X be a non-empty set. A maximal filter \mathcal{A} in X, i.e. a filter \mathcal{A} in X having the property that, if \mathcal{B} is a filter in X and $\mathcal{A} \subset \mathcal{B}$ then $\mathcal{A} = \mathcal{B}$, is said to be an **ultrafilter** in X.

Proposition 1.2.11.7 If $x \in X$, then the fundamental filter $\dot{x} = \{S : x \in S \subset X\}$ is an ultrafilter in X.

Proof: Let $\dot{x} \subset \mathcal{A}$, where \mathcal{A} is a filter in X and let $A \in \mathcal{A}$. Since $\{x\} \in \dot{x} \subset \mathcal{A}$, then $\{x\} \in \mathcal{A}$. Therefore $\{x\} \cap A \in \mathcal{A}$. But then $\{x\} \cap A \neq \emptyset$, so that $x \in A$, which proves that $A \in \dot{x}$. Thus $\mathcal{A} \subset \dot{x}$, so that $\dot{x} = \mathcal{A}$.

Fundamental filters in X are called **trivial ultrafilters** in X, whereas other ultrafilters are **non-trivial ultrafilters**.

Proposition 1.2.11.8 If A is an ultrafilter in X and $A \subset X$, then either $A \in A$ or $X - A \in A$.

Proof: If $X - A \notin \mathcal{A}$, then $U \cap A \neq \emptyset$ for every $U \in \mathcal{A}$. But then $\mathcal{A} \cap \{A\}$ is a filter base in X. If \mathcal{B} is a filter generated by this filter base, then $\mathcal{A} \cap \{A\} \subset \mathcal{B}$, so that $\mathcal{A} \subset \mathcal{B}$. Since \mathcal{A} is ultrafilter, then $\mathcal{A} = \mathcal{B}$, so that $A = X \cap A \in \mathcal{B}$. Therefore $A \in \mathcal{A}$.

Proposition 1.2.11.9 Let \mathcal{A} be a filter in X. If for every $A \subset X$, either $A \in \mathcal{A}$ or $X - A \in \mathcal{A}$ holds, then \mathcal{A} is an ultrafilter in X.

Proof: Let \mathcal{B} be any filter in X for which $\mathcal{A} \subset \mathcal{B}$ holds and let $B \in \mathcal{B}$. Since $B \cap (X - B) = \emptyset$, then $X - B \notin \mathcal{B}$, so that $X - B \notin \mathcal{A}$. But then by the previous proposition $B \in \mathcal{A}$ follows. Thus we proved that $\mathcal{B} \subset \mathcal{A}$, so that $\mathcal{A} = \mathcal{B}$.

Proposition 1.2.11.10 If \mathcal{A} is an ultrafilter in X, $A \in \mathcal{A}$ and $A = \bigcup_{i=1}^{n} A_i$, then there is some indices i such that $A_i \in \mathcal{A}$.

Proof: Otherwise it would be the case that $X - A_i \in \mathcal{A}$ for every *i*, and so $X - A = \bigcap_{i=1}^{n} (X - A_i) \in \mathcal{A}$ which contradicts the fact that $A \in \mathcal{A}$.

Proposition 1.2.11.11 If \mathcal{A} is an ultrafilter in X, \mathcal{C} a centered system in X and $\mathcal{A} \subset \mathcal{C}$, then $\mathcal{A} = \mathcal{C}$.

Proof: Any centered system C is a subbase of some filter \mathcal{B} in X, and $C \subset \mathcal{B}$. But then $\mathcal{A} \subset \mathcal{B}$, and since \mathcal{A} is an ultrafilter, then $\mathcal{A} = \mathcal{B}$. Therefore $\mathcal{A} = C$.

Proposition 1.2.11.12 Every centered system C in X can be included in an ultrafilter in X.

Proof: It suffices to show that every filter in X can be included in an ultrafilter in X. However, this follows from the Kuratowski-Zorn lemma, because the system of all filters in X is inductive. In fact, if $\{\mathcal{A}_i : i \in I\}$ is ordered with respect to the inclusion and every \mathcal{A}_i is a filter in X, then $\mathcal{A} = \bigcup_{i \in I} \mathcal{A}_i$ is a filter in X, since $A \in \mathcal{A}$ and $A \subset A' \subset X$ imply $A \in \mathcal{A}_i$ for some $i \in I$ and then $A' \in \mathcal{A}_i \subset \mathcal{A}$. Further, if $A_1, A_2 \in \mathcal{A}$, then $A_1 \in \mathcal{A}_i$, $A_2 \in \mathcal{A}_j$ for suitable $i, j \in I$ and e.g. $\mathcal{A}_i \subset \mathcal{A}_j$ implies $A_1, A_2 \in \mathcal{A}_j$, thus $A_1 \cap A_2 \in \mathcal{A}_j \subset \mathcal{A}$.

The application of ultrafilters to the theory of the proximity spaces is based on the following proposition:

Proposition 1.2.11.13 A filter in X is an ultrafilter if and only if it is compressed with respect to the discrete proximity on X.

Proof: If a filter \mathcal{A} is an ultrafilter and $A \cap B = \emptyset$, then according to Proposition 1.2.11.8 either $A \in \mathcal{A}$ or $X - A \in \mathcal{A}$, thus there exists in \mathcal{A} a set either not intersecting B or not intersecting A.

To prove the converse, suppose that the filter \mathcal{A} is compressed with respect to the discrete proximity of X. Since A and X - A far from each other, there exists in \mathcal{A} either a set which is a subset of A or a set which is a subset of X - A and then either $A \in \mathcal{A}$ or $X - A \in \mathcal{A}$. But then, on account of Proposition 1.2.11.8, filter \mathcal{A} is an ultrafilter.

Corollary 1.2.11.4 If (X, δ) is a proximity space and A an ultrafilter in X, then A is δ -compressed.

Proof: Immediately follows from the previously proposition and Corollary 1.2.11.1. ♣

Theorem 1.2.11.1 Let (X, U) be a uniform space. Then the following statements are equivalent:

- (a) \mathcal{U} is totally bounded;
- (b) the $\delta_{\mathcal{U}}$ -compressed filter bases coincide with the \mathcal{U} -Cauchy filter bases;
- (c) every $\delta_{\mathcal{U}}$ -compressed filter base is a \mathcal{U} -Cauchy filter base;
- (d) every ultrafilter in X is \mathcal{U} -Cauchy.

Proof: $(a) \Rightarrow (b)$: follows from Propositions 1.2.11.5 and 1.2.11.6.

 $(b) \Rightarrow (c)$: is evident.

 $(c) \Rightarrow (d)$: results from the previously corollary.

 $(d) \Rightarrow (a)$: Suppose that (d) is fulfilled, but \mathcal{U} is not totally bounded. Then there exists an entourage $U \in \mathcal{U}$ such that X cannot be decomposed into the union of a finite number of sets small of order U. Let us consider now the sets of the form $X - \bigcup_{1}^{n} A_{i}$, where $n \in \mathbb{N}$, and A_{i} is small of order U for every i. By hypothesis, these sets are non-empty and the intersection of two sets of this type has the same form, so that these sets constitute a filter base \mathcal{F} . On account of Proposition 1.2.11.12, \mathcal{F} can be included in an ultrafilter \mathcal{A} . This is \mathcal{U} -Cauchy and therefore there is in it a set $A \in \mathcal{A}$ small of order U. However, in this case, $X - A \in \mathcal{F} \subset \mathcal{A}$, which is impossible from Proposition 1.2.11.8.

A similar notion to the one of complete uniform spaces can be defined in the case of proximity spaces.

Definition 1.2.11.3 A proximity space (X, δ) (or proximity δ) is said to be **compact** if every compressed filter base is convergent. The uniform space (X, \mathcal{U}) and the uniformity \mathcal{U} are said to be **compact** if the proximity $\delta_{\mathcal{U}}$ is compact.

As the meaning of convergence is the same in the uniform space (X, \mathcal{U}) and the proximity space $(X, \delta_{\mathcal{U}})$ (viz. the convergence with respect to the topology $\tau_{\mathcal{U}} = \tau_{\delta_{\mathcal{U}}}$) by Proposition 1.2.11.5, it can be asserted that:

Proposition 1.2.11.14 Every compact uniform space is complete.

On the other hand, it follows from Theorem 1.1.3.4 that:

Proposition 1.2.11.15 A totally bounded uniform space is compact if and only if it is complete. \clubsuit

Therefore, instead of the term "totally bounded uniform space" and "totally bounded uniformity", the term a **precompact uniform space** is often used. In the following, we shall use this shorter expression.

The following important statement can be immediately obtained from Theorem 1.1.3.4:

Proposition 1.2.11.16 Every compact uniform space is precompact.

Proof: If (X, \mathcal{U}) is compact and \mathcal{F} is a $\delta_{\mathcal{U}}$ -compressed filter base, then \mathcal{F} is convergent, thus it is a \mathcal{U} -Cauchy filter base by Proposition 1.2.9.1. By Theorem 1.1.3.4, (X, \mathcal{U}) is a precompact uniform space.

As an immediate corollary of the last tree propositions there follows:

Corollary 1.2.11.5 A uniform space is compact if and only if it is precompact and complete.

Proposition 1.2.11.17 A compact proximity can be induced by a unique uniformity.

Proof: By the previous corollary, a compact proximity δ can be induced only by a precompact uniformity and exactly one of these types can be found by Corollary 1.2.10.8.

Let (x_n) be an arbitrary sequence in the topological space (X, τ) and let

(1)
$$R_k = \{x_n : n \ge k\}, \ n \in \mathbb{N},$$

and

(2)
$$\mathcal{F} = \{R_k : k \in \mathbb{N}\}.$$

Definition 1.2.11.4 The filter base assigned to the sequence (x_n) by formulae (1) and (2) are called the sequential filter base belonging to (x_n) .

Definition 1.2.11.5 Let (X, τ) be a topological space. A point $x \in X$ is said to be a cluster point of the filter base \mathcal{F} in X, if every neighborhood of x intersects every set of the filter base. We understand by the cluster point of a sequence of points (x_n) a cluster point of the sequential filter base belonging to it.

Proposition 1.2.11.18 Let (X, τ) be a topological space, $x \in X$, $\mathcal{N}(x)$ the neighborhood filter, $\mathcal{B}(x)$ a neighborhood base of the point x, and \mathcal{F} a filter base in X. Then the following statements are equivalent:

- (a) x is a cluster point of \mathcal{F} ;
- (b) $\emptyset \notin \mathcal{N}(x) \cap \mathcal{F};$
- $(c) \ \emptyset \not\in \mathcal{B}(x) \cap \mathcal{F};$
- (d) there exists a filter base finer than \mathcal{F} converging to x;
- $(e) \ x \in \cap \{\overline{F} : F \in \mathcal{F}\}.$

Proof: $(a) \Rightarrow (b) \Rightarrow (c)$: is evident.

 $(c) \Rightarrow (d)$: It is known from general topology that the filter base $\mathcal{F}' = \mathcal{B}(x) \cap \mathcal{F}$ is finer than \mathcal{F} and $\mathcal{F}' > \mathcal{B}(x)$. Since $\mathcal{N}(x)$ and $\mathcal{B}(x)$ are equivalent families, then $\mathcal{F}' \to x$.

 $(d) \Rightarrow (e)$: Let $\mathcal{F}' > \mathcal{F}$ and $\mathcal{F}' \to x$. If $F \in \mathcal{F}$ and $U \in \mathcal{N}(x)$, then there exists an $F'_1 \in \mathcal{F}'$ such that $F'_1 \subset U$, and an $F'_2 \in \mathcal{F}'$ such that $F'_2 \subset F$. Finally, there exists an $F'_3 \in \mathcal{F}'$ such that $F'_3 \subset F'_1 \cap F'_2 \subset U \cap F$. Thus $x \in \overline{F}$ for every $F \in \mathcal{F}$.

 $(e) \Rightarrow (a)$: is obvious.

Corollary 1.2.11.6 If $\mathcal{F} \to x$, then x is a cluster point of the filter base \mathcal{F} .

Corollary 1.2.11.7 If x is a cluster point of the filter base \mathcal{F} and $\mathcal{F}_1 < \mathcal{F}$, then x is a cluster point of the filter base \mathcal{F}_1 .

Proposition 1.2.11.19 Let (X, τ) be a topological space, $\emptyset \neq Y \subset X$, \mathcal{F} a filter base in Y and $x \in Y$. The point x is a $\tau | Y$ -cluster point of \mathcal{F} if and only if it is a τ -cluster point.

Proof: Under our conditions $\mathcal{N}(x) \cap \mathcal{F} = \mathcal{N}(x) \cap \{Y\} \bigcap \mathcal{F}$.

Proposition 1.2.11.20 If \mathcal{F} is a compressed filter base in a proximity space (X, δ) (in particular, if \mathcal{F} is a Cauchy filter base in a uniform space (X, \mathcal{U})) and x is a cluster point of \mathcal{F} , then $\mathcal{F} \to x$.

Proof: Let U be a neighborhood of x, i.e. $\{x\}\overline{\delta}X - U$. Then by Proposition 1.1.1.3 there exist C and D such that $C \cap D = \emptyset$, $\{x\}\overline{\delta}X - C$, $X - U\overline{\delta}X - D$. Hence C is a neighborhood of x and therefore intersects every set of the filter base \mathcal{F} . Since $C \subset X - D\overline{\delta}X - U$ and filter base \mathcal{F} is compressed, there exists $F \in \mathcal{F}$ which intersects one of the sets C and X - U at the most. Since C is the neighborhood of x, then F intersects C, so that $F \subset U$ which was to be proved.

The following statement is similar to the preceding one:

Proposition 1.2.11.21 Let (X, τ) be a topological space, \mathcal{A} an ultrafilter in X. If x is a cluster point of \mathcal{A} , then $\mathcal{A} \to x$.

Proof: Let U be any neighborhood of x. Then, by Proposition 1.2.11.8, either $U \in \mathcal{A}$ or $X - U \in \mathcal{A}$ holds. Since $X - U \in \mathcal{A}$ is in contradiction with the fact that x is a cluster point of \mathcal{A} , then $U \in \mathcal{A}$.

Now the following theorem can be proved:

Theorem 1.2.11.2 A proximity space is compact if and only if every filter base admits a cluster point.

Proof: Let (X, δ) be a compact space. If \mathcal{F} is a filter base in X, then by Proposition 1.2.11.12 there exists an ultrafilter \mathcal{A} in X containing \mathcal{F} (thus finer than \mathcal{F}). On account of Corollary 1.2.11.4, \mathcal{A} is compressed, hence convergent. If $\mathcal{A} \to x$, then by Proposition 1.2.11.18 x is a cluster point of \mathcal{A} .

Conversely, if any filter base in X has a cluster point, and \mathcal{F} is a compressed filter base, then \mathcal{F} converges to any of its cluster points according to Proposition 1.2.11.20.

This theorem shows that the compactness of a proximity space depends only on the topology of the space as the existence of a cluster point of a filter base is determined by the neighborhood filters of the points. Moreover, in connection with Theorem 1.2.11.2, there is the possibility of defining the compactness of topological spaces in the manner that is in accordance with the compactness of proximity spaces defined earlier: let us call the topological space (X, τ) and the topology τ **compact** if every filter base in X has a cluster point. Using this terminology, we can formulate Theorem 1.2.11.2 as follows: **Theorem 1.2.11.3** The proximity space (X, δ) is compact if and only if the topology τ_{δ} is compact.

In Theorem 1.1.3.4 we have proved that in every compact T_2 -space there exists a unique proximity compatible with the given topology. According to Proposition 1.2.11.17 following statement holds:

Theorem 1.2.11.4 For every compact T_2 -topology there exists a unique uniformity which generates the given topology.

In locally compact spaces there exists a proximity compatible with the given topology, but it is not unique defined. The proximity described in Definition 1.1.3.4 is only one among the topologies compatible with locally compact topology.

It can be seen that every locally compact S_2 -space is regular. Moreover, these spaces are completely regular. This will be shown by giving a proximity inducing the topology of the space:

Theorem 1.2.11.5 Let (X, τ) be a locally compact S_2 -space and let $A\delta B$ if and only if $\overline{A} \cap \overline{B} = \emptyset$ and at least one of the sets \overline{A} and \overline{B} is compact. Then δ is a proximity inducing the topology τ ; more precisely, it is the coarsest of the proximities inducing the topology τ .

Proof: (*B*₁) obviously holds. Let $A\overline{\delta}C$ and $B\overline{\delta}C$. Then $\overline{A} \cap \overline{C} = \overline{B} \cap \overline{C} = \emptyset$, so that from $\overline{A \cup B} = \overline{A} \cup \overline{B}$ it follows $\overline{A \cup B} \cap \overline{C} = \emptyset$. If \overline{C} is compact, then $A \cup B\overline{\delta}C$ evidently holds while, if \overline{C} is not compact, then \overline{A} and \overline{B} are compact, so that their union is also compact, i.e. $\overline{A \cup B}$ is compact, hence $A \cup B\overline{\delta}C$. Conversely, if $(A \cup B)\overline{\delta}C$, then $\overline{A \cup B} \cap \overline{C} = \emptyset$, so that $(\overline{A} \cap \overline{C}) \cup (\overline{B} \cap \overline{C}) = \emptyset$. Therefore $(\overline{A} \cap \overline{C}) = (\overline{B} \cap \overline{C}) = \emptyset$, and if \overline{C} is compact, then $A\overline{\delta}C$ and $B\overline{\delta}C$. If $\overline{A\cup B} = \overline{A}\cup \overline{B}$ is a compact set, then \overline{A} and \overline{B} are compact sets, so that $A\overline{\delta}C$ and $B\overline{\delta}C$. Thus we prove (B_2) . (B_3) holds on account to the fact that $\overline{\emptyset} = \emptyset$ is a compact set, while (B_4) is obviously fulfilled. To prove (B_5) , let us suppose that $\overline{A} \cap \overline{B} = \emptyset$, and, say, let A be compact. Then for every point $x \in A$ there exists a closed compact neighborhood K_x of x for which $K_x \cap \overline{B} = \emptyset$. The compact set \overline{A} is covered by finite number of sets $\operatorname{int} K_{x_i}$, $i = 1, 2, \ldots, n$: $\overline{A} \subset \bigcup_{1}^{n} \operatorname{int} K_{x_i}$, $x_i \in \overline{A}$. Let $K = \bigcup_{i=1}^{n} K_{x_i}$. Then K is compact, closed, $K \cap \overline{B} = \emptyset$, and $\overline{A} \subset \operatorname{int} K$. Let $P = \operatorname{int} K, Q = X - K.$ Then $P \cap Q = \emptyset, \overline{A} \cap \overline{X - P} = \emptyset$ and $\overline{B} \cap \overline{X - Q} = \emptyset.$ Finally, \overline{A} and $\overline{X-Q} = \overline{K} = K$ are compact. Thus (B_5) is fulfilled.

Let us prove that the proximity δ is compatible with the topology τ . Let U be a δ -neighborhood of $x \in X$. Then $\overline{x} \cap \overline{X - U} = \emptyset$, thus $x \notin$ $\overline{X-U}$ and $\overline{X-V} \subset U$ is a neighborhood of x. Conversely, if V is a neighborhood of x, then, by $x \in \operatorname{int} V$ there follows that $x \notin \overline{X-\operatorname{int} V}$, so that $\overline{x} \cap \overline{X-\operatorname{int} V} = \emptyset$ and \overline{x} is compact. Therefore $\{x\}\overline{\delta}X - \operatorname{int} V$, and a fortiory $\{x\}\overline{\delta}X - V$, V is a δ -neighborhood of x. Hence δ generates the topology τ .

Let δ_1 be any proximity on X compatible with the topology τ . If $A\overline{\delta}B$, then $\overline{A} \cap \overline{B} = \emptyset$, where at least one of the sets \overline{A} and \overline{B} is compact. By Lemma 1.1.3.1 we have $\overline{A}\overline{\delta}_1\overline{B}$, so that by Proposition 1.1.2.4 $A\overline{\delta}_1B$ holds. This proves that $\delta < \delta_1$.

Historical and bibliographic notes

The concept of a uniform space was introduced in 1936 implicitly by Dj. Kurepa in papers [179] and [180] (see also papers [177], [178] and [182]) and in 1938 by A. Weil in paper [334] explicitly. The first systematic exposition of the theory of uniform spaces was given by Burbaki, N. in 1940 (see [38]). A different but equivalent concept of a uniform space, defined in terms of a collection of covers, was introduced and studied by J. W. Tukey in [323]. J. R. Isbell's book [150], which contains an important development of the theory of uniform spaces, is written in terms of covers.

The uniform spaces can be also described in terms of pseudo-metrics. Such a description was given by N. Bourbaki in [38]. The "pseudo-metric" language is used in L. Gillman and M. Jerison's book [123].

Subspaces and Cartesian products of uniform spaces were defined by Weil in [334]. The notion of a totally bounded uniform space was introduced by N. Bourbaki in [38].

Interesting generalizations of a total boundedness in uniform spaces have been introduced and studied by Lj. Kochinac in [167] (see also [168]). For example, it was shown in [167] that Corollary 1.2.6.1 and Proposition 1.2.6.5 remain valid if "totally bounded" is replaced by "Hurewicz bounded".

The notion of a complete uniform space was introduced by A. Weil in paper [334].

1.3 Extensions of spaces and mappings

1.3.1 Extensions of topological spaces

It is well known that, through the omission of some points of a complete metric space, the space can lose its completeness. As the converse of this phenomenon, the question arises quite naturally whether a non-complete metric space, or more generally a non-complete uniform space, can be extended by adding points to it so as to make it complete. It is an analogous question whether a proximity space or a topological space which is not compact, because some filter bases in it have no cluster points, can be extended so as to become compact by adding further points to it. In this extension first the cluster points of the filter bases possessing no cluster points have to be procured, taking care at the same time that all the filter bases which can be constructed in the extended space have cluster points as well.

The task in all of these questions is to construct to the given space an extended space containing the original space as a subspace and fulfilling further prescribed conditions (e.g. to be complete or compact). In view of the last mentioned problems, we first look for an extended space containing the given space as a dense subspace; in fact, if e.g. a topological space can be included in an extended compact space, then the closure of the given space in the extended space is compact as well and the given space is dense in it.

With respect to these considerations, restricting ourselves for the moment to topological spaces, and let us introduce the following:

Definition 1.3.1.1 A topological space (X', τ') is said to be an extension of a topological space (X, τ) , if $X \subset X'$, $\tau'|X = \tau$ and X is τ' -dense in X'.

In this case it is also said that the topology τ' is an extension of the topology τ . First we shall be dealing with the question how such extensions can be constructed for a given topological space.

Thus let (X, τ) be a topological space and let $X \subset X'$. If (X', τ') is an extension of (X, τ) , then X is a dense subset in X', so that every τ' neighborhood of each point $x \in X'$ has a non-empty intersection with X. Hence a filter $\mathcal{F}(x) = \mathcal{N}'(x) \cap \{X\}$ can be constructed in X from the τ' neighborhood filter $\mathcal{N}'(x)$ of the point x. It will be called the **trace** in X of the neighborhood filter $\mathcal{N}'(x)$. If $x \in X$, the trace filter $\mathcal{F}(x)$ is identical with the τ -neighborhood filter $\mathcal{N}(x)$ of the point x.

It is an obvious idea to define the topology τ' on the set X' by joining to every point $x \in X'$ the trace filter $\mathcal{F}(x)$ belonging to it. In connection with these two questions it arises immediately: whether the trace filter $\mathcal{F}(x)$ can be arbitrarily chosen or must it fulfil some restrictions and whether the topology τ' is uniquely defined by the trace filters.

The first question can be answered immediately: for the points $x \in X$ the trace filter is given; it is identical with the neighborhood filter $\mathcal{N}(x)$, but the trace filters belonging to the points $x \in X' - X$ cannot be arbitrary, because, in a topological space, the neighborhood filter of a point possesses a base consisting of open sets and, since the intersection of a τ' -open set with X is τ -open, the same can be said of the trace filter \mathcal{F} as well. Therefore if say **open filter** for a filter in a topological space which possesses a base consisting of open sets, the foregoing can be summarized in the following:

Proposition 1.3.1.1 Let (X', τ') be an extension of a topological space (X, τ) . If $x \in X'$, let $\mathcal{N}'(x)$ be the τ' -neighborhood filter of the point x, and $\mathcal{F}(x) = \mathcal{N}'(x) \cap \{X\}$ the corresponding trace filter. In this case $\mathcal{F}(x)$ is a τ -open filter in X, in particular, if $x \in X$, $\mathcal{F}(x) = \mathcal{N}(x)$ is the τ -neighborhood filter of the point x.

In general, a negative answer is to be given to the second question as well.

Example 1.3.1.1 Let $X = \mathbb{Q}$, $X' = \mathbb{R}$, and let us consider on the one hand the Euclidean topology \mathcal{E} on \mathbb{R} , on the other hand the topology $\tau' \neq \mathcal{E}$ for which the \mathcal{E} -open sets and \mathbb{Q} itself constitute a subbase. It is clear that $\mathcal{E}|\mathbb{Q} = \tau'|\mathbb{Q}$, so that, denoting this topology by τ , both $(\mathbb{R}, \mathcal{E})$ and (\mathbb{R}, τ') are extensions of (\mathbb{Q}, τ) . The fact that \mathbb{Q} is not only \mathcal{E} -dense but τ' -dense as well follows from the fact that any τ' -neighborhood of a point $x \in \mathbb{R} - \mathbb{Q}$ is at the same time an \mathcal{E} -neighborhood of x and, since $\mathcal{E} < \tau'$, the \mathcal{E} -neighborhoods of x are at the same time τ' -neighborhoods too so that the \mathcal{E} -trace filters are identical to the τ' -trace filters.

However it will be shown that if the trace filters, with the restrictions given in Proposition 1.3.1.1, are arbitrarily given, then there exists always an extension furnishing the given trace filters, moreover, there is a coarsest one among them.

Theorem 1.3.1.1 Let (X, τ) be a topological space and $X \subset X'$. Let us also assign to every point $x \in X'$ a τ -open filter $\mathcal{F}(x)$ in X, in particular, if $x \in X$, then let $\mathcal{F}(x) = \mathcal{N}(x)$ be the τ -neighborhood filter of the point x. For $A \subset X$, let

(1)
$$s(A) = \{x : x \in X', A \in \mathcal{F}(x)\}.$$

Then

(2)
$$\mathcal{S} = \{s(G) : G \subset X \text{ is } \tau - open\}$$

constitutes a base for the topology τ' on X'. τ' is the coarsest topology on X' such that, for every point $x \in X'$, the given filter $\mathcal{F}(x)$ is the trace filter. Finally, $\tau'|X = \tau$ is true.

Proof: For every filter \mathcal{F} in X and sets $A, B \subset X, A \subset B, A \in \mathcal{F}$ imply $B \in \mathcal{F}$ and $A \cap B \in \mathcal{F}$ is true if and only if $A \in \mathcal{F}$ and $B \in \mathcal{F}$. Therefore

$$(3) \qquad A \subset B \Rightarrow s(A) \subset s(B) \,, \quad s(A \cap B) = s(A) \cap s(B) \,, \quad A, B \subset X \,,$$

and of course s(X) = X'. It can be easily seen that S is a base for a topology τ' on X'. The τ' -neighborhood filter $\mathcal{N}'(x)$ of the point $x \in X'$ is generated by the system of sets

$$\{s(G): G \text{ is } \tau - \text{ open}, x \in s(G)\}$$

and since $x \in s(A)$ is equivalent to $A \in \mathcal{F}(x)$ by (1), this can be also written in the form of

$$\{s(G): G \text{ is } \tau - \text{ open}, \ G \in \mathcal{F}(x)\}$$

The trace filter $\mathcal{N}'(x) \cap \{X\}$ will be generated by the system

(4)
$$\{s(G) \cap X : G \text{ is } \tau - \text{ open}, \ G \in \mathcal{F}(x)\}.$$

If $x \in X$, then $\mathcal{F}(x) = \mathcal{N}(x)$, so that $G \in \mathcal{F}(x)$ holds for a τ -open set G if and only if $x \in G$, i.e.

(5)
$$s(G) \cap X = G$$
, G is τ -open.

Therefore the system (4) is nothing other than the system of τ -open sets in $\mathcal{F}(x)$, which generates $\mathcal{F}(x)$, since $\mathcal{F}(x)$ is a τ -open filter. Therefore

$$\mathcal{N}'(x) \cap \{X\} = \mathcal{F}(x) \,,$$

in particular, if $x \in X$, then $\mathcal{N}'(x) \cap \{X\} = \mathcal{N}(x)$. This shows that τ' is indeed an extension of τ furnishing the given trace filters $\mathcal{F}(x)$.

Now let τ'_1 be another topology on X', $\mathcal{N}'_1(x)$ the τ'_1 -neighborhood filter of the point $x \in X'$ and let us suppose that, for each point $x \in X'$,

$$\mathcal{N}_1'(x) \cap \{X\} = \mathcal{F}(x) \,.$$

Let $x \in s(G)$ where G is a τ -open set. Then $G \in \mathcal{F}(x)$, x has a τ'_1 -neighborhood U'_1 such that $G = U'_1 \cap X$ and then there exists a τ'_1 -open set G'_1 such that $x \in G'_1 \subset U'_1$. If $y \in G'_1$, then $G'_1 \in \mathcal{N}'_1(y)$, so that

 $G'_1 \cap X \in \mathcal{F}(y)$. Now $G'_1 \cap X \subset U'_1 \cap X = G$ implies $G \in \mathcal{F}(y)$, and $y \in s(G)$. Therefore if G is τ -open, then s(G) contains a τ'_1 -neighborhood G'_1 of any point $x \in s(G)$, so that s(G) is τ'_1 -open. This shows that every τ' -open set is τ'_1 -open as well, $\tau' < \tau'_1$.

The extensions arising in the way described in the previous theorem are called **strict extensions**. More precisely:

Definition 1.3.1.2 A topological space (X', τ') is called a strict extension of a topological space (X, τ) (or the topology τ' is a strict extension of the topology τ) if $X \subset X'$, $\tau = \tau'|X, X$ is τ' -dense, and if, denoting by $\mathcal{F}(x)$ for $x \in X'$ the trace filter $\mathcal{N}'(x) \cap \{X\}$ of the τ' -neighborhood filter $\mathcal{N}'(x)$, and for $A \subset X$ by s(A), the set in (1), the system of sets S in (2) is a base for τ' .

Corollary 1.3.1.1 Let (X, τ) be a topological space, $X \subset X'$, and let us join a τ -open filter $\mathcal{F}(x)$ in X to every point $x \in X'$ and let us suppose that $\mathcal{F}(x)$ is the τ -neighborhood filter of x for $x \in X$. Then there exists a unique topology τ' on X' which is a strict extension of τ and furnishes the given filters $\mathcal{F}(x)$ as trace filters; this is the coarsest of all topologies on X' leading to the given trace filters.

In order to give a further characterization of strict extensions, let us pay attention to the following:

Proposition 1.3.1.2 Let (X', τ') be a topological space, $X \subset X' \tau'$ -dense, $\tau = \tau' | X, \mathcal{N}'(x)$ be τ' -neighborhood filter of $x \in X', \mathcal{F}(x) = \mathcal{N}'(x) \cap \{X\}$, and

$$s(A) = \{x : x \in X', A \in \mathcal{F}(x)\}$$

for $A \subset X$. Then:

(a) if $B \subset X$, then τ' -closure of the set B is X' - s(X - B);

(b) if $G \subset X$ is τ -open, then s(G) is the largest τ' -open set whose intersection with X is G.

Proof: (a) $x \in X'$ does not belong to the τ' -closure of B if and only if it is not a τ' -limit point of B, i.e. if and only if there exists in $\mathcal{N}'(x)$ a set not intersecting B which holds if and only if there exists in $\mathcal{F}(x)$ a set not intersecting B, i.e. if and only if $X - B \in \mathcal{F}(x)$, $x \in s(X - B)$.

(b) According to the foregoing, X' - s(G) is the τ' -closure of the set X - G, i.e. the smallest τ' -closed set whose intersection with E is X - G. Passing to the complements, we obtain the assertion.

Let us now introduce the following notion. A system \Im of the subsets of a topological space (X, τ) is a **closed base** if the system $\{X - F : F \in \Im\}$ is a base for the topology τ , i.e. if the sets $F \in \Im$ are τ -closed and every τ -closed set distinct from X is an intersection of sets belonging to \Im .

Proposition 1.3.1.3 Let a topological space (X', τ') be an extension of (X, τ) . The topology τ' is a strict extension of τ if and only if the τ' -closures of the $(\tau$ -closed) sets $B \subset X$ constitute a closed base for the topology τ' .

Proof: If τ' is a strict extension of τ , then the sets of the form s(G), where G is τ -open, constitute a base for τ' . By the previous proposition X' - s(G) is identical with the τ' -closure of X - G so that the τ' -closures of τ -closed sets constitute a closed base for τ' .

Conversely, if the τ' -closures of sets in X constitute a closed base for τ' , then the same is true even for the τ' -closures of the τ -closed sets as the τ' -closure of a set $B \subset X$ is identical with the τ' -closure of the τ -closure of B. Therefore, in this case, by Proposition 1.3.1.2, the sets of the form s(G), where G is τ -open, constitute a τ' -base, so that τ' is indeed a strict extension of τ .

Proposition 1.3.1.4 If, with the hypotheses and notations of Proposition 1.3.1.2, the topology τ' is regular, then it is a strict extension of τ .

Proof: For an arbitrary point $x \in X'$ and its τ' -neighborhood U' we can find a τ' -open set G' and a τ' -closed set F' such that $x \in G' \subset F' \subset U'$. Let $F = (X' - G') \cap X$. Then, by $F \subset X' - G'$, x cannot belong to the τ' -closure of F. On the other hand, if $y \in X' - U'$, then by $y \in X' - F'$, together with every τ' -neighborhood U'_1 of y, $(X' - F') \cap U'_1$ is also a τ' -neighborhood of y which intersects X since X is τ' -dense in X', i.e. U'_1 intersects the set $(X' - F') \cap X \subset (X' - G') \cap X = F$. Hence the complement of the τ' -closure of F is a τ' -neighborhood of x contained in U'. Therefore, by the previous proposition, we have the proof of proposition.

Proposition 1.3.1.5 Let $X \subset X' \subset X''$, and let τ , τ' , τ'' be the topologies on X, X', X'' respectively. If τ' is an extension of τ , and τ'' the one of τ' , then τ'' is an extension of τ .

Proof: If $\tau = \tau' | X, \tau' = \tau'' | X'$, then $\tau = \tau'' | X$. If X is τ' -dense in X', and X' is τ'' -dense in X'', then, for every τ'' -open neighborhood G'' of any point $x \in X'', G'' \cap X' \neq \emptyset$, thus $G'' \cap X'$ being τ' -open, $G'' \cap X' \cap X = G'' \cap X \neq \emptyset$, and X is τ'' -dense in X''.

Proposition 1.3.1.6 Let $X \subset X' \subset X''$, τ'' a topology on X'', $\tau' = \tau''|X'$, $\tau = \tau''|X = \tau'|X$. If τ'' is a (strict) extension of τ , then τ' is a (strict) extension of τ and τ'' the one of τ' .

Proof: If X is τ'' -dense, then its τ' -closure is equal to $X'' \cap X' = X'$ and it is therefore τ' -dense in X', too. Further, the τ'' -closure of X' is X'', thus X' is also τ'' -dense in X''.

If τ'' is a strict extension of τ , then the τ'' -closures of the subsets of X constitute a closed base for τ'' by Proposition 1.3.1.3. Hence the τ'' -closures of the subsets of X' constitute a closed base for τ'' and τ'' is a strict extension of τ' . On the other hand, the τ' -closure of $A \subset X$ is the intersection of X' with the τ'' -closure of A. But then these intersections constitute a closed base for τ' is a strict extension of τ .

Proposition 1.3.1.7 With the hypotheses and notations of Proposition 1.3.1.2, let τ' be a strict extension of τ and $x, y \in X'$. Then

(a) x and y are weakly separated if and only if $\mathcal{F}(x) \neq \mathcal{F}(y)$;

(b) x and y are separated if and only if neither of the filters $\mathcal{F}(x)$ and $\mathcal{F}(y)$ contains the other;

(c) x and y are disconnected if and only if $\emptyset \in \mathcal{F}(x) \cap \mathcal{F}(y)$.

Proof: (a) It can be easily seen that the points x and y are weakly separated if and only if $\mathcal{N}'(x) \neq \mathcal{N}'(y)$. If $\mathcal{F}(x) \neq \mathcal{F}(y)$, then of course $\mathcal{N}'(x) \neq \mathcal{N}'(y)$ holds as well. Conversely, if $\mathcal{N}'(x) \neq \mathcal{N}'(y)$, then e.g. x has a τ' neighborhood which is not a τ' -neighborhood of y and there exists a set of the form s(G), where G is τ -open, such that $x \in s(G)$, $y \notin s(G)$, i.e. $G \in \mathcal{F}(x), G \notin \mathcal{F}(y)$.

(b) If neither of filters $\mathcal{F}(x)$ and $\mathcal{F}(y)$ contains the other, then, as these filters are open, there exist τ -open sets G_1 and G_2 such that $G_1 \in \mathcal{F}(x)$, $G_1 \notin \mathcal{F}(y), G_2 \in \mathcal{F}(y)$ and $G_2 \notin \mathcal{F}(x)$. Then $s(G_1)$ is a τ' -neighborhood of x not containing y, while $s(G_2)$ is a τ' -neighborhood of y not containing x. On the other hand, if x has a τ' -neighborhood not containing y and yhas one not containing x, then these can be taken in the form of $s(G_1)$ and $s(G_2)$, where G_1 and G_2 are τ -open, and $G_1 \in \mathcal{F}(x), G_1 \notin \mathcal{F}(y), G_2 \in \mathcal{F}(y),$ $G_2 \notin \mathcal{F}(x)$.

(c) If x and y have disjoint τ' -neighborhoods U'_1 and U'_2 , then $U'_1 \cap X \in \mathcal{F}(x)$ and $U'_2 \cap X \in \mathcal{F}(y)$ are disjoint sets as well. Conversely, if $\emptyset \in \mathcal{F}(x) \cap \mathcal{F}(y)$, then as they are open, there exist τ -open sets G_1 and G_2 such that $G_1 \in \mathcal{F}(x), G_2 \in \mathcal{F}(y), G_1 \cap G_2 = \emptyset$. $s(G_1)$ and $s(G_2)$ will then be disjoint τ' -neighborhoods of x and y respectively, for which, by the proof of Theorem 1.1.3.4, $s(G_1) \cap s(G_2) = s(G_1 \cap G_2) = s(\emptyset) = \emptyset$ holds.

Definition 1.3.1.3 The extension (X', τ') of the topological space (X, τ) (or the extension τ' of the topology τ) is called a **reduced extension** if $x \in X' - X$, $y \in X'$, $x \neq y$ imply that x and y are weakly separated.

It follows directly from the definition:

Corollary 1.3.1.2 If the extension (X', τ') of the space (X, τ) is a T_0 -space, then it is a reduced extension.

Proposition 1.3.1.8 If (X, τ) is a T_0 -space, and (X', τ') is a reduced extension of (X, τ) , then (X', τ') is a T_0 -space.

Proof: It need only be shown that if $x, y \in X$, $x \neq y$, then x and y are weakly separated with respect to τ' . But in this case x and y are weakly separated with respect to τ , thus there exists a τ -open set G such that $x \in G$ and $y \notin G$. For a suitable τ' -open set G', $G = G' \cap X$ and then $x \in G'$, $y \notin G'$.

Proposition 1.3.1.9 Let (X', τ') be a strict extension of (X, τ) , let $\mathcal{F}(x)$ be a trace in X of the τ' -neighborhood filter of $x \in X'$. (X', τ') is a reduced extension of (X, τ) if and only if $x \in X' - X$, $y \in X'$, $x \neq y$ implies $\mathcal{F}(x) \neq \mathcal{F}(y)$.

Proof: There follows immediately from Proposition 1.3.1.7 (a).

Proposition 1.3.1.10 Let $X \subset X' \subset X''$, τ'' be a topology on X'', $\tau' = \tau''|X'$, $\tau = \tau'|X = \tau''|X$, and let X be τ'' -dense in X''. If τ'' is a reduced extension of τ , then τ' has the same property. Conversely, if τ' is a reduced extension of τ , while τ'' is the one of τ' , then τ'' is a reduced extension of τ as well.

Proof: If τ'' is a reduced extension of τ and $x \in X' - X$, $y \in X'$, $x \neq y$, then there exists a τ'' -open set G'' such that $x \in G''$, $y \notin G''$, so that $G'' \cap X'$ is a τ' -neighborhood of x not containing y.

Conversely, let us suppose now that τ' is a reduced extension of τ , τ'' the one of τ' , and let $x \in X'' - X$, $y \in X''$, $x \neq y$. If $x \in X'' - X'$, then x and y are weakly τ'' -separated. If $x \in X' - X$, $y \in X'' - X'$, the same can again be asserted because of the fact that τ'' is reduced with respect to τ' . Finally, if $x \in X' - X$, $y \in X'$, then there exists a τ' -open set G' such that $x \in G'$, $y \notin G'$ and choosing a τ'' -open set G'' such that $G' = G'' \cap X'$, then $x \in G''$ and $y \notin G''$ will hold.

The content of the following theorem is that strict extensions are essentially defined uniquely by prescribing the trace filters. In order to formulate this more precisely, let us give the following:

Definition 1.3.1.4 Let (X'_1, τ'_1) and (X'_2, τ'_2) be two extensions of the space $(X, \tau), f : X'_1 \to X'_2$ a mapping such that f(x) = x for each $x \in X$. A mapping of this kind will be called a **mapping fixing** X.

Theorem 1.3.1.2 Let (X'_1, τ'_1) and (X'_2, τ'_2) be two strict extensions of a topological space (X, τ) , $\mathcal{F}_1(x) = \mathcal{N}'_1(x) \cap \{X\}$, $\mathcal{F}_2(y) = \mathcal{N}'_2(y) \cap \{X\}$ for $x \in X'_1$, $y \in X'_2$, where $\mathcal{N}'_1(x)$ and $\mathcal{N}'_2(y)$ denote τ'_1 - and τ'_2 -neighborhood filter respectively, and let $f : X'_1 \to X'_2$ be an injection fixing X. The mapping $h = f|_{X'_1}^{f(X'_1)} : (X'_1, \tau'_1) \to (f(X'_1), \tau'_2|f(X'_1))$ is a homeomorphism if and only if y = f(x) implies $\mathcal{F}_1(x) = \mathcal{F}_2(y)$. If τ'_2 is a reduced extension of τ , and $f_1 : X'_1 \to X'_2$ as well as $f_2 : X'_1 \to X'_2$ are mappings fixing X such that $f_1|_{X'_1}^{f_1(X'_1)} : (X'_1, \tau'_1) \to (f_1(X'_1), \tau'_2|f_1(X'_1))$ and $f_2|_{X'_1}^{f_2(X'_2)} : (X'_1, \tau'_1) \to (f_2(X'_1), \tau'_2|f_1(X'_1))$ are homeomorphisms, then $f_1 = f_2$.

Proof: If $h : (X'_1, \tau'_1) \to (f(X'_1), \tau'_2|f(X'_1))$ is a homeomorphism, then $y = f(x) = h(x), x \in X'_1$ implies $f(\mathcal{N}'_1(x)) = \mathcal{N}'_2(y) \cap \{f(X'_1)\}$, thus

$$\mathcal{F}_1(x) = \mathcal{N}'_1(x) \bigcap \{X\} = f(\mathcal{N}'_1(x)) \bigcap \{X\} = \\ = \mathcal{N}'_2(y) \bigcap \{f(X'_1)\} \bigcap \{X\} = \mathcal{N}'_2(y) \bigcap \{X\} = \mathcal{F}_2(y) \,.$$

Let us suppose now $\mathcal{F}_1(x) = \mathcal{F}_2(y)$ whenever $x \in X'_1$ and y = f(x). With the usual notations

$$s_1(A) = \{x : x \in X'_1, A \in \mathcal{F}_1(x)\}, s_2(A) = \{y : y \in X'_2, A \in \mathcal{F}_2(y)\},\$$

the sets $s_1(G)$ constitute a τ'_1 -base, the sets $s_2(G)$ a τ'_2 -base, and the sets $s_2(G) \cap f(X'_1)$ a $\tau'_2|f(X'_1)$ -base; G always runs over all the τ -open sets. But, by the hypothesis, $x \in s_1(G)$, i.e. $G \in \mathcal{F}_1(x)$ holds if and only if $G \in \mathcal{F}_2(f(x))$, i.e. $f(x) \in s_2(G) \cap f(X'_1)$ and therefore $h: (X'_1, \tau'_1) \to (f(X'_1), \tau'_2|f(X'_1))$ is a homeomorphism.

If f_1 and f_2 are homeomorphisms corresponding to the hypothesis, then by the first statement $x \in X'_1$ implies $\mathcal{F}_2(f_1(x)) = \mathcal{F}_2(f_2(x)) = \mathcal{F}_1(x)$. But if $x \in X$, then $f_1(x) = f_2(x) = x$, and if $x \in X' - X$, then, since τ'_2 is reduced, $\mathcal{F}_2(f_1(x)) = \mathcal{F}_2(f_2(x))$ can hold only if $f_1(x) = f_2(x)$ by Proposition 1.3.1.9.

It was mentioned that the study of extensions of topological spaces is particularly important for the construction of compact extensions. For this
purpose strict extensions are very suitable since, if a compact extension of a space is known, then the strict extension belonging to the same trace filters is compact as well on account of Corollary 1.3.1.1. In connection to this, let us notice:

Proposition 1.3.1.11 With the hypothesis and notations introduced in Theorem 1.3.1.2 let τ' be a strict extension of τ . The topology τ' is compact if and only if from any system of τ -open sets $\{G_i : i \in I\}$ such that for every $x \in X'$ there exists a $G_i \in \mathcal{F}(x)$, a finite subsystem $\{G_{i_k} : k = 1, 2, ..., n\}$ having the same property can be selected.

Proof: The assertion immediately follows from the fact that the topology τ' is compact if and only if there can be select a finite cover from every cover of X' consisting of sets of the form s(G), where G is a τ -open set.

Definition 1.3.1.5 Every compact extension (X', τ') of a topological space (X, τ) is said to be a compactification of the space.

As the first application of strict extensions, let (X, τ) be an arbitrary non-compact space. Let us consider the complements of compact closed sets in X. These are non-empty as was supposed and constitute a filter base consisting of open sets. Denote by \mathcal{M} the (open) filter generated by this filter base, and construct that strict extension (X', τ') of (X, τ) in which X' arises by adding a single new point ω to X, and $\mathcal{F}(\omega) = \mathcal{M}$. By assigning as $\mathcal{F}(x)$ the τ -neighborhood filter $\mathcal{N}(x)$ to the point $x \in X$, the obtained extension will be compact. In fact, if $\{G_i : i \in I\}$ is a system of τ -open sets for which for every $x \in X'$ there exists an $i \in I$ such that $G_i \in \mathcal{F}(x)$, then this holds for $x = \omega$, i.e. there are an $i_o \in I$ and a compact τ -closed set $K \subset X$ such that $X - K \subset G_{i_0}$. To each point $x \in K$ there belongs an $i_x \in I$ such that $G_{i_x} \in \mathcal{N}(x)$, i.e. $x \in G_{i_x}$. Let us select a finite covering from the covering obtained in this way for the compact set K:

$$K \subset \bigcup_{j=1}^{n} G_{i_{x_j}}, \ x_j \in K, \ i_{x_j} \in I.$$

For the system $\{G_{i_0}, G_{i_{x_1}}, \ldots, G_{i_{x_n}}\}$ it is again the case that, if $x \in X'$, one of its members belongs to $\mathcal{F}(x)$, viz. G_{i_0} if $x = \omega$ or $x \in X - K$, or a $G_{i_{x_j}}$ if $x \in K$. Thus by Proposition 1.3.1.11 the following statement is proved:

Proposition 1.3.1.12 If (X, τ) is a non-compact topological space, $X' = X \cup \{\omega\}, \mathcal{F}(x)$ is the filter \mathcal{M} in X generated by the complements of the compact closed sets, then the strict extension (X', τ') corresponding to this choice is compact.

The extension described in the previous proposition is called the **Alex-androff compactification** of the space (X, τ) . It can be seen from Proposition 1.3.1.2 that the Alexandroff compactification of a space (X, τ) is uniquely determined up to a homeomorphism fixing X.

Proposition 1.3.1.13 With the hypothesis and notations of Proposition 1.3.1.12, $\mathcal{F}(x) \subset \mathcal{F}(\omega)$ cannot hold for any point $x \in X$.

Proof: Assuming that $\mathcal{F}(x) \subset \mathcal{F}(\omega)$, i.e. $\mathcal{N}(x) \subset \mathcal{M}$, let $X = \bigcup_{i \in I} G_i$, $G_i \tau$ -open. Then, for an index $i_0 \in I$, $x \in G_{i_0}$ and there exists a compact, closed set K such that $X - K \subset G_{i_0}$. The set K is covered by a finite number of G_i although X is not compact, which is a contradiction.

It follows from this on account of Proposition 1.3.1.8 and Corollary 1.3.1.2:

Proposition 1.3.1.14 The Alexandroff compactification of any space is a reduced extension. If the space is a T_0 -space, then its Alexandroff compactification has the same property.

Proposition 1.3.1.15 The Alexandroff compactification of the space (X, τ) is an S_2 -space if and only if (X, τ) is a locally compact S_2 -space.

Proof: If (X, τ) is a locally compact S_2 -space, then, again with the notations of Proposition 1.3.1.12, $\mathcal{F}(x) \neq \mathcal{F}(y)$ implies $\emptyset \in \mathcal{F}(x) \cap \mathcal{F}(y)$, which is true whenever $x, y \in X$ since τ fulfils (S_2) , and for $x \in X$ and $y = \omega$ as a consequence of the fact that in a locally compact (S_2) -space, every point has a neighborhood base consisting of compact closed sets, x has a compact closed neighborhood K and then $(X - K) \cap K = \emptyset$, $X - K \in \mathcal{F}(\omega)$. Hence by Proposition 1.3.1.7 (X', τ') is an S_2 -space, where $X' = X \cup \{\omega\}$.

On the other hand, if (X', τ') is an S_2 -space, then (X, τ) is an S_2 -space. Now $x \in X$ implies, by Proposition 1.3.1.13, $\mathcal{F}(x) \neq \mathcal{F}(\omega)$ thus $\emptyset \in \mathcal{F}(x) \cap \mathcal{F}(\omega)$ so that x has a neighborhood in (X, τ) which does not intersect the complement of a compact closed set and has therefore a compact neighborhood as well.

1.3.2 Extension of mappings

In connection with the question of the extension of topological spaces studied above, the following problem arises quite naturally. Let (X', τ') be an extension of a topological space (X, τ) , (Y, τ^*) a given topological space, $f: (X, \tau) \to (Y, \tau^*)$ a continuous mapping. The question can be raised whether there exists a continuous extension of f onto X', i.e. a mapping $g: (X', \tau') \to (Y, \tau^*)$ which is continuous and for which g|X = f.

A necessary condition for the existence of such a g can be formulated at once. For this purpose, let us denote as usual the τ' -neighborhood filter of $x \in X'$ by $\mathcal{N}'(x)$, and its trace filter in X by $\mathcal{N}'(x) \cap \{X\} = \mathcal{F}(x)$. Since $\mathcal{F}(x) > \mathcal{N}'(x)$, then $\mathcal{F}(x) \to x$ with respect to τ' . Thus, if g: $(X', \tau') \to (Y, \tau^*)$ is continuous, then $g(\mathcal{F}(x)) \to g(x)$ and g|X = f implies $f(\mathcal{F}(x)) \to g(x)$. According to this there follows:

Proposition 1.3.2.1 Let (X, τ) and (Y, τ^*) be two topological spaces, (X', τ') an extension of the space (X, τ) , $f: (X, \tau) \to (Y, \tau^*)$ a given mapping, $\mathcal{N}'(x)$ the τ' -neighborhood filter of $x \in X'$, $\mathcal{F}(x) = \mathcal{N}'(x) \cap \{X\}$. In order that a continuous mapping $g: (X', \tau') \to (Y, \tau^*)$ exists for which g|X = f, it is necessary that $f: (X, \tau) \to (Y, \tau^*)$ be continuous and $f(\mathcal{F}(x))$ $a \tau^*$ -convergent filter for each $x \in X'$.

Proposition 1.3.2.2 With the notations of the previous proposition, let τ^* be regular, $f: (X, \tau) \to (Y, \tau^*)$ a continuous mapping, and let us suppose that $f(\mathcal{F}(x))$ is a τ^* -convergent filter for every $x \in X'$. For $x \in X$, let g(x) = f(x), and if $x \in X' - X$, let us choose the point $g(x) \in Y$ such that $f(\mathcal{F}(x)) \to g(x)$ with respect to τ^* . Then $g: (X', \tau') \to (Y, \tau^*)$ is a continuous mapping and g|X = f.

Proof: Since the mapping $f: (X, \tau) \to (Y, \tau^*)$ is continuous, $f(\mathcal{F}(x)) \to f(x) = g(x)$ holds for $x \in X$, too. Let us prove that $g(\mathcal{N}'(x)) \to g(x)$ for $x \in X'$. Let U^* be an arbitrary τ^* -neighborhood of g(x), and $V^* \subset U^*$ a closed τ^* -neighborhood of g(x). By $f(\mathcal{F}(x)) \to g(x)$, there exists a τ' -open neighborhood G of x with $f(G \cap X) \subset V^*$. For an arbitrary point $y \in G$, G is a τ' -neighborhood of y, hence $G \cap X \in \mathcal{F}(y)$, and, as a consequence of $f(\mathcal{F}(y)) \to g(y)$, then $g(y) \in f(\overline{G \cap X}) \subset V^* \subset U^*$. Accordingly $g(G) \subset U^*$.

Concerning the uniqueness of the continuous extension, the following holds:

Proposition 1.3.2.3 With the notations of Proposition 1.3.2.1, let g_1 : $(X', \tau') \rightarrow (Y, \tau^*)$ and $g_2: (X', \tau') \rightarrow (Y, \tau^*)$ be continuous mappings, where $g_1|X = g_2|X = f$. If τ^* is a T₂-topology, then $g_1 = g_2$. **Proof:** If $\mathcal{F}(x) \to x, x \in X'$, then $g_1(\mathcal{F}(x)) \to g_1(x), g_2(\mathcal{F}(x)) \to g_2(x)$ and $g_1(\mathcal{F}(x)) = g_2(\mathcal{F}(x)) = f(\mathcal{F}(x))$. Since τ^* is a T_2 -topology, then $g_1(x) = g_2(x)$.

Now let (X, \mathcal{U}) and (Y, \mathcal{V}) be two uniform spaces, $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ a uniformly continuous mapping and examine the question whether f can be extended in a uniformly continuous way to an extension (X', \mathcal{U}') of the space (X, \mathcal{U}) , i.e. whether there can be found a uniformly continuous mapping $g : (X', \mathcal{U}') \to (Y, \mathcal{V})$ for which g|X = f. Of course, we have

Definition 1.3.2.1 A uniform space (X', U') is an extension of a uniform space (X, U) if $X \subset X', U'|_X = U$ and X is $\tau_{U'}$ -dense in X'.

A necessary condition for the existence of such a g is, by Proposition 1.2.5.2, that f has a continuous extension $g: (X', \tau_{\mathcal{U}'}) \to (Y, \mathcal{V})$ to X'. It is an important fact that this condition is also sufficient:

Proposition 1.3.2.4 Let (X', \mathcal{U}') be an extension of (X, \mathcal{U}) , (Y, \mathcal{V}) a uniform space, $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ uniformly continuous mapping, $g : (X', \tau_{\mathcal{U}'}) \to (Y, \tau_{\mathcal{V}})$ continuous and g|X = f. Then $g : (X', \mathcal{U}') \to (Y, \mathcal{V})$ is uniformly continuous as well.

Proof: Let $V \in \mathcal{V}$ be a given entourage, $V_1 \in \mathcal{V}$ an entourage such that $V_1 \circ V_1 \subset V$. Since $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is a uniformly continuous mapping, there exists an entourage $U \in \mathcal{U}$ such that $(x, y) \in U$ implies $(f(x), f(y)) \in V_1$. Further, let $U' \in \mathcal{U}'$ be an entourage such that $U' \cap (X \times X) \subset U$; finally let $U'_1 \in \mathcal{U}'$ be an entourage for which is $U'_1 \circ U'_1 \circ U'_1 \subset U'$. Let us prove that $(x, y) \in U'_1$ implies $(f(x), f(y)) \in V$.

Let $\mathcal{N}'(x)$ be the $\tau_{\mathcal{U}'}$ -neighborhood filter of the point $x \in X'$. Since $g: (X', \tau_{\mathcal{U}'}) \to (Y, \tau_{\mathcal{V}})$ is a continuous mapping, there exists for the $\tau_{\mathcal{V}'}$ neighborhood $V_1(g(x))$ of g(x) a $V'_1 \in \mathcal{N}'(x)$ such that $g(V'_1) \subset V_1(g(x))$. Similarly there exists $V'_2 \in \mathcal{N}'(y)$ such that $g(V'_2) \subset V_1(g(y))$. As $V'_1 \cap U'_1(x) \in \mathcal{N}'(x)$ and X is $\tau_{\mathcal{U}'}$ -dense, we can find a point $x_1 \in V'_1 \cap U'_1(x) \cap X$. In the same way we can see that there exists a point $y_1 \in V'_2 \cap U'_1(y) \cap X$. Now $(x_1, x) \in U'_1$, $(x, y) \in U'_1$ and $(y, y_1) \in U'_1$, so that $(x_1, y_1) \in U'$. But then $(x_1, y_1) \in V$, and hence $(f(x_1), f(y_1)) = (g(x_1), g(y_1)) \in V_1$. Since $g(x_1) \in V_1(g(x)), g(y_1) \in V_1(g(y))$, then $(g(x), g(y)) \in V$.

Lemma 1.3.2.1 If (X', \mathcal{U}') is an extension of a uniform space (X, \mathcal{U}) , then the topology $\tau_{\mathcal{U}'}$ is a (strict) extension of the topology $\tau_{\mathcal{U}}$. **Proof:** By hypothesis, X is $\tau_{\mathcal{U}'}$ -dense in X'. From $\mathcal{U}'|X = \mathcal{U}$, it follows by Proposition 1.2.4.3 that $\tau_{\mathcal{U}'}|X = \tau_{\mathcal{U}}$. Thus the topology $\tau_{\mathcal{U}'}$ is an extension of $\tau_{\mathcal{U}}$, namely a strict extension by 1.3.1.4, because the topology $\tau_{\mathcal{U}'}$ is regular by Proposition 1.1.2.7 and $\tau_{\mathcal{U}'} = \tau_{\delta_{\mathcal{U}'}}$.

Theorem 1.3.2.1 Let (X, \mathcal{U}) , (X', \mathcal{U}') , (Y, \mathcal{V}) be uniform spaces, (X', \mathcal{U}') an extension of (X, \mathcal{U}) , and let (Y, \mathcal{V}) be complete. If $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is a uniformly continuous mapping, then there is a uniformly continuous mapping $g : (X', \mathcal{U}') \to (Y, \mathcal{V})$ such that g|X = f.

Proof: According to Proposition 1.3.2.4, it suffices to show that f has a continuous extension and to prove this, by Proposition 1.3.2.2, we must show that if $\mathcal{F}(x)$ denotes the trace filter in X of the $\tau_{\mathcal{U}'}$ -neighborhood filter of the point $x \in X'$, then $f(\mathcal{F}(x))$ is $\tau_{\mathcal{V}}$ -convergent for every $x \in X'$. Since $\tau_{\mathcal{V}} = \tau_{\delta_{\mathcal{V}}}$, the topology $\tau_{\mathcal{V}}$ is regular from Proposition 1.1.2.7. The mapping $f : (X, \tau_{\mathcal{U}}) \to (Y, \tau_{\mathcal{V}})$ is continuous by Proposition 1.2.5.2. Furthermore, by the previous lemma, the space $(X', \tau_{\mathcal{U}'})$ is an extension of $(X, \tau_{\mathcal{U}})$. Now $\mathcal{F}(x) \to x$ with respect to $\tau_{\mathcal{U}'}$, thus by Proposition 1.2.9.1 it is a \mathcal{U} -Cauchy filter, and then it is, on account of Corollary 1.2.9.2, a \mathcal{U} -Cauchy filter. Hence, by Proposition 1.2.9.2, $f(\mathcal{F}(x))$ is a \mathcal{V} -Cauchy filter and, as \mathcal{V} is complete, it is $\tau_{\mathcal{V}}$ -convergent.

Theorems similar to the preceding ones can be proved in connection with proximally continuous mappings. For this purpose, the following terminology will be introduced:

Definition 1.3.2.2 A proximity space (X', δ') is said to be an extension of the proximity space (X, δ) if $X \subset X', \delta' | X = \delta$ and X is $\tau_{\delta'}$ -dense in X'. At the same time the proximity δ' will be called an extension of the proximity δ .

Proposition 1.3.2.5 If (X', \mathcal{U}') is an extension of (X, \mathcal{U}) , then $\delta_{\mathcal{U}'}$ is an extension of the proximity $\delta_{\mathcal{U}}$.

Proof: By Proposition 1.2.4.3, $\delta_{\mathcal{U}'}|X = \delta_{\mathcal{U}'|X} = \delta_{\mathcal{U}}$ holds, and since $\tau_{\mathcal{U}'} = \tau_{\delta_{\mathcal{U}'}}$, then X is $\tau_{\delta_{\mathcal{U}'}}$ -dense in X'.

Proposition 1.3.2.6 If (X', δ') is an extension of (X, δ) , then the topology $\tau_{\delta'}$ is a strict extension of the topology τ_{δ} .

Proof: Since, by Proposition 1.1.5.1, $\tau_{\delta'}|X = \tau_{\delta'|X} = \tau_{\delta}$, then, according to the hypothesis, X is $\tau_{\delta'}$ -dense. By Proposition 1.1.2.7 the topology $\tau_{\delta'}$ is regular, thus, it is a strict extension of τ_{δ} by Proposition 1.3.1.4.

Proposition 1.3.2.7 Let (X', δ') be an extension of (X, δ) , (Y, δ^*) a given proximity space, $f : (X, \delta) \to (Y, \delta^*)$ a δ -continuous mapping, and g : $(X', \tau_{\delta'}) \to (Y, \tau_{\delta^*})$ a continuous mapping for which g|X = f. Then g : $(X', \delta') \to (Y, \delta^*)$ is δ -continuous as well.

Proof: Let \mathcal{U}' and \mathcal{U}^* be precompact uniformities inducing the proximity relations δ' and δ^* . By Corollary 1.2.10.7 these uniformities exist. Let $\mathcal{U}'|X = \mathcal{U}$. Then on account of Proposition 1.2.4.3 $\delta_{\mathcal{U}} = \delta$, thus, $f: (X,\mathcal{U}) \to (Y,\mathcal{U}^*)$ is uniformly continuous by Theorem 1.2.6.1. Therefore, Proposition 1.3.2.4 can be applied to show that $g: (X',\mathcal{U}') \to (Y,\mathcal{U}^*)$ is uniformly continuous, so that by Proposition 1.2.5.2 $g: (X',\delta') \to (Y,\delta^*)$ is δ -continuous.

Proposition 1.3.2.8 Let (X, δ) , (X', δ') , (Y, δ^*) be three proximity spaces, (X', δ') an extension of (X, δ) , (Y, δ^*) compact. If $f : (X, \delta) \to (Y, \delta^*)$ is δ -continuous, then there exists a δ -continuous mapping $g : (X', \delta') \to (Y, \delta^*)$ for which $g|_X = f$.

Proof: Let us consider again the precompact uniformities \mathcal{U}' and \mathcal{U}^* inducing δ' and δ^* respectively. If $\mathcal{U} = \mathcal{U}'|X$, then $f : (X, \mathcal{U}) \to (Y, \mathcal{U}^*)$ is uniformly continuous, and uniformity \mathcal{U}^* is complete by Corollary 1.2.11.5. Then, by Theorem 1.3.2.1, for the mapping f there exists a uniformly continuous extension $g : (X', \mathcal{U}') \to (Y, \mathcal{U}^*)$, which is also δ -continuous by Proposition 1.2.5.2.

1.3.3 Extensions of uniform spaces

By former results, we can show that as well as in the case of strict extensions of topological spaces, the extensions of uniform spaces and proximity spaces are determined by prescribing the trace filters of the neighborhood filters:

Proposition 1.3.3.1 Let (X, U) be a uniform space, $X \subset X'$, \mathcal{U}'_1 and \mathcal{U}'_2 two uniformities on X' such that both (X', \mathcal{U}'_1) and (X', \mathcal{U}'_2) are extensions of (X, U), and let $\mathcal{N}'_1(x)$ and $\mathcal{N}'_2(x)$ be the $\tau_{\mathcal{U}'_1}$ - and $\tau_{\mathcal{U}'_2}$ -neighborhood filters of the point $x \in X'$ respectively, and let us suppose that, for every $x \in X'$,

(1)
$$\mathcal{N}'_1(x) \cap \{X\} = \mathcal{N}'_2(x) \cap \{X\} = \mathcal{F}(x) \,.$$

Then $\mathcal{U}'_1 = \mathcal{U}'_2$.

Proof: Let $g: X' \to X'$ be the identity mapping of the set X', and let f = g|X. Since by Lemma 1.3.2.1 topologies $\tau_{\mathcal{U}'_1}$ and $\tau_{\mathcal{U}'_2}$ are identical with the strict extension of $\tau_{\mathcal{U}}$ with respect to the trace filters $\mathcal{F}(x)$, then $\tau_{\mathcal{U}'_1} = \tau_{\mathcal{U}'_2}$ by Corollary 1.3.1.1. The mapping $g: (X', \tau_{\mathcal{U}'_1}) \to (X', \tau_{\mathcal{U}'_2})$ is therefore continuous and $f: (X, \mathcal{U}) \to (X', \mathcal{U}'_2)$ is evidently uniformly continuous. Hence, Proposition 1.3.2.4 can be applied so that $g: (X', \mathcal{U}'_1) \to (X', \mathcal{U}'_2)$ is uniformly continuous. An analogous reasoning shows that $g: (X', \mathcal{U}'_2) \to (X', \mathcal{U}'_1)$ is uniformly continuous. Hence, by Proposition 1.2.5.3 $\mathcal{U}'_1 < \mathcal{U}'_2 < \mathcal{U}'_1$, so that $\mathcal{U}'_1 = \mathcal{U}'_2$.

We can prove by the same reasoning the following:

Proposition 1.3.3.2 Let (X, δ) be a proximity space, $X \subset X'$, δ'_1 and δ'_2 two proximities on X', (X', δ'_1) and (X', δ'_2) extensions of (X, δ) , $\mathcal{N}'_1(x)$ and $\mathcal{N}'_2(x)$ the $\tau_{\delta'_1}$ - and $\tau_{\delta'_2}$ -neighborhood filters of the point $x \in X'$ respectively, and let us assume that (1) holds for every $x \in X'$. Then $\delta'_1 = \delta'_2$.

Two questions arise now quite naturally. Let a proximity space (X, δ) or a uniform space (X, \mathcal{U}) and a set $X' \supset X$ be given and let us assign to every point $x \in X'$ a filter $\mathcal{F}(x)$ in X. What conditions do the filters $\mathcal{F}(x)$ have to fulfil in order that there exist a proximity δ' or a uniformity \mathcal{U}' on X' which is an extension of δ or \mathcal{U} respectively and for which $\mathcal{F}(x)$ is equal, for every $x \in X'$, to the trace filter in X of the $\tau_{\delta'}$ - or $\tau_{\mathcal{U}'}$ -neighborhood filter of the point x? Propositions 3.2.2.6 and 3.5.1.2 show that there exists at most one δ' or \mathcal{U}' having this property, but the question is whether it exists at all.

In order to look for necessary conditions, let us consider first the case of proximities; the conditions found will, of course, be necessary in the case of uniformities as well, since, by Proposition 1.3.2.5, $\delta_{\mathcal{U}'}$ is an extension of $\delta_{\mathcal{U}}$ whenever \mathcal{U}' is an extension of \mathcal{U} .

In order to formulate such a condition, let us introduce the following definition:

Definition 1.3.3.1 Let (X, δ) be a proximity space. A filter \mathcal{F} in X is said to be **round** if $F \in \mathcal{F}$ implies the existence of an $F_1 \in \mathcal{F}$ such that $F \in \mathcal{P}(F_1)$, where $\mathcal{P}(F_1)$ is a δ -filter of the set F_1 .

Proposition 1.3.3.3 In a proximity space (X, δ) , the δ -filter $\mathcal{P}(A)$ of any set $\emptyset \neq A \subset X$, in particular, the τ_{δ} -neighborhood filter $\mathcal{N}(x)$ of any point $x \in X$, is a round filter.

Proof: Assertion immediately follows from Proposition 1.1.1.5 (e) as well as the fact that $\mathcal{N}(x) = \mathcal{P}(\{x\})$.

Proposition 1.3.3.4 In a proximity space (X, δ) , every round filter is a τ_{δ} -open filter.

Proof: Let \mathcal{F} be a round filter, $F \in \mathcal{F}$. Then there exists $F_1 \in \mathcal{F}$ such that $F \in \mathcal{P}(F_1)$, so that F is a δ -neighborhood of F_1 , i.e. $F_1\overline{\delta}X - F$. Then, by Proposition 1.1.2.4, we have $\overline{F}_1\overline{\delta}\overline{X} - \overline{F} = X - \operatorname{int} F$, thus $F_1 \subset \overline{F}_1 \subset \operatorname{int} F$. But then $\operatorname{int} F \in \mathcal{F}$.

Proposition 1.3.3.5 If \mathcal{F} is a round filter in the proximity space (X, δ) , $Y \subset X$, and $\emptyset \notin \mathcal{F} \cap \{Y\}$, then $\mathcal{F}_Y = \mathcal{F} \cap \{Y\}$ is a round filter in the subspace $(Y, \delta | Y)$.

Proof: First of all let us notice that \mathcal{F}_Y is a filter in Y. Let $F_Y \in \mathcal{F}_Y$. Then $F_Y = Y \cap F$, where $F \in \mathcal{F}$; thus, there exists an $F_1 \in \mathcal{F}$ such that $F \in \mathcal{P}(F_1)$. Let $F_2 = F_1 \cap Y$. Then $F_2 \in \mathcal{F}_Y$. Furthermore, $F \in \mathcal{P}(F_2)$ holds by Proposition 1.1.1.5 (c), so that, by Proposition 1.1.5.1, $S_Y = S \cap Y \in \mathcal{P}_Y(F_2)$, where $\mathcal{P}_Y(F_2)$ denotes the $(\delta|Y)$ -filter.

Proposition 1.3.3.6 If \mathcal{F} is a round filter in the proximity space (X, δ) and $Y \subset X$ is τ_{δ} -dense, then $\emptyset \notin \mathcal{F} \cap \{Y\}$.

Proof: If $F \in \mathcal{F}$, then by Proposition 1.3.3.4 there exists a τ_{δ} -open set $G \subset F$ such that $G \in \mathcal{F}$. For any $x \in G$, G is a τ_{δ} -neighborhood of x for which $G \cap Y \neq \emptyset$, so that $S \cap Y \neq \emptyset$.

Proposition 1.3.3.7 Let \mathcal{F}_1 and \mathcal{F}_2 be round filters in the proximity space (X, δ) . If $\emptyset \notin \mathcal{F}_1 \cap \mathcal{F}_2$, then $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2$ is a round filter.

Proof: First let us notice that \mathcal{F} is a filter. Let $F = F_1 \cap F_2 \in \mathcal{F}$, where $F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2$. Then there exist $F'_1 \in \mathcal{F}_1$ and $F'_2 \in \mathcal{F}_2$ such that $F_1 \in \mathcal{P}(F'_1), F_2 \in \mathcal{P}(F'_2)$. On account to Proposition 1.1.1.5 (f), we have $F = F_1 \cap F_2 \in \mathcal{P}(F')$, where $F' = F'_1 \cap F'_2 \in \mathcal{F}$, hence \mathcal{F} is a round filter.

Proposition 1.3.3.8 Let (X, δ) be a proximity space. If \mathcal{B} is a filter base in X, then the collection of all δ -neighborhoods of all sets $R \in \mathcal{B}$ constitutes a round filter called the δ -filter of the filter base \mathcal{B} and is denoted by $\mathcal{P}(\mathcal{F})$.

Proof: If $A_1 \in \mathcal{P}(B_1)$, $A_2 \in \mathcal{P}(B_2)$, where $B_1, B_2 \in \mathcal{B}$, then there exists a $B \in \mathcal{B}$ such that $B \subset B_1 \cap B_2$. But then by Proposition 1.1.1.5 (f) and (c) $A_1 \cap A_2 \in \mathcal{P}(B_1 \cap B_2) \subset \mathcal{P}(B)$. Now it is obvious that $\mathcal{P}(\mathcal{B})$ is a filter in X. Let $A \in \mathcal{P}(\mathcal{B})$, i.e. $A \in \mathcal{P}(B)$ for some $B \in \mathcal{B}$. Then by Proposition 1.1.1.5 (e) there exists an $A_1 \in \mathcal{P}(B)$ such that $A \in \mathcal{P}(A_1)$. Therefore $A_1 \in \mathcal{P}(\mathcal{B})$, so that $\mathcal{P}(\mathcal{B})$ is a round filter. \clubsuit

Proposition 1.3.3.9 If \mathcal{B} is a compressed filter base in the proximity space (X, δ) , then $\mathcal{P}(\mathcal{B})$ is also a compressed filter base.

Proof: If $A\overline{\delta}B$, let $C, D \subset X$ be such that $C \cap D = \emptyset$, $A\overline{\delta}X - C$ and $B\overline{\delta}X - D$. Further, let $C_1, D_1 \subset X$ be sets such that $C_1 \cap D_1 = \emptyset$, $A\overline{\delta}X - C_1$ and $X - C\overline{\delta}X - D_1$. Since \mathcal{B} is a compressed filter base, there exists either an $F \in \mathcal{B}$ such that $F \cap (X - C) = \emptyset$, i.e. $F \subset C$, or an $F \in \mathcal{B}$ such that $F \cap (X - D_1) = \emptyset$, i.e. $F \subset D_1$. If $F \in \mathcal{B}$ is a set for which $F \subset C$, then $C \subset X - D\overline{\delta}B$ implies $X - B \in \mathcal{P}(F) \subset \mathcal{P}(\mathcal{B})$. On the other hand, if $F \in \mathcal{B}$ is such that $F \subset D_1$, then from $D_1 \subset X - C_1\overline{\delta}A$ follows that $F\overline{\delta}A$, so that $X - A \in \mathcal{P}(F) \subset \mathcal{P}(\mathcal{B})$.

Proposition 1.3.3.10 If $\mathcal{F} \to x$ in the proximity space (X, δ) , then $\mathcal{P}(\mathcal{F}) \to x$.

Proof: By Proposition 1.2.11.1, \mathcal{F} is compressed, thus $\mathcal{P}(\mathcal{F})$ has the same property according to the previous proposition. Of course, $\mathcal{P}(\mathcal{F}) < \mathcal{F}$; thus by Proposition 1.2.11.18 *x* is a cluster point of $\mathcal{P}(\mathcal{F})$ so the statement follows from Proposition 1.2.11.20.

It is an important fact that among the round filters, the compressed ones are identical with the maximal ones. More precisely, the following statement holds:

Theorem 1.3.3.1 Let \mathcal{F} be a round filter in the proximity space (X, δ) . If \mathcal{F} is compressed, \mathcal{F}_1 is a round filter for which $\mathcal{F} \subset \mathcal{F}_1$, then $\mathcal{F} = \mathcal{F}_1$. Conversely, if there is no round filter distinct from \mathcal{F} and containing it, then \mathcal{F} is compressed.

Proof: Let us suppose that \mathcal{F} is compressed and $\mathcal{F} \subset \mathcal{F}_1$, where \mathcal{F}_1 is a round filter. If $A \in \mathcal{F}_1$, then there exists a $B \in \mathcal{F}_1$ such that $B\overline{\delta}X - A$. Then there exists an $F \in \mathcal{F}$ such that either $F \subset A$ or $F \subset X - B$. The second case is impossible from the fact that $F \in \mathcal{F}_1$, so that $A \in \mathcal{F}$, and $\mathcal{F} = \mathcal{F}_1$.

Let us suppose now that \mathcal{F} is not compressed. Then there exist the sets $A, B \subset X$ such that $A\overline{\delta}B$, $X - A \notin \mathcal{F}$, $X - B \notin \mathcal{F}$. Hence $A \neq \emptyset$, $X - B \in \mathcal{P}(A)$ and applying Proposition 1.1.3.2 (c) a sequence (C_n) can be constructed such that $C_1 \subset X - B$, $C_n \in \mathcal{P}(A)$, $C_n \in \mathcal{P}(C_{n+1})$ for every $n \in \mathbb{N}$. The sets C_n obviously constitute a filter base. Let \mathcal{F}_0 be a filter in X generated by it. \mathcal{F}_0 is a round filter. Indeed, if $F_0 \in \mathcal{F}_0$, then there exists an $n \in \mathbb{N}$ such that $C_n \subset F_0$ and then $C_{n+1} \in \mathcal{F}_0$, $F_0 \in \mathcal{P}(C_{n+1})$.

Finally, let $\mathcal{F}_1 = \mathcal{F} \cap \mathcal{F}_0$. By Proposition 1.3.3.7 \mathcal{F}_1 is a round filter, and since $X - A \notin \mathcal{F}$, then each set from \mathcal{F} intersects the set A, and thus a fortiori each C_n . Evidently $\mathcal{F} \subset \mathcal{F}_1$, and $\mathcal{F} \neq \mathcal{F}_1$, since $X - B \in \mathcal{F}_1$, but $X - B \notin \mathcal{F}$.

Let us introduce now a notion which gives a characterization of maximal round filters, i.e. compressed round filters.

Definition 1.3.3.2 A collection \mathcal{F} of subsets of a proximity space (X, δ) is said to be an **end** if

(a) for arbitrary two sets $B, C \in \mathcal{F}$ there exists a non-void subset $A \in \mathcal{F}$ for which $A \ll B$ and $A \ll C$;

(b) if $A \ll B$, then either $X - A \in \mathcal{F}$ or $B \in \mathcal{F}$.

If a collection \mathcal{F} of subsets of X is an end, then $\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$. It is easy to see that the system $\mathcal{N}(x)$ of δ -neighborhood of any point $x \in X$ is an end in X. P. S. Alexandroff introduced first the notion of an end in the following way:

Definition 1.3.3.3 A collection of sets \mathcal{F} from a proximity space (X, δ) is said to be a centered δ -system if two following conditions are satisfied:

(a) if $A, B \in \mathcal{F}$, then $A \cap B \neq \emptyset$;

(b) if $A \in \mathcal{F}$, then there exists a $B \in \mathcal{F}$ such that $B \ll A$.

A maximal centered δ -system is said to be an end.

It is easy to prove that these definitions are equivalent.

Proposition 1.3.3.11 Every end is a maximal round filter.

Proof: Let \mathcal{F} be an end. Let us first prove that \mathcal{F} is a filter. From condition (a) of the definition of an end and the fact that $X \in \mathcal{F}$, there follows that \mathcal{F} is a non-empty filter base. Let $C \in \mathcal{F}$ and $C \subset D$. We must show that $D \in \mathcal{F}$. By condition (a) of the definition of an end there exists a set $A \in \mathcal{F}$ such that $A \ll C$. Thus by Theorem 1.1.1.1 $A \ll D$. Condition (b) of Definition 1.3.3.3 demands that either $X - A \in \mathcal{F}$ or $D \in \mathcal{F}$. Condition (a) excluded the first possibility since $A \in \mathcal{F}$, so that $D \in \mathcal{F}$, which was to be proved.

That \mathcal{F} is a round filter follows immediately from condition (a) of the definition of an end.

Finally, we must show that the round filter \mathcal{F} is maximal. Let \mathcal{G} be a round filter for which $\mathcal{F} \subset \mathcal{G}$ and let $B \in \mathcal{G}$. Then by the definition of round

filter there exists a set $A \in \mathcal{G}$ such that $A \ll B$. \mathcal{G} is a filter, $A \in \mathcal{G}$, so that $X - A \notin \mathcal{F}$. Hence by condition (b) of the definition of an end, there follows that $B \in \mathcal{F}$, which proves that $\mathcal{F} = \mathcal{G}$.

Proposition 1.3.3.12 Let \mathcal{F} be a round filter in the proximity space (X, δ) and $A \ll B$. If A intersects every member of \mathcal{F} , then B belongs to some round filter finer than \mathcal{F} .

Proof: Let $\mathcal{G} = \{A \cap F : F \in \mathcal{F}\}$. Let us prove that the family $\mathcal{G}^{\circ} = \{E \subset X : \text{ there is an } A \in \mathcal{G} \text{ such that } A \ll E\}$ is a round filter finer than \mathcal{F} and that it contains B. Let P and Q be arbitrary elements of \mathcal{G}° . Then, by definition of the family \mathcal{G}° , there exist elements C and D of the family \mathcal{F} such that $A \cap C \ll P$ and $A \cap D \ll Q$. Since \mathcal{F} is a filter, $E = C \cap D \in \mathcal{F}$. From Theorem 1.1.1.1, it is evident that $A \cap E \ll P$ and $A \cap E \ll Q$, so that $A \cap E \ll P \cap Q$. Since $A \cap E \in \mathcal{G}$, it follows that $P \cap Q \in \mathcal{G}^{\circ}$. Furthermore, it is obvious that the supersets of elements of family \mathcal{G}° are also contained in \mathcal{G}° , so that \mathcal{G}° is a filter. By Theorem 1.1.1.1, there exists a set R such that $A \cap E \ll R \ll P \cap Q$. By taking P = Q and noting that $A \cap E \in \mathcal{G}$ implies $R \in \mathcal{G}^{\circ}$, we can see that \mathcal{G}° is a round filter.

Since $A \ll B$, by Theorem 1.1.1.1 there follows $A \cap E \ll B$, so that $B \in \mathcal{G}^{\circ}$. To prove that \mathcal{G}° is finer than \mathcal{F} , let us suppose that $E \in \mathcal{F}$. Since \mathcal{F} is a round filter, there exists an $F \in \mathcal{F}$ such that $F \ll E$. Then $A \cap F \ll E$ holds by Theorem 1.1.1.1 and so $E \in \mathcal{G}^{\circ}$.

Theorem 1.3.3.2 \mathcal{F} is an end if and only if it is a maximal round filter.

Proof: On account of Proposition 1.3.3.11 it is sufficient to show that every maximal round filter \mathcal{F} is an end. Condition (a) of the definition of an end is clearly satisfied by any round filter. In verifying condition (b), let us suppose $A \ll B$ and $B \notin \mathcal{F}$. Since \mathcal{F} is maximal, by the previous proposition there exists a set $E \in \mathcal{F}$ for which $A \cap E = \emptyset$. Therefore $E \subset X - A$ and $X - A \in \mathcal{F}$ since \mathcal{F} is a filter, thus condition (b) of the definition of an end is satisfied.

Proposition 1.3.3.13 If \mathcal{F} is a round filter in the proximity space (X, δ) and $\mathcal{F} \to x$, then $\mathcal{F} = \mathcal{N}(x)$, where $\mathcal{N}(x)$ is the τ_{δ} -neighborhood filter of the point x.

Proof: By Proposition 1.3.3.3 $\mathcal{N}(x)$ is a round filter, and, by Proposition 1.2.11.1, it is compressed. Hence, if \mathcal{F} is a round filter and $\mathcal{F} > \mathcal{N}(x)$, i.e. $\mathcal{N}(x) \subset \mathcal{F}$, then $\mathcal{F} = \mathcal{N}(x)$.

In a uniform space (X, \mathcal{U}) the notions "round filter" and "proximity filter of a filter base" are always understood with respect to the proximity $\delta_{\mathcal{U}}$.

Proposition 1.3.3.14 If \mathcal{F} is a Cauchy filter in a uniform space (X, \mathcal{U}) , then its δ -filter is a Cauchy filter.

Proof: Let $U \in \mathcal{U}$ be an arbitrary entourage and $U_1 \in \mathcal{U}$ an entourage such that $U_1 \circ U_1 \circ U_1 \subset U$. If $F \in \mathcal{F}$ is a set small of order U_1 , then $U_1[F] \in \mathcal{P}(\mathcal{F})$ since $(F \times (X - U_1[F])) \cap U_1 = \emptyset$. Therefore $F\overline{\delta}_{\mathcal{U}}X - U_1[F]$ and $U_1[F]$ is small of order U.

Now the question raised concerning uniform spaces can be answered.

Proposition 1.3.3.15 If (X', \mathcal{U}') is an arbitrary extension of the uniform space $(X, \mathcal{U}), \mathcal{N}'(x)$ denotes the $\tau_{\mathcal{U}'}$ -neighborhood filter of $x \in X'$, and $\mathcal{F}(x) = \mathcal{N}'(x) \cap \{X\}$ is its trace filter in X, then $\mathcal{F}(x)$ is a round Cauchy filter in (X, \mathcal{U}) ; in particular, if $x \in X, \mathcal{F}(x)$ is identical with the $\tau_{\mathcal{U}}$ neighborhood filter $\mathcal{N}(x)$ of x.

Proof: Since $\tau_{\mathcal{U}'}$ is an extension of $\tau_{\mathcal{U}}$ by Lemma 1.3.2.1, $\mathcal{F}(x)$ is identical to $\mathcal{N}(x)$ if $x \in X$. $\mathcal{N}'(x)$ is a $\delta_{\mathcal{U}'}$ -round filter by Proposition 1.3.3.3 and, on account of Proposition 1.3.2.5, $\delta_{\mathcal{U}'}$ is an extension of $\delta_{\mathcal{U}}$. Therefore $\mathcal{F}(x)$ is a $\delta_{\mathcal{U}}$ -round filter by Proposition 1.3.3.5. Finally $\mathcal{N}'(x)$, being $\tau_{\mathcal{U}'}$ -convergent, is a \mathcal{U}' -Cauchy filter by Proposition 1.2.9.1. Hence $\mathcal{F}(x)$ is a \mathcal{U}' -Cauchy filter base by Corollary 1.2.9.3 and a \mathcal{U} -Cauchy filter by Corollary 1.2.9.2.

Theorem 1.3.3.3 Let (X, \mathcal{U}) be a uniform space, $X \subset X'$, and let us assign to every point $x \in X'$ a round Cauchy filter $\mathcal{F}(x)$ in X, in particular, if $x \in X$, let $\mathcal{F}(x) = \mathcal{N}(x)$ be the $\tau_{\mathcal{U}}$ -neighborhood filter of x. Then there exists exactly one uniformity \mathcal{U}' on X' such that \mathcal{U}' is an extension of \mathcal{U} and $\mathcal{F}(x) = \mathcal{N}'(x) \cap \{X\}$ for every $x \in X'$, where $\mathcal{N}'(x)$ is the $\tau_{\mathcal{U}'}$ -neighborhood filter of the point x.

Proof: From Proposition 1.3.3.1 there exists one uniformity \mathcal{U}' with this property at the most. It will be shown that there exists indeed at least one.

For an arbitrary entourage $U \in \mathcal{U}'$, let a subset $U' \subset X' \times X'$ be defined as follows: $(x, y) \in U'$ if and only if $(P \times Q) \cap U \neq \emptyset$ for every $P \in \mathcal{F}(x)$ and every $Q \in \mathcal{F}(y)$. It will be shown that the sets U' obtained in this way constitute a uniform base $\mathcal{B}' = \{U' : U \in \mathcal{U} \text{ is an entourage}\}$ on X'.

If x = y, then $P \cap Q \neq \emptyset$ for any sets $P \in \mathcal{F}(x)$, $Q \in \mathcal{F}(y)$, and if $z \in P \cap Q$, then $(z, z) \in (P \times Q) \cap U$ so that $(x, x) \in U'$.

From the definition, it is evident that $(x, y) \in U'$ implies $(y, x) \in U'$ since $U = U^{-1}$.

It is also clear that $U_1 \subset U_2$ implies $U'_1 \subset U'_2$. From this, and by the filter base property of the entourages in \mathcal{U} , it follows that \mathcal{B}' is a filter base as well.

Finally, for the entourage $U \in \mathcal{U}$, let $U_1 \in \mathcal{U}$ be an entourage such that $U_1 \circ U_1 \circ U_1 \subset U$. It will be shown that $U'_1 \circ U'_1 \subset U'$. Indeed if $(x, y) \in U'_1$, $(y, z) \in U'_1$, let $P \in \mathcal{F}(x)$ and $R \in \mathcal{F}(z)$ be arbitrary and $Q \in \mathcal{F}(y)$ a set small of order U_1 . There are then $x_1 \in P$, $y_1 \in Q$ such that $(x_1, y_1) \in U_1$ and $y_2 \in Q$, $z_2 \in R$ such that $(y_2, z_2) \in U_1$. Then $(y_1, y_2) \in U_1$ implies $(x_1, z_2) \in U$, so that $(P \times R) \cap U \neq \emptyset$.

Thus \mathcal{B}' is a uniform base on X' and it generates a uniformity \mathcal{U}' on X'. It will be shown that $\mathcal{U}'|X = \mathcal{U}$. For this purpose, let us select first, for an entourage $U \in \mathcal{U}$, the entourage $U_1 \in \mathcal{U}$ as before. It will be verified that $U'_1 \cap (X \times X) \subset U$. Indeed if $x, y \in X$, $(x, y) \in U'_1$, then by $U_1[x] \in \mathcal{F}(x)$, $U_1[y] \in \mathcal{F}(y)$ there exist $x_1 \in U_1[x]$, $y_1 \in U_1[y]$ such that $(x_1, y_1) \in U_1$ and then $(x, y) \in U$. On the other hand, for any entourage $U \in \mathcal{U}$, we have that $U \subset U' \cap (X \times X)$ as $(x, y) \in U$ implies $(x, y) \in (P \times Q) \cap U$ for all sets $P \in \mathcal{F}(x)$ and $Q \in \mathcal{F}(y)$.

Now let $\mathcal{N}'(x)$ be the $\tau_{\mathcal{U}'}$ -neighborhood filter of $x \in X'$. If $V' \in \mathcal{N}'(x)$, then there exists an entourage $U \in \mathcal{U}$ such that $U'[x] \subset V'$. Let $F \in \mathcal{F}(x)$ be a set small of order U. If $y \in F$, $P \in \mathcal{F}(x)$ and $Q \in \mathcal{F}(y)$ are arbitrary, then $x_1 \in P \cap F$ implies $(x_1, y) \in (P \times Q) \cap U$, thus $(x, y) \in U'$ and $y \in U'[x] \cap X$. Therefore $F \subset U'[x] \cap X \subset V' \cap X$, and $V' \cap X \in \mathcal{F}(x)$. On the other hand, if $F \in \mathcal{F}(x)$, then there exists an $F_1 \in \mathcal{F}(x)$ such that $F_1 \overline{\delta}_{\mathcal{U}} X - F$. Hence, for a suitable entourage $U \in \mathcal{U}$ we have $(F_1 \times (X - F)) \cap U = \emptyset$. Let $U_1 \in \mathcal{U}$ be an entourage for which $U_1 \circ U_1 \subset U$. Then $U'_1[x] \cap X \subset F$, so that $F \in \mathcal{N}'(x) \cap \{X\}$. Indeed, if $y \in U'_1[x] \cap X$, then, by $(x, y) \in U'_1$, there are in the sets $F_1 \in \mathcal{F}(x)$ and $U_1[y] \in \mathcal{F}(y)$ two points $x_1 \in F_1$ and $y_1 \in U_1[y]$ such that $(x_1, y_1) \in U_1$. Hence $(x_1, y) \in U$, thus $y \in U[F_1] \subset F$.

According to this, $\mathcal{N}'(x) \cap \{X\} = \mathcal{F}(x)$ for all $x \in X'$. It is clear that X is $\tau_{\mathcal{U}'}$ -dense in X' so that \mathcal{U}' is an extension of \mathcal{U} .

From the above remark it is evident that $U_1, U_2 \in \mathcal{U}, U_1 \subset U_2$ imply $U'_1 \subset U'_2$:

Corollary 1.3.3.1 Under the hypotheses and with the notations of Theorem 1.3.3.3, let \mathcal{B} be a uniform base generating \mathcal{U} . Then the entourages U' constructed from the entourages $U \in \mathcal{B}$ constitute a uniform base generating the uniformity \mathcal{U}' .

From Proposition 1.3.3.1 and Proposition 1.3.3.15 we obtain:

Corollary 1.3.3.2 Every extension of a uniform space (X, U) can be obtained by means of the construction described in Theorem 1.3.3.3.

From this and on account of Corollary 1.3.3.1 and Theorem 1.2.10.3 the following holds:

Corollary 1.3.3.3 Every extension of a pseudo-metrizable uniform space is pseudo-metrizable as well.

Proposition 1.3.3.16 Under the hypotheses and with the notations of Theorem 1.3.3.3, let $X \subset X'_1 \subset X'$. If \mathcal{U}'_1 is the extension of \mathcal{U} corresponding to the trace filters $\mathcal{F}(x)$ ($x \in X'_1$) on X'_1 , then $\mathcal{U}'_1 = \mathcal{U}'|X'_1$.

Proof: It follows from Proposition 1.2.4.3 and Corollary 1.2.4.2 that $\mathcal{U}'|X'_1$ is also an extension of \mathcal{U} , namely precisely that one corresponding to the trace filters $\mathcal{F}(x), x \in X'_1$. Hence by Theorem 1.3.3.3 and Proposition 1.3.3.1 $\mathcal{U}'_1 = \mathcal{U}'|X'_1$.

Proposition 1.3.3.17 Let (X', U') be an extension of the uniform space (X, U) and, if $x \in X'$, $\mathcal{F}(x)$ the trace filter in X of the $\tau_{U'}$ -neighborhood filter of the point x. Then the following statements are equivalent:

(a) $\tau_{\mathcal{U}'}$ is a reduced extension of $\tau_{\mathcal{U}}$;

(b) if $x \in X' - X$, then $\mathcal{F}(x)$ is a non-convergent filter with respect to $\tau_{\mathcal{U}}$ and $x, y \in X' - X$, $x \neq y$ imply $\mathcal{F}(x) \neq \mathcal{F}(y)$;

(c) for $x \in X' - X$, $\mathcal{F}(x) \to y \in X'$ holds with respect to $\tau_{\mathcal{U}'}$ if (and only if) y = x.

Proof: $(a) \Rightarrow (b)$: By Lemma 1.3.2.1 and Proposition 1.3.1.9 (a) means that if $x \in X' - X$, $y \in X'$ and $x \neq y$, then $\mathcal{F}(x) \neq \mathcal{F}(y)$. However, this implies (b); for, if $x \in X' - X$, $\mathcal{F}(x) \to y \in X$ would hold with respect to $\tau_{\mathcal{U}}$, then, on account of Proposition 1.3.3.13, $\mathcal{F}(x) = \mathcal{N}(y) = \mathcal{F}(y)$ would follow since $\mathcal{F}(x)$ is a round filter in (X, \mathcal{U}) by Proposition 1.3.3.15.

 $(b) \Rightarrow (c)$: If $x \in X' - X$, then by (b) $\mathcal{F}(x) \to y \in X$ cannot hold with respect to $\tau_{\mathcal{U}'}$, because then the same would hold for $\tau_{\mathcal{U}}$. On the other hand, if $\mathcal{F}(x) \to y \in X' - X$ with respect to $\tau_{\mathcal{U}'}$, then $\mathcal{F}(x)$ is finer than the $\tau_{\mathcal{U}'}$ -neighborhood filter of y and, of course, than its trace filter $\mathcal{F}(y)$ as well. However, $\mathcal{F}(y) \subset \mathcal{F}(x)$ can hold by Theorem 1.3.3.1 only if $\mathcal{F}(x) = \mathcal{F}(y)$ as $\mathcal{F}(x)$ and $\mathcal{F}(y)$ are round, compressed filters in (X, \mathcal{U}) by Proposition 1.3.3.15. Hence x = y on account of (b). Of course, $\mathcal{F}(x) \to x$ holds in any case.

 $(c) \Rightarrow (a)$: It is to be shown that if (c) is fulfilled, then $x \in X' - X$, $y \in X', x \neq y$ imply $\mathcal{F}(x) \neq \mathcal{F}(y)$. However $\mathcal{F}(x) = \mathcal{F}(y)$ would imply $\mathcal{F}(x) \to y$ with respect to $\tau_{\mathcal{U}'}$.

Definition 1.3.3.4 The extension (X', \mathcal{U}') of the uniform space (X, \mathcal{U}) is **reduced** if $\tau_{\mathcal{U}'}$ is a reduced extension of $\tau_{\mathcal{U}}$, i.e. if one of conditions (b) or (c) of the preceding theorem is fulfilled.

Proposition 1.3.3.18 If (X, U) is a separated uniform space and (X', U') is a reduced extension of (X, U), then (X', U') is separated as well.

Proof: By Corollary 1.2.2.4 the property of \mathcal{U} or \mathcal{U}' of being separated is equivalent to the property of being T_0 of the topology $\tau_{\mathcal{U}}$ or $\tau_{\mathcal{U}'}$ respectively. Thus Proposition 1.3.1.8 furnishes the statement.

The following theorem is of fundamental importance in the theory of uniform spaces.

Theorem 1.3.3.4 Let (X, \mathcal{U}) be an arbitrary uniform space, $X'_c \supset X$ a set such that the points $x \in X'_c - X$ can be associated in a one-to-one manner with all round Cauchy filters $\operatorname{non-\tau_{\mathcal{U}}}$ -convergent in (X, \mathcal{U}) . Denoting by $\mathcal{F}(x)$ the filter associated in this way with the point $x \in X'_c - X$ and making $\mathcal{F}(x)$ equal to the $\tau_{\mathcal{U}}$ -neighborhood filter $\mathcal{N}(x)$ of x whenever $x \in X$, let \mathcal{U}'_c be the uniformity on X'_c constructed in the proof of Theorem 1.3.3.3. Then (X'_c, \mathcal{U}'_c) is complete and a reduced extension of (X, \mathcal{U}) .

Proof: Only the completeness of (X'_c, \mathcal{U}'_c) should be proved; the rest follows from Proposition 1.3.3.17. Thus let \mathcal{I}' be a \mathcal{U}'_c -Cauchy filter base. Let us consider the δ -filter $\mathcal{P}'(\mathcal{I}')$ of \mathcal{I}' with respect to the proximity $\delta_{\mathcal{U}'_c}$. By Proposition 1.3.3.17 this is a \mathcal{U}'_c -Cauchy filter, and, on account of Proposition 1.3.3.8, also a $\delta_{\mathcal{U}'_c}$ -round filter. By Proposition 1.3.3.6, we can speak of the filter $\mathcal{F} = \mathcal{P}'(\mathcal{I}') \cap \{X\}$ which is, by Proposition 1.3.3.5, a round filter with respect to the proximity $\delta_{\mathcal{U}'_c}|X = \delta_{\mathcal{U}}$ (this equality follows from Proposition 1.3.2.5). \mathcal{F} is a \mathcal{U}'_c -Cauchy filter base by Corollary 1.2.9.3 and hence a \mathcal{U} -Cauchy filter on account of Corollary 1.2.9.2. Therefore there exists an $x \in X'_c$ such that $\mathcal{F} = \mathcal{F}(x)$. Namely, if \mathcal{F} converges to a point $x \in X$ with respect to $\tau_{\mathcal{U}}$, then $\mathcal{F} = \mathcal{F}(x)$ by Proposition 1.3.3.13, while, if \mathcal{F} is not $\tau_{\mathcal{U}}$ convergent, then $\mathcal{F} = \mathcal{F}(x)$ will hold for some point $x \in X'_c - X$. However, denoting the $\tau_{\mathcal{U}'}$ -neighborhood filter of x by $\mathcal{N}'(x)$, $\mathcal{F}(x) = \mathcal{N}'(x) \cap \{X\} >$ $\mathcal{N}'(x)$ implies $\mathcal{F}(x) \to x$ with respect to $\tau_{\mathcal{U}'_c}$, hence x is a $\tau_{\mathcal{U}'_c}$ -cluster point of the filter $\mathcal{P}'(\mathcal{I}')$ by Proposition 1.2.11.18. On account of Proposition 1.2.11.20, $\mathcal{P}'(\mathcal{I}') \to x$, and by $\mathcal{I}' > \mathcal{N}'(\mathcal{I}')$ we have that $\mathcal{I}' \to x$ with respect to $\tau_{\mathcal{U}'_c}$.

Proposition 1.3.3.19 Let (X, U) be an arbitrary uniform space, (X'_c, U'_c) its extension constructed in Theorem 1.3.3.4, $X \subset X' \subset X'_c$. If $(X', U'_c|X')$ is complete, then $X' = X'_c$.

Proof: Lat us suppose that $x \in X'_c - X'$. Then $\mathcal{F}(x) \to x$, but $\mathcal{F}(x)$ is also a $(\mathcal{U}'_c|X')$ -Cauchy filter base by Corollary 1.2.9.2 so that there is a $y \in X'$ such that $\mathcal{F}(x) \to y$ with respect to $\tau_{\mathcal{U}'_c|X'} = \tau_{\mathcal{U}'_c}|X'$, i.e. with respect to $\tau_{\mathcal{U}'_c}$ on account of Proposition 1.2.4.3. However, x and y are weakly separated, thus disconnected by the regularity of $\tau_{\mathcal{U}'_c}$ which is a contradiction. Thus $X'_c = X'$.

Proposition 1.3.3.20 Let (X, U) be a uniform space, (X'_c, U'_c) its extension constructed in Theorem 1.3.3.4 and (X', U') an arbitrary reduced extension of (X, U). Then there exists a uniquely defined isomorphism h fixing X which maps (X', U') into a subspace of (X'_c, U'_c) . U' is complete if and only if $h(X') = X'_c$.

Proof: If $x \in X'$, let $\mathcal{N}'(x)$ be the $\tau_{\mathcal{U}'}$ -neighborhood filter of x. Its trace filter $\mathcal{F}'(x) = \mathcal{N}'(x) \cap \{X\}$ is a round Cauchy filter in (X, \mathcal{U}) by Proposition 1.3.3.15 and $\mathcal{F}'(x)$ is not $\tau_{\mathcal{U}}$ -convergent if $x \in X' - X$ on account of Proposition 1.3.3.17. Moreover $x, y \in X' - X$ and $x \neq y$ imply $\mathcal{F}'(x) \neq \mathcal{F}'(y)$. Therefore it can be given a uniquely defined bijection $h: X' \to X''$ onto a suitable set $X \subset X'' \subset X'_c$ such that h(x) = x for $x \in X$ and $\mathcal{F}'(x) =$ $\mathcal{F}(h(x))$ for $x \in X' - X$, where $\mathcal{F}(h(x))$ denotes the filter belonging to the point $h(x) \in X'_c$ according to Theorem 1.3.3.4. Let us denote by \mathcal{U}'' the extension of \mathcal{U} constructed on the set X'' by means of Theorem 1.3.3.3 starting from the trace filters $\mathcal{F}(y), y \in X''$. In this case $\tau_{\mathcal{U}''}$ and $\tau_{\mathcal{U}'}$ are, by Lemma 1.3.2.1, strict extensions of $\tau_{\mathcal{U}}$ with respect to the trace filters $\mathcal{F}''(y)$ and $\mathcal{F}'(y)$ respectively, so that by Theorem 1.3.1.2 $h: (X', \tau_{\mathcal{U}'}) \to (X'', \tau_{\mathcal{U}''})$ is a homeomorphism. Since $h|X: X \to X''$ is the canonical injection of X into X'' and $\mathcal{U} = \mathcal{U}''|X, h|X: (X, \mathcal{U}) \to (X'', \mathcal{U}'')$ is uniformly continuous, thus Proposition 1.3.2.4 can be applied and shows that $h: (X', \mathcal{U}') \to (X'', \mathcal{U}'')$ is uniformly continuous. Interchanging the roles of X' and X'', we get in the same way that $h^{-1}: (X'', \mathcal{U}'') \to (X', \mathcal{U}')$ is uniformly continuous. Since by Proposition 1.3.3.16 $\mathcal{U}'' = \mathcal{U}'_c|X', h: (X', \mathcal{U}') \to (X'', \mathcal{U}'')$

is the required isomorphism. The uniqueness of h results from the fact that uniform isomorphism $h: (X', \mathcal{U}') \to (X'', \mathcal{U}'_c | X'')$ is a homeomorphism $h: (X', \tau_{\mathcal{U}'}) \to (X'', \tau_{\mathcal{U}'_c} | X'')$ if $X \subset X'' \subset X'_c$. Thus Theorem 1.3.1.2 can be applied.

On account of Corollary 1.2.9.4, \mathcal{U}' and $\mathcal{U}'_c|h(X')$ are simultaneously complete, namely by Proposition 1.3.3.19 if and only if $h(X') = X'_c$.

Definition 1.3.3.5 The uniform space (X', U') is called a completion of the uniform space (X, U) if U' is a reduced complete extension of U.

Corollary 1.3.3.4 Every uniform space (X, \mathcal{U}) has a completion: the space (X'_c, \mathcal{U}'_c) constructed in the proof of Theorem 1.3.3.4 is of this kind. Two completions of the space (X, \mathcal{U}) can be mapped onto each other by means of a uniquely defined isomorphism fixing X.

Proof: It follows from Theorem 1.3.3.4 and Proposition 1.3.3.20.

Corollary 1.3.3.5 Let (X, U) be an arbitrary uniform space, (X', U') its reduced extension and (X'', U'') a completion of (X', U'). Then (X'', U'') is a completion of the space (X, U).

Proof: It needs only to be proved that (X'', \mathcal{U}'') is a reduced extension of (X, \mathcal{U}) which follows from Corollary 1.3.2.1 and Proposition 1.3.1.10.

Corollary 1.3.3.6 If the uniform space (X'', \mathcal{U}'') is a completion of (X, \mathcal{U}) , $X \subset X' \subset X''$ and $\mathcal{U}' = \mathcal{U}''|X'$, then (X'', \mathcal{U}'') is a completion of (X', \mathcal{U}') .

Proof: Corollary 1.3.2.1 and Proposition 1.3.1.10 can be applied again.

Corollary 1.3.3.7 The completion of a separated uniform space is separated as well.

Proof: The proof follows immediately from Proposition 1.3.3.18.

Corollary 1.3.3.8 The completion of a pseudo-metrizable (metrizable) uniform space is pseudo-metrizable (metrizable) as well.

Proof: By means of Corollary 1.3.3.7, it follows from Corollary 1.3.3.3.

Corollary 1.3.3.9 Let (X, U) be a uniform space. The following statements are equivalent:

- (a) (X, \mathcal{U}) is precompact;
- (b) the completion of (X, \mathcal{U}) is compact;
- (c) uniform space (X, \mathcal{U}) has a compact extension.

Proof: $(a) \Rightarrow (b)$: If (X', \mathcal{U}') denotes the completion of (X, \mathcal{U}) , then by Proposition 1.2.6.5 \mathcal{U}' is precompact as well, hence, by Proposition 1.2.11.15, it is compact.

 $(b) \Rightarrow (c)$: Evident.

 $(c) \Rightarrow (a)$: If (X', \mathcal{U}') is a compact extension of the space (X, \mathcal{U}) , then \mathcal{U}' is precompact on account of Proposition 1.3.2.6. Thus, by Corollary 1.2.6.1, \mathcal{U} is precompact as well.

1.3.4 Extensions of proximity spaces

Proposition 1.3.4.1 If (X', δ') is an arbitrary extension of the proximity space (X, δ) , $\mathcal{N}'(x)$ denoting for $x \in X'$ the $\tau_{\delta'}$ -neighborhood filter of xand $\mathcal{F}(x) = \mathcal{N}'(x) \cap \{X\}$ is its trace filter in X, then $\mathcal{F}(x)$ is a round compressed filter in (X, δ) and, in particular, if $x \in X$, it is identical with the τ_{δ} -neighborhood filter $\mathcal{N}(x)$ of x.

Proof: $\tau_{\delta'}$ is an extension of τ_{δ} by Proposition 1.3.2.6. Thus $\mathcal{F}(x) = \mathcal{N}(x)$ whenever $x \in X$. On account of Proposition 1.3.3.3, $\mathcal{N}'(x)$ is δ' -round filter for every $x \in X'$, thus $\mathcal{F}(x)$ is a δ -round filter by Proposition 1.3.3.5. According to Proposition 1.2.11.1, $\mathcal{N}'(x)$ is δ' -compressed, thus by Corollary 1.2.11.3 $\mathcal{F}(x)$ is δ' -compressed, and then by Corollary 1.2.11.2 δ -compressed, too.

Proposition 1.3.4.2 Let (X', δ') be an extension of the proximity space $(X, \delta), \mathcal{U}$ and \mathcal{U}' the precompact uniformities inducing δ and δ' respectively. Then \mathcal{U}' is an extension of \mathcal{U} .

Proof: $\delta_{\mathcal{U}'|X} = \delta_{\mathcal{U}'}|X = \delta'|X = \delta$ by Proposition 1.2.4.3 and $\mathcal{U}'|X$ is precompact on account of Corollary 1.2.6.1. $\mathcal{U}'|X = \mathcal{U}$ according to Corollary 1.2.6.4; further, by $\tau_{\mathcal{U}'} = \tau_{\delta_{\mathcal{U}'}} = \tau_{\delta'}$, X is $\tau_{\mathcal{U}'}$ -dense in X'.

The following theorem corresponds to Theorem 1.3.3.3.

Theorem 1.3.4.1 Let (X, δ) be a proximity space, $X \subset X'$, and to every point $x \in X'$ let us assign a round compressed filter in X, in particular, for $x \in X$, let $\mathcal{F}(x) = \mathcal{N}(x)$ be the τ_{δ} -neighborhood filter of x. Then there exists exactly one proximity δ' on X' such that δ' is an extension of δ and, for all $x \in X'$, $\mathcal{F}(x) = \mathcal{N}'(x) \cap \{X\}$ where $\mathcal{N}'(x)$ is the $\tau_{\delta'}$ -neighborhood filter of the point x. **Proof:** The uniqueness of the proximity δ' with the given property follows from Proposition 1.3.3.2. Let us show that a δ' of this kind exists.

Let \mathcal{U} be the precompact uniformity inducing δ ; this exists by Corollary 1.2.10.5. The filters $\mathcal{F}(x)$ are \mathcal{U} -Cauchy filters by Proposition 1.2.11.6 so that the existence of an extension \mathcal{U}' of \mathcal{U} which furnished the given trace filters $\mathcal{F}(x)$ is guaranteed by Theorem 1.3.3.3. Then the proximity $\delta' = \delta_{\mathcal{U}'}$ will do, as on account of Proposition 1.3.2.5, $\delta_{\mathcal{U}'}$ is an extension of $\delta_{\mathcal{U}} = \delta$.

In the preceding theorem the construction of δ' was done by means of the extension of the precompact uniformity inducing δ . However, we δ' can also be obtained by extending the topology induced by δ . More precisely, there exists:

Proposition 1.3.4.3 Under the hypotheses of the previous theorem, let τ' be the strict extension of the topology τ_{δ} with respect to the trace filters $\mathcal{F}(x)$. Then, for $A', B' \subset X', A'\overline{\delta'}B'$ holds if and only if there are $A, B \subset X$ such that $A\overline{\delta}B, A' \subset \overline{A}, B' \subset \overline{B}$, where \overline{Y} denotes the τ' -closure of the set $Y \subset X'$.

Proof: On account of Proposition 1.3.2.6, $\tau_{\delta'}$ is a strict extension of τ_{δ} and by Corollary 1.3.1.1 $\tau' = \tau_{\delta'}$. If now $A\overline{\delta}B$, then $A\overline{\delta'}B$, and by Proposition 1.1.2.4 $\overline{A}\overline{\delta'}\overline{B}$ so that $A' \subset \overline{A}$ and $B' \subset \overline{B}$ implies $A'\overline{\delta'}B'$. Conversely, if $A'\overline{\delta'}B'$, then $X' - B' \in \mathcal{P}'(A')$ (denoting by this the δ' -proximity filter of A'), hence by Proposition 1.1.1.5 (e) there exist $C', D' \subset X'$ such that $C' \in \mathcal{P}'(A'), D' \in \mathcal{P}'(C'), X' - B' \in \mathcal{P}'(D')$, i.e. $C'\overline{\delta'}X' - D'$ and by Proposition 1.1.1.5 (b) $X' - D' \in \mathcal{P}'(B')$. Let $A = C' \cap X, B = (X' - D') \cap X$. Then $A\overline{\delta'}B$, and, at the same time, $A\overline{\delta}B$, further $A' \subset \overline{A}, B' \subset \overline{B}$. In fact, if $x \in A'$, taking an arbitrary set $V \in \mathcal{N}'(x) = \mathcal{P}'(\{x\})$, then $V \cap C' \in \mathcal{P}'(\{x\})$ since $C' \in \mathcal{P}'(\{x\})$ from Proposition 1.1.1.5 (c). Since X is τ' -dense, then $V \cap C' \cap X \neq \emptyset$, i.e. $V \cap A \neq \emptyset$. It can be similarly proved that $B' \subset \overline{B}$.

Proposition 1.3.4.4 Under the hypotheses of Theorem 1.3.4.1, let $X \subset X'_1 \subset X'$ and δ'_1 be the extension of δ on X'_1 constructed with the help of the trace filters $\mathcal{F}(x)$, $x \in X'_1$. Then $\delta'_1 = \delta' |X'_1$.

Proof: $\delta'|X'_1$ is also an extension of δ by Proposition 1.1.5.1, namely that one with the trace filters $\mathcal{F}(x)$.

Proposition 1.3.4.5 The following two statements are equivalent under the hypotheses of Theorem 1.3.4.1:

(a) if x ∈ X' - X, then F(x) is non-convergent with respect to τ_δ and x, y ∈ X' - X, x ≠ y imply F(x) ≠ F(y);
(b) τ_{δ'} is a reduced extension of τ_δ.

Proof: On account of Propositions 1.3.2.6 and 1.3.1.9 (b) means that if $x \in X' - X$, $y \in X'$ and $x \neq y$, then $\mathcal{F}(x) \neq \mathcal{F}(y)$ which is the same as (a) by Proposition 1.3.3.13.

Definition 1.3.4.1 The extension (X', δ') of the proximity space (X, δ) is said to be **reduced** if it fulfils the statement (b) of the previous theorem (hence condition (a)).

According to Proposition 1.3.3.18, we obtain:

Corollary 1.3.4.1 Every reduced extension of a separated proximity space is also separated.

The following corresponds now to Theorem 1.3.3.4.

Theorem 1.3.4.2 Let (X, δ) be an arbitrary proximity space, $X'_k \supset X$ a set such that, with the points $x \in X'_k - X$, there are associated in a one-to-one manner all non- τ_{δ} -convergent, compressed, round filters in (X, δ) . Let us denote by $\mathcal{F}(x)$ the filter associated in this way with the point $x \in X'_k - X$, while $\mathcal{F}(x) = \mathcal{N}(x)$ for $x \in X$, where $\mathcal{N}(x)$ is a τ_{δ} -neighborhood filter of x. Then the proximity δ'_k constructed according to Theorem 1.3.4.1 on X'_k is compact and is a reduced extension of δ .

Proof: If \mathcal{U} denotes the precompact uniformity inducing δ , then the nonconvergent round compressed filters in (X, δ) are by Theorem 1.1.3.4 the same as the non-convergent round Cauchy filters in (X, \mathcal{U}) . Accordingly, with the notation of Theorem 1.3.3.4, we can write $X'_k = X'_c$ and $\delta'_k = \delta_{\mathcal{U}'_c}$ by Theorem 1.3.4.1. Since \mathcal{U}'_c is compact by Corollary 1.3.3.9, all statements are proved by Theorem 1.3.3.4.

Proposition 1.3.4.6 Let (X, δ) be an arbitrary proximity space, (X'_k, δ'_k) its extension constructed in the proof of Theorem 1.3.4.2, $X \subset X' \subset X'_k$. If $(X', \delta'_k | X')$ is compact, then $X' = X'_k$.

Proof: Let \mathcal{U}' be the precompact uniformity inducing δ'_k . \mathcal{U}' is complete by Proposition 1.3.2.6, and it is evidently a reduced extension of $\mathcal{U}'|X = \mathcal{U}$, hence, it is identical with the complete extension of \mathcal{U} . If $\delta'_k|X'$ is compact, then $\mathcal{U}'|X'$ is complete, thus the statement follows from Proposition 1.3.3.19 and Corollary 1.3.3.4. **Proposition 1.3.4.7** Let (X'_k, δ'_k) be the extension of the proximity space (X, δ) described in Theorem 1.3.4.2 and (X', δ') an arbitrary reduced extension of the same space. Then (X', δ') can be mapped by means of a uniquely determined δ -homeomorphism h fixing X onto a suitable subspace (X'', δ'') of (X'_k, δ'_k) , where $X \subset X'' \subset X'_k$, $\delta'' = \delta'_k |X''$. δ' is compact if and only if $h(X') = X'' = X'_k$.

Proof: Let $\mathcal{U}, \mathcal{U}'_c$ and \mathcal{U}' be the precompact uniformities inducing the proximities δ, δ'_k and δ' respectively. \mathcal{U}'_c and \mathcal{U}' are extensions of \mathcal{U} by Proposition 1.3.4.2, namely reduced extensions, \mathcal{U}'_c is also complete according to Proposition 1.3.2.6, thus there exists by Proposition 1.3.3.20 a uniform isomorphism $h: (X', \mathcal{U}') \to (X'', \mathcal{U}'_c | X'')$ fixing $X, X \subset X'' \subset X'_k$. According to Proposition 1.2.5.2, $h: (X', \delta') \to (X'', \delta''_k | X'')$ is a δ -homeomorphism. On the other hand, if, for a set $X \subset X'' \subset X'_k$, $g: (X', \delta') \to (X'', \delta'_k | X'')$ is a δ -homeomorphism fixing X, then g is by Corollary 1.2.6.1 also a uniform isomorphism with respect to the uniformity \mathcal{U}' and $\mathcal{U}'_c | X''$, since, by Corollary 1.2.6.1, $\mathcal{U}'_c | X''$ is a precompact uniformity. Hence, according to Proposition 1.3.3.20, g = h. δ' and $\delta'_k | X''$ are simultaneously compact, namely by the previous proposition if and only if $X'' = X'_k$.

Definition 1.3.4.2 The proximity space (X', δ') is said to be the **Smirnoff** compactification of the proximity space (X, δ) if δ' is a compact and reduced extension of δ .

The Smirnoff compactification of the proximity space (X, δ) will be denoted by uX. The set uX consists, by Theorem 1.3.4.1, from all compressed, round filters in X, i.e. from maximal round filters by Theorem 1.3.3.1, i.e. from ends in X by Theorem 1.3.3.2. Smirnoff defined the proximity in uX with the help of the operator $O\langle \rangle$, which corresponds to each set $A \subset X$ the set of all ends $\xi \in uX$ which contain the set A as an element: $O\langle A \rangle = \{\xi \in uX : A \in \xi\}.$

Now let C and D be any two sets of uX. We will say they are far apart if and only if there are two sets A and B in X which are far apart such that $C \subset O\langle A \rangle$ and $D \subset O\langle B \rangle$. It can be proved that a relation defined in such a way is a proximity on uX which is equivalent to the proximity defined in Proposition 1.3.4.3. Operator $O\langle \rangle$ has the following simple, but important properties:

- (a) $O\langle A \rangle \cap O\langle B \rangle = O\langle A \cap B \rangle$ for any $A, B \subseteq X$;
- (b) $\cup_{\lambda} O\langle A_{\lambda} \rangle \subseteq O\langle \cup A_{\lambda} \rangle$ for any $\{A_{\lambda}\} \subseteq P(X)$;
- (c) if X A and X B are far in X, then $O\langle A \rangle \cup O\langle B \rangle = uX$;

(d) $O(\operatorname{int} A) = O(A)$ holds for any $A \subseteq X$;

(e) $O\langle A \rangle$ is open in uX for any $A \subseteq X$;

(f) the sets $O\langle\Gamma\rangle$, where Γ is any open set in X, form a basis for compact uX;

(g) for any set A of the proximity space X, the set $O\langle A \rangle$ is the largest among the open sets of the space uX which trace is int A in X;

(h) $\overline{O\langle H \rangle}^{uX} = \overline{H}^{uX}$ for each $H \in \tau_{\delta}$;

(i) $\overline{A}^{uX} = uX - O\langle X - A \rangle$ for each $A \subseteq X$.

Let us prove some of these properties.

(c) For this it is enough to prove that any end in $\xi \in uX$ contains either A or B, if X - A and X - B are far in X. In case that A = X it is clear that $X \in \xi$. Otherwise X - A = D is non-empty, so the system ξ_D of all δ -neighborhoods of D is a centered by property (O_6) formulated in Theorem 1.1.1.1. If every H of an end ξ meets D, then the union $\xi \cup \xi_D$ of the system ξ_D with the end ξ will be centered δ -system, so $\xi_D \subseteq \xi$. But $D\overline{\delta}X - B$, which means $B \in \xi_D$ and consequently $B \in \xi$. Finally, in the remaining case when D is non-empty and the end ξ has an element H which does not meet D, it follows that $H \subseteq A$, and so $A \in \xi$, which proves our assertion.

(e) On account of Proposition 1.1.2.5, it suffices to prove that $O\langle A \rangle$ is a δ -neighborhood of each end $\xi \in O\langle A \rangle$. Indeed, if $\xi \in O\langle A \rangle$, i.e. if $A \in \xi$, then there are sets B and C in ξ such that $C \ll B \ll A$. Since B and X - A are far, by (c) it follows that $uX = O\langle X - B \rangle \cup O\langle A \rangle$, whence $uX - O\langle A \rangle \subseteq O\langle X - B \rangle$. But, as a matter of fact, $\xi \in O\langle C \rangle$ and the sets C and X - B are far apart. This means by our definition of proximity in uX that ξ is far from $uX - O\langle A \rangle$, which was to be proved.

We know that the system ξ_x of all δ -neighborhood of any point $x \in X$ is an end. It is easy to prove that a mapping $f : X \to uX$, assigning to each point $x \in X$ the end $\xi_x \in uX$, is a δ -homeomorphism of X into uX. However, it can be proved that this mapping is at the same time a δ -homeomorphism for which $f^{-1}(O\langle A \rangle) = \operatorname{int} A$, where A is an arbitrary set in X. Identifying each point $x \in X$ with the end ξ_x , we can see that the proximity space X is a subspace of the proximity space uX.

In other words, the operator $O\langle P \rangle = O_P \langle \rangle$, considered only on the system of all open sets of proximity space X, is, in fact, the well-known operator $O() = O_{uX}^X()$, which corresponds to each open set Γ of X the largest open set H of uX excises Γ from X (see [294]).

Theorem 1.3.4.2 and Proposition 1.3.4.7 give

Corollary 1.3.4.2 Every proximity space (X, δ) has a compactification; (X'_k, δ'_k) in Theorem 1.3.4.2 is one of them. Two compactificatons of the

space (X, δ) can be mapped onto each other by means of a uniquely determined δ -homeomorphism fixing X.

By Proposition 1.3.4.7 we have:

Corollary 1.3.4.3 The compactification of a separated proximity space is separated.

Proposition 1.3.4.8 Let (X', δ') be a compactification of the proximity space (X, δ) . If \mathcal{U} is a uniformity inducing δ , then there corresponds to it a uniquely determined set $X \subset X'_{\mathcal{U}} \subset X'$ and a uniquely determined uniformity $\mathcal{U}'_{\mathcal{U}}$ on it such that $(X'_{\mathcal{U}}, \mathcal{U}'_{\mathcal{U}})$ is the completion of (X, \mathcal{U}) and $\delta_{\mathcal{U}'_{\mathcal{U}}} = \delta' | X'_{\mathcal{U}}$ is fulfilled. If \mathcal{U}_1 and \mathcal{U}_2 are uniformities inducing δ and if $\mathcal{U}_1 < \mathcal{U}_2$, then $X'_{\mathcal{U}_1} \supset X'_{\mathcal{U}_2}$.

Proof: If $\delta_{\mathcal{U}} = \delta$ and (X'_c, \mathcal{U}'_c) is the completion of the space (X, \mathcal{U}) , then $(X'_c, \delta_{\mathcal{U}'_c})$ is evidently a reduced extension of (X, δ) . Hence there exists a uniquely determined set $X \subset X'_{\mathcal{U}} \subset X'$ and uniquely determined δ -homeomorphism $h: (X'_c, \delta_{\mathcal{U}'_c}) \to (X'_{\mathcal{U}}, \delta'|X'_{\mathcal{U}})$ fixing X. If f denotes its inverse and $\mathcal{U}'_{\mathcal{U}} = f^{-1}(\mathcal{U}'_c)$, then $\mathcal{U}'_{\mathcal{U}}$ is the required uniformity on $X'_{\mathcal{U}}$ since fis a uniform isomorphism with respect to $\mathcal{U}'_{\mathcal{U}}$ and \mathcal{U}'_c by Proposition 1.2.5.10, $\delta_{\mathcal{U}'_{\mathcal{U}}} = f^{-1}(\delta_{\mathcal{U}'_c})$ by Proposition 1.2.4.7 and hence $\delta_{\mathcal{U}'_{\mathcal{U}}} = \delta'|X'_{\mathcal{U}}|$ by Proposition 1.1.6.11. On account of Theorem 1.3.3.4 and Proposition 1.3.3.20, $X'_{\mathcal{U}}$ consists evidently of those points $x \in X'$ whose $\tau_{\delta'}$ -neighborhood filter in Xfurnishes a trace filter $\mathcal{F}(x)$ which is a \mathcal{U} -Cauchy filter. From this and from Corollary 1.2.9.1 it follows that $\mathcal{U}_1 < \mathcal{U}_2$ implies $X'_{\mathcal{U}_1} \supset X'_{\mathcal{U}_2}$.

Let us consider compactification of completely regular spaces. First, let us introduce the following notion.

Definition 1.3.4.3 The space (X', τ') is said to be an ordinary compactification of the completely regular space (X, τ) if (X', τ') is a compact S_2 -space and a reduced extension of (X, τ) .

Definition 1.3.4.4 Let (X'_1, τ'_1) and (X'_2, τ'_2) be two ordinary compactifications of the space (X, τ) . We say that (X'_1, τ'_1) is a coarser compactification than (X'_2, τ'_2) , or that (X'_2, τ'_2) is a finer compactification than (X'_1, τ'_1) , if there exists a continuous surjection $f : X'_2 \to X'_1$ fixing X. We say that (X'_1, τ'_1) and (X'_2, τ'_2) are equivalent compactifications if there exists a homeomorphism from X'_1 onto X'_2 fixing X.

The latter relation is obviously reflexive, symmetric and transitive.

Proposition 1.3.4.9 Let (X, τ) be a completely regular space, δ a proximity relation inducing τ , and (X', δ') a compactification of the space (X, τ) . Then $(X', \tau_{\delta'})$ is an ordinary compactification of the space (X, τ) .

Proof: $(X', \tau_{\delta'})$ is a reduced extension of (X, τ) , it is compact, an S_3 -space and a fortiory S_2 -space.

Proposition 1.3.4.10 Let (X, τ) be a completely regular space, (X', τ') an ordinary compactification of it. Then there exists exactly one proximity δ' on X' inducing τ' such that (X', δ') is the compactification of $(X, \delta'|X)$ and $\tau = \tau_{\delta'|X}$.

Proof: By Theorem 1.1.3.4 there exists exactly one proximity δ' inducing τ' and, on account of Proposition 1.1.5.1, $\tau = \tau'|X = \tau_{\delta'|X}$. Therefore (X', τ') is a reduced, compact extension of $(X, \delta'|X)$.

Theorem 1.3.4.3 (Smirnoff's theorem) Let (X, τ) be a completely regular space. Every ordinary compactification of this space can be obtained by constructing the compactification (X', δ') of (X, δ) for a proximity δ inducing τ and choosing τ' equal to $\tau_{\delta'}$. In this way an ordinary compactification of (X, τ) is obtained from any proximity δ inducing τ . Let δ_1 and δ_2 be two proximities inducing τ , and (X'_1, τ'_1) and (X'_2, τ'_2) ordinary compactifications corresponding to them in the way mentioned above. The compactification (X'_1, τ'_1) is coarser than the compactification (X'_2, τ'_2) if and only if $\delta_1 < \delta_2$. These two compactifications are equivalent if and only if $\delta_1 = \delta_2$.

Proof: The first statements are repetitions of Propositions 1.3.4.9 and 1.3.4.10. Let us suppose that δ_1 , δ_2 , (X'_1, τ'_1) and (X'_2, τ'_2) have the properties described in the theorem, and δ'_1 and δ'_2 are extensions of δ_1 and δ_2 inducing τ'_1 and τ'_2 respectively. If the compactification (X'_1, τ'_1) is coarser than (X'_2, τ'_2) , then there exists a continuous surjection $f : (X'_2, \tau'_2) \to (X'_1, \tau'_1)$ fixing X. By Proposition 1.1.6.10, f is at the same time δ -continuous with respect to δ'_2 and δ'_1 and then on account of Proposition 1.1.6.6 and Proposition 1.1.6.11 $f|_X^X$ is δ -continuous with respect to δ_2 and δ_1 ; thus by Corollary 1.1.6.4 $\delta_1 < \delta_2$. Conversely, if $\delta_1 < \delta_2$, then the identity mapping of X is δ -continuous with respect to δ'_2 and δ'_1 . Hence Proposition 1.3.2.8 guarantees the existence of a δ -continuous mapping $f : X'_2 \to X'_1$ such that $f|_X^X = g$. Accordingly, $f : (X'_2, \tau'_2) \to (X'_1, \tau'_1)$ is a continuous mapping fixing X on account of

Proposition 1.1.6.8. Since f is continuous, the topology $\tau'_1|f(X'_2)$ is compact and since $X \subset f(X'_2) \subset X'_1$, on account of Proposition 1.3.4.6 and Corollary 1.3.4.2 we have that $f(X'_2) = X'_1$.

If the compactifications (X'_1, τ'_1) and (X'_2, τ'_2) are equivalent, then evidently either of them is coarser than the other one and hence, by the foregoing $\delta_1 < \delta_2 < \delta_1$, which implies $\delta_1 = \delta_2$. On the other hand, if $\delta_1 = \delta_2$, then (X'_1, δ'_1) and (X'_2, δ'_2) are two compactifications of the proximity space (X, δ) , where $\delta = \delta'_1 = \delta'_2$. Thus by Corollary 1.3.4.2 there exists a δ -homeomorphism with respect to the proximities δ'_1 and δ'_2 fixing X which is a homeomorphism with respect to the topologies τ'_1 and τ'_2 by Proposition 1.1.6.8.

Proposition 1.3.4.11 The ordinary compactifications of a Tychonoff space coincide with the T_2 -compactifications of the space.

Proof: If (X, τ) is a completely regular T_0 -space, then its ordinary compactifications are by the previous theorem simultaneous T_0 - and S_2 -spaces, thus T_2 -spaces. Conversely a T_2 -compactification is an S_2 -space and T_0 -space; thus by Corollary 1.3.1.2 it is a reduced S_2 -compactification.

The proximity belonging to a given ordinary compactification according to Theorem 1.3.4.3 can be obtained directly:

Proposition 1.3.4.12 Let (X, τ) be a completely regular space, δ a proximity inducing τ , (X', δ') a compactification of the space (X, δ) and $\tau' = \tau_{\delta'}$. Then, for $A, B \subset X$, $A\delta B$ holds if and only if $\overline{A}^{\tau'} \cap \overline{B}^{\tau'} \neq \emptyset$.

Proof: $A\overline{\delta}B$ holds if and only if $A\overline{\delta'}B$, and this is valid by Proposition 1.1.2.4 if and only if $\overline{A}^{\tau'}\overline{\delta'}\overline{B}^{\tau'}$. The latter is by Theorem 1.1.3.4 equivalent to $\overline{A}^{\tau'}\cap\overline{B}^{\tau'}=\emptyset$.

With the help of this result, the following can be easily proved:

Theorem 1.3.4.4 Let (X, τ) be a non-compact, completely regular space. The Alexandroff compactification of this space is an ordinary compactification if and only if the space is locally compact and then this is the coarsest ordinary compactification of the space.

Proof: By Proposition 1.3.1.14 the Alexandroff compactification (X', τ') is always a reduced extension and it is on account of Proposition 1.3.1.15 - for a completely regular topology τ - an S_2 -space if and only if τ is locally compact. If (X, τ) is a space with this property, then the proximity

 δ corresponding to (X', τ') on X holds between A and B if and only if the τ' -closures of A and B intersect each other; in other words $A\overline{\delta}B$ holds if and only if $\overline{A}^{\tau'} \cap \overline{B}^{\tau'} = \emptyset$. The latter is equivalent to the fact that the τ -closures of A and B do not intersect and moreover - with the usual notation $X' = X \cup \{\omega\}$ - one of $\overline{A}^{\tau'}$ and $\overline{B}^{\tau'}$ at the most contains ω . But if $\omega \notin \overline{A}^{\tau'}$, then $\overline{A} = \overline{A}^{\tau'} \cap X$ is the τ -closure of A and it is compact. On the other hand, if $\overline{A}^{\tau'} \cap X$ is compact, then $X - \overline{A}^{\tau'}$ belongs to the trace filter $\mathcal{F}(\omega)$, so that $(X - \overline{A}^{\tau'}) \cup \{\omega\}$ is a τ' -neighborhood of ω not intersecting A and $\omega \notin \overline{A}^{\tau'}$. Finally, we can say that δ is identical with the proximity defined in Theorem 1.2.11.5 from which it is known that it is the coarsest proximity inducing the topology τ .

It is known that there exists the finest among the proximities inducing the completely regular topology τ ; this is called the Czech-Stone proximity. By Theorem 1.3.4.3, the ordinary compactification corresponding to it is the finest ordinary compactification. On account of Proposition 1.2.10.11 we have:

Proposition 1.3.4.13 Let (X, τ) be a completely regular space, δ the proximity on X for which $A\overline{\delta}B$ holds if and only if A and B are separated by a τ -continuous function, (X', δ') the compactification of the space (X, δ) and $\tau' = \tau_{\delta'}$. Then (X', τ') is the finest ordinary compactification of the space (X, τ) .

Definition 1.3.4.5 The space (X', τ') in Proposition 1.3.4.13 and the topology τ' are called the **Czech-Stone compactification** of the space (X, τ) and of the topology τ respectively.

It follows from Theorem 1.3.4.3 that two Czech-Stone compactifications of the space (X, τ) can be mapped onto each other by means of a homeomorphism fixing X.

Important characteristic properties of the Czech-Stone compactification are contained in the following theorem:

Theorem 1.3.4.5 Let (X', τ') be an ordinary compactification of the completely regular space (X, τ) . The following statements are equivalent:

(a) (X', τ') is the Czech-Stone compactification of (X, τ) ;

(b) if (Y, τ_1) is a compact S_2 -space and $f : (X, \tau) \to (Y, \tau_1)$ is a continuous mapping, then there exists a continuous mapping $g : (X', \tau') \to (Y, \tau_1)$ such that g|X = f; (c) if f is a τ -continuous bounded function, then there exists a τ' -continuous function g such that g|X = f.

Proof: $(a) \Rightarrow (b)$: Let δ be the Czech-Stone proximity of the topology τ , δ' its extension inducing τ' , δ_1 the proximity inducing τ_1 (it is unique by Theorem 1.1.3.4). If $f: (X, \tau) \to (Y, \tau_1)$ is a continuous mapping, then it is δ -continuous by Proposition 1.1.6.9. Thus on account Proposition 1.3.2.8 there exists a δ -continuous mapping $g: (X', \delta') \to (Y, \delta_1)$ such that g|X = f. Then $g: (X', \tau') \to (Y, \tau_1)$ is continuous as well.

 $(b) \Rightarrow (c)$: Let $I \subset \mathbb{R}$ be a finite closed interval such that $f(X) \subset I$. Then $h = f|_X^I : X \to I$ is continuous with respect to τ and $\mathcal{E}|I$, where $\mathcal{E}|I$ is a compact T_2 -topology. Furthermore there exists continuous function $k : (X', \tau') \to (I, \mathcal{E}|I)$ such that k|X = h. If $m : I \to \mathbb{R}$ is the canonical injection and $g = m \circ k$, then $g : (X', \tau') \to (\mathbb{R}, \mathcal{E})$ is the required continuous extension of f.

 $(c) \Rightarrow (a)$: Let δ'_0 be the proximity inducing τ' , $\delta_0 = \delta'_0 | X$. It is to be shown that δ_0 is identical with the Czech-Stone proximity δ of τ , i.e. that $\delta < \delta_0$, on account of $\delta_0 < \delta$. However, if $A\overline{\delta}B$, i.e. if by Proposition 1.2.10.11 A and B are separated by a τ -continuous function f, then let $g: X' \to \mathbb{R}$ be a τ' -continuous function for which g|X = f holds. Then $g(\overline{A}^{\tau'}) = 0, g(\overline{B}^{\tau'}) = 1$, so that $\overline{A}^{\tau'} \cap \overline{B}^{\tau'} = \emptyset$. But then according to Proposition 1.3.4.12 it follows that $A\overline{\delta}_0 B$.

Proposition 1.3.4.14 Let (X, τ) be a completely regular space, (X_1, τ_1) one of its completely regular, reduced extensions. If (X_1, τ_1) is a subspace of the Czech-Stone compactification of the space (X, τ) , then every bounded τ -continuous function has a τ_1 -continuous extension. If (X_1, τ_1) has the latter property, then any Czech-Stone compactification (X_2, τ_2) of the space (X_1, τ_1) is at the same time the Czech-Stone compactification of (X, τ) .

Proof: The first statement follows directly from the previous proposition. To prove the second part of the statement, let us notice that (X_2, τ_2) is a reduced extension by Proposition 1.3.1.10, and hence an ordinary compactification of (X, τ) . If f is a bounded τ -continuous function, then it has a τ_1 -continuous extension which is itself bounded. Hence it can be τ_2 -continuous extended over X_2 . Thus the statement follows from Theorem 1.3.4.5.

The Czech-Stone compactification was originally defined with the help of the Czech-Stone proximity. However, it can be constructed by means of a uniformity as well. **Proposition 1.3.4.15** Let (X, τ) be a completely regular space, Φ^* the function family consisting of all bounded, τ -continuous functions. If (X', \mathcal{U}') is the completion of the uniform space $(X, \mathcal{U}_{\Phi^*})$, then $(X', \tau_{\mathcal{U}'})$ is the Czech-Stone compactification of (X, τ) .

Proof: The proximity $\delta_{\mathcal{U}_{\Phi^*}} = \delta_{\Phi^*}$ is identical with the Czech-Stone proximity of τ by Proposition 1.2.10.11 and \mathcal{U}_{Φ^*} is precompact on account of Proposition 1.2.10.7. (X', \mathcal{U}') is a compact, reduced extension of $(X, \mathcal{U}_{\Phi^*})$ by Corollary 1.3.3.9, thus $(X', \delta_{\mathcal{U}'})$ is identical with the compactification (X, δ_{Φ^*}) while $(X', \tau_{\delta_{\mathcal{U}'}}) = (X', \tau_{\mathcal{U}'})$ is identical with the Czech-Stone compactification of (X, τ) .

Therefore compact spaces can be also characterized among the completely regular spaces as follows:

Proposition 1.3.4.16 Let (X, τ) be a completely regular space, Φ^* the function family consisting of bounded, τ -continuous functions. The following statements are equivalent:

(a) (X, τ) is compact;

(b) \mathcal{U}_{Φ^*} is complete;

(c) (X, τ) is the Czech-Stone compactification of itself.

Proof: $(a) \Rightarrow (b)$: By Proposition 1.2.10.11 $\tau_{\mathcal{U}_{\Phi^*}} = \tau$, so that on account of Proposition 1.3.2.6 the compactness of τ implies the completeness of \mathcal{U}_{Φ^*} .

 $(b) \Rightarrow (c)$: If \mathcal{U}_{Φ^*} is complete, then $(X, \mathcal{U}_{\Phi^*})$ is a completion of itself and by the previous proposition the Czech-Stone compactification of (X, τ) is $(X, \tau_{\mathcal{U}_{\Phi^*}}) = (X, \tau)$.

 $(c) \Rightarrow (a)$ Obvious.

Historical and bibliographic notes

Specific examples of extensions of spaces, such as the completion of rational numbers by means of real numbers, or the compactification of the complex plane by adding the "point in infinity," have been known for a long time. The work on "prime ends" by C. Caratheodory in 1913 gives further impetus to the development of general theory of extensions (see [42]). The beginnings of such a theory can be found in articles of H.Tietze in 1924, who used the concept of "one-point compactification" and "absolute H-closure," (see [322]), P. S. Alexandroff in 1924 (see [3]) and P.S. Urysohn in 1924 (see [11]), who, besides using these concepts, introduce "bicompactness". A. Tychonoff in 1930 made further significant advances, among others, by pointing out the importance of complete regularity in this context (see [324]). Elaborating and analyzing Tychonoff's ideas further, E. Czech proved in 1937 that the compactification, now known as the Czech-Stone, is maximal in the set of all compactification of a Tychonoff space (see [62]). M. H. Stone also obtained the same results, as well as many other results for extensions (see [312]). A large number of results concerning the Czech-Stone compactification is collected in R. C. Walker's book [328].

The centred system is used by the Soviet school instead of the filter. An excellent survey of centred systems in topological spaces has recently been published by S. Iliadis and S. V. Fomin [149]. The concept of an end was originated by Alexandroff, while both H. Freudentahl (see [112]) and P. S. Alexandroff (see [5], p. 244) defined a round filter. Ju. M. Smirnoff used these devices in his proximal extension theory. The results on proximal extensions are due to Smirnoff [294], who was the first to explain the relationship between proximities and compactifications.

1.4 Connectedness of uniform and proximity spaces

1.4.1 Definition and basic properties

Connectedness of topological spaces can be defined in terms of continuous function to a discrete space. We will consider similar properties for proximity and uniform spaces obtained by replacing continuous functions by δ -continuous or uniformly continuous functions.

Definition 1.4.1.1 A proximity space (X, δ) is **\delta-connected** if every δ -continuous function on X to a discrete space is constant. A subset A of X is δ -connected if it is δ -connected as a proximity subspace.

Since each δ -continuous function, by Proposition 1.1.6.8, is τ_{δ} -continuous, then every τ_{δ} -connected space is also δ -connected. The converse in general case is not true.

Example 1.4.1.1 To prove that the converse is not true, let us consider the set \mathbb{Q} of rational numbers as a proximity subspace of the space of real numbers with the metric proximity δ_d . As a proximity subspace of \mathbb{R} , the space $(Q, \mathcal{E}|\mathbb{Q})$ is not connected.

However, it is δ_d -connected. Indeed, let f be any δ -continuous function from \mathbb{Q} to discrete space $\{0, 1\}$. If it is not constant, then the sets $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are far, because the sets $\{0\}$ and $\{1\}$ are far in the space $\{0, 1\}$. But then one of them must be empty. Indeed, in the opposite case these sets will be downright subsets of \mathbb{Q} whose union gives the space \mathbb{Q} ; but since they are far, it follows that $d(f^{-1}(\{0\}), f^{-1}(\{1\})) = \eta > 0$. This means that there exists an interval $(\alpha, \beta) \subset \mathbb{R}$ with the property that $|\beta - \alpha| > 0$, and $(\alpha, \beta) \cap \mathbb{Q} = \emptyset$, which is a contradiction, because the set \mathbb{Q} is dense in \mathbb{R} . Thus, one of the sets $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ must be empty, which proves that the function f is constant.

Proposition 1.4.1.1 In any proximity space (X, δ) the following statements are equivalent:

- (a) the space (X, δ) is δ -connected;
- (b) $A\delta(X A)$ for each subset A of X, $\emptyset \neq A \neq X$;
- (c) if $X = A \cup B$ and $A\overline{\delta}B$, then one of the sets A and B is empty.

Proof: $(a) \Rightarrow (b)$: Let us suppose that there exists a non-empty set $A \neq X$ such that $A\overline{\delta}(X - A)$. Let us define the function $f : X \to \{0, 1\}$ in the following manner: $f(A) = \{0\}, f(X - A) = \{1\}$. Since $\{0, 1\}$ is a discrete space, the sets $\{0\}$ and $\{1\}$ are only sets which are far. Also, we have that $f^{-1}(\{0\})\overline{\delta}f^{-1}(\{1\})$, and therefore f is a δ -continuous function which is not constant.

 $(b) \Rightarrow (c)$: Let us suppose that there are non-empty sets $A, B \subset X$ such that $X = A \cup B$ and $A\overline{\delta}B$. Since $X = A \cup B$, then $X - A \subset B$. Therefore, by Proposition 1.1.1.2 (b) $A\overline{\delta}X - A$ holds.

 $(c) \Rightarrow (a)$: Let $f: X \to \{0, 1\}$ be a δ -continuous function which is not constant. Then $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$ are non-empty sets for which it is obvious that $A \cup B = X$, and since f is a δ -continuous function, we have that $A\overline{\delta}B$.

Corollary 1.4.1.1 A proximity space (X, δ) is δ -connected if and only if it can not be presented as the union of two non-empty far sets.

Proposition 1.4.1.2 A subspace (Y, δ_Y) of the proximity space (X, δ_X) is δ -connected if and only if for each two sets A and B for which $Y = A \cup B$ and $A\overline{\delta}_X B$ holds, one of them is empty.

Proof: Let us suppose that the subspace Y is δ -connected and let A and B be the sets for which $Y = A \cup B$ and $A\overline{\delta}_X B$ holds. Then, $A\overline{\delta}_Y B$, and therefore, by the previous proposition, $A = \emptyset$ or $B = \emptyset$ holds.

To prove the converse, let us suppose that Y is not a δ -connected subspace of X. Then by previous proposition there are non-empty sets A and B such that $Y = A \cup B$ and $A\overline{\delta}_Y B$. But then for the sets A and B we have that $Y = A \cup B$ and $A\overline{\delta}_X B$.

Proposition 1.4.1.3 Let (Y, δ_Y) be a δ -connected subspace of proximity space (X, δ_X) . If the sets A and B are far in X and $Y \subset A \cup B$, then either $Y \subset A$ or $Y \subset B$.

Proof: The sets $A \cap Y$ and $B \cap Y$ are far in X by Proposition 1.1.1.2 (b), while their union is equal to Y. Therefore, by the previous proposition, one of them must be empty. But then the set Y is contained in the other.

Proposition 1.4.1.4 Let Y be a δ -connected subspace of the δ -connected space X. If A and B are far subsets in the space X and if $X - Y = A \bigcup B$, then $A \cup Y$ and $B \cup Y$ are δ -connected sets.

Proof: Let us suppose that the set $A \cup Y$ is not δ -connected. Then by Proposition 1.4.1.1 there exist non-empty far sets M and N for which $A \cup Y$ $= M \cup N$ holds. Since $Y \subset A \cup Y = M \cup N$, the set Y is contained in exactly one of the sets M or N by Proposition 1.4.1.3. Let us suppose that $Y \subset N$. Then $Y \cap M = \emptyset$. So, from $M \subset A \cup Y$ follows $M \subset A$. Since $A\overline{\delta}B$, we have that $M\overline{\delta}B$. Now, from $M\overline{\delta}N$ and $M\overline{\delta}B$ we have that $M\overline{\delta}(B \cup N)$. It is obvious that $X = Y \cup A \cup B = (M \cup N) \cup B = M \cup (N \cup B)$, so X is not δ -connected, which is a contradiction. \clubsuit

Proposition 1.4.1.5 Let $\{Y_s : s \in S\}$ be a family of δ -connected subspaces of the proximity space (X, δ) . If there exists an $s_0 \in S$ such that the set Y_{s_0} is near to each of the sets Y_s , then the union $\cup \{Y_s : s \in S\}$ is a δ -connected subspace of the space X.

Proof: Let us suppose that $Y = \bigcup \{Y_s : s \in S\} = A \cup B$, where A and B are far subsets of X. Then, by the previous proposition, the set Y_{s_0} is contained in one of the sets A or B. Let us suppose that $Y_{s_0} \subset A$ holds. Then also $Y_s \subset A$ for each Y_s . Indeed, if $Y_s \subset B$ for some Y_s , then by Proposition 1.1.1.2 (b) we have that $Y_{s_0}\overline{\delta}Y_s$, contrary to the supposition. Therefore we have that $Y_s \subset A$ for each $s \in S$. Hence $Y \subset A$. But then $B = \emptyset$ holds, and thus, by Proposition 1.4.1.2, Y is a δ -connected subspace of the space X.

Corollary 1.4.1.2 If the family $\{Y_s : s \in S\}$ of δ -connected subspaces of the proximity space X has a non-empty intersection, then the union $\bigcup_{s \in S} Y_s$ is a δ -connected subspace of the space X.

Corollary 1.4.1.3 If a subspace Y of the proximity space X is δ -connected, then every subspace Z of X which satisfies the condition $Y \subset Z \subset \overline{Y}$ is also δ -connected.

Proof: The family $\{Y \cup \{x\} : x \in Z\}$ satisfies the condition in Proposition 1.4.1.5, with $Y_{s_0} = Y$.

Corollary 1.4.1.4 If the proximity space X contains a δ -connected dense subspace, then X is a δ -connected space.

Corollary 1.4.1.5 If any two points of the proximity space X can be joined by a δ -connected subspace of X, then the space X is δ -connected.

Proof: Let $x_0 \in X$ be a fixed point of the space X. For every point $x \in X$ let Y_x denote a connected subspace of X joining x_0 and x. Then the family $\{Y_x : x \in X\}$ satisfies the assumptions of Consequence 1.4.1.2, which implies that $\bigcup_{x \in X} Y_x = X$ is a δ -connected space.

Proposition 1.4.1.6 The Smirnoff compactification (X^*, δ^*) of the proximity space (X, δ) is δ -connected if and only if the proximity space X is δ -connected.

Proof: Let us suppose first that the space (X, δ) is δ -connected. If the space (X^*, δ^*) is not δ^* -connected, then it is not τ_{δ^*} -connected. Therefore there exist two non-empty sets A and B which are simultaneously open and closed in X^* , different from X^* , such that

$$A \cup B = X^*, \qquad A \cap B = \emptyset.$$

But then

$$(A \cap X) \cup (B \cap X) = X, \qquad (A \cap X) \cap (B \cap X) = \emptyset,$$

for which we have that $A \cap X \neq \emptyset$ and $B \cap X \neq \emptyset$. Indeed, if the equality $A \cap X = \emptyset$ is true, then $X = B \cap X \subset B$, and since X is dense in X^* , and B is closed in X^* , we have that $B = X^*$, which is in contradiction with the choice of the sets A and B. In an analogous manner we can prove that $B \cap X \neq \emptyset$. We can also see that $A \neq X \neq B$. Indeed, if X = A, then $B \cap X = \emptyset$, which is a contradiction.

Let us prove now that

$$\overline{A \cap X}^{\tau^*} = A, \qquad \overline{B \cap X}^{\tau^*} = B,$$

where $\tau^* = \tau_{\delta^*}$. To do this, let us first note that $\overline{A}^{\tau^*} = A$ and $\overline{B}^{\tau^*} = B$. If $x \in \overline{A \cap X}^{\tau^*}$, then it is obvious that $x \in \overline{A}^{\tau^*} = A$. To prove the converse inclusion, let us suppose that there exists some point $x \in A$ which is not contained in the set $\overline{A \cap X}^{\tau^*}$. In this case there exists a neighborhood U_x of the point x in the space X^* with the property $U_x \cap (A \cap X) = \emptyset$, which is impossible, because X is dense in X^* . In an analogous manner it can be proved that $\overline{B \cap X}^{\tau^*} = B$. Now from $A \cap B = \emptyset$ follows the equality $\overline{A \cap X}^{\tau^*} \cap \overline{B \cap X}^{\tau^*} = \emptyset$. Thus, by Theorem 1.1.3.4 and Proposition 1.1.2.4 we have that $(A \cap X)\overline{\delta}^*(B \cap X)$, i.e. $(A \cap X)\overline{\delta}(B \cap X)$. Since $X - (A \cap X) \subset B \cap X$, then $(A \cap X)\overline{\delta}(X - (A \cap X))$, from which, according to Proposition 1.4.1.1, there follows that the space (X, δ) is not δ -connected, which is in contradiction to the supposition.

To prove the converse, let us first note that the discrete space $\{0, 1\}$, as a subspace of the space \mathbb{R} of real numbers, is close and compact. Let $f: X \to \{0, 1\}$ be any δ -continuous function. Then by Proposition 1.3.2.8 there exists a unique determined δ -continuous extension $f^*: X^* \to \{0, 1\}$ of f from X to the compactification X^* . Since X^* is δ^* -connected, the function f^* is constant. But then the function f is also constant, which proves that the space X is δ -connected.

Proposition 1.4.1.7 The Czech-Stone compactification (X^*, δ^*) of a proximity space (X, δ) is δ -connected if and only if the proximity space (X, δ) is δ -connected.

Proof: Let us suppose that (X^*, δ^*) is δ -connected and let $f : X \to \{0, 1\}$ be any δ -continuous function. Then by Proposition 1.3.2.8 there exists a δ -continuous extension $f^* : X^* \to \{0, 1\}$ such that $f^*|X = f$. Since X^* by supposition is δ -connected, the f^* is a constant function. But then f is also a constant function, so that X is a δ -connected space.

Conversely, if X is a δ -connected space, then, by Corollary 1.4.1.4, the space X^* is δ -connected.

Proposition 1.4.1.8 If a space (X, τ) is δ -connected with respect to any proximity relation on X which is compatible with the topology τ , then the space X is τ -connected.

Proof: Let S be a set of all proximity relations on X compatible with the topology τ . Then by Corollary 1.1.4.2 between them there exists the finest proximity δ and this is exactly the Czech-Stone proximity. It is compatible with the topology τ , so that (X, δ) is a δ -connected space. According

to Theorem 1.3.4.3, Smirnoff compactification (X^*, δ^*) is the finest ordinal compactification. Therefore (X^*, δ^*) is the Czech-Stone compactification of proximity space (X, τ) . Since, by the previous proposition, (X^*, δ^*) is a δ -connected space, it is, on account of the compactness of the space X^* and by Proposition 1.1.6.10, τ_{δ^*} -connected. Therefore (X, τ) is τ -connected by the well known theorem of general topology.

Proposition 1.4.1.9 Let f be a δ -continuous mapping from a δ -connected proximity space (X, δ_X) onto a proximity space (Y, δ_Y) . Then the space Y is δ -connected.

Proof: If the proximity space (Y, δ) is not δ -connected, then by Proposition 1.4.1.1 there exists a set $B, \emptyset \neq B \neq Y$, such that $B\overline{\delta}_Y(Y-B)$. Since f is a δ -continuous mapping, we have that $f^{-1}(B)\overline{\delta}_X f^{-1}(Y-B) = X - f^{-1}(B)$. The set $f^{-1}(B)$ is non-empty and different from X, so that the space X is not δ -connected by Proposition 1.4.1.1.

Proposition 1.4.1.10 The Cartesian product $\prod_{i \in I} X_i$, where $X_i \neq \emptyset$ for each $i \in I$, is δ -connected if and only if all spaces X_i are δ -connected.

Proof: If the Cartesian product $X = \prod_{i \in I} X_i$ is δ -connected, then all spaces X_i are δ_i -connected by previous proposition, because the projection $p_i : X \to X_i$ is a δ -continuous mapping of X onto X_i .

We shall now prove that Cartesian product of δ_i -connected spaces $(X_i, \delta_i), i \in I$, is a δ -connected space. To begin with, let us consider the Cartesian product $X \times Y$ of two δ -connected proximity spaces. Any two points (x_1, y_1) and (x_2, y_2) of the space $X \times Y$ can be joined by the set $(X \times \{y_1\}) \cup (\{x_2\} \times Y)$, which is δ -connected as the union of two δ -connected sets with a non-empty intersection. Hence, the space $X \times Y$ is δ -connected by Corollary 1.4.1.5.

By induction one can readily show that any finite Cartesian product of δ -connected spaces is also a δ -connected space.

Let us consider the family $\{X_i\}_{i \in I}$ of non-empty δ -connected spaces. For every $i \in I$ let us choose a point $a_i \in X_i$. Let us denote by \mathcal{I} the family of all finite subsets of the set I and for every $L \in \mathcal{I}$ let $C_L = \prod_{i \in I} A_i$, where $A_i = \{a_i\}$ if $i \notin L$, and $A_i = X_i$ if $i \in L$. By the finite case of our theorem, the family $\{C_L\}_{L \in \mathcal{I}}$ consists of δ -connected subspaces of the space X. Since $a = (a_i) \in \bigcap_{L \in \mathcal{I}} C_L \neq \emptyset$, it follows from Corollary 1.4.1.2 that the union $C = \bigcup_{L \in \mathcal{I}} C_L$ is δ -connected. But C is a dense subspace of X, so that we conclude the proof by applying Corollary 1.4.1.4. \clubsuit **Definition 1.4.1.2** A δ -continuous mapping f from a proximity space X to a proximity space Y is called δ -monotone if for every $y \in Y$ the set $f^{-1}(y)$ is a δ -connected set.

Definition 1.4.1.3 A δ -continuous mapping $f : (X, \delta_X) \to (Y, \delta_Y)$ is δ quotient if for each $C, D \subset Y, C\delta_Y D$ if and only if $f^{-1}(C)\delta_X f^{-1}(D)$.

Proposition 1.4.1.11 If f is a δ -monotone and δ -quotient mapping from a proximity space X onto a proximity space Y, then $f^{-1}(C)$ is a δ -connected subset of X for each δ -connected subset C of Y.

Proof: Let us suppose that $f^{-1}(C)$ is not δ -connected. Then by Corollary 1.4.1.1 there exist non-empty far sets A and B such that $f^{-1}(C) = A \cup B$. Since the mapping f is δ -monotone, we have that for each $y \in C$ the set $f^{-1}(y)$ is δ -connected and contained in one of the sets A or B by Proposition 1.4.1.3. Let us define the sets P and Q in the following way:

$$P = \{ y \in C : f^{-1}(y) \subset A \}, \quad Q = \{ y \in C : f^{-1}(y) \subset B \}.$$

It is obvious that $A = f^{-1}(P)$, $B = f^{-1}(Q)$ and $C = P \cup Q$. Since f is a δ -quotient and $A\overline{\delta}B$, then $P\overline{\delta}Q$, i.e. C is not connected, which is a contradiction.

Definition 1.4.1.4 A finite sequence of the subsets A_1, A_2, \ldots, A_n of a proximity space X is a δ -chain if $A_i \delta A_{i+1}$ for each $i = 1, 2, \ldots, n-1$. A family A of subsets of the proximity space X is called δ -chained if for every two elements A and B of A, there exists a δ -chain consisting of the elements of the family A which joins the sets A and B.

Proposition 1.4.1.12 If A_1, A_2, \ldots, A_n is a δ -chain and if the sets A_i , $i = 1, 2, \ldots, n$, are δ -connected, then the union $\cup \{A_i : i = 1, 2, \ldots, n\}$ is a δ -connected set.

Proof: For n = 2 the assertion is true by Proposition 1.4.1.5. Now, the assertion of the proposition can be proved easily by induction.

Proposition 1.4.1.13 Let $\mathcal{A} = \{A_s : s \in S\}$ be a δ -chained family. If A_s is a δ -connected set for each $s \in S$, then the set $A = \bigcup_{s \in S} A_s$ is δ -connected.

Proof: Let a and b be any two points of the set A. Let us suppose that $a \in A_{s_1}$ and $b \in A_{s_2}$. Since \mathcal{A} is a δ -chained family, there exists a δ -chain consisting of (δ -connected) elements of the family \mathcal{A} which joins the sets A_{s_1}

and A_{s_2} . According to the previous proposition the union of all the sets of this δ -chain is a δ -connected set. But then by Corollary 1.4.1.5 the set A is δ -connected.

Definition 1.4.1.5 A cover \mathcal{U} of a proximity space X is called a **proximity** cover if for any two near sets A and B there exists a set $U \in \mathcal{U}$ such that $A \cap U \neq \emptyset$ and $B \cap U \neq \emptyset$.

Proposition 1.4.1.14 Every proximity cover of a δ -connected proximity space X is a δ -chained family.

Proof: Let $\mathcal{U} = \{U_s : s \in S\}$ be a proximity cover of the δ -connected space X. Let us assume that there are the sets U_{s_1} and U_{s_2} in \mathcal{U} which cannot be joined by a δ -chain composed from elements of the cover \mathcal{U} . Let us denote with A the union of all elements of \mathcal{U} which can be joined with U_{s_1} by some δ -chain $\mathcal{C} \subset \mathcal{U}$ and let B be the union of all other elements of \mathcal{U} . It is obvious that $X = A \cup B$. Let us prove that $A\overline{\delta}B$. Indeed, if $A\delta B$, then there exists a set $U \in \mathcal{U}$ such that $U \cap A \neq \emptyset$ and $U \cap B \neq \emptyset$. Therefore there exist the sets $U_k \subset A$ and $U_m \subset B$ for which we have $U \cap U_k \neq \emptyset$ and $U \cap U_m \neq \emptyset$. But then the set U_m can be joined with U_{s_1} by some δ -chain $S \subset \mathcal{U}$, which is impossible. Thus $A\overline{\delta}B$ holds, so that X is not δ -connected. This contradiction proves the proposition.

Definition 1.4.1.6 A uniformity space (X, U) is uniformly or U-connected if every uniformly continuous mapping from X to a discrete space $\{0, 1\}$ is constant.

Proposition 1.4.1.15 Let (X, U) be a uniform space and let $\delta = \delta_U$ be a proximity on X generated by the uniformity U. Then the following conditions are equivalent:

(a) the proximity space (X, δ) is δ -connected;

(b) for every δ -continuous function $f: X \to \mathbb{R}$ the set f(X) is dense in some interval of \mathbb{R} ;

(c) the uniform space (X, \mathcal{U}) is \mathcal{U} -connected;

(d) for every uniformly continuous function $f: X \to \mathbb{R}$ the set f(X) is dense in some interval of \mathbb{R} ;

(e) the uniform space X is \mathcal{U} -chain connected, i.e. for every pair $(p,q) \in X \times X$ and every $U \in \mathcal{U}$ there exists $n \in \mathbb{N}$ such that $(p,q) \in U^n$.

Proof: $(a) \Rightarrow (b)$ Let us suppose that the set f(X) is not dense in the interval $(\inf f(X), \sup f(X))$. In this case there exists a point x in this
interval which is not in closure $\overline{f(X)}$ of the set f(X). Therefore there exists some finite interval $(a,b) \subset (\inf f(X), \sup f(X))$ which contains point x such that $f(X) \cap (a,b) = \emptyset$. Let us define the function g on the set $(-\infty, a] \cup [b, +\infty)$ into the discrete space $\{0, 1\}$ in the following way:

$$g(x) = \begin{cases} 0, & x \in (-\infty, a], \\ 1, & x \in [b, +\infty). \end{cases}$$

It is obvious that the function g is δ -continuous, so that $g \circ f : X \to \{0, 1\}$ by Corollary 1.1.6.3 is a δ -continuous function which is not constant. But this is in contradiction with the supposition that the proximity space X is δ -connected.

 $(b) \Rightarrow (d)$ According to Proposition 1.2.11.18 every uniformly continuous function is δ -continuous, and $\tau_{\mathcal{U}} = \tau_{\delta_{\mathcal{U}}}$ holds. If the condition (b) holds for a δ -continuous function f, then it also holds for f as a uniformly continuous function.

 $(d) \Rightarrow (c)$ Let $f: X \to \{0, 1\}$ be a uniformly continuous function. According to the supposition the set f(X) is dense in some interval of the real line, i.e. the closure $\overline{f(X)}$ is a segment of the real line which contains the points 0 and 1, which is impossible. Therefore we have that f(x) = 0 for each $x \in X$, or f(x) = 1 for each $x \in X$. Consequently, the function f is constant on X, so that the uniform space (X, \mathcal{U}) is uniformly connected.

 $(c) \Rightarrow (e)$ Let us suppose that the uniform space (X, \mathcal{U}) is not \mathcal{U} -chain connected. Then there exist a pair $(p,q) \in X \times X$ and a set $U \in \mathcal{U}$ such that $(p,q) \notin U^n$ for each $n \in \mathbb{N}$. Since $(p,p) \in U$, the set of all the points $x \in X$ for which $(p,x) \in U^n$ for some $n \in \mathbb{N}$ holds is not empty. Let us define the function $f: X \to \mathbb{R}$ in the following way:

$$f(x) = \begin{cases} 0, \text{ if there exists some } n \in \mathbb{N} \text{ such that } (p, x) \in U^n, \\ 1, \text{ in others cases.} \end{cases}$$

It is obvious that f(p) = 0 and f(q) = 1. Let us prove that the function f is uniformly continuous. To do this we can note that the discrete space $\{0,1\}$ for the base of uniformity has the set $\{(0,0), (1,1)\}$. Let us prove that $U \subset f^{-1}(\Delta)$, i.e. $f^{-1}(\Delta) \in \mathcal{U}$, from which follows that the function f is uniformly continuous. Let $(x, y) \in U$. If f(x) = 0, then $(p, x) \in U^n$ for some $n \in \mathbb{N}$, so that $(p, y) \in U \circ U^n = U^{n+1}$ for some $n \in \mathbb{N}$, from which follows that f(y) = 0. On the other hand, if we suppose that f(y) = 1, then there must be f(x) = 1. Indeed, if f(x) = 0, then $(p, x) \in U^n$ for some n, from which follows that $(p, y) \in U \circ U^n = U^{n+1}$, i.e. f(y) = 0, which is in contradiction with the supposition. Hence $U \subset f^{-1}(\Delta)$, i.e. $f^{-1}(\Delta) \in \mathcal{U}$. In

this manner we have proved that the function $f: X \to \{0, 1\}$ is uniformly continuous. Since it is different from the constant function, X is not a uniformly connected space. A contradiction obtained in such a way proves the implication $(c) \Rightarrow (e)$.

 $(e) \Rightarrow (a)$ Let X be a \mathcal{U} -chain connected uniform space and let A be a non-empty subset of X different from X. Let us prove that $A\overline{\delta}X - A$, from which by Proposition 1.4.1.1 there follows δ -connectedness of the space X. Let us choose any point $p \in A$ and let q be an arbitrary point of the set X-A. Then for every $U \in \mathcal{U}$ there exists a natural number n such that $(p,q) \in U^n$. Thus there exists a sequence of points $p = x_0, x_1, \ldots, x_{k-1}, x_k, \ldots, x_n = q$ with the property that $(x_{k-1}, x_k) \in U$ for each $k = 1, 2, \ldots, n$. Since $q \in X - A$, there exists in this sequence the first point x_i for which $x_i \in X - A$ holds and therefore we have that $x_{i-1} \in A$. So we prove that for each $U \in \mathcal{U}$ there exists a pair (x_{i-1}, x_i) of the points x_{i-1} and x_i of the set X with the property that $(x_{i-1}, x_i) \in U$. Hence $(A \times (X - A)) \cap U \neq \emptyset$ for every $U \in \mathcal{U}$, which proves that $A\overline{\delta}X - A$. The proof of the proposition is completed.

In connection with parts (b) and (d), let us notice that one cannot replace the condition given on the range of the function f by the requirement that the range of f is an interval as the example of the rationales with f(x) = xshows. It is natural to ask what would happen if one would require that all δ -continuous functions on a proximity space have the Darboux property. The answer is rather unexpected.

Proposition 1.4.1.16 Let (X, δ) be a Lindelöf space. If every real-valued function on X has the Darboux property, then X is connected (in topological sense).

Proof: Let us suppose that X is not connected and let $X = A \cup B$, where A and B are closed and disjoint sets. Let us denote by X^* the Smirnoff compactification of X associated with the proximity δ and let us set $Z = \overline{A} \cap \overline{B}$, where the closures of the sets A and B are taken in the space X^* . By the theorem of Smirnoff (see [293]), there exists a real-valued continuous function f on X^* for which f(p) = 0 holds if $p \in Z$, but f(p) > 0 for $p \in X$. Let us define a real-valued function g on X^* by setting g(p) = f(p) for $p \in \overline{A}$ and g(p) = -f(p) for $p \in \overline{B}$. Clearly, the range of the function g|X is not an interval since g|X is never 0, but does take on both positive and negative values. Since f is continuous on \overline{A} , -f is continuous on \overline{B} , and f and -f agree on $\overline{A} \cap \overline{B}$, g is continuous on X^{*}. Hence the restriction g|X of g to X is δ -continuous. However, the function g does not have the Darboux property.

1.4.2 δ -components

Definition 1.4.2.1 The δ -component $C_{\delta}(x)$ of a point x in a proximity space X is the union of all δ -connected subsets of X which contain the point x.

By Proposition 1.4.1.2 it follows that the δ -component of each point is a δ connected set. It is easy to see that the δ -components of two distinct points of a proximity space X either coincide or are far sets in X. In this way all δ -components in X constitute a decomposition of the space X into pairwise far δ -connected subsets, which are called the δ -components of the proximity space X.

Let us point out that the δ -components of a proximity space (X, δ) in general does not coincide with the components with respect to the topology τ_{δ} generated by the proximity δ . For example, the set \mathbb{Q} of rational numbers is a δ -connected space, as is well known, and therefore \mathbb{Q} is the δ -component of every point $x \in \mathbb{Q}$, while the τ_{δ} -component of the point $x \in \mathbb{Q}$ is the set $\{x\}$.

Since each τ_{δ} -connected set is also δ -connected, each τ_{δ} -component is contained in some δ -component. However, if X is a compact proximity space, then the δ -component $C_{\delta}(x)$ of a point $x \in X$ is contained in the quasi-component Q_x of the point x. Indeed, let F be both an open and closed set (in the topology τ_{δ}) such that $x \in F$. Since X is a compact space, the sets F and X - F are far. Therefore the sets $C_{\delta}(x) \cap F$ and $C_{\delta}(x) - F$ are also far. Since $C_{\delta}(x) \cap F \neq \emptyset$ and the set $C_{\delta}(x)$ is δ -connected, we have that $C_{\delta}(x) - F = \emptyset$, i.e. $C_{\delta}(x) \subset F$. Thus $C_{\delta}(x) \subset Q_x$.

Proposition 1.4.2.1 The δ -components of a proximity space (X, δ) are closed sets in topology τ_{δ} .

Proof: The proof follows from Corollary 1.4.1.3.

Proposition 1.4.2.2 Let X be a δ -connected proximity space. If A is a δ -connected subset of X and $C \subset X - A$ is a δ -component in X - A, then the set X - C is δ -connected.

Proof: Let us suppose conversely that X - C is not δ -connected. Then, by Corollary 1.4.1.1, it can be presented as $X - C = M \cup N$, where M and N are non-empty, far sets. Since $A \subset X - C = M \cup N$, then according to Proposition 1.4.1.3, $A \subset M$ or $A \subset N$. Let us suppose that $A \subset N$. Then $A \cap (C \cup M) = \emptyset$, and hence $C \cup M \subset X - A$. But then the set $C \cup M$ is δ -connected by Proposition 1.4.1.4, and since C is δ -component in the set X - A, we have that $C = C \cup M$. This implies $M = \emptyset$, which is a contradiction. This proves that the set X - C is a δ -connected set.

Proposition 1.4.2.3 The δ -component of a point $x = (x_s)$ in the product $(X, \delta) = \prod\{(X_s, \delta_s) : s \in S\}$ coincides with the product $\prod\{C_{\delta_s}(x_s) : s \in S\}$, where $C_{\delta_s}(x_s)$ is the δ_s -component of the point x_s in the space X_s .

Proof: Let us denote the δ -component of the point x in X with $C_{\delta}(x)$. Then the product $\prod \{C_{\delta_s}(x_s) : s \in S\}$ is a δ -connected set according to Proposition 1.4.1.10, and therefore it is contained in $C_{\delta}(x)$. Conversely, by Proposition 1.4.1.9, the projection $p_s C_{\delta}(x)$ is a δ -connected set for every $s \in S$ and hence $p_s C_{\delta}(x) \subset C_{\delta_s}(x_s)$. Therefore $C_{\delta}(x) \subset \prod \{p_s C_{\delta}(x) : s \in S\} \subset \prod \{C_{\delta_s}(x_s) : s \in S\}$.

Proposition 1.4.2.4 If $f: X \to Y$ is a δ -monotone and δ -quotient mapping from a proximity space X onto a proximity space Y, then C is a δ -component of some set $B \subset Y$ if and only if $f^{-1}(C)$ is a δ -component of the set $f^{-1}(B)$.

Proof: Let us suppose first that the set C is a δ -component of $B \subset Y$ (as a subspace of the space Y). Let us suppose that there exists a δ -connected set K in $f^{-1}(B)$ satisfying $f^{-1}(C) \subset K \subset f^{-1}(B)$. Then $C \subset f(K) \subset B$ and since f(K) is a δ -connected set, according to Proposition 1.4.1.9, there follows that C = f(K), because C is a δ -component of the set B. It also holds that $f^{-1}(C) = f^{-1}(f(K)) \supset K$, so that $K = f^{-1}(C)$, which proves that $f^{-1}(C)$ is a δ -component in the set $f^{-1}(B)$.

Let us suppose now that $f^{-1}(C)$ is a δ -component of the set $f^{-1}(B)$, and let us suppose that there exists a δ -connected set L for which $C \subset L \subset B$ holds. Since the set $f^{-1}(L)$ is δ -connected by Proposition 1.4.1.11, from inclusion $f^{-1}(C) \subset f^{-1}(L) \subset f^{-1}(B)$ and the fact that $f^{-1}(C)$ is a δ component in the set $f^{-1}(B)$ as a subspace of the space X, the equality $f^{-1}(C) = f^{-1}(L)$ follows, i.e. C = L. This proves that C is a δ -component of the set B.

1.4.3 δ -quasi-components

Let us define a relation \sim on a proximity space (X, δ) in the following way:

 $x \sim y$ if and only if there are not the sets A and B far in X for which $x \in A$, $y \in B$ and $X = A \cup B$ hold. It is easy to check that \sim is an equivalence relation on X. Therefore it determines a decomposition of X into disjoint sets - the equivalence classes of the relation \sim .

Definition 1.4.3.1 We shall call the equivalence class of a point $x \in X$ with respect to the relation $\sim \delta$ -quasi-component of the point x in proximity space (X, δ) and denote it by $Q_{\delta}(x)$.

Proposition 1.4.3.1 The δ -quasi-components are closed sets in topology τ_{δ} .

Proof: Let $Q_{\delta}(x)$ be the δ -quasi-component of a point x and let us suppose that $y \notin Q_{\delta}(x)$. Then $y \nsim x$ and hence there exist the sets A and B far in X for which $x \in A$, $y \in B$ and $X = A \cup B$ hold. If $z \in B$, then $z \nsim x$ and therefore $B \cap Q_{\delta}(x) = \emptyset$. From $Q_{\delta}(x) \subset A$ it follows that $B\overline{\delta}Q_{\delta}(x)$, and therefore by Proposition 1.1.1.2 we have that $y\overline{\delta}Q_{\delta}(x)$. In this way we have proved that $Q_{\delta}(x)$ is a closed set in the topology τ_{δ} .

Proposition 1.4.3.2 In a compact proximity space X the quasi-component Q_x of the point x coincides with the δ -quasi-component $Q_{\delta}(x)$ of the point x.

Proof: First we shall prove the inclusion $Q_{\delta}(x) \subset Q_x$. Let us suppose that $y \notin Q_x$. Then there exists a set F which is simultaneously open and closed and containing the point x, but not containing the point y. Since X is a compact space, by Theorem 1.1.3.4 $F\overline{\delta}X - F$ and hence $y \not\sim x$. This proves that $y \notin Q_{\delta}(x)$.

To prove the inclusion $Q_x \,\subset Q_\delta(x)$, let us suppose that $y \notin Q_\delta(x)$. Then there exist the sets A and B such that $x \in A$, $y \in B$, $A\overline{\delta}B$ and $X = A \cup B$. Since X is compact, by Theorem 1.1.3.4 we have that $\overline{A} \cap \overline{B} = \emptyset$. Now from $A \cup \overline{B} = X$ and $A \cap \overline{B} = \emptyset$ it follows $A = X - \overline{B}$, which proves that the set A is open. In a similar way it can be proved that the set B is open, too. Thus from $A \cup B = X$ and $A \cap B = \emptyset$ it follows that the sets A and B are closed. Since the point x belongs to the set A which is both open and closed and $y \notin A$, we have that $y \notin Q_x$. The proposition has been proved.

Proposition 1.4.3.3 If (X, δ) is a proximity space, then $C_{\delta}(x) \subset Q_{\delta}(x)$ for every $x \in X$.

Proof: Let us suppose that there exists a point $y \in C_{\delta}(x)$ such that $y \notin Q_{\delta}(x)$. Then there are two far sets A and B such that $x \in A$, $y \in B$ and $X = A \cup B$. The sets $C_{\delta}(x) \cap A$ and $C_{\delta}(x) \cap B$ are non-empty far subsets of a δ -connected set $C_{\delta}(x)$, which is a contradiction.

Corollary 1.4.3.1 If a separated proximity space X is compact, then $C_x = C_{\delta}(x) = Q_x = Q_{\delta}(x)$ for each $x \in X$.

Proof: This corollary follows immediately from Propositions 1.4.3.2 and 1.4.3.3, the comment after Definition 1.4.2.1 and the fact that in a compact T_2 -space the component of a point coincides with the quasi-component of that point.

Proposition 1.4.3.4 The δ -quasi-component of a point $x = (x_s)$ in the product (X, δ) of the proximity spaces (X_s, δ_s) , $s \in S$, coincides with the product $\prod \{Q_{\delta_s}(x_s) : s \in S\}$, where $Q_{\delta_s}(x_s)$ is a δ -quasi-component of the point x_s in X_s .

Proof: Let $x = (x_s)$ and $y = (y_s)$ be any two points of the space X. Let us prove that $x \sim y$ if and only if $x_s \sim y_s$ for each $s \in S$.

If $x_s \not\sim y_s$ for some $s \in S$, then there exist two sets A and B which are far in X_s such that $x_s \in A$, $y_s \in B$ and $X_s = A \cup B$. Since the projection $p_s : X \to X_s$ is a δ -continuous mapping, the sets $p_s^{-1}(A)$ and $p_s^{-1}(B)$ are far in X. Now from $x \in p_s^{-1}(A)$, $y \in p_s^{-1}(B)$ and $p_s^{-1}(A) \cup p_s^{-1}(B) = X$ there follows that $x \not\sim y$.

Let us suppose now that $x \not\sim y$ and let A and B be two far subsets of X for which $x \in A$, $y \in B$ and $A \cup B = X$ hold. From $A\overline{\delta}B$ it follows that there exist covers $\{A_1, A_2, \ldots, A_m\}$ and $\{B_1, B_2, \ldots, B_n\}$ of the sets A and B respectively, and some index $s \in S$ for which $p_s(A_i)\overline{\delta}_s p_s(B_j)$ holds for each $i \in J_m$ and each $j \in J_n$. It is obvious that $x_s \in \bigcup\{p_s(A_i) : i \in J_m\} = M$, $y_s \in \bigcup\{p_s(B_j) : j \in J_n\} = N$, $M\overline{\delta}_s N$ and $M \cup N = X_s$. This means that $x_s \not\sim y_s$.

1.4.4 Locally δ -connected spaces

Definition 1.4.4.1 A proximity space X is locally δ -connected at the point x if every δ -neighborhood of the point x contains some δ -connected δ -neighborhood of the point x. The space X is locally δ -connected if it is locally δ -connected at each of its points. A subset $Y \subset X$ is locally δ -connected if Y is locally δ -connected as a proximity subspace of X.

If a proximity space X is locally connected with respect to the topology τ_{δ} , then it is also locally δ -connected. Indeed, if $x \in X$ is an arbitrary point and U is any δ -neighborhood of x, then $x \in \text{int } U$ by virtue of Proposition

1.1.2.5. Since X is locally connected, there is a connected neighborhood V of the point x for which $x \in V \subset \operatorname{int} U$ holds. But V is also a δ -connected set, and since $x \in \operatorname{int} V$, it follows that $x\overline{\delta}X - V$. This proves that V is a δ -neighborhood of the point x which is δ -connected.

The following example shows that the converse in general is not valid.

Example 1.4.4.1 The space $\mathbb{Q} \cap ([0,1) \cup (2,3])$ is locally δ -connected, but it is not locally connected (and δ -connected).

Proposition 1.4.4.1 If $x \in A \cap B$ and if the sets A and B are locally δ -connected at the point x, then the set $A \cup B$ is locally δ -connected at the point x.

Proof: Let U be a δ -neighborhood of the point x in the set $A \cup B$. Then $U_A = U \cap A$ and $U_B = U \cap B$ are δ -neighborhoods of the point x in A and B respectively. Since A and B are locally δ -connected at the point x, there exist δ -connected δ -neighborhoods V_A and V_B of the point x such that $x \in V_A \subset U_A$ and $x \in V_B \subset U_B$. Then $V = V_A \cup V_B \subset U_A \cup U_B = U$ is a δ -connected set. On the other hand, from $x\overline{\delta}A - V_A$ and $x\overline{\delta}B - V_B$ it follows that $x\overline{\delta}(A - V_A) \cup (B - V_B) \subset (A \cup B) - V$, and hence $x\overline{\delta}(A \cup B) - V$. Thus V is a δ -neighborhood of the point x in $A \cup B$.

Proposition 1.4.4.2 A proximity space X is locally δ -connected if and only if the δ -component of every open subspace of the space X is open.

Proof: Let U be an open subspace of a locally δ -connected space X and let C be a δ -component of the set U. If $x \in C$, then $x\overline{\delta}X - U$ because U is open. But then U is a δ -neighborhood of the point x, and therefore (since Xis locally δ -connected) there exists a δ -connected δ -neighborhood V of the point x which is contained in U. Since C is the δ -component of the point x, we have that $V \subset C$. But since each δ -neighborhood of the point x is also a topological neighborhood of the point x with respect to the topology τ_{δ} , the set C is open.

Conversely, let us suppose that the δ -components of any open subspace of the space X are open and let U be a δ -neighborhood of an arbitrary point $x \in X$. The δ -component $C_{\delta}(x)$ of the point x in U is an open set in X and thus it is a δ -connected δ -neighborhood of the point x which is contained in the set U. Therefore X is a locally δ -connected space.

Corollary 1.4.4.1 The δ -components in a locally δ -connected proximity space are open and closed sets (in the induced topology τ_{δ}).

Corollary 1.4.4.2 In a locally δ -connected proximity space the quasi-component of a point is contained in the δ -component of this point.

Corollary 1.4.4.3 If a locally δ -connected proximity space is compact, then it has a finite number of δ -components.

Proposition 1.4.4.3 Let $\{(X_s, \delta_s) : s \in S\}$ be a family of proximity spaces. The product $(X, \delta) = \prod\{(X_s, \delta_s) : s \in S\}$ is locally δ -connected if and only if all the spaces X_s are locally δ -connected and there exists a finite set $F = \{s_1, s_2, \ldots, s_k\} \subset S$ such that X_s is δ -connected for each $s \in S - F$.

Proof: By virtue of Proposition 1.4.4.2 it is enough to prove that the δ components of any open subspace of the space X are open sets. Let U be
an arbitrary open set in X and let $x = (x_s) \in X$. Since the topology τ_{δ} is equal to the product topology of (X, δ) , we can assume without a loss of
generality that $U = \prod \{U_s : s \in S\}$, where U_s are open sets in X_s , $U_s = X_s$ for $s \in S - F$, and all X_s , $s \in S - F$, are δ -connected. Let $C_{\delta}(x)$ be a δ -component of the point x in the set U. We shall prove that $x \in \operatorname{int} C_{\delta}(x)$.
Let $C_{\delta_s}(x_s)$ be a δ -component of the point x_s in U_s . Then by Proposition
1.4.2.3 $C_{\delta}(x) = \prod \{C_{\delta_s}(x_s) : s \in S\}$ holds. For $s \in S - F$ we have that $U_s = X_s$. But then $C_{\delta_s}(x_s) = X_s$ holds, because X_s is a δ -connected set.
Now from $x_s \in \operatorname{int} C_{\delta_s}(x_s)$ and $\operatorname{int} C_{\delta}(x) = \prod \{\operatorname{int} C_{\delta_s}(x_s) : s \in S\}$ it follows
that $x \in \operatorname{int} C_{\delta}(x)$, which completes the proof of the proposition.

1.4.5 Treelike proximity spaces

Let (X, δ) be a proximity space. If it can be present as the union of the far sets A and B, then for the sets A and B we shall say that they **separate** the space X and write X = A + B. If the sets A and B separate the proximity space X, so that A contains a set P, and B contains a set Q, we shall write X = A(P) + B(Q). Especially, if $P = \{x_1, x_2, \ldots, x_m\}$, and $Q = \{y_1, y_2, \ldots, y_n\}$, we shall write $X = A(x_1, x_2, \ldots, x_m) + B(y_1, y_2, \ldots, y_n)$.

Lemma 1.4.5.1 If X is a δ -connected proximity space and if $X - \{x\} = A + B$, then $x\delta A$ and $x\delta B$.

Proof: Since $X - \{x\} = A + B$, then $X - \{x\} = A \cup B$, where $A\overline{\delta}B$. Let us suppose that $x\overline{\delta}A$. Since $A\overline{\delta}B$, i.e. $B\overline{\delta}A$, by axiom (B_2) we have that $A\overline{\delta}B \cup \{x\}$, i.e. $A\overline{\delta}X - A$. Therefore the space X is not δ -connected by Proposition 1.4.1.1, which is contrary to the supposition. **Definition 1.4.5.1** A proximity space (X, δ) is called δ -**treelike** if it is δ connected and for each two distinct points x and y from X there exists δ -connected set $K \subset X$ such that X - K = A(x) + B(y).

An example of the rational numbers with the usual proximity shows that there exists a δ -treelike proximity space which is not (topologically) treelike.

Proposition 1.4.5.1 Every δ -treelike proximity space (X, δ) is separated.

Proof: Let us suppose that there exist two distinct points $x, y \in X$ for which $x\delta y$ follows. Then the set $\{x, y\}$ is δ -connected. On the other hand, if K is a δ -connected set which separates the points x and y: X - K = A(x) + B(y), then the sets $A \cap \{x, y\}$ and $B \cap \{x, y\}$ make a disconnection of the set $\{x, y\}$ in non-empty far sets. But this is a contradiction.

Definition 1.4.5.2 A subset S of a proximity space (X, δ) is called a δ -segment (of the point x) if S is the δ -component of $X - \{x\}$ for some $x \in X$.

Proposition 1.4.5.2 If (X, δ) is a δ -treelike proximity space, then each δ -segment on X is open in the topology τ_{δ} .

Proof: Let *C* be an arbitrary δ -segment in *X*. Then there exists a point $x \in X$ such that *C* is a δ -component of the set $X - \{x\}$. Let us suppose that the δ -segment *C* is not an open set. Then there exists a point $y \in C - int C$; for this point we have that $\{y\}\delta X - C$. Let *K* be a δ -connected set which separates the points *x* and *y*: X - K = A(x) + B(y). The set $B \cup K$ is δ -connected in $X - \{x\}$ and intersects the set *C*. Since *C* is a δ -component in $X - \{x\}$, we have the inclusion $B \cup K \subset C$. But then $X - C \subset A$, and therefore by Proposition 1.1.1.2 (a) it follows that $\{y\}\delta A$. Since $y \in B$, then again by virtue of Proposition 1.1.1.2 (a) we have that $A\delta B$ which is a contradiction. This means that the set *C* must be open.

Proposition 1.4.5.3 In a δ -treelike proximity space (X, δ) any two distinct points x and y can be separated by an open δ -connected set.

Proof: Let $x, y \in X$ be any two distinct points and let K be a δ -connected set which separates the points x and y: X - K = M(x) + N(y). Let $X - \{x\} = B(y) + C$, where B is a δ -connected set and let $B - \{y\} = \cup \{C_a : a \in A\}$ be a decomposition of the set $B - \{y\}$ in δ -components. Since K is a δ -connected set which is contained in union of two far sets B and C,

it must be contained in one of them by Proposition 1.4.1.3. Let B is the set with this property. Then the set $K \cup N$ is δ -connected by Proposition 1.4.1.4, and since $y \in K \cup N$, we have that $K \cup N \subset B$. Therefore $K \subset B$, or more precisely, $K \subset B - \{y\}$. Consequently, the set K is contained in some δ -component C_{a_0} of the set $B - \{y\}$. According to Proposition 1.4.2.2 the set $B - C_{a_0}$ is δ -connected in the set B and therefore the set $\overline{B - C_{a_0}}$ is δ -connected by Corollary 1.4.1.3. Since $y \in B - C_{a_0}$, we have that $x\overline{\delta}B - C_{a_0}$, because in contrary case the sets $\overline{B - C_{a_0}} \cap M$ and $\overline{B - C_{a_0}} \cap N$ would make a disconnection of $\overline{B - C_{a_0}}$ on non-empty far sets, which is impossible. Since $X - \{x\} = B(y) + C$, then, by Lemma 1.4.5.1, $x\delta B$ and $x\delta C$, i.e. $x \in \overline{B}$ and $x \in \overline{C}$. Now from the fact that $x\overline{\delta B} - C_{a_0}$, i.e. $x \notin \overline{B - C_{a_0}} \supset \overline{B} - \overline{C_{a_0}}$, it follows that $x\delta C_{a_0}$. The set $C_{a_0} \cup \{x\} \cup C$ is δ -connected. Moreover, it is a δ -segment of the point y. This immediately follows from the fact that C_{a_0} is δ -component in the set $B - \{y\}$. But then the set $C \cup \{x\} \cup C_{a_0}$ is open according to the above proposition. Moreover, the set $C \cup \{x\}$ is closed. Indeed, since the space X is δ -connected, we have that $C \cup \{x\} \delta X - (C \cup \{x\})$. But then $y\delta C \cup \{x\}$ for every $y \in X - (C \cup \{x\})$, which shows that the set $C \cup \{x\}$ is closed. Therefore the set $C_{a_0} = (C_{a_0} \cup \{x\} \cup C) - (C \cup \{x\})$ is open. It is obvious that

$$X - C_{a_0} = (M - C_{a_0})(x) + (N - C_{a_0})(y),$$

which proves the statement of the proposition. \clubsuit

Proposition 1.4.5.4 Let (X, δ) be a δ -treelike proximity space. Then, among any three distinct points of the space X, there exists at least one which is contained in an open δ -connected set which separates the other two points.

Proof: Let us suppose that there exist three distinct points $x_1, x_2, x_3 \in X$ for which the assertion of the proposition is not true. Let K and L be δ -connected sets such that

 $X - K = A_1(x_1) + B_1(x_2, x_3)$ and $X - L = A_2(x_2) + B_2(x_1, x_3)$.

First we shall prove that the set $A_2 \cup L \cup K$ is not δ -connected. Let us suppose contrary, that it is δ -connected. Then from

$$X - (A_2 \cup K \cup L) = X - ((X - B_2) \cup K) = B_2 \cap (A_1 \cup B_1) = (B_1 \cap B_2) \cup (A_1 \cap B_2),$$

 $x_1 \in A_1 \cap B_2$, $x_2 \in A_2 \cup K \cup L$, $x_3 \in B_1 \cap B_2$ and $(B_1 \cap B_2)\overline{\delta}(A_1 \cap B_2)$ it follows that $A_2 \cup K \cup L$ is a δ -connected set which contains the point x_2 and

separates the points x_1 and x_3 . But this contradicts the assumption about these points. So $A_2 \cup K \cup L$ is not a δ -connected set.

Since the set $A_2 \cup L$ is δ -connected, according to Proposition 1.4.1.4 we can conclude that $K\overline{\delta}(A_2 \cup L)$. Indeed, if $K\delta(A_2 \cup L)$, then the set $A_2 \cup K \cup L$ would be δ -connected, which is impossible. In a similar way we obtain that $L\overline{\delta}(A_1 \cup K)$.

Now we shall prove that $A_1\overline{\delta}A_2$. From $L\overline{\delta}K$, i.e. $L \subset X - K$ it follows that the set L is contained in the union of far sets A_1 and B_1 , so that it is contained in one of them by virtue of Proposition 1.4.1.3. But the set L can not be contained in the set A_1 . Indeed, if $L \subset A_1$, then, by δ -connectedness of the set $A_2 \cup L$, we have that $A_2 \cup L \subset A_1$. But then $A_2 \subset A_1$ implies $A_2\overline{\delta}B_1$, which is impossible because $x_2 \in A_2$ and $x_2 \in B_1$. Therefore $L \subset B_1$. So we have proved that $A_2 \cup L \subset B_1$, from which follows that $A_2 \subset B_1$. Now $A_1\overline{\delta}B_1$ implies $A_1\overline{\delta}A_2$.

Now we are going to prove that the set $(B_1 \cap B_2) \cup K \cup L$ is not δ -connected. Let us suppose that it is δ -connected. From the equality

$$X - ((B_1 \cap B_2) \cup K \cup L) = (A_1 \cap (X - L)) \cup (A_2 \cap (X - K))$$

and the facts that $A_1 \cap (X - L)\overline{\delta}A_2 \cap (X - K)$ (which follows from $A_1\overline{\delta}A_2$), $x_1 \in A_1 \cap (X - L), x_2 \in A_2 \cap (X - K)$ and $x_3 \in (B_1 \cap B_2) \cup K \cup L$, it follows that $(B_1 \cap B_2) \cup K \cup L$ is a δ -connected set which contains the point x_3 and separates the points x_1 and x_2 . This contradicts the assumption about the points x_1, x_2 and x_3 . Thus the set $(B_1 \cap B_2) \cup K \cup L$ cannot be δ -connected and hence there exists a disconnection if this set of the form

$$(B_1 \cap B_2) \cup K \cup L = P(L) + Q$$

The δ -connected set K is contained in the union of the far sets P and Q and hence it is contained in one of them by Proposition 1.4.1.3.

If $K \subset P$, then $X = (A_1 \cup A_2 \cup P) + Q$. Indeed, since $K \cup L \subset P$, we have that $Q \subset B_1 \cap B_2$, so that $Q\overline{\delta}A_1$ and $Q\overline{\delta}A_2$, which, together with $Q\overline{\delta}P$, implies $Q\overline{\delta}(A_1 \cup A_2 \cup P)$. But this contradicts the fact that the space X is δ -connected.

If $K \subset Q$, then $X = (A_2 \cup P) + (A_1 \cup Q)$. To prove this assertion, let us first note that $A_1\overline{\delta}(B_1 \cap B_2) \cup L$; this immediately follows from the fact that $A_1\overline{\delta}L$ and $A_1\overline{\delta}(B_1 \cap B_2)$. Since $P \subset (B_1 \cap B_2) \cup L$, then $A_1\overline{\delta}P$. From this and $A_1\overline{\delta}A_2$ we obtain that $A_1\overline{\delta}A_2 \cup P$. We have also that $A_2\overline{\delta}K$ and $A_2\overline{\delta}(B_1 \cap B_2)$, so that $A_2\overline{\delta}(B_1 \cap B_2) \cup K$. Since $Q \subset (B_1 \cap B_2) \cup K$, then $A_2\overline{\delta}Q$. But now we have that $Q\overline{\delta}A_2 \cup P$, because $Q\overline{\delta}P$. Finally, from the facts that $A_1\overline{\delta}A_2 \cup P$ and $Q\overline{\delta}A_2 \cup P$, it follows that $(A_1 \cup Q)\overline{\delta}(A_2 \cup P)$, which again obtains a contradiction.

In both cases we have a contradiction, which completes the proof of the proposition. \clubsuit

Historical and bibliographic notes

The notion of a δ -connectedness of a proximity space was introduced by S. Mrówka and W. J. Pervin in 1964 as an equiconnectedness of a proximity space (see [228]). In this paper they also introduced the notion of a \mathcal{U} connectedness of uniform space as a uniform connectedness. Propositions 1.4.1.15 and 1.4.1.16 were also proved there. The Example 1.4.1.1 is given in [207]. The notions of local δ -connectedness, δ -components and δ -quasicomponents were introduced 1987 by R. Dimitriyević and Lj. Kochinac in [83]. All the results of subsections 4.1., 4.2. and 4.3. were proved in paper [83]. The notion of a treelike space was introduced by R. Dimitriyevic and Lj. Kochinac 1987 in [85] (see also [84]). All the results of subsection 4.5. were proved in paper [85]. Connectedness in syntopogenous spaces was considered by Z. Mamuzic [206] (see also [207]) and J. L. Sieber and W. J. Pervin [285].

1.5 Dimension functions of proximity spaces

1.5.1 Covering dimension of proximity spaces

Definition 1.5.1.1 Let (X, δ) be a proximity space. We call a finite covering $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$ of X a δ -covering if there exist sets A_1, A_2, \ldots, A_n such that $\bigcup_{i \leq n} A_i = X$ and $A_i \overline{\delta} X - \Gamma_i$ (i.e. $A_i \ll \Gamma_i$) for each $i \leq n$.

We shall note further that, if a proximity space X is compact, then any two disjoint closed sets are far, which means that in this case, for any closed set A and any neighborhood Γ of $A, A \ll \Gamma$ holds. From this, in turn, it follows that for any compact proximity space X each finite open covering $\{\Gamma_1, \ldots, \Gamma_k\}$ is a δ -covering, since, by a well-known lemma of P. S. Aleksandroff, there exist closed sets A_1, \ldots, A_k , such that $\bigcup_{i \leq k} A_i = X$ and each A_i is contained in Γ_i . **Lemma 1.5.1.1** For each δ -covering $\alpha = \{A_1, A_2, \ldots, A_n\}$ of a proximity space X the sets $O\langle A_1 \rangle, O\langle A_2 \rangle, \ldots, O\langle A_n \rangle$ form a δ -covering $O\langle \alpha \rangle$ of the compact space uX.

Proof: Let $\alpha = \{A_i\}$ be a δ -covering of X and let $\beta = \{B_i\}$ be a covering of X consisting of the elements for which $B_i \ll A_i$. According to the property (c) of the operator $O\langle \rangle$, for each i the equality $uX = O\langle X - B_i \rangle \cup O\langle A_i \rangle$ is true. By the property (g) of the operator $O\langle \rangle$ and the property of O() which is given on page 122, for every set B_i the following equality holds:

$$\overline{B_i}^{uX} = \overline{\overline{B_i}^{X}}^{uX} = \overline{X - \operatorname{int} (X - B_i)}^{uX} = uX - O(\operatorname{int} (X - B_i)) = uX - O\langle X - B_i \rangle$$

Now it follows that $\overline{B_i}^{uX} \subseteq O\langle B_i \rangle$. Therefore, $uX = \bigcup_i \overline{B_i}^{uX} \subseteq \bigcup_i O\langle B_i \rangle$.

Definition 1.5.1.2 The δ -dimension δ -dim X of the proximity space X is the smallest natural number $n \ge 0$ such that every δ -covering of X can be refined by a δ -covering of order $\le n + 1$; if there is no such n, we set δ -dim $X = +\infty$. For the empty space \emptyset we set δ -dim X = -1.

It is clear from the definition that δ -dimension is δ -invariant, i.e. it is unchanged by δ -homeomorphic mappings. Also we can conclude that for the dimension δ -dim of the proximity space X the inequality δ -dim $X \leq n$ $(n \text{ is an integer } \geq 1)$ holds if and only if every δ -covering of X can be refined by a δ -covering of order n + 1 at the most.

Theorem 1.5.1.1 The δ -dimension of any proximity space X coincides with the topological dimension of its compact (absolutely closed) δ -extension $uX: \delta$ -dim $X = \dim uX$.

Proof: Let us prove first that $\delta \dim X \leq \dim uX$. Let $\dim uX = n$. Take any δ -covering $\gamma = {\Gamma_1, \Gamma_2, \ldots, \Gamma_k}$ of X. Let us apply to it the operator $O\langle \rangle$, which associates to each set $\Gamma \subseteq X$ the largest open set $O\langle\Gamma\rangle$ of uXwhose intersection with X is the interior int Γ_i of the set Γ_i ; this yields, by Lemma 1.5.1.1, an open covering $O_{\gamma} = {O\langle\Gamma_1\rangle, O\langle\Gamma_2\rangle, \ldots, O\langle\Gamma_k\rangle}$ of the compact space uX. Let $\tilde{\omega} = {\tilde{U}_i}$ be a finite open refinement of O_{γ} of order $\leq n + 1$. Since $\tilde{\omega}$ is a δ -covering of uX, the restriction ${X \cap \tilde{U}_i}$ of $\tilde{\omega}$ to Xis a δ -covering ω . The order of the restriction ω is again $\leq n + 1$. For each $i \leq k$ we have $X \cap O\langle\Gamma_i\rangle = \operatorname{int} \Gamma_i \subseteq \Gamma_i$. Consequently ω is a refinement of γ , since $\tilde{\omega}$ was a refinement of O_{γ} . Thus we see that δ -dim $X \leq \dim uX = n$. To prove the converse, let δ -dim X = n, and let $\tilde{\omega}$ be any finite open covering of uX. Let $\tilde{\eta}$ be some finite refinement of $\tilde{\omega}$ consisting of regular open sets \tilde{H}_j . Then $\tilde{\eta}$ is a δ -covering of uX and consequently, its restriction to $X, \eta = \{H_j : H_j = X \cap \tilde{H}_j\}$, is a δ -covering of X. Let us refine η by a δ -covering γ of order n+1 at the most and let us consider the open covering O_{γ} of uX obtained from γ by the application of the operator $O\langle \rangle$. From the multiplicativity of this operator (see the property (a) on the page 122) we conclude that the order of O_{γ} cannot exceed the order of γ . Further, using the multiplicativity of operator $O\langle \rangle$ again and the fact that for regular open sets H_j of X one has $O\langle H_j \rangle = O(H_j) = \tilde{H}_j$ (see [297], p. 210), we find that if $\Gamma \subseteq H_i$ then $O\langle \Gamma \rangle \subseteq O\langle H_i \rangle = \tilde{H}_i$. This means that the covering O_{γ} of uX is a refinement of $\tilde{\eta}$ and hence also of $\tilde{\omega}$. Since O_{γ} has order $\leq n+1$, we have dim $uX \leq n = \delta$ -dim X, as was to be shown. \clubsuit

Corollary 1.5.1.1 For compact proximity spaces, δ -dimension coincides with the topological dimension dim.

Proposition 1.5.1.1 For any subspace A of a proximity space X, δ -dim A $\leq \delta$ -dim X.

Proof: Let us consider the compact δ -extension uX of X. Since the closure \overline{A}^{uX} of the set A in uX is its compact δ -extension uA, δ -dim $A = \dim uA = = \dim \overline{A}^{uX} \leq \dim uX = \delta$ -dim X holds, as it is required.

Proposition 1.5.1.2 If A is a dense subspace of the proximity space X, then δ -dim $A = \delta$ -dim X.

Proof: Let us consider again the compact extension uX. We can see that it is also a compact δ -extension of A. Therefore, according to Theorem 1.5.1.1, it follows that δ -dim $A = \delta$ -dim X, as was to be shown.

Corollary 1.5.1.2 The dimension δ -dim A of any subspace A of the proximity space X coincides with the dimension δ -dim \overline{A} of the closure \overline{A} of A in X.

The importance of Theorem 1.5.1.1 is clear from the last two proofs; numerous propositions concerning the δ -dimension of a proximity space can be proved by reducing them to already known facts from the dimension theory of compact spaces. We can also see that the δ -dimension of even a "good" δ -space need not coincide with the topological dimension; taking in the *n*-dimensional cube Q^n (or the Hilbert parallelotope Q^{∞}) a countable dense set A^n (respectively, A^{∞}) we find that the δ -dimension of A^n (or of A^{∞}) is *n* (or ∞), while, at the same time, the topological dimension of each of these is zero. Observe thirdly that, since a countable metric space can have a big δ -dimension too, the δ -dimension is not at all similar to the topological dimension. In particular, the known sum theorem for countable many closed sets cannot be generalized. None the less there follows

Proposition 1.5.1.3 The δ -dimension of the union of any finite number of the subsets of a given proximity space X is equal to the largest of the δ -dimension of the summands.

Proof: Let A_1, A_2, \ldots, A_n be any *n* subsets of the proximity space *X*. Taking the compact δ -extension uX of *X*, we can see that the closure $\overline{A_i}^{uX}$ of each summand A_i is its compact δ -extension uA_i . But since the closure of the union of finitely many sets is equal to the union of their closures, we obtain $u(\bigcup_i A_i) = \overline{(\bigcup_i A_i)}^{uX} = \bigcup_i \overline{A_i}^{uX} = \bigcup_i uA_i$. Therefore the topological dimension of the set $u(\bigcup_i A_i)$ is equal to the maximum of the dimension of the summands uA_i . Then from Theorem 1.5.1.1 we immediately obtain the required result.

Let us consider next the case of the proximity space X embedded in some proximity space Y. In this case it is convenient to define δ -dimension of X not with respect to its own δ -coverings, but in terms of the systems δ -covering of X in the following sense:

Definition 1.5.1.3 We call a finite system of sets $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ in the proximity space Y an exterior δ -covering of the subspace X relative to the space Y, if there exist sets B_1, B_2, \ldots, B_k such that $X \subseteq \bigcup_{i \leq k} B_i$ and $B_i \ll_Y \Gamma_i$ for each $i \leq k$.

To avoid any misunderstanding, recall that under the inclusion $B_i \ll_Y \Gamma_i$ we understand strong inclusion with respect to the space Y (i.e. $B_i \overline{\delta} Y - \Gamma_i$), not with respect to X. Accordingly the δ -coverings of X are not obligated to be, generally speaking, exterior δ -coverings of X (relative to Y). We remark further that by Proposition 1.1.2.5 (b) the interiors int Γ_i of the sets Γ_i , taken in Y, also constitute an exterior δ -covering of the subspace X.

Lemma 1.5.1.2 If X is a compact subspace of a proximity space Y then every exterior open covering of X is an exterior δ -covering of X. **Proof:** Let us suppose $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ are open sets of the space Y whose union contains the compact set X. By force of a known lemma of P. S. Alexandroff, there exist closed subsets of $X, B_1 \subseteq \Gamma_1, B_2 \subseteq \Gamma_2, \ldots, B_k \subseteq \Gamma_k$, whose union is X. Since each B_i is compact, its neighborhood Γ_i is a δ neighborhood in Y. The lemma has been proved. \clubsuit

It is not difficult to see that every exterior δ -covering of the subspace X relative to a δ -space $Y \supseteq X$ intersects X in a δ -covering of X itself. It turns out that the converse is also true: every δ -covering γ of a subspace X of a proximity space Y can be extended to an exterior δ -covering of X, in the sense that for each $\Gamma_i \in \gamma$ one can select a Γ'_i so that $X \cap \Gamma'_i = \Gamma_i$ for each i and the system of all Γ'_i form an exterior δ -covering of X. Indeed, a somewhat stronger proposition is true:

Lemma 1.5.1.3 Each δ -covering γ' of a subspace X of a proximity space Y can be extended to a δ -covering of all of Y.

Proof: Let the sets $\Gamma'_1, \Gamma'_2, \ldots, \Gamma'_k$ constitute a δ -covering of the subspace X of the proximity space Y. This means that there exist sets $B'_i \ll_X \Gamma'_i$ such that $\bigcup_{i \leq k} B'_i = X$. This means that $B'_i \overline{\delta} X - \Gamma'_i$ for each i, and therefore $B'_i \ll_Y Y - (X - \Gamma'_i) = (Y - X) \cup \Gamma'_i$. It is also clear from this that the sets $\Gamma_i = (Y - X) \cup \Gamma'_i$ form an exterior δ -covering of X. Let us prove that they form a δ -covering of all of Y. To this end, let us note that there exist sets B_i , for each i such that $B'_i \ll_Y B_i \ll_Y \Gamma_i$. This implies that $X \subseteq \bigcup_{i \leq k} B'_i \ll_Y \prod_{i \leq k} B_i \ll_Y \Gamma_i$. But $B_1 \cup (Y - \bigcup_{i \leq k} B_i) \cup \bigcup_{2 \leq i \leq k} B_i = Y$. Therefore $\{\Gamma_i\}$ is a δ -covering of Y, as was to be shown.

Proposition 1.5.1.4 For any subspace X of a proximity space Y, the δ -dimension δ -dim X is the least of the integers n = 0, 1, 2, ..., such that every exterior δ -covering (relative to Y) of the subspace X can be refined by an exterior δ -covering of X (relative to Y) of order $\leq n + 1$.

Proof: It suffices to show that δ -dim $X \leq n$ if and only if every exterior δ -covering of X can be refined by an exterior δ -covering of order $\leq n + 1$.

We shall prove first the sufficiency of this condition. Let us assume that every exterior δ -covering of the subspace X can be refined by an exterior δ covering of order $\leq n+1$. Let us take an arbitrary δ -covering γ' of X and let us extend it to an exterior δ -covering according to the previous lemma. For the latter, we take a finer exterior δ -covering γ of order $\leq n+1$. Evidently, the exterior δ -covering γ intersects X in a δ -covering of order $\leq n+1$, a refinement of γ' , as required. We shall now prove the necessity of the condition. Let $\delta \operatorname{-dim} X \leq n$. Let us take an arbitrary exterior δ -covering $\gamma = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_k\}$ of the subspace X. From the definition, there exist sets B_i such that $X \subseteq \bigcup_{i \leq k} B_i$ and $B_i \ll_Y \Gamma_i$ for each $i \leq k$. Let us take now the compact δ -extension uYof the proximity space Y, and in it the closure \overline{X}^{uY} of X, which is, as is known, the compact δ -extension uX of X. Since $B_i \overline{\delta} Y - \Gamma_i$ for each i, then by Proposition 1.1.2.4, $\overline{B_i}^{uY} \overline{\delta} Y - \Gamma_i$ also holds. Consequently, just as in the above lemma, we have $\overline{B_i}^{uY} \ll_{uY} (uY - Y) \cup \Gamma_i$. Since $X \subseteq \bigcup_{i \leq k} B_i$, $uX = \overline{X}^{uY} \subseteq \bigcup_{i \leq k} \overline{B_i}^{uY}$. Hence the sets $\widetilde{\Gamma}_i = (uY - Y) \cup \Gamma_i$, $i \leq k$, form an exterior δ -covering $\widetilde{\gamma}$ of the compact set uX relative to uY. But the sets $\operatorname{int}_{uY} \widetilde{\Gamma}_i$ also form an exterior δ -covering of uX; so, we may assume that the sets $\widetilde{\Gamma}_i$, and thus also the sets $\Gamma_i = Y \cap \widetilde{\Gamma}_i$, are open in uY (respectively, in the space Y).

Since $\dim uX = \delta - \dim X \leq n$, there exists a finite closed covering ϕ of uX, finer that $\tilde{\gamma}$, of order $\leq n + 1$. From the compactness of uX, we can associate to each $\Phi_j \in \phi$ an open set \widetilde{U}_j in uY so that $\Phi_j \subseteq \widetilde{U}_j \subseteq \widetilde{\Gamma}_{i(j)}$ (where $\widetilde{\Gamma}_{i(j)}$ is one of the sets $\widetilde{\Gamma}_i$ containing Φ_j) and that these sets \widetilde{U}_j form a family $\widetilde{\omega}$ similar to ϕ , therefore having order $\leq n + 1$. By Lema 1.5.1.2, $\widetilde{\omega}$ is an exterior δ -covering of the compact set uX relative to uY. Consequently it intersects Y in the family ω of the open sets $U_j = Y \cap \widetilde{U}_j$ forming an exterior δ -covering of X relative to Y. It is easy to see that ω is a refinement of γ of order $\leq n + 1$ and the proof is then complete.

Let us now consider proximity spaces of the δ -dimension zero. Let us recall that topological spaces of the dimension zero are characterized by the condition that the open-closed sets form a basis for closed sets, i.e. that for any closed set A and neighborhood U of A there exists an open-closed set H such that $A \subseteq H \subseteq U$. In the proximity spaces the open-closed sets evidently must be replaced by those sets which are distant from their complements, i.e. those which constitute δ -neighborhoods of themselves. We call such sets δ -isolated.

Theorem 1.5.1.2 A non-empty proximity space X has δ -dimension zero if and only if for every (closed) set $F \subseteq X$ and every δ -neighborhood U of F there exists a δ -isolated set H such that $A \subseteq H \subseteq U$.

The proof of this theorem is not difficult; it proceeds just like the proof of the analogous theorem for topological dimension (see [325], &1, chapter IV).

Definition 1.5.1.4 Let there be given a proximity space X, a compact space Φ and a δ -covering α of X. We call a δ -continuous mapping f of X into Φ an α -mapping if for each point x of Φ one can find a neighborhood O_x whose complete inverse image under f is contained in one element of the covering α .

Since every continuous mapping from one compact space to another is a δ -mapping, and every open covering of a compact space is a δ -covering, it follows that the concept of an α -mapping just introduced coincides in the case of a compact space X with the standard topological definition of an α -mapping.

Lemma 1.5.1.4 Let there be given a proximity space X, a δ -covering α of X, and a δ -continuous mapping f of X into an arbitrary compact space Φ . The mapping f is an α -mapping if and only if its continuous extension \tilde{f} over the compact δ -extension uX is an O_{α} -mapping.

Proof: Let f be an α -mapping of the space X into a compact space Φ . For each point $x \in \Phi$ there is a neighborhood O_x whose complete inverse image under f is contained in some element A_i of α . The complete inverse image of each neighborhood O_x under the mapping \tilde{f} is an open subset of uX, which intersects X in the inverse image of O_x under f, which is itself an open set in X. But $f^{-1}(O_x)$ is contained in some $A_i \in \alpha$, and thus, in the interior int A_i of A_i too. It is known that $O\langle A_i \rangle$ is the largest open subset of uX whose intersection with X is the interior of A_i . That is, the complete inverse image $\tilde{f}^{-1}(O_x)$ is contained in $O\langle A_i \rangle$. Consequently, the extension \tilde{f} is an O_{α} -mapping.

Conversely, let us suppose that the continuous extension f of f is an O_{α} -mapping of the compact extension uX into Φ . We choose for each point $x \in \Phi$ a neighborhood O_x whose inverse under \tilde{f} is contained in some $O\langle A_i \rangle$. But then, using the inclusion $O\langle A_i \rangle \cap X \subseteq A_i$, we conclude that the complete inverse image of the neighborhood O_x of x under f is contained in $O\langle A_i \rangle \cap X$, and therefore in A_i . Consequently the mapping f is an α -mapping, as was to be shown.

Theorem 1.5.1.3 The δ -dimension δ -dim X of a proximity space X is the smallest integer $n \ge 0$ such that for every δ -covering α of X there exists an α -mapping of the space X into an n-dimensional finite polyhedron.

Proof: Let the proximity space X have δ -dimension δ -dim X = n. Then by Theorem 1.5.1.1 the topological dimension of uX is also n. Therefore,

by the well known theorem of dimension theory (see Theorem 4.3 in [6]), for any δ -covering α of the space X, there exists an O_{α} -mapping \tilde{f} of uX into some *n*-dimensional polyhedron Π . By the above lemma, the δ -continuous mapping f obtained by restricting \tilde{f} to X is an α -mapping of X into Π .

Now it remains to prove that for neither of any δ -covering α of X does there exist an α -mapping of this space into a compact space of the topological dimension less than n. Let us suppose, on the contrary, that there is such a mapping $f: X \to \Phi$ for every α . Then the continuous extension $\tilde{f}: uX \to \Phi$ is an O_{α} -mapping by the above lemma. But for each open covering α' of uX consisting of regular open sets, the covering O_{α} , obtained from the restriction α of α' to X by applying the operator $O\langle \rangle$, is finer than α' . That is, for every open covering α' of uX there is an α' -mapping of uXinto a compact space of the topological dimension less than n, which is a contradiction. The proof is complete.

Definition 1.5.1.5 We call a mapping f of a space X into a space Y dense if the image of X under f is a dense subset of Y.

Lemma 1.5.1.5 In order that a δ -continuous mapping f of proximity space X into a proximity space Y should be dense, it is necessary and sufficient that the continuous extension \tilde{f} of f over the compact δ -extension uX of X, with values in the compact δ -extension uY of Y, should be onto.

Proof: Let the δ -continuous mapping f of the proximity space X into the proximity space Y be dense. Then its extension \tilde{f} is also a dense mapping of uX into uY, since $\tilde{f}(X)$ is dense in Y and Y is dense in uY. But every dense mapping of one compact space into another is an onto mapping.

Conversely, let the extension f of f be an onto mapping of the compact extension uX on uY. The continuous image of a dense set is a dense set. Therefore $\tilde{f}(X)$ is dense in uY. Since $f(X) = \tilde{f}(X) \subseteq Y$, f(X) is also dense in Y.

Theorem 1.5.1.4 The δ -dimension δ -dim X of a proximity space X is the smallest $n \ge 0$ such that for every δ -covering α of X there exists a dense α -mapping of X into some finite polyhedron of dimension n.

Proof: In view of the preceding theorem there is only left to prove that for every δ -covering α of an *n*-dimensional proximity space X, there exists a dense α -mapping f of X into some *n*-dimensional polyhedron II. Indeed, since the compact space uX has dimension n, it follows that for every δ -covering α of X there exists an O_{α} -mapping \tilde{f} of uX onto some *n*-dimensional polyhedron Π (Theorem 4.3 in [6]). But then, by the last two lemmas, the restriction f of \tilde{f} to X is a dense α -mapping of X into Π , as required.

We remark here that, generally speaking, one cannot construct mappings of an *n*-dimensional proximity space X onto *n*-dimensional polyhedra. As an example of this we may take the countable set $\mathbb{Q} \cap [0, 1]$ of all rational points in the segment [0, 1]. This set has δ -dimension 1. Nevertheless, its mapping onto a 1-dimensional polyhedron is impossible, for the simple reason that every 1-dimensional polyhedron has the power of the continuum and $\mathbb{Q} \cap [0, 1]$ is countable.

Theorem 1.5.1.5 The δ -dimension of a proximity space X is the smallest integer $n \ge 0$ such that every δ -continuous mapping of an arbitrary (closed) subset A of X into the n-sphere S^n can be extended to a δ -continuous mapping of X into S^n .

Proof: Let δ -dim X = n and let there be given any subset A of X and any δ -continuous mapping f of A into the sphere S^n . In the compact δ -extension uX of X the closure of A is its compact δ -extension uA. Therefore the mapping f can be extended to a continuous mapping \tilde{f} of the closed subset uA of uX into S^n . But the compact space uX is n-dimensional by Theorem 1.5.1.1. It follows that \tilde{f} can be extended to a continuous mapping \tilde{F} to X, we obtain the desired δ -continuous mapping F of X into S^n , extending f.

It remains now to prove that for any non-negative number m less than n there exists a δ -continuous mapping of some closed subset of X into the sphere S^m which cannot be extended to a δ -continuous mapping on X. Let us observe that uX is an n-dimensional compact space. This means that there exists a continuous mapping h of some closed subset A into the sphere S^m which cannot be continuously extended over uX (see Theorem 4.5 in [6]). Since the closed ball Q^{m+1} can be homemorphically mapped upon the cube of the same dimension so that its boundary S^m goes onto the boundary of the cube, it follows from a known theorem of P. S. Urysohn that the mapping h, considered as a mapping into the ball Q^{m+1} , can be continuously extended to a mapping H of uX into Q^{m+1} . The complete inverse image U of the open set Q^{m+1} which is obtained from Q^{m+1} by deleting the center, will be a neighborhood of A since no point of A is mapped to the center point of Q^{m+1} . From the normality of the compact space uX, there exists a neighborhood O_A of A such that $\overline{O_A}^{uX} \subseteq U$.

Let π denote the continuous mapping of Q^{m+1} upon S^m which associates to each non-central point of Q^{m+1} its projection upon the boundary. The mapping $H' = \pi \circ H$ is a continuous mapping of U into the sphere S^m . Let \tilde{f} denote the restriction of H' to the regular closed set $\tilde{\Phi} = \overline{O_A}^{uX}$. It is a continuous extension of h. Consequently, the mapping \tilde{f} cannot be extended to a continuous mapping from uX into S^m either.

It is known that the set $\Phi = \Phi \cap X$ is a regular closed set in the space Xand that, therefore, $\overline{\Phi}^{uX} = \widetilde{\Phi}$ (see [297], heading 2, remarks and Theorem 5). Therefore $u\Phi = \widetilde{\Phi}$ also holds. Finally, let f denote the restriction of \widetilde{f} to Φ . It is a δ -continuous mapping of Φ into S^m . We show that it is the desired mapping which is not extensible over X. To this end, let us suppose, on the contrary, that there exists a δ -continuous mapping F of all X into S^m which extends f. Then F can be extended to a continuous function \widetilde{F} defined on uX. Since each value of the continuous extension \widetilde{F} of F at each point $x \in uX - X$ is uniquely determined, thus at points $x \in u\Phi - \Phi = \widetilde{\Phi} - \Phi$ it coincides with the extension \widetilde{f} of f. Therefore the mapping \widetilde{F} is a continuous extension of \widetilde{f} , which is, as we know, impossible. The contradiction here concludes the proof of the theorem.

Definition 1.5.1.6 We shall call a δ -continuous mapping of a proximity space into a closed ball Q^n essential if there is no δ -continuous mapping g of X into the boundary S^{n-1} of Q^n which coincides with f on $f^{-1}(S^{n-1})$.

It is easy to see that a continuous extension of an essential δ -continuous mapping of a proximity space X over the compact δ -extension uX is also essential. But the first natural formulation of the converse proposition is false: take the proximity space X_i consisting of the interior of the ball Q^i together with some boundary point p, with the proximity structure naturally defined by the metric. Then Q^i is itself the compact δ -extension uX_i of the proximity space X_i . The identity mapping h of X_i into Q^i is an inessential δ -continuous mapping, for the "null mapping" h_0 taking all of X_i to the point p coincides with h on $h^{-1}(S^{i-1})$. Nevertheless the identity mapping \tilde{h} of uX_i onto itself, the continuous extension of the δ -continuous mapping h, is essential.

Theorem 1.5.1.6 The δ -dimension δ -dim X of a proximity space X is the largest integer $n \ge 0$ for which there exists an essential mapping into the closed n-dimensional ball.

Proof: Let the proximity space X have δ -dimension δ -dim X = n. Then by the previous theorem there exists a δ -continuous mapping f of some subset

A of X into the sphere S^{n-1} which cannot be extended to a δ -continuous mapping of the whole space X into S^{n-1} . At the same time, by Proposition 1.3.2.8, the mapping f considered as a mapping into the boundary of the cube Q^n can be extended to a δ -continuous mapping F of the whole space X into Q^n . This mapping F will be essential; otherwise f could be extended to a δ -continuous mapping of X into S^{n-1} .

It remains to show that for each m > n every δ -continuous mapping f of X into Q^m is inessential. In order to do this, every δ -continuous mapping f of X into Q^m , restricted to $f^{-1}(S^{m-1})$, can be extended, by the previous theorem, to a δ -continuous mapping of X into S^{m-1} . Therefore f is inessential, and the proof is complete.

Before going on to the further study of the δ -dimension of arbitrary proximity spaces, we shall test the theory so far developed on some important special cases, where the spaces are subsets lying in Euclidean or Hilbert spaces.

Proposition 1.5.1.5 Each open set in Euclidean or Hilbert space \mathbb{R}^n $(n = 1, 2, ..., +\infty)$ has δ -dimension n.

Proof: Let us observe first that every open set Γ in \mathbb{R}^n completely contains some *n*-dimensional closed parallelotope Q^n . Therefore, by Corollary 1.5.1.1 and Proposition 1.5.1.1, we conclude that δ -dim $\Gamma \ge \delta$ -dim $Q^n = \dim Q^n =$ n. In the case $n = +\infty$ this is all there is to prove. In the remaining case reverse inequality is yet to be proved. In view of the inclusion $\Gamma \subseteq \mathbb{R}^n$ it suffices to prove that δ -dim $\mathbb{R}^n \le n$. In order to do this we need the simple geometric

Lemma 1.5.1.6 Let there be given in \mathbb{R}^n a collection of parallelepipeds with faces parallel to the coordinate hyperplanes. If one of them intersects all the remaining ones and if these remaining ones have a non-empty intersection, then the intersection of all the given parallelepipeds is also non-empty.

This lemma is easy to prove by induction which we leave to the reader.

We shall now prove the inequality $\delta - \dim \mathbb{R}^n \leq n$.

Let us observe first that in Euclidean space \mathbb{R}^n of any dimension n, for each $\varepsilon > 0$ there exists a so called Lebesque tessellation ϕ , a covering of \mathbb{R}^n of order n + 1 consisting of cubes Q_α which are non-intersecting or intersecting only in faces (of various dimensions), with faces parallel to the coordinate hyperplanes, and with each O_α having edges of length ε (see [194], p. 266). From the construction of this tessellation it is clear that the distance between any two non-intersecting cubes of ϕ is bounded below by some positive number $2\tau < \varepsilon$. This means that if each cube of the tessellation ϕ is expanded by a similarity (with the center of similarity at the center of the cube) in the ratio $1 + \tau/\varepsilon\sqrt{n}$, no new intersections are obtained. If some cube Q_{α_0} is disjoint from the non-empty intersection of some cubes Q_{α_i} , where $i = 1, 2, \ldots, n+1$, then by the above lemma, it is disjoint from at least one of them. But then the similar cube Q'_{α_0} will also be disjoint from one of the similar cubes Q'_{α_i} (similar to Q_{α_i}). Consequently the cube Q'_{α_0} will not meet the intersection of the cubes Q'_{α_i} . Thus the covering ω_{ε} , consisting of the cubes $Q_{\alpha'}$ expanded by similarity in the ratio $1 + \tau/\varepsilon\sqrt{n}$, will have just the same order n+1. Though it is an infinite covering, it still satisfies the defining condition for (finite) δ -coverings. The diameter of each cube Q'_{α} of this coverings is $\varepsilon \sqrt{n} + 2\tau$. From the definition of a δ -covering it follows that for every δ -covering γ of \mathbb{R}^n there is a positive ε such that every covering of \mathbb{R}^n whose elements have a diameter not exceeding ε is a refinement of γ . Therefore an arbitrary δ -covering $\gamma = \{\Gamma_i\}$ of the space \mathbb{R}^n can be refined by a sufficiently fine (in the sense of the diameters of its elements) infinite covering ω_{ε} of order n+1. Let us consider next the union U_1 of all the cubes in the covering ω_{ε} which lie completely in $\Gamma_1 \in \gamma$; afterwards - the union U_2 of all those cubes in ω_{ε} which lie in Γ_2 but not in Γ_1 ; the union U_3 of all those cubes in ω_{ε} which lie in Γ_3 but neither in Γ_1 nor in Γ_2 , and so on. Since the covering γ is finite, and ω_{ε} is a refinement of γ , we obtain in this way a finite cover $\omega = \{U_i\}$ refining γ and having the same order n + 1 as ω_{ε} does. Clearly ω is a δ -covering. Therefore every δ -covering of the space \mathbb{R}^n can be refined by a δ -covering of order n+1, as was to be proved.

Corollary 1.5.1.3 Every set which is somewhere dense in \mathbb{R}^n , n = 1, 2, ..., has δ -dimension n.

Corollary 1.5.1.4 For every open subset of \mathbb{R}^n , n = 1, 2, ..., the δ -dimension is equal to the topological dimension.

The first corollary can be strengthened for bounded sets.

Proposition 1.5.1.6 A bounded subset A of Euclidean space \mathbb{R}^n has δ dim A = n if and only if it is somewhere dense in \mathbb{R}^n .

Proof: Evidently it suffices to prove that if the bounded subset A of \mathbb{R}^n has δ -dim A = n, then it is dense in some open set. Let us take the closure \overline{A} of the set A. Since A is bounded, \overline{A} is compact, and therefore \overline{A} coincides

with uA. Thus the dimension of the closure \overline{A} is equal to n. From this it follows that \overline{A} contains some open set Γ of \mathbb{R}^n . Since A is dense in \overline{A} , we conclude easily that A is dense in Γ , and this is what was to be proved.

Let us notice here that for unbounded, even closed, sets in \mathbb{R}^n the assertion of this theorem is not true. Also the assertion of Corollary 1.5.1.4, though it is clearly true for closed bounded sets, is false for unbounded closed sets.

Let us consider in \mathbb{R}^n a sequence of balls Q_j^n having the same radius r, the pairwise distances between which all are r too. In each ball Q_j^n let us take some finite set A_j which forms a 1/n-net in Q_j^n . The set $A = \bigcup_j A_j$ will be a countable closed set of the δ -dimension n. This follows from the following theorem:

Theorem 1.5.1.7 In order that a set $A \subseteq \mathbb{R}^n$ should have δ -dimension δ dim A = n, it is necessary and sufficient that there exists a positive number r such that for each $\varepsilon > 0$ one can find a sphere of radius r in which Aforms an ε -net.

This is a basic theorem concerning the dimension of the subsets of Euclidean space \mathbb{R}^n . The proof of this theorem is omitted, since it is complicated and too long. (see [304], &3, Theorem 4.)

From this example it is clear that the δ -dimension of a non-compact δ -space depends strongly on how it "recedes to infinity". Of course this "recession to infinity" is not yet clearly expressed.

Definition 1.5.1.7 *Let there be given a proximity space* X*. We shall define a* δ *-bordering of the space* X *as a finite family* γ *of sets* $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$, *for which one can select the sets* $B_1 \ll \Gamma_1, \ldots, B_k \ll \Gamma_k$, *so that the closure of the complement* $X - \bigcup_i B_i$ *is compact.*

Definition 1.5.1.8 We shall define the **boundary** δ -dimension of the proximity space X as the smallest of the integers n = -1, 0, 1, 2, ..., such that every δ -bordering of X can be refined by a δ -bordering of order $\leq n+1$. Let us agree to denote δ -dim^{∞}X for the boundary δ -dimension.

Lemma 1.5.1.7 Let there be given in the proximity space X a set B and some exterior δ -covering γ of B. Then the family O_{γ} of the sets $O\langle \Gamma \rangle$, $\Gamma \in \gamma$, is an exterior δ -covering of the set $uB = \overline{B}^{uX}$ in the compact space uX. **Proof:** Let $\gamma = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_k\}$ be an arbitrary exterior δ -covering of B with respect to the proximity space X. This means that there exist sets $B_1 \ll_X \Gamma_1, \ldots, B_k \ll_X \Gamma_k$, such that $B \subseteq \bigcup_{i \leq k} B_i$. By Proposition 1.1.2.5 we may suppose that each Γ_i is open in X. Since for each i the set B_i is far from $X - \Gamma_i$, the closures \overline{B}_i^{uX} and $\overline{X - \Gamma_i}^{uX}$ of these sets do not meet. Since $O\langle\Gamma_i\rangle$ is the largest open subset of uX whose intersection with X is Γ_i , $O\langle\Gamma_i\rangle = uX - \overline{X - \Gamma_i}^{uX}$ holds for each i. This means that $\overline{B_i}^{uX} \subseteq O\langle\Gamma_i\rangle$ for each i. Consequently $\overline{B}^{uX} \subseteq \bigcup_{i \leq k} \overline{B}_i^{uX} \subseteq \bigcup_{i \leq k} O\langle\Gamma_i\rangle$. From Lema 1.5.1.2, the family $O(\gamma) = \{O\langle\Gamma_i\rangle\}$ constitutes an exterior δ -covering of \overline{B}^{uX} , as was to be shown.

Theorem 1.5.1.8 For every proximity space X the boundary δ -dimension is equal to the δ -dimension of the set uX - X: δ -dim $^{\infty}X = \delta$ -dim (uX - X).

Proof: Let X be a proximity space for which δ -dim (uX - X) = n. We shall show that δ -dim^{∞}X = n. In order to do this we shall first prove that every δ -bordering of the space X can be refined by a δ -bordering of order $\leq n + 1$. Let $\gamma = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_k\}$ be an arbitrary δ -bordering of X. This means that there exist sets $B_i \ll \Gamma_i$ such that the set $\Phi = \overline{X - \bigcup_{i \leq k} B_i}^X$ is compact.

Evidently the δ -bordering γ is an exterior δ -covering of the set $B = \bigcup_{i \leq k} B_i$ with respect to X. This implies, by means of the above lemma, that the sets $O\langle \Gamma_1 \rangle, \ldots, O\langle \Gamma_k \rangle$ form an exterior δ -covering O_{γ} of uB with respect to uX. But since $X = B \cup \Phi$, and Φ is compact, thus $uX = \overline{B}^{uX} \cup \overline{\Phi}^{uX} = uB \cup \Phi$, and therefore $uX - X \subseteq uX - \Phi \subseteq uB$. Consequently, O_{γ} is an exterior δ -covering of the set uX - X (with respect to uX) as well.

Since δ -dim (uX - X) = n, there exists an exterior δ -covering $\widetilde{\omega} = {\widetilde{U}_1, \ldots, \widetilde{U}_s}$ of uX - X (with respect uX) which refines O_{γ} and has an order $\leq n + 1$. For each $i \leq k$ the intersection $O\langle \Gamma_i \rangle \cap X \subseteq \Gamma_i$ holds, and the family $\widetilde{\omega}$ is a refinement of O_{γ} ; this means that the family ω of the sets $U_j = \widetilde{U}_j \cap X, \ j \leq s$, is a refinement of the δ -covering γ . The order of ω is evidently $\leq n+1$. Therefore it remains only to prove that ω is a δ -bordering of the proximity space X.

For this we associate to the exterior δ -covering $\widetilde{\omega}$ of uX - X the sets $\Phi_j \ll_{uX} \widetilde{U}_j$ such that $uX - X \subseteq \bigcup_{j \leq s} \Phi_j$, and to these again we associate open sets \widetilde{V}_j of uX such that $\Phi_j \subseteq \widetilde{V}_j \ll_{uX} \widetilde{U}_j$ (from the property (O_6) in Theorem 1.1.1.1 and Proposition 1.1.2.5 (b)). Since $uX - X \subseteq \bigcup_{j \leq s} \Phi_j$, thus $uX - X \subseteq \bigcup_{j \leq s} \widetilde{V}_j$. Consequently the sets $V_j = \widetilde{V}_j \cap X$, open in X,

satisfy the equation $\overline{X - \bigcup_{j \leq s} V_j}^X = X - \bigcup_{j \leq s} V_j = uX - \bigcup_{j \leq s} \widetilde{V}_j$. But each \widetilde{V}_j is open in uX, and therefore the closure $\overline{X - \bigcup_{j \leq s} V_j}^X$ is compact. But since each $\widetilde{V}_j \ll_{uX} \widetilde{U}_j$, consequently $V_j \ll_X U_j$ for each $j \leq s$.

Thus we obtain the family $\omega = \{U_j\}$ which is a δ -bordering of X of order $\leq n+1$ and a refinement of the δ -bordering γ .

It remains to prove that there exists a δ -bordering of X which is not refined by a bordering of order $\leq n$. In order to do this we shall take an exterior δ -covering $\tilde{\gamma} = \{\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_k\}$ of uX - X with respect to uX which cannot be refined by any exterior δ -covering of uX - X of order $\leq n$. Then, as we have just seen, the sets $\Gamma_i = X \cap \tilde{\Gamma}_i$ constitute a δ -bordering γ of the space X. We may as well suppose that the sets $\tilde{\Gamma}_i$ are already regular open sets in uX. But then the sets Γ_i are also open subsets of X, and for each $i \ O\langle \Gamma_i \rangle = \tilde{\Gamma}_i$ holds. If there were a δ -bordering $\omega = \{U_j\}$ of X of order $\leq n$ and refining γ , then the sets $O\langle U_j \rangle$, according to the first part of the proof, would constitute an exterior δ -covering of uX - X (with respect to uX) which refines $\tilde{\gamma} = O_{\gamma}$ and has the same order as ω itself, which is impossible. The proof is complete. \clubsuit

Corollary 1.5.1.5 For every proximity space X there follows δ -dim^{∞}X $\leq \delta$ -dim X.

Proof: It immediately follows, because $\delta - \dim^{\infty} X = \delta - \dim (uX - X) \leq \delta - \dim uX = \delta - \dim X$.

Definition 1.5.1.9 The relative dimension of a completely regular space X is the largest integer n = 0, 1, ..., for which it contains a compact subspace of the dimension n.

The relative dimension of the space X will be denoted as rdX. From Theorem 1.5.1.1 and Proposition 1.5.1.1 it follows that $rdX \leq \delta$ -dim X for every proximity space.

Theorem 1.5.1.9 The δ -dimension δ -dim X of the proximity space X is the largest of the dimensions δ -dim^{∞}X and rd X.

Proof: Since $\delta - \dim^{\infty} X \leq \delta - \dim X$ and $rdX \leq \delta - \dim X$, we need only to prove that if $\delta - \dim^{\infty} X \leq n$ and $rdX \leq n$, then $\delta - \dim X = \dim uX \leq n$, as well. For this, by Theorem 1.5.1.5, it suffices to prove that every continuous mapping f of any closed subset $A \subseteq uX$ into S^n can be extended to a continuous mapping \tilde{f} of the compact space uX into S^n . For this we can notice that $\dim uX - X = \delta - \dim(uX - X) = \delta - \dim^{\infty} X \leq n$. This means that the mapping f, restricted to the closed set $A \cap \overline{uX - X}$, can be extended to a continuous mapping f' of all of $\overline{uX - X}$ into S^n . Evidently, the mapping F, which coincides with f on A and coincides with f' on $\overline{uX - X}$, will be a continuous mapping of the closed subset $A' = A \cup \overline{uX - X}$ of uX into S^n . Now, in just the same way as in the proof of Theorem 1.5.1.5, we shall extend this mapping F to a continuous mapping F' of some neighborhood U of A' into S^n . Since uX and A' are compact, there exists a neighborhood U' of A' such that $\overline{U'} \subseteq U$. Therefore we may suppose immediately that F' is defined on the closure of the neighborhood U.

The set uX - U does not meet $\overline{uX - X}$ and therefore the compact set $\Phi = uX - U$ is contained in X. Since $rdX \leq n$, then $\dim \Phi \leq n$. Therefore the mapping F', restricted to the closed set $\Phi \cap \overline{U}$, can be extended to a continuous mapping F'' of the whole Φ into S^n . Evidently the function \tilde{f} , which coincides with F' on \overline{U} and with F'' on Φ , is a continuous mapping of the whole compact space uX into S^n . At the same time, as can easily be seen, it coincides with f on A. This proves the theorem.

Corollary 1.5.1.6 If the proximity space X satisfies the inequality

 $\delta - \dim^{\infty} X < \delta - \dim X < \infty \,,$

then there exists a compact subset Φ of X of the dimension dim $\Phi = \delta$ -dim X.

Returning to the example of the countable closed set A lying in \mathbb{R}^n and having δ -dimension n, we can see that here rd A = 0 and δ -dim^{∞}A = n. To conclude, "recession to infinity" in this or that proximity space is to be understood as "recession beyond any compact subspace".

1.5.2 Definition and basic properties of the δ -large inductive dimension

Definition 1.5.2.1 Let (X, δ) be a proximity space. Then we say that a set $L \subseteq X$ is a δ -partition in X between A and B and denote this with (L; U, V), if there exist open sets $U, V \subseteq X$ such that

$$A \ll U$$
, $B \ll V$, $U \cap V = \emptyset$ and $U \cup V = X - L$.

It is clear that if L is a partition between A and B, then $L\overline{\delta}(A \cup B)$. Also, the proof of the following lemma is obvious. **Lemma 1.5.2.8** Let (X, δ) be a proximity space and let F_1 and F_2 be two of its closed subsets. If $F_1\overline{\delta}F_2$ and ψ^* is a partition between \overline{F}_1^{uX} , \overline{F}_2^{uX} in uX (in the topological sense), then $\psi = \psi^* \cap X$ is a δ -partition between F_1 and F_2 in X.

Definition 1.5.2.2 We say that the proximity space (X, δ) is **perfect** if $\overline{r_X H}^{uX} = r_{uX} O\langle H \rangle$ for each $H \in \tau_{\delta}$, where $r_X H$ denotes the boundary of the set H in X.

Lemma 1.5.2.9 The proximity space (X, δ) is perfect if and only if for every two disjoint open sets $H_1, H_2 \in \tau_{\delta}, O\langle H_1 \cup H_2 \rangle = O\langle H_1 \rangle \cup O\langle H_2 \rangle$ holds.

Proof: Let (X, δ) be a perfect proximity space and let $H_1, H_2 \in \tau_{\delta}, H_1 \cap H_2 = \emptyset$. To prove that $O\langle H_1 \cup H_2 \rangle = O\langle H_1 \rangle \cup O\langle H_2 \rangle$, it suffices, by properties (b) and (d) of the operator $O\langle \rangle$ and the complementation, to prove that if $F_1, F_2 \in \tau_{\delta}^c$ and $F_1 \cup F_2 = X$, then $\overline{F_1}^{uX} \cap \overline{F_2}^{uX} \subseteq \overline{F_1 \cap F_2}^{uX}$.

properties (b) and (a) of the operator $O(\gamma)$ and the complementation, to prove that if $F_1, F_2 \in \tau_{\delta}^c$ and $F_1 \cup F_2 = X$, then $\overline{F_1}^{uX} \cap \overline{F_2}^{uX} \subseteq \overline{F_1 \cap F_2}^{uX}$. Let us suppose that $\zeta \in \overline{F_1}^{uX} \cap \overline{F_2}^{uX}$ but $\zeta \notin \overline{F_1 \cap F_2}^{uX}$. Then $\zeta \in r_u X \overline{F_1}^{uX}$, otherwise ζ has a neighborhood V (in uX) such that $V \subseteq \overline{F_1}^{uX}$ and $V \cap F_1 \cap F_2 = \emptyset$. But $V \cap F_2$ is a non-empty subset of X contained in $\overline{F_1}^{uX} \cap X = F_1$, contradicting $V \cap F_1 \cap F_2 = \emptyset$. Thus, applying the perfectness of (X, δ) to $H = X - F_1, \zeta \in \overline{r_X F_1}^{uX}$ holds. But $r_X F_1 \subseteq F_1 \cap F_2$, proving $\zeta \in \overline{F_1 \cap F_2}^{uX}$, which is a contradiction.

Conversely, let us assume the condition of the lemma. Let $H \in \tau_{\delta}$ and let $H^* = X - \overline{H}$. Then it is clear that $r H = X - (H \cup H^*)$. Consequently,

(1)
$$\overline{r H}^{uX} = \overline{X - (H \cup H^*)}^{uX} = uX - O\langle H \cup H^* \rangle = uX - (O\langle H^* \rangle \cup O\langle H \rangle).$$

Moreover, $uX - \overline{O\langle H \rangle}^{uX} = uX - \overline{\overline{H}^X}^{uX} = O\langle H^* \rangle$, i.e.

(2)
$$r_{uX}O\langle H\rangle = uX - (O\langle H\rangle \cup O\langle H^*\rangle).$$

From (1) and (2) $\overline{r H}^{uX} = r_{uX} O \langle H \rangle$ holds.

Corollary 1.5.2.1 Every compact proximity space (X, δ) is perfect.

Proof: The proof is immediate if we note that $O\langle H \rangle = int H$ for every $H \subseteq X$.

Lemma 1.5.2.10 If the closed subset ψ of a perfect proximity space (X, δ) is a δ -partition between two closed sets $F_1, F_2 \subset X$, then $\overline{\psi}^{uX}$ is a partition between \overline{F}_1^{uX} and \overline{F}_2^{uX} in uX.

Proof: Let ψ be a δ -partition between F_1 and F_2 . Then, by definition, there exist the sets U_1 , $U_2 \in \tau_{\delta}$ such that

$$X - \psi = U_1 \cup U_2$$
, $U_1 \cap U_2 = \emptyset$ and $U_i \gg F_i$, $i = 1, 2$.

Let $\psi_i = \psi \cup U_i$, i = 1, 2. Then $\psi_i \in \tau_{\delta}^c$ and $\psi_1 \overline{\delta} F_2$, $\psi_2 \overline{\delta} F_1$. Consequently, $\overline{F}_1^{uX} \subset uX - \overline{\psi_2}^{uX} = O\langle U_1 \rangle$, $\overline{F}_2^{uX} \subset uX - \overline{\psi}_1^{uX} = O\langle U_2 \rangle$. Since $U_1 \cap U_2 = \emptyset$, then $O\langle U_1 \rangle \cap O\langle U_2 \rangle = O\langle U_1 \cap U_2 \rangle = \emptyset$. Now from $X - \psi = U_1 \cup U_2$, (X, δ) is perfect and by property (h) of operator $O\langle \rangle$ we have $uX - \overline{\psi}^{uX} = O\langle X - \psi \rangle = O\langle U_1 \cup U_2 \rangle = O\langle U_1 \rangle \cup O\langle U_2 \rangle$, i.e. $u\psi$ is a partition between uF_1 and uF_2 in uX, where $u\psi = \overline{\psi}^{uX}$.

Definition 1.5.2.3 To every proximity space (X, δ) one assigns the δ -large inductive dimension of X, denoted by δ -Ind X, which is an integer larger then -1 or "infinite number" $+\infty$. The definition of dimension function δ - Ind X consists in the following conditions:

 $(LID_1) \ \delta - Ind X = -1$ if and only if $X = \emptyset$;

 (LID_2) δ - Ind $X \leq n$, where n = 0, 1, ..., if for every two far closed sets F_1 and F_2 there exists a δ -partition L between F_1 and F_2 such that δ - Ind $L \leq n - 1$;

 $(LID_3) \ \delta - Ind \ X = n \text{ if and only if } \delta - Ind \ X \leq n \text{ and } \delta - Ind \ X > n-1,$ i.e. the inequality $\delta - Ind \ X \leq n-1$ does not hold ;

 $(LID_4) \ \delta - Ind X = +\infty$ if and only if $\delta - Ind X > n$, for each $n = -1, 0, 1, \ldots$

Theorem 1.5.2.1 For every proximity subspace (Y, δ_Y) of a proximity space (X, δ) we have $\delta - Ind Y \leq \delta - Ind X$.

Proof: The theorem is obvious if $\delta - Ind X = +\infty$, so that one can suppose that $\delta - Ind X < +\infty$. We shall apply induction with respect to $\delta - Ind X$. Clearly, the inequality holds if $\delta - Ind X = -1$. Let us assume the theorem is proved for all proximity spaces whose δ -large inductive dimension is $\leq n-1$. Let us consider a proximity space (X, δ) with $\delta - Ind X = n$, a subspace (Y, δ_Y) and A, B two far closed subsets of Y. Then $A\overline{\delta}B$ and therefore $\overline{A} \overline{\delta B}$. Since $\delta - Ind X = n$, there exists a δ -partition (L; U, V) in X between \overline{A} and \overline{B} such that $\delta - Ind L \leq n-1$. It is easy to see that the triple $(L \cap Y; U \cap Y, V \cap Y)$ is a δ -partition in Y between A and B. Hence by the inductive assumption, $\delta - Ind L \cap Y \leq n-1$ which, together with (LID_2) in Definition 1.5.2.3, yields the inequality $\delta - Ind Y \leq n = \delta - Ind X$.

Theorem 1.5.2.2 A proximity space (X, δ) satisfies the inequality δ -Ind $X \leq n$ if and only if for every closed set $F \subseteq X$ and each δ -neighborhood U_F of F there exists an open δ -neighborhood U_F^* of F such that $F \ll U_F^* \ll U_F$ and δ - Ind $rU_F^* \leq n - 1$.

Proof: Let (X, δ) be a proximity space satisfying $\delta - Ind X \leq n, n \geq 0$, and let us consider a closed subset F of X and an open δ -neighborhood U_F of F. Then $F \ll U_F$, and, by definition of the relation \ll , $F\overline{\delta}X - U_F$ holds. Let (L; U, V) be the δ -partition between F and $X - U_F$ in X, satisfying $\delta - Ind L \leq n - 1$, then we have: $X - L = U \cup V, U \cap V = \emptyset, F \ll U$ and $X - U_F \ll V$. By Theorem 1.1.1.1 there follows that $F \ll U \subseteq X - V \ll U_F$. Since $rU \subseteq (X - U) \cap (X - V) = X - (U \cup V) = L$, then $\delta - Ind \, rU \leq n - 1$ by Theorem 1.5.2.1.

Now, let us assume that a proximity space (X, δ) satisfies the conditions of the theorem. Let us consider two far closed sets $A, B \subseteq X$. By the definition of relation \ll , $A \ll X - B$ holds. From the conditions of the theorem, there exists U_A^* such that $A \ll U_A^* \ll X - B$ and $\delta - Indr U_A^* \leqslant$ n-1. Using the property of the relation \ll it is easy to see that the triple $(r U_A^*; U_A^*, X - U_A^*)$ is a δ -partition between A and B, so that $\delta - Ind X \leqslant n$.

Theorem 1.5.2.3 If (X, δ) is a proximity space and $\delta - Ind X = n, n > 1$, then for k = 0, 1, 2, ..., n - 1 the proximity space (X, δ) contains a closed proximity subspace Y_k such that $\delta - Ind Y_k = k$.

Proof: It suffices to show that X contains a closed subspace Y_{n-1} such that $\delta - Ind Y_{n-1} = n-1$. As $\delta - Ind X > n-1$, there exists a closed set $F \subseteq X$ and an open set U_F , $F \ll U_F$, such that for every open set U_F^* , $F \ll U_F^*$, satisfying the condition $F \ll U_F^* \ll U_F$ we have $\delta - Ind r U_F^* > n-2$. On the other hand, since $\delta - Ind X \leq n$, there exists an open set U_F^{**} , $F \ll U_F^{**}$, satisfying the above condition and such that $\delta - Ind r U_F^{**} \leq n-1$. The closed subspace $Y_{n-1} = r U_F^{**} \subseteq X$ has the required property.

Theorem 1.5.2.4 For every proximity space (X, δ) we have that

$$\delta - \operatorname{Ind} X \leqslant \operatorname{Ind} u X,$$

where InduX is the topological dimension of the Smirnoff compactification uX of X.

Proof: We shall apply induction with respect to Ind uX. If Ind uX = -1, then $uX = \emptyset = X$ and our inequality holds. Let us assume that the inequality holds for all proximity spaces X with Ind uX < n for some $n \ge 0$, and let us consider a proximity space X such that Ind uX = n.

Let F_1 and F_2 be far closed sets in X. Then the sets uF_1 and uF_2 are disjoint in uX so that there exists a partition $\tilde{\psi}$ in uX between uF_1 and uF_2 such that $Ind \tilde{\psi} \leq n-1$. From Lemma 1.5.2.10 we can see that $\psi = \tilde{\psi} \cap X$ is a δ -partition in X between F_1 and F_2 . Since $u\psi = \overline{\psi}^{uX}$, it follows from Theorem 2.2.1 in [96] and the inductive assumption that $\delta - Ind \psi \leq n-1$, so that $\delta - Ind X \leq n = Ind uX$.

Definition 1.5.2.4 A perfect proximity space is called a **strongly perfect** proximity space (or S-perfect proximity space) if every closed subspace of (X, δ) is perfect.

The following statements may be easily proved.

Lemma 1.5.2.11 Every compact proximity space, and every normal fine proximity space is an S-perfect space.

Proposition 1.5.2.1 Every proximity space is homeomorphic with a closed subset of a fine proximity space.

Theorem 1.5.2.5 For every S-perfect proximity space X we have

$$\delta - Ind X = Ind uX.$$

Proof: From Theorem 1.5.2.4 it suffices to show that $Ind \, uX \leq \delta - Ind \, X$. As in the proof of Theorem 1.5.2.4, we shall suppose that $\delta - Ind \, X < \delta$

 $+\infty$ and apply induction with respect to $\delta - Ind X$.

The inequality holds if $\delta - Ind X = -1$.

Let us assume that the inequality is proved for all S-perfect proximity spaces with dimension $\delta - Ind X$ smaller than $n \ge 0$, and let us consider an S-perfect proximity space X such that $\delta - Ind X = n$. Let \widetilde{F}_1 and \widetilde{F}_2 be disjoint closed sets in uX. Then there exist open sets \widetilde{V}_1 and \widetilde{V}_2 in uX such that $\widetilde{F}_i \subseteq \widetilde{V}_i$, i = 1, 2 and $\overline{\widetilde{V}_1}^{uX} \cap \overline{\widetilde{V}_2}^{uX} = \emptyset$. The sets $V_i = \overline{\widetilde{V}_i}^{uX} \cap X$ are closed in X and far, so that there exists a δ -partition ψ in X between V_1 and V_2 such that $\delta - Ind \psi \le n - 1$. From Lemma 1.5.2.10 the set $u\psi$ is a partition between uV_1 and uV_2 in uX. And from the inductive assumption $Ind X \leq n-1$ follows.

Since $\widetilde{F}_i \subseteq \overline{\widetilde{V}_i}^{uX}$, then $u\psi$ is a partition between F_1 and F_2 ; consequently $Ind \, uX \leq \delta - Ind \, X$.

Corollary 1.5.2.2 For every compact proximity space X, the topological dimension Ind X coincides with the δ -dimension $\delta - Ind X$.

Proof: This follows immediately from Theorem 1.5.2.5 and Corollary 1.5.2.1. ♣

Corollary 1.5.2.3 *Every normal fine proximity space* X *has* δ – *Ind* X = *Ind* βX .

Proof: The proof of this corollary follows immediately from Theorem 1.5.2.5 and Lemma 1.5.2.11. \clubsuit

Corollary 1.5.2.4 If X is an S-perfect proximity space and F is a closed subset of X, then $\delta - Ind F \leq \delta - Ind X$.

Proof: From the above theorem we have that $\delta - Ind F = Ind uF = Ind \overline{F}^{uX} \leq Ind uX = \delta - Ind X$.

Corollary 1.5.2.5 For every S-perfect proximity space X we have $\delta - \dim X \leq \delta - \operatorname{Ind} X$.

Proof: From Theorem 1.5.1.1 we have that $\delta - \dim X - \dim uX$. From Theorem 1.5.2.5 $\delta - \operatorname{Ind} X = \operatorname{Ind} uX$ holds, and from Theorem 3.1.28 in [96] we have that $\dim uX \leq \operatorname{Ind} uX$. Thus, for every S-perfect space, $\delta - \dim X \leq \delta - \operatorname{Ind} X$ holds.

Corollary 1.5.2.6 If (X, δ) is an S-perfect proximity space, and A and B are closed subsets of (X, δ) , then

$$\delta - Ind(A \cup B) \leqslant \delta - Ind A + \delta - Ind B + 1.$$

Proof:

$$\delta - Ind(A \cup B) = Ind \overline{A \cup B}^{uX} = Ind (\overline{A}^{uX} \cup \overline{B}^{uX}) \leqslant \\ \leqslant Ind \overline{A}^{uX} + Ind \overline{B}^{uX} + 1 = (\text{see } [96]) \\ = \delta - Ind A + \delta - Ind B + 1.$$

Theorem 1.5.2.6 The perfect proximity space X has δ – Ind X = 0 if and only if for every closed set $F \subseteq X$ and for every δ -neighborhood U of F there exists a δ -isolated set H such that $F \subseteq H \subseteq U$.

Proof: Let $\delta - Ind X = 0$ and let F be a closed subset of proximity space (X, δ) , and let $U \gg F$; then $F\overline{\delta}X - U$. Therefore, the empty set \emptyset is a δ -partition between F and X - U. Thus, there exist the sets $U_1, U_2 \in \tau_{\delta}$ such that $X = U_1 \cup U_2, U_1 \cap U_2 = \emptyset$ and $U_1 \gg F, U_2 \gg X - U$. But $uX = O\langle X \rangle = O\langle U_1 \cup U_2 \rangle = O\langle U_1 \rangle \cup O\langle U_2 \rangle$, and $O\langle U_1 \rangle \cap O\langle U_2 \rangle = \emptyset$, because $U_1 \cap U_2 = \emptyset$. Then $O\langle U_1 \rangle$ and $O\langle U_2 \rangle$ are open-closed sets in uX, i.e. $O\langle U_1 \rangle \cap X\overline{\delta}O\langle U_2 \rangle \cap X$, which implies that $U_1\overline{\delta}U_2$, i.e. $U_1\overline{\delta}X - U_1$. It is clear that $U \gg U_1 \gg F$. The converse is obvious.

Corollary 1.5.2.7 For every perfect proximity space X the conditions δ – Ind X = 0 and δ – dim X = 1 are equivalent.

Proof: This follows immediately from the above theorem and Theorem 1.5.1.2. \clubsuit

Definition 1.5.2.5 A subfamily β of the power set PX of X is said to be a δ -base of a proximity space (X, δ) if for every two subsets $A, B \subseteq X, A\overline{\delta}B$, there exist sets $U, V \in \beta$ such that $A \subseteq U, B \subseteq V$ and $U\overline{\delta}V$.

Lemma 1.5.2.12 A family $\beta \subseteq PX$ is a δ -base for a proximity space (X, δ) if and only if for every subset B of X and every δ -neighborhood A of B there exists $H \in \beta$ such that $B \ll H \ll A$.

Proof: Let β be a δ -base for (X, δ) and let $A, B \subseteq X$ such that $B \ll A$. Then $B\overline{\delta}X - A$. By Proposition 1.1.1.3 there are sets $C, D \subseteq X$ such that $B \ll C, X - A \ll D$ and $C\overline{\delta}D$. Since β is a δ -base, there exist sets $H, H^* \in \beta$ such that $C \subseteq H, D \subseteq H^*$ and $H\overline{\delta}H^*$. Hence $B \ll C \subseteq H, X - A \ll D \subseteq H^*$ and $H \ll X - H^*$. From Theorem 1.1.1.1 it follows that $B \ll H, X - H^* \ll A$ and $H \ll X - H^*$. Hence $B \ll H \ll A$.

Conversely, let $\beta \subseteq PX$ such that there exists $H \in \beta$ for which $B \ll H \ll A$ whenever $B \ll A$. Assuming that $A\overline{\delta}B$, we have $B \ll X - A$. Thus there is $H \in \beta$ such that $B \ll H \ll X - A$. Since $H \ll X - A$, then $A \ll X - H$ and hence there exists $H^* \in \beta$ such that $A \ll H^* \ll X - H$. Now it is clear that $H\overline{\delta}H^*$, $A \subseteq H^*$ and $B \subseteq H$. Therefore β is a δ -base for the proximity space (X, δ) .

From Proposition 1.5.2.2 and the above lemma one can easily prove the following:

Theorem 1.5.2.7 A proximity space (X, δ) satisfies the inequality δ -Ind $X \leq n$ if and only if it has a δ -base β consisting of open sets such that δ - Ind $rH \leq n - 1$ for every $H \in \beta$.

From Theorem 1.5.1.2 and the fact that every δ -isolated set is an openclosed set, one may easily obtain the following:

Theorem 1.5.2.8 If a proximity space (X, δ) has $\delta - \dim X = 0$, then it has $\delta - \operatorname{Ind} X = 0$ as well.

Example 1.5.2.1 The converse of the last theorem, in general, is not true, e.g., the space $(\mathbb{Q} \cap [0,1], \delta)$, where \mathbb{Q} is the set of all rational numbers and $A\overline{\delta}B$ if and only if $\overline{A}^{[0,1]} \cap \overline{B}^{[0,1]} = \emptyset$, has $\delta - \dim(\mathbb{Q} \cap [0,1]) = 1$ and $\delta - Ind(\mathbb{Q} \cap [0,1]) = 0$.

Lemma 1.5.2.13 Let (X, δ) be a proximity space such that (X, τ_{δ}) is a hereditarily normal space, and let (Y, δ_Y) be a proximity subspace of (X, δ) such that $\delta - \operatorname{Ind} Y \leq n, n \geq 0$. Then for every two far closed subsets F_1, F_2 of X, there exists a δ -partition (L; U, V) in X between F_1 and F_2 such that $\delta - \operatorname{Ind} (L \cap Y) \leq n - 1$.

Proof: Let F_1 and F_2 be far closed subsets of X. By Propositions 1.1.1.3 and 1.1.2.5 there exist two open subsets U_1 and U_2 of X such that $F_i \ll U_i$ and $U_1 \overline{\delta} U_2$. Since $Y \subseteq X$, then $\overline{U}_1 \cap Y \overline{\delta}_Y \overline{U}_2 \cap Y$, and consequently, there is a δ -partition $(L^*; V_1^*, V_2^*)$ in Y between $\overline{U}_1 \cap Y$ and $\overline{U}_2 \cap Y$ such that $\delta - Ind L^* \leq n-1$. It is easy to see that $\overline{U}_1 \cap V_1^*$ and $\overline{U}_2 \cap V_2^*$ are separated, so that there exist two open subsets V_1 and V_2 of X such that $\overline{U}_i \cap V_i^* \subseteq V_i$, $V_1 \cap V_2 = \emptyset$. The triple $(L = X - (V_1 \cup V_2), V_1, V_2)$ is a δ -partition in X between F_1 and F_2 for which $L \cap Y \subset L^*$. Hence $\delta - Ind (L \cap Y) \leq$ $\delta - Ind L^* \leq n-1$.

Theorem 1.5.2.9 For every pair of proximity subspaces X_1, X_2 of a hereditarily normal proximity space (X, δ) , we have

 $\delta - Ind \left(X_1 \cup X_2 \right) \leqslant \delta - Ind X_1 + \delta - Ind X_2 + 1.$

Proof: The theorem is obvious if one of X_1 or X_2 has $\delta - Ind = +\infty$, so that we can suppose that $I(X_1, X_2) = \delta - Ind X_1 + \delta - Ind X_2 < +\infty$. We shall apply induction with respect to $I(X_1, X_2)$. If $I(X_1, X_2) = -2$, then $X_1 = X_2 = \emptyset$ and our inequality holds. Let us assume that the inequality holds for every pair of subspaces, the sum of which $\delta - Ind$ is smaller than

 $n, n \ge -1$. Let us consider a pair of proximity subspaces X_1, X_2 such that $I(X_1, X_2) = n$. Clearly, we can suppose that $\delta - Ind X_1 \ge 0$. Let F_1 and F_2 be far closed subsets of $X_1 \cup X_2$. By virtue of Lemma 1.5.2.13 there exists a δ -partition (L; U, V) in $X_1 \cup X_2$ between F_1 and F_2 such that $\delta - Ind (L \cap X_1) \le \delta - Ind X_1 - 1$.

Since $I(L \cap X_1, L \cap X_2) \leq \delta - Ind X_1 - 1 + \delta - Ind X_2 = n - 1$, it follows by the inductive assumption that $\delta - Ind L \leq n$. This implies

$$\delta - Ind \left(X_1 \cup X_2 \right) \leqslant \delta - Ind X_1 + \delta - Ind X_2 + 1.$$

Corollary 1.5.2.8 If a hereditarily normal proximity space (X, δ) can be represented as the union of n+1 proximity subspaces X_1, \ldots, X_{n+1} such that $\delta - \operatorname{Ind} X_i \leq 0$ for $i = 1, 2, \ldots, n+1$, then $\delta - \operatorname{Ind} X \leq n$ holds.

Lemma 1.5.2.14 If (Y, δ_Y) is an open proximity subspace of a proximity space (X, δ) such that $\delta - \operatorname{Ind} Y \leq n$, $n \geq 0$, then for every two far closed subsets F_1 and F_2 of X, there exists a δ -partition (L; U, V) in X between F_1 and F_2 such that $\delta - \operatorname{Ind} (L \cap Y) \leq n - 1$.

Proof: Since $F_1\overline{\delta}F_2$, then by propositions 1.1.1.3 and 1.1.2.5 there exist two open subsets U_1 and U_2 of X such that $F_i \ll U_i$ and $U_1\overline{\delta}U_2$. Let us notice that $\overline{U}_1 \cap Y$ and $\overline{U}_2 \cap Y$ are far closed subsets of Y, hence there is a δ -partition (L^*, U_1^*, U_2^*) in Y between $\overline{U}_1 \cap Y$ and $\overline{U}_2 \cap Y$ such that $\delta - Ind L^* \leq n-1$. Since Y is open in X, then U_1^* and U_2^* are also open in X.

Let us consider $V_i = U_i^* \cup U_i$ and $L = X - (V_1 \cup V_2)$. It is easy to see that the triple $(L; V_1, V_2)$ is a δ -partition in X between F_1 and F_2 for which $L \cap Y = L^*$. Hence $\delta - Ind(L \cap Y) = \delta - IndL^* \leq n - 1$.

In a similar way, to that used to proving Theorem 1.5.2.9, and by taking into consideration Lemma 1.5.2.14, one can prove the following

Theorem 1.5.2.10 If a proximity space (X, δ) can be represented as the union of two proximity subspaces Y and Z, one of which being open, then $\delta - \operatorname{Ind} X \leq \delta - \operatorname{Ind} Y + \delta - \operatorname{Ind} Z + 1$ holds.

Corollary 1.5.2.9 If (X, δ) is a proximity space such that $X = Y \cup Z$ and Y is closed, then

$$\delta - \operatorname{Ind} X \leqslant \delta - \operatorname{Ind} Y + \delta - \operatorname{Ind} Z + 1.$$

Proof: Since $X = Y \cup Z = Y \cup (X - Y)$, then $X - Y \subseteq Z$ and X - Y is open in X. By Theorem 1.5.2.10 it follows that

 $\delta - Ind\left(Y \cup Z\right) = \delta - Ind\left(Y \cup (X - Y)\right) \leqslant \delta - Ind\left(Y + \delta - Ind\left(X - Y\right) + 1.$

But from Theorem 1.5.2.1 we have $\delta - Ind(X - Y) \leq \delta - IndZ$. Hence $\delta - Ind(Y \cup Z) \leq \delta - IndY + \delta - IndZ + 1$.

1.5.3 Definition and basic properties of the δ -small inductive dimension

Definition 1.5.3.1 To every proximity space (X, δ) one assigns the δ small inductive dimension of X, denoted by δ – ind X, which is an integer larger then or equal to -1 or "infinite number" + ∞ . The definition of dimension function δ – ind X consists in the following conditions:

 $(SID_1) \ \delta - ind X = -1$ if and only if $X = \emptyset$;

 (SID_2) δ - ind $X \leq n$, where n = 0, 1, ..., if for every point $x \in X$ and every closed set $F \subset X$ not containing x, there is a δ -partition (L; U, V)between x and F such that δ - ind $L \leq n - 1$;

 $(SID_3) \ \delta - ind \ X = n \text{ if and only if } \delta - ind \ X \leq n \text{ and } \delta - ind \ X > n-1,$ i.e. the inequality $\delta - ind \ X \leq n-1$ does not hold ;

 (SID_4) δ - ind $X = +\infty$ if and only if δ - ind $X > n, n = -1, 0, 1, \dots$

Modifying slightly the proof of Theorems 1.5.2.1, 1.5.2.2 and 1.5.2.3 we obtain the following parallel three theorems:

Theorem 1.5.3.1 For every proximity subspace (Y, δ_Y) of a proximity space (X, δ) $\delta - ind Y \leq \delta - ind X$ holds.

Theorem 1.5.3.2 A proximity space (X, δ) satisfies the inequality δ -ind $X \leq n$ if and only if for every $x \in X$ and each open neighborhood O_x of x there exists an open neighborhood O_x^* of x such that $x \in O_x^* \ll O_x$ and δ - ind $rO_x^* \leq n-1$.

Theorem 1.5.3.3 If (X, δ) is a proximity space and $\delta - ind X = n \ge 1$, then for k = 0, 1, 2, ..., n - 1 there exists $Y_k \subseteq X$ such that $\delta - ind Y_k = k$.

Theorem 1.5.3.4 For every proximity space (X, δ) ind $X \leq \delta$ - ind X holds.
Proof: We shall apply induction with respect to $\delta - ind X$. Clearly, the inequality holds if $\delta - ind X = -1$. Let us assume that the theorem proved for all the proximity spaces (X, δ) whose $\delta - ind X \leq n - 1$. Let us consider the proximity space (X, δ) with $\delta - ind X = n$, a point $x \in X$ and a closed set $F \subseteq X$ not containing x. Since $x\delta F$, there exists a δ -partition (L; U, V) between x and F such that $\delta - ind L \leq n - 1$. Using Proposition 1.1.2.5 it is easy to see that the triple (L; U, V) is also a topological partition between x and F. Hence, by the induction, $ind L \leq n - 1$, which together with Proposition 1.1.4 in [96] yields the inequality $ind X \leq n = \delta - ind X$.

Theorem 1.5.3.5 For every proximity space (X, δ) it follows that δ - ind $X \leq ind uX$.

Proof: We shall apply induction with respect to ind uX. Clearly, the inequality holds if ind uX = -1.

Let us consider the proximity space (X, δ) with ind uX = n, a point $x \in X$ and a closed set $F \subseteq X, x \in F$. Since $x\overline{\delta}F$, we have that $\overline{x}^{uX} \cap \overline{F}^{uX} = \emptyset$ and therefore $x \notin \overline{F}^{u\overline{X}}$. From the definition of ind uX, there exists a partition (L; U, V) in uX between x and \overline{F}^{uX} such that $ind L \leq n-1$.

It is easy to see that $(L \cap X; U \cap X, V \cap X)$ is a δ -partition in X between x and F. Hence, by the inductive assumption and the fact that $u(L \cap X) = \overline{L \cap X}^{uX}$, $\delta - ind(L \cap X) \leq indu(L \cap X) = ind(\overline{L \cap X})^{uX} = indL \leq n-1$, which, together with (SID_2) , yields the inequality $\delta - indX \leq induX$.

Corollary 1.5.3.1 For every compact proximity space (X, δ) the δ -small inductive dimension coincides with the topological small inductive dimension.

Proof: From Theorem 1.5.3.4 it follows that $ind X \leq \delta - ind X$. From Theorem 1.5.3.5 it follows that $\delta - ind X \leq ind uX = ind X$. Therefore $ind X = \delta - ind X$.

1.5.4 Some relations between dimension functions

Theorem 1.5.4.1 δ – ind X = ind X for any proximity space (X, δ) .

Proof: It suffices to show that $\delta - ind X \leq ind X$, because in Theorem 1.5.3.4 we have proved that $ind X \leq \delta - ind X$ for any proximity space X. Clearly we can assume that $ind X < +\infty$. We shall apply induction

with respect to ind X. The inequality holds if ind X = -1. Assuming the inequality valid for all proximity spaces of $ind X \leq n, n \geq 0$, we consider a proximity space (X, δ) with ind X = n. Let $x \in X$ and F be a closed subset of X with $x \notin F$; then $x\overline{\delta}F$. From Propositions 1.1.1.3 and 1.1.2.5 there exists an open subset U of X such that $x \notin U$ and $F \ll U$. Since ind X = n, there exists a topological partition $(L; U_1, U_2)$ between x and \overline{U} such that $\delta - ind L \leq n-1$ (by 1.1.4. in [96]). It follows from the inductive assumption that $ind L \leq n-1$. Since $F \ll \overline{U} \subseteq U_2$, then $F \ll U_2$ and hence $(L; U_1, U_2)$ is a δ -partition in X between x and F. Thus $\delta - ind X \leq n = ind X$, and the proof of the theorem is complete.

Modifying slightly the above proof, we obtain the following:

Theorem 1.5.4.2 $\delta - Ind X \leq Ind X$ for every normal proximity space (X, δ) .

Theorem 1.5.4.3 For every proximity space (X, δ) it follows that δ - ind $X \leq \delta$ - Ind X.

Proof: It is easy to prove by applying the induction with respect to $\delta - Ind X$.

Corollary 1.5.4.1 For every proximity space (X, δ) there follows ind $X \leq \delta - Ind X \leq Ind uX$.

Proof: It follows from Theorems 1.5.3.4, 1.5.4.3 and 1.5.2.4.

Corollary 1.5.4.2 ind $X \leq \delta - Ind X \leq Ind X$ for every normal proximity space (X, δ) .

The above corollary shows that $\delta - Ind X = Ind X$ for each normal proximity space (X, δ) having the property ind X = Ind X.

Thus using 1.7.7 in [96] we have:

Corollary 1.5.4.3 For a separable metric space (X, τ) the equality δ – Ind X = Ind X holds for each proximity δ on X compatible with τ .

Also, using 4.8.2 in [8] and 8.10. in [238], we have

Corollary 1.5.4.4 δ – Ind X = Ind X for every normal proximity space (X, δ) having a countable base (a countable δ -base).

Using 1.6.5 in [96], we have:

Corollary 1.5.4.5 δ -Ind X = 0 if and only if Ind X = 0 for every Lindeöf proximity space (X, δ) .

Using 2.2.4 in [96], we have:

Corollary 1.5.4.6 $\delta - Ind X = Ind X$ for every strongly paracompact strongly hereditarily normal proximity space.

By 2.4.2 and 2.4.3 in [96], we have:

Corollary 1.5.4.7 δ – Ind X = 0 if and only if Ind X = 0 and δ – Ind X = 1 if and only if Ind X = 1 for every strongly paracompact proximity space (X, δ) .

Using 3.1.4 in [96], we have

Corollary 1.5.4.8 $\delta - Ind X \leq \dim X$ for every metric proximity space (X, δ) .

Corollary 1.5.4.9 δ -ind X = Ind X for every normal fine proximity space (X, δ) .

Proof: By 2.2.9 in [96] it follows that $Ind X = Ind \beta X$, where βX is the Czech-Stone compactification of (X, τ_{δ}) . By Corollary 1.5.2.3 we have $\delta - Ind X = Ind \beta X$. Hence $\delta - Ind X = Ind \beta X = Ind X$.

Example 1.5.4.1 Dowker constructed a compact space Z, which contains a normal subspace X such that Ind X = 1 (see example 2.2.11 in [96]). The pair (X, \overline{X}^Z) defines a proximity δ on X as follows:

For $A, B \subseteq X$, $A\overline{\delta}B$ if and only if $\overline{A}^Z \cap \overline{B}^Z = \emptyset$. By Theorem 1.5.2.4 it follows that $\delta - Ind X = 0 \neq Ind X$.

Historical and bibliographic notes

The definition of the covering dimension δ -dim of proximity spaces was formulated by Ju. M. Smirnoff in 1954 [296]. All the results of subsection 5.1. were proved by Smirnoff in paper [302] (see also [303] and [305]). The notion of the large inductive dimension δ -Ind of proximity spaces was introduced by A. Kandil in 1983 [160]. In the same year he introduced the notion of the small inductive dimension δ -ind of proximity spaces in paper [162]. All the results in the other subsection of this section were proved by Kandil in his papers [159], [160], [161] and [162].

Chapter 2

Semi-proximity spaces and semi-uniform spaces

2.1 Semi-uniform spaces

2.1.1 Semi-uniformities and semi-pseudometrics

Definition 2.1.1.1 A semi-uniformity on a set X is a filter \mathcal{U} on $X \times X$ satisfying the following two conditions:

 $(SU_1) \Delta_X \subseteq U$ for each $U \in \mathcal{U}$;

 (SU_2) if $U \in \mathcal{U}$, then U^{-1} contains an element of \mathcal{U} .

A semi-uniform space is a pair (X, \mathcal{U}) , where X is a set and \mathcal{U} is a semi-uniformity on X.

Since \mathcal{U} is a filter, the condition (SU_2) can be replaced by the following formally stronger condition:

 (SU'_2) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$.

The elements of semi-uniformity are called entourages of $X \times X$ or entourages on X.

Definition 2.1.1.2 A base for a semi-uniformity \mathcal{U} is a subcollection \mathcal{V} of \mathcal{U} such that each element of \mathcal{U} contains an element of \mathcal{V} . A sub-base for a semi-uniformity \mathcal{U} is a subcollection \mathcal{W} of \mathcal{U} such that the collection of all finite intersections of elements of \mathcal{W} is a base of \mathcal{U} .

It is obvious that a base for a semi-uniformity \mathcal{U} is a filter base for the filter \mathcal{U} , while a sub-base for a semi-uniformity \mathcal{U} is a filter sub-base for the filter \mathcal{U} .

Proposition 2.1.1.1 Conditions (SU_1) and (SU_2) are necessary and sufficient for a filter base on $X \times X$ to be a base for a semi-uniformity for X. These conditions are also sufficient (but not necessary) for a filter sub-base on $X \times X$ to be a sub-base for a semi-uniformity for X.

Proposition 2.1.1.2 A collection W of sets is a sub-base for a semi-uniformity on a set X if and only if $W \neq \emptyset$, each element of W being an entourage of diagonal of X, and if $W \in W$, then W^{-1} contains a finite intersection of the elements of W.

Proof: Let us consider the collection \mathcal{V} consisting of all finite intersections of the elements of \mathcal{W} . If \mathcal{W} is a sub-base for a semi-uniformity, then \mathcal{V} is a base and therefore, by Proposition 2.1.1.1, if $V \in \mathcal{V}$ then $V' \subseteq V^{-1}$ for some $V' \in \mathcal{V}$; it follows that for each $U \in \mathcal{W}$ the set U^{-1} contains a finite intersection of elements of \mathcal{W} . It is obvious that each element of \mathcal{W} contains the diagonal and that $\mathcal{W} \neq \emptyset$. Conversely, if the conditions of proposition are satisfied, one can show without difficulty that \mathcal{V} is a filter base satisfying conditions (SU_1) and (SU_2) . Now by Proposition 2.1.1.1 \mathcal{V} is a base for semi-uniformity and finally, by definition, \mathcal{W} is a sub-base for a semi-uniformity.

Corollary 2.1.1.1 The collection of all symmetric elements of a given semi-uniformity \mathcal{U} is a base for \mathcal{U} .

Proof: Indeed, if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$ by (SU'_2) and thus $U \cap U^{-1} \in \mathcal{U}$. But $U \cap U^{-1}$ is symmetric and contained in U.

If \mathcal{U} is a semi-uniformity for a set X, then $\mathcal{U}[x] = \{U[x] : U \in \mathcal{U}\}$ is a filter on X and $x \in U[x]$ for each $x \in X$. Then there exists a unique closure u for X such that $\mathcal{U}[x]$ is a local base at x in the closure space (X, u) for each $x \in X$. This closure is defined to be **closure induced or generated by semi-uniformity** \mathcal{U} . If \mathcal{V} is a base (sub-base) for \mathcal{U} , then $\mathcal{V}[x]$ is a local base (a local sub-base) at x in the closure space (X, u) for each $x \in X$.

Example 2.1.1.1 (a) The collection \mathcal{U} of all subsets of $X \times X$ containing the diagonal is clearly a semi-uniformity on a set X. The collection consisting of only one element, namely the diagonal of $X \times X$, is a base for \mathcal{U} . Clearly \mathcal{U} is the largest semi-uniformity on X, that is, if \mathcal{V} is a semi-uniformity for X, then $\mathcal{V} \subseteq \mathcal{U}$. It is obvious that \mathcal{U} generates the discrete closure.

Let us consider now the collection \mathcal{V}_1 of all subsets $U \subseteq X \times X$ of the form $\bigcup_i (G_i \times G_i)$, where $\{G_i\}$ is a finite cover of X. Obviously \mathcal{V}_1 is a filter

base and satisfies conditions (SU_1) and (SU_2) . Thus \mathcal{V}_1 is a base for some semi-uniformity \mathcal{V} for X. Clearly \mathcal{V} generates the discrete closure operation for X. If X is infinite, then the diagonal of $X \times X$ does not belong to \mathcal{V} and hence $\mathcal{V} \neq \mathcal{U}$. Thus, if X is infinite, then \mathcal{U} and \mathcal{V} are distinct semi-uniformities generating the same closure operation.

(b) Let d be a semi-pseudometric for a set X. Then the collection of all sets of the form $U_r = \{(x, y) : d(x, y) < r\}, r > 0$, is a filter base on $X \times X$ satisfying conditions (SU_1) and (SU_2) . By Proposition 2.1.1.1 this collection is a base for a semi-uniformity \mathcal{U} which will be said to be **induced** or **generated by semi-pseudometric** d. The semi-pseudometric d induces also a closure for X. It is almost self-evident that these closures coincide. Indeed, the family $\{U_r[x] : r > 0\}$ is a local base at x with respect to the closure induced by the semi-uniformity and the same family is a local base at x with respect to the semi-pseudometric closure because $U_r[x]$ is the open r-sphere about x.

Definition 2.1.1.3 A semi-uniformity is semi-pseudometrizable if it is induced by a semi-pseudometric.

Theorem 2.1.1.1 A semi-uniformity \mathcal{U} is semi-pseudometrizable if and only if it has a countable base.

Proof: If \mathcal{U} is generated by a semi-pseudometric d, and M is a set of positive real numbers the infimum of which is zero, then evidently, the collection of all sets $\{(x, y) : d(x, y) < r\}, r \in M$, is a base for \mathcal{U} . Since M can be taken as countable, the "only if" part is proved. Conversely, let $\{U_n : n \in \mathbb{N}\}$ be a base for \mathcal{U} . Without loss of generality we may assume that $U_0 = X \times X$ and $U_n = U_n^{-1} \supseteq U_{n+1}$ for each $n \in \mathbb{N}$. Putting $d(x, y) = 2^{-n}$ if and only if $(x, y) \in U_n - U_{n+1}$ and d(x, y) = 0 if and only if $(x, y) \in \bigcap_n U_n$, we obtain a semi-pseudometric d for X which generates \mathcal{U} .

Definition 2.1.1.4 A semi-pseudometric d for a semi-uniform space (X, \mathcal{U}) is said to be **uniformly continuous** if the semi-uniformity induced by d is contained in \mathcal{U} , i.e. $\{(x, y) : d(x, y) < r\} \in \mathcal{U}$ for each positive real number r. A **uniform collection of semi-pseudometrics** is the collection of all uniformly continuous semi-pseudometrics for a semi-uniform space.

Proposition 2.1.1.3 Let \mathcal{M} be a non-void collection of semi-pseudometrics for a set X and let \mathcal{V} be the collection of all sets of the form $\{(x, y) :$ $d(x,y) < r\}, d \in \mathcal{M}, r > 0$. Then \mathcal{V} is a sub-base for a semi-uniformity and if \mathcal{M} fulfils condition:

(a) if $d_1, d_2 \in \mathcal{M}$, then $d_1 + d_2 \in \mathcal{M}$,

then \mathcal{V} is a base for a semi-uniformity. If \mathcal{V} is a base for a semi-uniformity \mathcal{U} and \mathcal{M} fulfils condition:

(b) if d is a semi-pseudometric for X and if for each r > 0 there exists a $d' \in \mathcal{M}$ and an s > 0 such that d'(x, y) < r implies d(x, y) < r, then $d \in \mathcal{M}$; then \mathcal{M} is the set of all uniformly continuous semi-pseudometrics for (X, \mathcal{U}) .

Proof: Every element of the collection \mathcal{V} is a symmetric entourage of the diagonal of $X \times X$ and therefore, by Proposition 2.1.1.1 \mathcal{V} is a sub-base for a semi-uniformity. Let us suppose that the condition (a) is satisfied. It will be shown that \mathcal{V} is a filter base. If $V_i = \{(x, y) : d_i(x, y) < r_i\} \in \mathcal{V}$, i = 1, 2, where $d_i \in \mathcal{M}$ and $r_i > 0$, then $V_1 \cap V_2$ contains the entourage $\{(x, y) : (d_1 + d_2)(x, y) < r\}$, where $r = \min(r_1, r_2)$. Finally, if \mathcal{V} is a base for a semi-uniformity \mathcal{U} and if d is a uniformly continuous pseudometric for (X, \mathcal{U}) , then clearly d fulfils the supposition of condition (b). Thus if \mathcal{M} fulfils (b), then every uniformly continuous semi-pseudometric for (X, \mathcal{U}) belongs to \mathcal{M} .

Theorem 2.1.1.2 A collection \mathcal{M} of semi-pseudometrics is a uniform collection of semi-pseudometrics if and only if \mathcal{M} is non-void, all elements of \mathcal{M} are semi-pseudometrics for the same set, say X, and the following two conditions are fulfilled:

(a) if $d_1, d_2 \in \mathcal{M}$, then $d_1 + d_2 \in \mathcal{M}$;

(b) if d is a semi-pseudometric for X and if for each r > 0 there exists a $d' \in \mathcal{M}$ and an s > 0 such that d'(x, y) < s implies d(x, y) < r, then $d \in \mathcal{M}$.

Proof: Let us first suppose that \mathcal{M} is the collection of all uniformly continuous semi-pseudometrics for a semi-uniform space (X, \mathcal{U}) . It is obvious that $\{(x, y) \to 0 : (x, y) \in X \times X\} \in \mathcal{M}$ and hence $\mathcal{M} \neq \emptyset$. Evidently every $d \in \mathcal{M}$ is a semi-pseudometric for X and hence all $d \in \mathcal{M}$ are for the same set. If $d_1, d_2 \in \mathcal{M}$, $d = d_1 + d_2$, r > 0 is a positive real number and 0 < s < r/2, then

$$\{(x,y): d(x,y) < r\} \supseteq \{(x,y): d_1(x,y) < s\} \cap \{(x,y): d_2(x,y) < s\} \in \mathcal{U},$$

which shows that d is a uniformly continuous semi-pseudometric for (X, \mathcal{U}) , i.e. $d \in \mathcal{M}$. Condition (b) is an immediate consequence of the definition of uniformly continuous semi-pseudometrics. The second part of the proof is an immediate consequence of Proposition 2.1.1.3. If \mathcal{M} is a non-void collection of semi-pseudometrics for a set X, then by Proposition 2.1.1.3 the set of all $\{(x, y) : d(x, y) < r\}, d \in \mathcal{M}, r > 0$ is a sub-base for a semi-uniformity which is defined to be the **semi-uniformity** generated by \mathcal{M} .

Theorem 2.1.1.3 If a semi-uniformity \mathcal{U} is generated by a non-void collection \mathcal{M} of semi-pseudometrics for a set X, then $U \in \mathcal{U}$ if and only if $U \subset X \times X$ and there exits a finite sequence $\{d_i : i \leq n\}$ in \mathcal{M} and a positive real number r such that $\sum_{i \leq n} d_i(x, y) < r$ implies $(x, y) \in U$.

Proof: The set \mathcal{M}_1 of all finite sums of semi-pseudometrics from \mathcal{M} contains with each d_1 and d_2 their sum $d_1 + d_2$. Now the statement follows from Proposition 2.1.1.3.

Let \mathcal{U} be a semi-uniformity for a set X, \mathcal{M} be the set of all uniformly continuous semi-pseudometrics for (X, \mathcal{U}) and let \mathcal{V} be the semi-uniformity induced by \mathcal{M} . Obviously, \mathcal{V} is contained in \mathcal{U} . Now we shall prove that $\mathcal{U} = \mathcal{V}$.

Proposition 2.1.1.4 If \mathcal{U} is a semi-uniformity for a set X, then \mathcal{U} is generated by the set \mathcal{M} of all uniformly continuous semi-pseudometrics for (X, \mathcal{M}) which take only two values, 0 and 1.

Proof: If U is a symmetric element of \mathcal{U} and if d(x, y) = 0 for $(x, y) \in U$ and d(x, y) = 1 otherwise, then it is clear that $d = \{(x, y) \to d(x, y) : (x, y) \in X \times X\}$ is a uniformly continuous semi-pseudometric for (X, \mathcal{U}) .

As a corollary we obtain the following result which shows that a semiuniform space is uniquely determined by the collection of all uniformly continuous semi-pseudometrics, and that a semi-uniformity \mathcal{U} is the smallest semi-uniformity containing every semi-uniformity induced by a uniformly continuous semi-pseudometric for (X, \mathcal{U}) .

Theorem 2.1.1.4 If (X, U) is a semi-uniform space, then $U \in U$ if and only if $U \subset X \times X$ and there exists a uniformly continuous semi-pseudometric d for (X, U) such that d(x, y) < 1 implies $(x, y) \in U$.

2.1.2 Semi-uniform closure operation

In this subsection we shall consider various descriptions of the closure u induced by a semi-uniformity \mathcal{U} .

Definition 2.1.2.1 A continuous semi-uniformity for a closure space (X, u) is a semi-uniformity for X such that the closure induced by \mathcal{U} is coarser than u. A closure operation u will be called **semi-uniformizable** if u is induced by a semi-uniformity.

Let us recall that if X is a closure space, then a semi-neighborhood of the diagonal of the product space $X \times X$ is a neighborhood of the diagonal in $\operatorname{ind}(X \times X)$, i.e. a subset U of $X \times X$ such that $U[x] \cap U^{-1}[x]$ is a neighborhood of x in X for each $x \in X$.

Proposition 2.1.2.1 If \mathcal{U} is a continuous semi-uniformity for a closure space (X, u), then each element of \mathcal{U} is a semi-neighborhood of the diagonal in $(X, u) \times (X, u)$. The set of all semi-neighborhoods of the diagonal of $(X, u) \times (X, u)$ is a continuous semi-uniformity for (X, u).

Proof: Let v be the closure induced by \mathcal{U} . If $U \in \mathcal{U}$, then U[x] is a neighborhood of x in (X, v) for each $x \in X$, and v being coarser than u, U[x] is also a neighborhood of x in (X, u). Since U^{-1} belongs to $\mathcal{U}, U^{-1}[x]$ is also a neighborhood of x in (X, u). Thus U is a semi-neighborhood of the diagonal of $(X, u) \times (X, u)$. Now let \mathcal{W} be the set of all semi-neighborhoods of the diagonal $(X, u) \times (X, u)$. Since \mathcal{W} is the neighborhood of the diagonal in $ind((X, u) \times (X, u))$, \mathcal{W} is a filter consisting of entourages of the diagonal, and clearly $U \in \mathcal{W}$ implies $U^{-1} \in \mathcal{W}$. Thus \mathcal{W} is a semi-uniformity which is, evidently, continuous.

Corollary 2.1.2.1 Let (X, u) be a closure space and let \mathcal{U} be the set of all semi-neighborhoods of the diagonal of $(X, u) \times (X, u)$. Then \mathcal{U} is the largest continuous semi-uniformity for (X, u) and the closure induced by \mathcal{U} is the finest semi-uniformizable closure coarser than u. Finally, d is a continuous semi-pseudometric for (X, u) if and only if d is a uniformly continuous semi-pseudometric for (X, \mathcal{U}) .

Theorem 2.1.2.1 A closure space (X, u) is semi-uniformizable if and only if $x \in u(y)$ implies $y \in u(x)$.

Proof: Let us suppose that u is induced by a semi-uniformity \mathcal{U} and let \mathcal{V} be the set of all symmetric elements of \mathcal{U} . \mathcal{V} is a base for \mathcal{U} and thus $x \in u(A)$ if and only if $V[x] \cap A \neq \emptyset$ for each $V \in \mathcal{V}$. Now, if $x \in u(y)$, then $y \in V[x]$ for each $V \in \mathcal{V}$, and each $V \in \mathcal{V}$ being symmetric, we obtain $x \in V[y]$ for each $V \in \mathcal{V}$, which means that $y \in u(x)$.

Conversely let us assume the condition and let us consider the largest continuous semi-uniformity \mathcal{U} for (X, u). We shall prove that \mathcal{U} induces u. It is sufficient to show that, for each $x \in X$ and each neighborhood W of x, there exists an element $U \in \mathcal{U}$ such that $U[x] \subseteq W$. Let us choose a family $\{V_y : y \in X\}$ such that V_y is a neighborhood of y in (X, u) for each $y, V_x \subset W$, and if $y \notin u(x)$ then $x \in X - V_y$. Let us put $V = \bigcup_{y \in X} V_y$, $U = V \cup V^{-1}$. Obviously U is a semi-neighborhood of the diagonal and hence $U \in \mathcal{U}$. It will be show that $U[x] = V_x$ and hence that U is the required element of \mathcal{U} . Clearly $U[x] \supset V_x$. If $y \in (U[x] - V_x)$, then $y \in V^{-1}[x]$, because $V[x] = V_x$ and hence $x \in V[y] = V_y$. Thus by construction $y \in u(x)$ and by our condition there follows $x \in u(y)$. Hence $y \in V_x$ because V_x is a neighborhood of x. But this contradicts our assumption $y \notin V_x$.

Theorem 2.1.2.2 Let X be a closure space. A symmetric subset U of $X \times X$ is a semi-neighborhood of the diagonal of $X \times X$ if and only if $\overline{A} \subseteq U[A]$ for each $A \subseteq X$.

Proof: Let us first suppose that a symmetric subset U of $X \times X$ is a semineighborhood of diagonal and let $A \subseteq X$. If $x \in \overline{A}$, then $U[x] \cap A \neq \emptyset$, so that $y \in U[x]$ for some $y \in A$. Since U is a symmetric set, we have that $x \in U[y]$. Thus $\overline{A} \subseteq U[A]$.

Conversely, let us suppose that the inclusion $\overline{A} \subseteq U[A]$ holds for each $A \subseteq X$. Since U is symmetric, to show that U is a semi-neighborhood of the diagonal, it is sufficient to prove that U[x] is a neighborhood of x in X for each $x \in X$. But by our condition there follows $\overline{X - U[x]} \subseteq U[X - U[x]] = X - \{x\}$ and hence U[x] is indeed a neighborhood of x.

Let (X, u) be a closure space induced by a semi-pseudometric d and let $U_r = \{(x, y) : d(x, y) < r\}, r > 0$. For each $A \subset X$ the set $U_r[A]$ is the open r-sphere about the set A in (X, d) and therefore $uA \subseteq U_r[A]$. Furthermore $uA = \bigcap_{r>0} U_r[A]$ since uA is the set of all $x \in X$ which have zero distance from A. Now we shall prove that the same formula is true for every semi-uniformity inducing the closure u.

Theorem 2.1.2.3 Let us suppose that a closure u for a set X is induced

by a semi-uniformity \mathcal{U} , and \mathcal{V} is a base of \mathcal{U} . Then

$$uA = \bigcap \{ U[A] : U \in \mathcal{U} \} = \bigcap \{ \mathcal{V}[\mathcal{A}] : \mathcal{V} \in \mathcal{V} \}$$

for each $A \subseteq X$.

Proof: By Proposition 2.1.2.1 each element of \mathcal{U} is a semi-neighborhood of the diagonal of $(X, u) \times (X, u)$ and therefore, by Theorem 2.1.2.2, $uA \subseteq U[A]$ for each symmetric $U \in \mathcal{U}$ and hence each $U \in \mathcal{U}$. To prove the converse inclusion, let us suppose that $x \in X - uA$. Then $V[x] \cap A = \emptyset$ for some $V \in \mathcal{V}$. Selecting any element $V_1 \in \mathcal{V}$ contained in $V \cap V^{-1}$, we obtain $x \notin V_1[A]$ which establishes the inverse inclusion.

Lemma 2.1.2.1 If U and V are relations on a set X, then

(*)
$$U \circ V \circ U = \bigcup \{ U^{-1}[x] \times U[y] : (x, y) \in V \},$$

and if U is a symmetric relation, then

$$(**) \qquad \qquad U \circ V \circ U = \bigcup \{ U[x] \times U[y] : \ (x,y) \in V \} \, .$$

Proof: To prove (*) it is sufficient to observe that the left side of (*) is the set of all pairs (z,t) such that $(z,x) \in U$ and $(y,t) \in U$ for some $(x,y) \in V$, i.e. the set $\{(z,t): z \in U^{-1}[x], t \in U[y] \text{ for some } (x,y) \in V\}$ which is the set on the right side of (*). Formula (**) follows immediately from (*).

Now we shall give an interesting description of the product $u \times u$, where u is a semi-uniform closure.

Theorem 2.1.2.4 Let us suppose that a closure operation u for a set X is induced by a semi-uniformity \mathcal{U} and $(X \times X, u \times u)$ is the product space $(X, u) \times (X, u)$. Then

$$(u \times u)V = \bigcap \{U \circ V \circ U : U \in \mathcal{U}\}$$

for each subset V of $X \times X$.

Proof: Let \mathcal{V} be the collection of all symmetric elements of \mathcal{U} . Thus \mathcal{V} is a base of \mathcal{U} and $\mathcal{V}[x]$ is a local base at x in the closure space (X, u) for each $x \in X$. Then the collection consisting of all sets $W[x] \times W[y]$, $W \in \mathcal{V}$, is a local base at (x, y) in the product space $(X \times X, u \times u)$. Since the relations W are symmetric, we have $(z, t) \in W[x] \times W[y]$ if and only if

 $(x, y) \in W[z] \times W[t]$. But $(z, t) \in (u \times u)V$ if and only if $V \cap (W[z] \times W[t]) \neq \emptyset$ for each $W \in \mathcal{V}$, i.e. for each $W \in \mathcal{V}$ there exits a pair (x, y) in V such that $(z, t) \in W[x] \times W[y]$. By virtue of formula (**) of Lema 2.1.2.1 we obtain $(z, t) \in (u \times u)V$ if and only if $(z, t) \in W \circ V \circ W$ for each $W \in \mathcal{V}$.

In concluding part of this subsection we shall describe semi-uniform closure in terms of uniformly continuous semi-pseudometrics.

Theorem 2.1.2.5 Let us suppose that a closure u for a set X is induced by a semi-uniformity \mathcal{U} and \mathcal{U} is generated by a collection \mathcal{M} of semipseudometrics. Finally, let \mathcal{M}_1 be the set of all finite sums of semi-pseudometrics from \mathcal{M} . Then

(a) $x \in uA$ if and only if the distance from x to A is zero in (X, d) for each d in \mathcal{M}_1 ;

(b) A subset U of X is a neighborhood of $x \in X$ in (X, u) if and only if U contains an open r-sphere about x in (X, d) for some $d \in \mathcal{M}_1$;

(c) a net $\{x_a\}$ converges to x in (X, u) if and only if the net $\{d(x_a, x)\}$ converges to zero in \mathbb{R} for each d in \mathcal{M} .

Proof: Statements (a) and (b) are evident by Theorem 2.1.1.3. Statement (c), with \mathcal{M} replaced by \mathcal{M}_1 , is also evident (e.g. one can use (b)). It remains to notice that if the net $\{d(x_a, x)\}$ converges to zero in \mathbb{R} for each d in \mathcal{M} , then this net converges to zero for each d in \mathcal{M}_1 .

2.1.3 Uniformly continuous mappings

A mapping f of a semi-pseudometric space (X_1, d_1) into another one (X_2, d_2) is uniformly continuous if for each r > 0 there exists an s > 0 such that $d_1(x, y) < s$ implies $d_2(f(x), f(y)) < r$, stated in other words, if \mathcal{U}_i is the semi-uniformity induced by d_i , then for each $U_2 \in \mathcal{U}_2$ there exists a $U_1 \in \mathcal{U}_1$ such that $(x, y) \in V_1$ implies $(f(x), f(y)) \in U_2$, i.e. that $f_2(U_1) \subseteq U_2$ holds, where $f_2((x, y)) = (f(x), f(y))$.

Definition 2.1.3.1 A mapping f of a semi-uniform space (X, U) into a semi-unform space (Y, V) is said to be **uniformly continuous** if for each $V \in V$ there exits a $U \in U$ such that $(x, y) \in U$ implies $(f(x), f(y)) \in V$. A semi-uniformity U is said to be **uniformly finer** than a semi-uniformity V, and V is said to be **uniformly coarser** than U, if they are for the same set, say X, and the identity mapping of (X, U) onto (X, V) is uniformly continuous. A **uniform homeomorphism** is a one-to-one mapping f of a

semi-uniform space (X, \mathcal{U}) onto a semi-uniform space (Y, \mathcal{V}) such that both f and f^{-1} are uniformly continuous.

Thus a mapping $f: (X_1, d_1) \to (X_2, d_2)$ between semi-pseudometric spaces is uniformly continuous if and only if $f: (X_1, \mathcal{U}) \to (X_2, \mathcal{U}_2)$ is uniformly continuous, where \mathcal{U}_i is the semi-uniformity generated by d_i .

Theorem 2.1.3.1 Let us suppose that f is a mapping of a semi-uniform space (X, \mathcal{U}) into a semi-uniform space $(Y, \mathcal{V}), \mathcal{U}'$ is a base for \mathcal{U} and \mathcal{V}' is a sub-base for \mathcal{V} . Each of the following conditions is equivalent to the uniform continuity of f:

(a) for each $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $f_2(U) \subseteq V$;

(b) $f_2^{-1}(V) \in \mathcal{U}$ for each $V \in \mathcal{V}$; (c) $f_2^{-1}(V) \in \mathcal{U}$ for each $V \in \mathcal{V}'$;

(d) for each $V \in \mathcal{V}'$ there exists a $U \in \mathcal{U}$ such that $f_2(U) \subseteq V$, i.e. $f(U[x]) \subseteq V[f(x)]$ for each $x \in X$.

Proof: Since the implication $(x, y) \in U \Rightarrow (f(x), f(y)) \in V$ is equivalent to $f_2(U) \subseteq V$, condition (a) is a restatement of the definition. Since \mathcal{U} is a filter on $X \times X$ and $f_2(U) \subseteq V$ if and only if $f_2^{-1}(V) \supseteq U$, conditions (a) and (b) are equivalent. It is obvious that (b) implies (c). If (c) is fulfilled and V is an element of V, then there exits a finite family $\{V_i : i \leq n\}$ in \mathcal{V}' such that $\bigcap_i V_i \subseteq V$. By $(c) f_2^{-1}(V_i) \in \mathcal{U}$ for each *i*, there holds $\bigcap_i f_2^{-1}(V_i) \in \mathcal{U}$ and finally $f_2^{-1}(V) \in \mathcal{U}$ because \mathcal{U} is a filter on $X \times X$ and $f_2^{-1}(U) \supseteq f_2^{-1}(\bigcap_i V_i) = \bigcap_i f_2^{-1}(V_i)$. It is obvious that (a) implies (d). Indeed, if $f_2(U) \subseteq V$ for some $U \in \mathcal{U}$, then we can choose a $U' \in \mathcal{U}'$ with $U' \subseteq U$ and hence $f_2(U') \subseteq V$. Conversely, if $V \in \mathcal{V}$, we can choose finite families $\{V_i\}$ in \mathcal{V}' and $\{U_i\}$ in \mathcal{U}' such that $\bigcap_i V_i \subseteq V$ and $f_2(U_i) \subseteq V_i$ for each *i*. Clearly $U = \bigcap_i U_i \in \mathcal{U}$ and $f_2(U) \subseteq V$, which establishes $(d) \Rightarrow (a)$. ÷

Proposition 2.1.3.1 A semi-uniformity \mathcal{V} is uniformly coarser than a semi-uniformity \mathcal{U} if and only if $\mathcal{V} \subset \mathcal{U}$.

Proposition 2.1.3.2 The composition of two uniformly continuous mappings is a uniformly continuous mapping.

Proof: Let $f: (X, \mathcal{U}) \to (Y, \mathcal{V})$ and $g: (Y, \mathcal{V}) \to (Z, \mathcal{W})$ be uniformly continuous mappings, $h = g \circ f$ their composition and $W \in \mathcal{W}$. Then $V = g_2^{-1}(W) \in \mathcal{V}$ because g is uniformly continuous and $U = f_2^{-1}(V) \in \mathcal{U}$ because f is uniformly continuous. But then $U = h_2^{-1}(W)$, which shows that h is uniformly continuous.

Proposition 2.1.3.3 The identity mapping of a semi-uniform space onto itself is a uniform homeomorphism. If f is a uniform homeomorphism then f^{-1} is also a uniform homeomorphism. If f and g are uniform homeomorphisms, then $g \circ f$ is also a uniform homeomorphism.

Proof: The first two statements are obvious. To prove the third one it is sufficient to observe that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ and to apply Proposition 2.1.3.2 to both $g \circ f$ and $f^{-1} \circ g^{-1}$.

Corollary 2.1.3.1 The relation $\{(X, Y) : \text{there exists a uniform homeomorphism of X onto Y} is an equivalence relation on the class of all semi$ $uniform spaces. <math>\clubsuit$

Let us recall that, if we say that a semi-pseudometric space (X, d) has a property for closure space, it is to be understood that the induced closure space (X, u_d) has this property, and if a mapping f for semi-pseudometric space has a property defined for closure spaces, it should be understood that f transposed to a mapping for closure spaces has this property. Also, if we say that a semi-uniform space (X, \mathcal{U}) has a property defined for closure space it is to be understood that the induced closure space has this property, e.g. a semi-uniform space (X, \mathcal{U}) is discrete means that the induced closure space is discrete. Similarly, a semi-uniformity \mathcal{U} is finer than a semi-uniformity \mathcal{V} means that the closure induced by \mathcal{U} is finer than the closure induced by \mathcal{V} . If f is a mapping of a semi-uniform space (X_1, \mathcal{U}_1) into a semi-uniform space (X_2, \mathcal{U}_2) , then the mapping $f: (X_1, u_1) \to (X_2, u_2)$, where u_i is the closure induced by \mathcal{U}_i , is termed f transposed to a mapping for closure spaces, and, if we say that the mapping f for semi-uniform spaces has a property defined for mapping for closure spaces, it should be understood that f transposed to a mapping for closure spaces has this property. Finally, if we say that a semi-pseudometric space has a property defined for semi-uniform spaces, it is to be understood that the induced semi-uniform space has this property, and a similar convention is used for mappings.

Proposition 2.1.3.4 Every uniformly continuous mapping is continuous and every uniform homeomorphism is a homeomorphism.

Corollary 2.1.3.2 If a semi-uniformity \mathcal{U} is uniformly finer than a semiuniformity \mathcal{V} , then \mathcal{U} is finer than \mathcal{V} .

Proof: It is sufficient to show that every uniformly continuous mapping is continuous. Let us suppose that $f: (X, \mathcal{U}) \to (X, \mathcal{V})$ is uniformly continuous. We have to show that the mapping $f: (X, u) \to (X, v)$ is continuous,

where u and v are closures induced with \mathcal{U} and \mathcal{V} respectively. Since f is a uniformly continuous mapping, then $U = f_2^{-1}(V) \in \mathcal{U}$ for each $V \in \mathcal{V}$. But then $U[x] = f^{-1}(V[f(x)])$ holds for each $x \in X$. Since the sets V[f(x)], $V \in \mathcal{V}$, form a neighborhood system at f(x) in (X, v), and the sets U[x], $U \in \mathcal{U}$, form a neighborhood system at x in (X, u), f is continuous.

If \mathcal{U} and \mathcal{V} are distinct semi-uniformities inducing the same closure u for a set X, then the identity mapping $J : (X, \mathcal{U}) \to (X, \mathcal{V})$ is a homeomorphism, but neither $J : (X, \mathcal{U}) \to (X, \mathcal{V})$ nor its inverse $J : (X, \mathcal{V}) \to (X, \mathcal{U})$ is uniformly continuous. Thus a homeomorphism need not be uniformly continuous.

2.1.4 Subspaces and products

Definition 2.1.4.1 If (X, U) is a semi-uniform space and $Y \subset X$, then the collection $\{U \cap (Y \times Y) : U \in U\}$ is obviously a semi-uniformity for Y which is called the **relativization of** U to Y. The corresponding semi-uniform space is said to be a **subspace** of (X, U). A class of semi-uniform spaces is said to be hereditary if, with each space X, it contains all subspaces of X.

Proposition 2.1.4.1 Let us suppose that (Y, \mathcal{V}) is a subspace of a semiuniform space (X, \mathcal{U}) . Then

(a) the closure induced by \mathcal{V} is a relativization of the closure induced by \mathcal{U} ;

(b) \mathcal{V} is the unique uniformly coarsest semi-uniformity for Y which renders the identity mapping of Y into (X, \mathcal{U}) uniformly continuous;

(c) if $Z \subset Y$, then (Z, W) is a subspace of (Y, V) if and only if (Z, W) is a subspace of (X, U).

Proposition 2.1.4.2 If (Y, v) is a subspace of a semi-uniformizable closure space (X, u) and if a semi-uniformity \mathcal{V} induces v, then \mathcal{V} is a relativization of a semi-uniformity inducing u.

Proof: Let \mathcal{U}_1 be the largest continuous semi-uniformity for (X, u) and let us put $\mathcal{U} = \mathcal{V} \cup \mathcal{U}_1 = \{V \cup U_1 : V \in \mathcal{V}, U_1 \in \mathcal{U}_1\}$. It is easily seen that \mathcal{U} has the required properties.

Proposition 2.1.4.3 Every restriction of a uniformly continuous mapping is a uniformly continuous mapping.

Proposition 2.1.4.4 A mapping f of a semi-uniform space X into a semiuniform space Y is uniformly continuous if and only if the restriction of fto a mapping of X onto subspace f(X) of Y is uniformly continuous.

Proposition 2.1.4.5 If g is a restriction of a mapping f for semi-uniform space, and g_1 and f_1 are the transposes of g and f to mappings for closure spaces, then g_1 is a restriction of f_1 .

Definition 2.1.4.2 The product of a family $\{(X_a, U_a) : a \in A\}$ of semi-uniform spaces, denoted by $\prod\{(X_a, U_a) : a \in A\}$, is defined to be the semi-uniform space (X, U), where X is the product of the family $\{X_a\}$ of the underlying sets, and U, called the **product semi-uniformity**, is the collection of all subsets of $X \times X$ containing a set of the form

 $(*) \qquad \qquad \{(x,y):\, (x,y)\in X\times X\,,\ a\in F\Rightarrow (pr_ax,pr_ay)\in U_a\}\,,$

where F is a finite subset of A and $U_a \in \mathcal{U}_a$ for each $a \in A$. The sets of the form (*) are then called the **canonical elements** of the product semiuniformity.

It must be shown that the collection of all canonical elements of \mathcal{U} is a base for a semi-uniformity. It is sufficient to show that the collection of all sets of the form (*) with F one-point form a sub-base for a semi-uniformity; this follows from Proposition 2.1.1.1. The main properties of products are summarized in the following

Theorem 2.1.4.1 Let (X, U) be the product of a family $\{(X_a, U_a) : a \in A\}$ of semi-uniform spaces. Then

(a) the product closure is induced by \mathcal{U} , more precisely, if u_a is induced by \mathcal{U}_a for each $a \in A$, then the product closure $\prod_a u_a$ is induced by \mathcal{U} ;

(b) each mapping $pr_a : (X, \mathcal{U}) \to (X_a, \mathcal{U}_a)$ is uniformly continuous;

(c) \mathcal{U} is the uniformly coarsest semi-uniformity such that all the mappings $pr_a: (X, \mathcal{U}) \to (X_a, \mathcal{U}_a)$ are uniformly continuous;

(d) a mapping f of a semi-uniform space (Y, \mathcal{V}) into (X, \mathcal{U}) is uniformly continuous if and only if all the mappings $pr_a \circ f : (Y, \mathcal{V}) \to (X_a, \mathcal{U}_a), a \in A$, are uniformly continuous;

(e) if the projection $pr_a : (X, U) \to (X_a, U_a)$, where a is a fixed element in A, is surjective, then a mapping h of (X_a, U_a) into a semi-uniform space (Z, W) is uniformly continuous if and only if the composition $h \circ pr_a : (X, U) \to (Z, W)$ is uniformly continuous.

Proof: (a) If U_a is any subset of $X_a \times X_a$ and x is any point of X, then the set

(1)
$$\{y: y \in X, \ pr_a y \in U_a[pr_a x]\}$$

coincides with the set

(2)
$$\{(x,y): (x,y) \in X \times X, (pr_a x, pr_a y) \in U_a\}[x].$$

Indeed, given $x \in X$, the sets (1) with $a \in A$ and $U_a \in \mathcal{U}_a$ form a local subbase at x in $(X, \prod_a u_a)$, because $\mathcal{U}_a[pr_a x]$ is a neighborhood system at $pr_a x$ in (X_a, u_a) and the sets (2) with $a \in A$ and $U_a \in \mathcal{U}_a$ form a local sub-base at x in (P, u), because the sets $\{(x, y) : (x, y) \in X \times X, (pr_a x, pr_a y) \in U_a\}$ form a sub-base for \mathcal{U} .

(b) Let f_a be the projection of (X, \mathcal{U}) into (X_a, \mathcal{U}_a) . It follows that $(f_a \times f_a)^{-1}(U_a) = \{(x, y) : (x, y) \in X \times X, (pr_a x, pr_a y) \in U_a\} \in \mathcal{U}$ for each $U_a \in \mathcal{U}_a$ and this means that each f_a is uniformly continuous and establishes the statement.

(c) If \mathcal{U}' is any semi-uniformity such that all the mappings $pr_a : (X, \mathcal{U}') \to (X_a, \mathcal{U}_a)$, $a \in A$, are uniformly continuous, then every set $\{(x, y) : (x, y) \in X \times X, (pr_a x, pr_a y) \in U_a\}$ with $a \in A$ and $U_a \in \mathcal{U}_a$ necessarily belongs to \mathcal{U}' . But these sets form a sub-base for \mathcal{U} and hence $\mathcal{U} \subset \mathcal{U}'$. This shows that \mathcal{U}' is uniformly finer than \mathcal{U} and establishes the statement.

(d) If f is uniformly continuous, then each mapping in question is uniformly continuous as the composition of two uniformly continuous mappings. Conversely, let us suppose that all the mappings in question are uniformly continuous. Let \mathcal{U}_1 be the sub-base for \mathcal{U} consisting of all the sets

$$U'_{a} = \{(x,y) : (x,y) \in X \times X, (pr_{a}x, pr_{a}y) \in U_{a}\}, a \in A, U_{a} \in \mathcal{U}_{a}.$$

By Theorem 1.1.3.4 it is sufficient to show that $f_2^{-1}(U'_a) \in \mathcal{V}$ for each $a \in A$ and $U_a \in \mathcal{U}_a$. But this is almost self-evident since $f_2^{-1}(U'_a) = (pr_a \circ f)_2^{-1}(U_a)$ and $pr_a \circ f$ is a uniformly continuous mapping of (Y, \mathcal{V}) into (X_a, \mathcal{U}_a)

(e) If h is uniformly continuous then the mapping $h \circ pr_a$ of (X, \mathcal{U}) into (Z, \mathcal{W}) is uniformly continuous as the composition of two uniformly continuous mappings, namely of the projection of (X, \mathcal{U}) into (X_a, \mathcal{U}_a) and h. Conversely, let us suppose that $k = h \circ pr_a : (X, \mathcal{U}) \to (Z, \mathcal{W})$ is uniformly continuous and the projection f_a into (X_a, \mathcal{U}_a) is surjective. Clearly $k_2^{-1}(W) = (f_a)_2^{-1}(h_2^{-1}(W))$ for each $W \in \mathcal{W}$. Now the proof will be accomplished if we show that $U_a \subset X_a \times X_a$, $(f_a)_2^{-1}(U_a) \in \mathcal{U}$ implies $U_a \in \mathcal{U}_a$ provided that f_a is surjective. But this is evident. **Proposition 2.1.4.6** If $\{X_a\}$ and $\{Y_a\}$ are families of semi-uniform spaces such that Y_a is a subspace of X_a for each $a \in A$, then the product of Y_a is a subspace of the product of X_a .

Let us recall that a pseudometric d for a closure space (X, u) is continuous (i.e. the closure induced by d is coarser than u) if and only if the function $d : (X, u) \times (X, u) \to \mathbb{R}$ is continuous. The following theorem asserts a similar result for the uniform continuity.

Theorem 2.1.4.2 A pseudometric d for a semi-uniform space (X, U) is uniformly continuous if and only if the function $d : (X, U) \times (X, U) \to \mathbb{R}$ is uniformly continuous.

Proof: If $d: (X, \mathcal{U}) \times (X, \mathcal{U}) \to \mathbb{R}$ is uniformly continuous, then for each r > 0 there exists a $U \in \mathcal{U}$ such that $(x_1, x_2) \in U$, $(y_1, y_2) \in U$ implies $|d(x_1, x_2) - d(y_1, y_2)| < r$. In particular, if $y_1 = y_2$, then $(y_1, y_2) \in U$ and $d(y_1, y_2) = 0$, and hence $(x_1, x_2) \in U$ implies $d(x_1, x_2) < r$ which proves that d is a uniformly continuous semi-pseudometric for (X, \mathcal{U}) . Let us notice that the triangle inequality has not been used.

Conversely, let us suppose that d is a uniformly continuous pseudometric. We must show that for each r > 0 there exists a $U \in \mathcal{U}$ and a $V \in \mathcal{U}$ so that $(x_1, y_1) \in U, (x_2, y_2) \in V$ implies $|d(x_1, x_2) - d(y_1, y_2)| < r$. Let us choose a positive s such that $2s \leq r$ and a $U \in \mathcal{U}$ such that $(z_1, z_2) \in U$ implies $d(z_1, z_2) < s$. Now, if $(x_1, y_1) \in U$ and $(x_2, y_2) \in U$, then

$$|d(x_1, x_2) - d(y_1, y_2)| \leq d(x_1, y_1) + d(x_2, y_2) < 2s \leq r$$

which establishes the uniform continuity of the function d on $(X, \mathcal{U}) \times (X, \mathcal{U})$.

Historical and bibliographic notes

The concept of a semi-uniform space was introduced by M. Hushek in 1964 (see [146] and [147]). The first systematic exposition of theory of semiuniform spaces was given by M. Katetov and Z. Frolik in the revised edition of E. Czech's book "Topological spaces".

2.2 Semi-proximity spaces

2.2.1 Definition and basic properties of semi-proximity relation

Definition 2.2.1.1 A relation δ on the family P(X) of all subsets of a set X is called a **semi-proximity** or **basic proximity** if δ satisfies the following conditions:

 $(SP_1) \ \emptyset \overline{\delta} X;$

 (SP_2) $A\delta B$ implies $B\delta A$;

 $(SP_3) A \cap B \neq \emptyset$ implies $A\delta B$;

 (SP_4) $(A \cup B)\delta C$ if and only if either $A\delta C$ or $B\delta C$.

The semi-proximity satisfying the following condition:

 (SP_5) {x} δ {y} implies x = y,

is said to be a separated or Hausdorff semi-proximity. The pair (X, δ) is called a space of basic proximity or semi-proximity space. (X, δ) is said to be a separated semi-proximity space if the condition (SP_5) holds.

Proposition 2.2.1.1 If δ is a semi-proximity for X, then the following statements hold:

- (a) if $A \subseteq B \subseteq X$ and $A\delta C$, then $B\delta C$;
- (b) if $A \subseteq B \subseteq X$ and $B\delta C$, then $A\delta C$;
- (c) $A\overline{\delta}\emptyset$ for every $A \subseteq X$;
- (d) if $\{A_i\}$ and $\{B_k\}$ are finite families of subsets of X for which

$$(\bigcup_j A_j)\delta(\bigcup_k B_k)$$

then $A_j \delta B_k$ for some indices j and k.

Proof: (a) If $A\delta C$ and $A \subseteq B$, then on account of (SP_4) it follows that $(A \cup B)\delta C$, and since $A \subseteq B$, then $A \cup B = B$, so that $B\delta C$.

(b) Follows from (a).

(c) Follows by (SP_1) and (b).

(d) Let $\{A_j\}$ be a finite family of subsets of X for which $(\bigcup_j A_j)\delta B$ holds. Then on account of (SP_4) , by induction, it can be easily proved that $A_j\delta B$ for some j. But then $B\delta A_j$ for some j according to (SP_2) , i.e. $(\cup B_k)\delta A_j$ for some j. From this fact, there exists some k for which $B_k\delta A_j$, from where again, by property (SP_2) , it follows that $A_j\delta B_k$ holds for some j and k. **Example 2.2.1.1** Let d be a semi-pseudometric for a set X, \mathcal{U} the semiuniformity induced by d and δ semi-proximity induced by \mathcal{U} . It is almost self-evident that

 $A\delta B$ if and only if d(A, B) = 0.

We shall say that this semi-proximity has been **induced** or **generated** by d.

Example 2.2.1.2 Let $X = \{r \in \mathbb{Q} : r > 0\}$. It is easy to see that the following functions

$$d_1(x,y) = x^{-1} + y^{-1}, \quad d_2(x,y) = 1 \quad \text{if } x \neq y$$

are semi-pseudometrics on X. Clearly both d_1 and d_2 induce the discrete closure for X. On the other hand, d_1 and d_2 induce distinct proximities. Indeed, $A\delta_{d_1}B$ if and only if $A \cap B \neq \emptyset$ or both A and B are infinite, but $A\delta_{d_2}B$ if and only if $A \cap B \neq \emptyset$.

Definition 2.2.1.2 A set $B \subseteq X$ of a semi-proximity space (X, δ) is a δ -neighborhood of $A \subseteq X$ if $A\overline{\delta}X - B$.

It is easy to prove the following proposition:

Proposition 2.2.1.2 Let (X, δ) be a semi-proximity space. Then the relation \ll has the following properties:

(a) $\emptyset \ll A$ for each $A \subseteq X$;

(b) if $A \ll B$, then $A \subseteq B$;

(c) if $A \subseteq A_1 \ll B_1 \subseteq B$, then $A \ll B$;

(d) if $A \ll B_i$, i = 1, 2, then $A \ll (B_1 \cap B_2)$;

(e) if $A \ll B$, then $X - B \ll X - A$.

If a relation \ll defined on the power set P(X) of X is satisfying conditions (a) – (e), then there exists a unique semi-proximity δ on P(X) such that $A \ll B$ if and only if B is a δ -neighborhood of A.

Let (X, δ) be a proximity space and let u_{δ} be the closure induced by δ . Every subset of X is a proximal neighborhood of the empty set. If A is a non-empty subset of semi-proximity space X, then the family $\mathfrak{N}(\delta, A)$ of all δ -neighborhoods of A is a proper filter on X.

If a set B is a δ -neighborhood of a set A in a semi-proximity space (X, δ) , then it is a neighborhood of a set A in the space (X, u_{δ}) . Let us suppose that B is a δ -neighborhood of a set A. Then $A \ll B$, or equivalently, $A\overline{\delta}X - B$. If $x \in A$, then by Proposition 2.2.1.1 (b) $x\overline{\delta}X - B$ holds. But then $x \notin \overline{X - B}$. Therefore it holds that $x \in X - \overline{X - B}$, which implies that $A \subseteq X - \overline{X - B}$. This proves that B is a neighborhood of A in the closure space (X, u_{δ}) . The converse, in general case, need not be true. However, every neighborhood of x in the space (X, u_{δ}) is a δ -neighborhood of x in (X, δ) . Really, if U is a neighborhood of the point x, then $x \in X - \overline{X - U}$. Therefore it holds that $x \notin \overline{X - U}$ from which it follows that $x \delta \overline{X} - U$. This proves that the set U is a δ -neighborhood of the point x.

Proposition 2.2.1.3 Let (X, δ) be a semi-proximity space. Then the mapping $u: P(X) \to P(X)$ defined by

$$u(A) = \{x \in X : x\delta A\}$$

is a closure operation which is said to be **induced** by δ .

Proof: First, let us notice that, by Proposition 2.2.1.1 (d), $u\emptyset = \emptyset$ holds. If $x \in A$, then on the basis of (SP_3) it follows that $x\delta A$, so that $x \in uA$. Thus we have proved that $A \subseteq uA$ for every $A \subseteq X$. Let $x \in u(A \cup B)$. Then $x\delta(A \cup B)$, so that by (SP_4) either $x\delta A$ or $x\delta B$. Therefore, either $x \in uA$ or $x \in uB$ is true, so that $x \in (uA \cup uB)$, hence $u(A \cup B) \subseteq uA \cup uB$. The converse inclusion obviously holds, which proves that u is a closure operation.

For a closure space (X, u) or a neighborhood space (i.e. for the operator of closure u) described in the previous proposition it is said to be **induced** by a semi-proximity δ and this space (semi-proximity) is denoted by $(X, u(\delta))$ or (X, u_{δ}) $(u(\delta)$ or u_{δ}).

Note that u_{δ} is completely determined by the family $\{\delta(\{x\}) : x \in X\}$. Here $\delta(A) = \{B : B\delta A\}$. It is also true that $\delta(\{x\})$ is completely determined by c_{δ} , since

$$\delta(\{x\}) = \{A : x \in u_{\delta}(A)\}.$$

Thus the following proposition has been established.

Proposition 2.2.1.4 If two semi-proximities on X, δ and δ^* are such that for every $x \in X$, $\delta(\{x\}) = \delta^*(\{x\})$, then $u_{\delta} = u_{\delta^*}$. Conversely, if $u_{\delta} = u_{\delta^*}$, then $\delta(\{x\}) = \delta^*(\{x\})$ for all $x \in X$.

Proposition 2.2.1.5 $u_{\delta}(A) = \cap \{N_A : N_A \in \mathfrak{N}(\delta, A)\}.$

Proof: $x \notin u_{\delta}(A)$ implies that $\{x\} \notin \{B : B\delta A\}$. Hence $X - \{x\} \in \mathfrak{N}(\delta, A)$ and $\cap \{N_A : N_A \in \mathfrak{N}(\delta, A)\} \subset u_{\delta}(A)$. If there exists $N_A \supset c_{\delta}(A)$, then there is a $y \in X - N_A$ such that $\{y\} \in \{B : B\delta A\}$. It follows that $X - N_A \in \{B : B\delta A\}$, which is a contradiction. **Definition 2.2.1.3** If \mathcal{U} is a semi-uniformity for a set X, then

$$\{(A,B): A, B \subset X, \ U \in \mathcal{U} \Rightarrow U[A] \cap B \neq \emptyset\}$$

is a semi-proximity δ for X which is said to be **induced by** \mathcal{U} .

Proposition 2.2.1.6 Let \mathcal{U} be a semi-uniformity for a set X, δ a semiproximity induced by \mathcal{U} and u closure induced by δ . Then u is induced by \mathcal{U} .

Proof: By definition $x \in uA$ if and only if $x\delta A$, which means, by the definition of induced proximities, that $U[x] \cap A \neq \emptyset$ for each $U \in \mathcal{U}$. It follows that, for each $x \in X$, the collection $\mathcal{U}[x]$ is a local base at x in (X, u). By the definition of semi-uniform closure the closure u is induced by \mathcal{U} .

If δ is induced by a semi-uniformity \mathcal{U} on $X \neq \emptyset$, then $\mathcal{U}[A] = \{U[A] : U \in \mathcal{U}\}$ is a base for the filter of all proximal neighborhoods of A in (X, δ) . Moreover, $\mathcal{U}[A]$ coincides with this filter.

2.2.2 δ -continuous mappings

Definition 2.2.2.1 A mapping f of a semi-proximity space (X, δ_X) into a semi-proximity space (Y, δ_Y) is said to be δ -continuous if $A\delta_X B$ implies $f(A)\delta_Y f(B)$. A one-to-one mapping f of a semi-proximity space (X, δ_X) onto a semi-proximity space (Y, δ_Y) is a δ -homeomorphism if f, as well as its inverse f^{-1} , is δ -continuous. A proximity space (X, δ_X) is a δ homeomorphic to a proximity space (Y, δ_Y) if there exists a δ -homeomorphism of X onto Y.

Definition 2.2.2.2 A semi-proximity δ_1 is said to be **finer** than a semiproximity δ_2 , and δ_2 is said to be **coarser** than δ_1 , if the identity mapping of (X, δ_1) onto (X, δ_2) is δ -continuous.

Proposition 2.2.2.1 If f is δ -continuous mapping from a semi-proximity space X onto a semi-proximity space Y, and g is a δ -continuous mapping from Y to a semi-proximity space Z, then $h = g \circ f$ is δ -continuous mapping. If f and g are δ -homeomorphisms, then $g \circ f$ is also a δ -homeomorphism. The identity mapping of a semi-proximity space onto itself is a δ -homeomorphism, and finally, if f is a δ -homeomorphism, then so is f^{-1} . **Corollary 2.2.2.1** The relation $\{(\delta_1, \delta_2) : \delta_1 \text{ is finer than } \delta_2\}$ is an order on the class of all proximities, and the relation $\{(X, Y) : X \text{ and } Y \text{ are } \delta - homeomorphic}$ is an equivalence on the class of all semi-proximity spaces.

Proposition 2.2.2.2 A mapping f of a semi-proximity space (X, δ_X) into a semi-proximity space (Y, δ_Y) is δ -continuous if and only if the following condition is fulfilled: if B is a δ -neighborhood of A in Y, then $f^{-1}(B)$ is a δ -neighborhood of $f^{-1}(A)$ in X.

Proof: Let us suppose that f is δ -continuous mapping and let B be a δ -neighborhood of A in Y. We must prove that $f^{-1}(B)$ is a δ -neighborhood of $f^{-1}(A)$ in X, i.e. that $f^{-1}(A)\overline{\delta}_X X - f^{-1}(B)$. Assuming the contrary, we obtain $ff^{-1}(A)\delta_Y f(X-f^{-1}(B))$. Since $ff^{-1}(A) \subseteq A$ and $f(X-f^{-1}(B)) \subseteq Y-B$, then on account of Proposition 2.2.1.1 it follows that $A\delta_Y Y - B$. But this is in contradiction which our supposition that B is a δ -neighborhood of A in Y. To prove the converse, let us suppose that the condition is fulfilled and that $A\delta_X B$. We have to show that $f(A)\delta_Y f(B)$. Assuming the contrary, we find that Y - f(B) is a δ -neighborhood of f(A) in Y and by the condition, $f^{-1}(Y - f(B)) = X - f^{-1}(f(B))$ is a δ -neighborhood of $f^{-1}(f(A))$. But then $f^{-1}(f(A))\overline{\delta}f^{-1}(f(B))$, which contradicts our assumption $A\delta B$ because $A \subseteq f^{-1}(f(A))$ and $B \subseteq f^{-1}(f(B))$.

Corollary 2.2.2.2 A mapping f of a semi-proximity space X into a semiproximity space Y is δ -continuous if and only if, for each subset A of X and each δ -neighborhood U of f(A) in Y, there exists a δ -neighborhood V of Ain X such that $f(V) \subseteq U$.

Proposition 2.2.2.3 Let f be a mapping of a semi-proximity space (X_1, δ_1) into a semi-proximity space (X_2, δ_2) . If δ_i is induced by a semi-uniformity \mathcal{U}_i , and the mapping $f : (X_1, \mathcal{U}_1) \to (X_2, \mathcal{U}_2)$ is uniformly continuous, then the mapping $f : (X_1, \delta_1) \to (X_2, \delta_2)$ is δ -continuous. If u_i is the closure induced by the semi-proximity δ_i and $f : (X_1, \delta_1) \to (X_2, \delta_2)$ is δ -continuous, then $f : (X_1, u_1) \to (X_2, u_2)$ is continuous.

Proof: Let us suppose that $f: (X_1, \mathcal{U}_1) \to (X_2, \mathcal{U}_2)$ is uniformly continuous and $A\delta_1 B$. If $f(A)\overline{\delta}_2 f(B)$, then $U_2[f(A)] \cap f(B) = \emptyset$ for some $U_2 \in \mathcal{U}_2$, and consequently $U_1[A] \cap B = \emptyset$, where $U_1 = f_2^{-1}(U_2)$. But $U_1 \in \mathcal{U}_1$ by the uniform continuity of f, and hence $A\overline{\delta}_1 B$ which contradicts our assumption and establishes the proximal continuity of f. Now let $f: (X_1, \delta_1) \to (X_2, \delta_2)$ be δ -continuous. If $x \in u_1 A$, then $x \delta_1 A$ and hence $f(x) \delta_2 f(A)$ by the δ -continuity. But then $f(x) \in u_2 f(A)$ holds, which proves the continuity of f.

Corollary 2.2.2.3 Let f be a Lipschitz continuous mapping of a semipseudometric space (X_1, d_1) into another one (X_2, d_2) . If δ_i is the proximity induced by d_i , i = 1, 2, then the mapping $f : (X_1, \delta_1) \to (X_2, d_2)$ is δ -continuous.

Proof: Let us suppose that $A\delta_1B$. Then we have that $d_1(A, B) = 0$, i.e. $inf_{x \in A, y \in B}d_1(x, y) = 0$. Since f is a Lipschitz continuous mapping, there exists some L > 0 such that $d_2(f(x), f(y)) \leq L d_1(x, y)$ for each $x, y \in X_1$. Therefore $inf_{x \in A, y \in B} d_2(f(x), f(y)) = 0$, i.e. $d_2(f(A), f(B)) = 0$, and hence $f(A)\delta_2f(B)$ holds. This proves δ -continuity of f.

Definition 2.2.2.3 The transpose of a mapping $f : (X, U) \to (Y, V)$ for semi-uniform spaces to a mapping for proximity spaces is the mapping $f : (X, \delta_U) \to (f(X), \delta_V|_{f(X)})$. The transpose of a mapping $f : (X, \delta_X) \to$ (Y, δ_Y) for proximity spaces to a mapping for closure spaces is the mapping $f : (X, u_{\delta_X}) \to (f(X), u_{\delta_Y}|_{f(X)})$.

If we say that a semi-uniform space (proximity space) has a property defined for proximity spaces (closure spaces), it should be understood that the induced proximity space (closure space) has this property. The same conventions are made for mappings, i.e. if we say that a mapping f for a semi-uniform spaces has a property defined for mappings for proximity spaces, e.g. that f is proximally continuous, it should be understood that the transpose of f to a mapping for proximity spaces has this property, and if we say that a mapping f for proximity spaces has a property defined for closure spaces, e.g. f is continuous, it should be understood that the transpose of f to a mapping for proximity spaces has a property defined for closure spaces, e.g. f is continuous, it should be understood that the transpose of f to a mapping for closure spaces has this property.

Now Proposition 2.2.2.3 and its corollary can be restated as follows:

Proposition 2.2.2.4 Every Lipschitz continuous mapping and every uniformly continuous mapping is proximally continuous. Every proximally continuous mapping is continuous.

Corollary 2.2.2.4 Every uniform homeomorphism (uniform embedding) is a proximal homeomorphism (proximal embedding). Every uniformly continuous pseudometric is a proximally continuous pseudometric. We recall that a uniformly continuous mapping for semi-pseudometric spaces need not be Lipschitz continuous, a proximally continuous mapping for semi-uniform spaces need not be uniformly continuous and a continuous mapping for proximity spaces need not be proximally continuous. The following theorem gives the conditions under which a proximally continuous mapping is uniformly continuous.

Theorem 2.2.2.1 A proximally continuous mapping of a pseudometrizable uniform space into a pseudometrizable uniform space is uniformly continuous.

Proof: Let us suppose that f is a proximally continuous but not uniformly continuous mapping of a pseudometric space (X', d') into another one (X, d); we have to derive a contradiction. The mapping f is not uniformly continuous and therefore there exists a positive real r and sequences (ξ_n) and (η_n) in X' such that the sequence $(d'(\xi_n, \eta_n))$ converges to zero but $d(f(\xi_n), f(\eta_n)) \ge r$ for each $n \in \mathbb{N}$. If n_i is an unbounded sequence in \mathbb{N} , then the distance from $\{\xi_{n_i}\}$ to $\{\eta_{n_i}\}$ is zero in (X', d') and consequently, f being proximally continuous, the distance from $\{f(\xi_{n_i})\}$ to $\{f(\eta_{n_i})\}$ in (X, d) is zero. We write $x_n = f(\xi_n), y_n = f(\eta_n)$ so that

(a) $d(x_n, y_n) \ge r > 0$ for each $n \in \mathbb{N}$, and

(b) the distance from $\{x_n : n \in M\}$ to $\{y_n : n \in M\}$ is zero for each infinite subset M of \mathbb{N} .

We shall derive a contradiction.

I. If the net $\{d(x_n, x_m) : (n, m) \in \mathbb{N} \times \mathbb{N}\}$ converges to zero where $\mathbb{N} \times \mathbb{N}$ is endowed with the product order, then a contradiction is obtained as follows. Let us choose $n_0 \in \mathbb{N}$ such that $n \ge n_0$, $m \ge n_0$ implies $d(x_n, x_m) < r/2$. The distance from $\{x_k : k \ge n_0\}$ to the set $\{y_k : k \ge n_0\}$ is zero and therefore, by (b), we can choose $m \ge n_0$ and $n \ge n_0$ such that $d(x_n, x_m) < r/2$. Now $d(x_m, y_m) \le d(x_m, x_n) + d(x_n, y_m) < r/2 + r/2 = r$ which contradicts our assumption (a).

II. If there exists an infinite subset M of \mathbb{N} such that the net $\{d(x_n, x_m) : (n,m) \in M \times M\}$ converges to zero, then a contradiction is obtained as in I.

III. If there exists an infinite subset M of \mathbb{N} such that the net $\{d(y_n, y_m) : (n,m) \in M \times M\}$ converges to zero, then a contradiction is obtained by applying the argument of I. with x_n and y_n interchanged.

IV. In the remaining case there exists no infinite subset M of \mathbb{N} such that the net $\{d(x_n, x_m) : (n, m) \in M \times M\}$ or the net $\{d(y_n, y_m) : (n, m) \in M \times M\}$ converges to zero. Consequently, there exists a positive real s and an infinite subset M of \mathbb{N} such that

(c) $d(x_n, x_m) \ge s, d(y_n, y_m) \ge s$

for each $n \in M$, $m \in M$, $n \neq m$. Let us choose a positive real t such that $t \leq s/2$ and $t \leq r$. It is easily seen that there exists an infinite subset L of M such that the distance from x_n to $\{y_k : k \in L\}$ as well as the distance from y_n to $\{x_k : k \in L\}$ is smaller than t for each $n \in L$. Indeed, assuming the contrary, we can construct an infinite subset K of M such that the distance from $\{x_n : n \in K\}$ to the set $\{y_n : n \in K\}$ is at least s, which contradicts our assumption (b). Let ρ be the relation consisting of all $(n,m) \in L \times L$ such that $d(x_n, y_m) < t$. We have $\rho[n] \neq t$ $\emptyset \neq \rho^{-1}[n]$ for each n. It follows from (c) that the relations ρ and ρ^{-1} are single-valued. Indeed, if $d(x_n, y_k) < t$, $d(x_m, y_k) < t$, $k, m, n \in L$, then $d(x_n, x_m) < d(x_n, y_k) + d(x_m, y_k) < 2t \leq s$ which contradicts (c) and proves that ρ^{-1} is single-valued. The same argument with x and y interchanged yields that ρ is single-valued. Thus $\rho: L \to L$ is a bijective mapping. If $n \in L$, then $n \in \mathbb{N}$ and hence $d(x_n, y_n) \ge r \ge t$ (by (a)) which shows that $\rho n \neq n$ for each n. Now it is easily seen that there exists an infinite subset K of L such that $\rho[K] \cap K = \emptyset$. (Take a maximal element K of the ordered subset of $(P(L), \subset)$ consisting of all H such that $H \cap \rho[H] = \emptyset$ and show that K is infinite). Evidently the distance from $\{x_n : n \in K\}$ to the set $\{y_n: n \in K\}$ is s at the most, which contradicts our assumption (b). The proof is complete. 🐥

Corollary 2.2.2.5 Two pseudo-metrics are uniformly equivalent if and only if they are proximally equivalent; in other words, if d_1 and d_2 are pseudometrics for a set X, \mathcal{U}_i is the semi-uniformity induced by d_i and δ_i is the semi-proximity induced by d_i , i = 1, 2, then $\mathcal{U}_1 = \mathcal{U}_2$ if and only if $\delta_1 = \delta_2$.

Proof: Any uniform homeomorphism is a proximal homeomorphism by Corollary 2.2.2.4 and therefore $\mathcal{U}_1 = \mathcal{U}_2$ implies $\delta_1 = \delta_2$. It follows immediately from Theorem 2.2.2.1 that $\delta_1 = \delta_2$ implies $\mathcal{U}_1 = \mathcal{U}_2$.

Definition 2.2.2.4 The class of all semi-proximities ordered by the relation $\{(\delta_1, \delta_2) : \delta_1 \text{ is proximally finer than } \delta_2\}$ will be denoted by \mathbf{P} , and, given a set X, the ordered subset of \mathbf{P} consisting of all semi-proximities for X will be denoted by $\mathbf{P}(X)$. The set of all proximally continuous mappings of a semi-proximity space (X, δ_X) into a semi-proximity space (Y, δ_Y) will be denoted by $\mathbf{P}(X, Y)$.

If (X, \mathcal{U}) and (Y, \mathcal{V}) are semi-uniform spaces, then $\mathbf{U}(X, Y)$ denotes the set of all uniformly continuous mappings of X into Y. In accordance with

our convention, the symbol $\mathbf{P}(X, Y)$ will denote the set of all proximally continuous mappings of X into Y. Similarly, if (X, δ_X) and (Y, δ_Y) are semi-proximal spaces, then $\mathbf{C}(X, Y)$ will denote the set of all continuous mapping of X into Y. Our earlier results can be restated as follows:

(*)
$$\mathbf{C}(X,Y) \supset \mathbf{P}(X,Y) \supset \mathbf{U}(X,Y)$$

for all semi-uniform spaces X and Y. The first inclusion holds for all proximity spaces (X, δ_X) and (Y, δ_Y) whereas $\mathbf{U}(X, Y)$ is not always defined. Roughly speaking, inclusions (*) are true whenever the symbols are defined. Theorem 2.2.2.1 asserts that $\mathbf{P}(X, Y) \subset \mathbf{U}(X, Y)$ for all pseudometric spaces (X, d_X) and (Y, d_Y) . Earlier, we have introduced the concept of a continuous semi-uniformity and a continuous semi-pseudometric for a closure space, and of a uniformly continuous semi-pseudometric for a semi-uniform space. In a similar way we shall define a continuous proximity for a closure space, and a proximally continuous semi-uniformity and a proximally continuous semipseudometric for a proximity space. Although the definitions are evident we give the precise formulations.

Definition 2.2.2.5 A continuous semi-proximity for a closure space (X, u) is a semi-proximity δ for X such that the closure induced by δ is coarser than u, i.e. the identity mapping of (X, u) onto (X, δ) is continuous. A proximally continuous semi-pseudometric (a proximally continuous semi-uniformity) for a semi-proximity (X, δ) is a semi-pseudometric (semi-uniformity) ξ for X such that the semi-proximity induced by ξ is proximally coarser than δ , i.e. the identity mapping of (X, δ) onto (X, ξ) is proximally continuous.

It is to be noted that, according to earlier results, if d is proximally continuous semi-pseudometric for a proximity space (X, δ) and if \mathcal{U} is the semiuniformity induced by d, then \mathcal{U} is a proximally continuous semi-uniformity for proximity space, and similarly, for continuous semi-pseudometrics, semiuniformities and proximities for closure space.

Example 2.2.2.1 Let us suppose that X is a closure space.

(a) The relation $\delta_s = \{(A, B) : A, B \subset X, (\overline{A} \cap B) \cup (A \cap \overline{B}) \neq \emptyset\}$ is a proximally finest continuous semi-proximity for X. If δ is any continuous semi-proximity for X and $A\delta_s B$, then $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$. But $\overline{A} \cap B \neq \emptyset$ implies $y \in \overline{A}$ for some $y \in B$, and δ being a continuous proximity, we obtain $y\delta A$ and hence $B\delta A$ and thus $A\delta B$. Similarly $A \cap \overline{B} \neq \emptyset$ yields $A\delta B$. Thus $A\delta B$ there always holds whenever $A\delta_s B$, which shows that δ_s is proximally finer than δ . On the other hand, if $x \in \overline{A}$, then clearly $x\delta_s A$, which means that δ_s is a continuous proximity for the closure space X.

(b) The relation $\delta_c = \{(A, B) : A\delta_s B \text{ or both } A \text{ and } B \text{ are infinite}\}$ is a continuous proximity for X, and if some semi-proximity induces the closure structure of X, then δ_c is the proximally coarsest semi-proximity inducing the closure structure of X.

(c) The relation $\delta_w = \{(A, B) : \overline{A} \cap \overline{B} \neq \emptyset\}$ is a continuous semiproximity. The semi-proximity δ_w is called the Wallman semi-proximity.

Definition 2.2.2.6 If (X, δ_X) is a semi-proximity space and $Y \subset X$, then $\delta_Y = \delta_X \cap (P(X) \times P(X))$ is a semi-proximity for Y which will be called the **relativization** of δ_X to Y, and the space (Y, δ_Y) will be called a **subspace** of (X, δ_X) .

The verification of the fact that δ_Y is actually a semi-proximity for Y is left to the reader. One can prove that δ_Y is the proximally coarsest proximity for Y such that the identity mapping $J : (Y, \delta_Y) \to (X, \delta_X)$ is proximally continuous. Now we have the following result:

Proposition 2.2.2.5 Let Y be a subset of a set X. If \mathcal{V} is the relativization to Y of a semi-uniformity \mathcal{U} for X, then the semi-proximity induced by \mathcal{V} is the relativization of the semi-proximity induced by \mathcal{U} . If δ_1 is the relativization to Y of a semi-proximity δ for X, then the closure induced by δ_1 is the relativization of the one induced by δ .

2.2.3 Semi-proximities and grills

In this subsection we shall present a new approach to semi-proximity structures based on the recognition that many of the entities important in this theory are grills, a concept introduced by Choquet in 1947. Not only clusters and bunches are grills but all the families $\delta(A)$, $A \in P(X)$, and $\delta(\mathcal{U})$, \mathcal{U} being an ultrafilter, are also grills. A grill is dual of filter and one of its important properties is that it is a union of ultrafilters.

Definition 2.2.3.1 A stack S on X is a family of subsets of X satisfying the condition

$$B \supset A \in \mathcal{S} \Rightarrow B \in \mathcal{S}$$
.

A grill \mathcal{G} on X is a stack on X satisfying $\emptyset \notin \mathcal{G}$ and

$$A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G} \text{ or } B \in \mathcal{G}.$$

 \mathcal{G} is a **proper grill** if \mathcal{G} is a grill and $\mathcal{G} \neq \emptyset$.

For a fixed set X we shall denote by $\Sigma(X)$, $\Phi(X)$ and $\Gamma(X)$ the set of all stacks on X, filters on X and grills on X, respectively. Finally, by $\Omega(X)$ we shall denote the set of all the ultrafilters on X. We shall use the convention that \mathcal{F} always denotes a filter, \mathcal{G} a grill. \mathcal{U} , \mathcal{A} and \mathcal{B} are used for ultrafilters.

Definition 2.2.3.2 For all $\mathcal{G} \in \Sigma(X)$ we define functions

$$c(\mathcal{G}) = \{B : X - B \notin \mathcal{G}\},\ d(\mathcal{G}) = \{B : B \cap S \neq \emptyset \text{ for each } S \in \mathcal{G}\}$$

Proposition 2.2.3.1 For all $\mathcal{G} \in \Sigma(X)$, $c(\mathcal{G}) \in \Sigma(X)$, $d(\mathcal{G}) = c(\mathcal{G})$ and $c(c(\mathcal{G})) = \mathcal{G}$ holds. Moreover, c is a bijection from Σ to Σ , from Γ to Φ and from Φ to Γ . Finally, $c(\cup \mathcal{G}_i) = \cap c(\mathcal{G}_i)$, $c(\cap \mathcal{G}_i) = \cup c(\mathcal{G}_i)$ and $c(\mathcal{A}) = \mathcal{A}$ holds for all $\mathcal{A} \in \Omega(X)$.

The proofs of these assertions are straight forward.

Making use of the well known result that every filter \mathcal{F} is the intersection of all ultrafilters containing it, there follows that

Proposition 2.2.3.2 If \mathcal{G} is a grill on X, then

 $\mathcal{G} = \cup \{ \mathcal{A} : \mathcal{A} \in \Omega(X), \, \mathcal{A} \supset c(\mathcal{G}) \}.$

Thus every grill is the union of all ultrafilters contained in it.

Proof: $\mathcal{G} = c(c(\mathcal{G})) = c(\cap \mathcal{A}) = \cup c(\mathcal{A}) = \cup \mathcal{A}.$

Proposition 2.2.3.3 If $\mathcal{G}_i \in \Gamma(X)$ for all $i \in I$, then $\cup \{\mathcal{G}_i : i \in I\} \in \Gamma(X)$.

Proof: $\cup_i \mathcal{G}_i = c(c(\cup_i \mathcal{G}_i)) = c(\cap_i c(\mathcal{G}_i))$. Since the sets $c(\mathcal{G}_i)$ are filters, $\cap(c(\mathcal{G}_i))$ is a filter and hence $\cup_i \mathcal{G}_i$ is the image of a filter under c, and hence it is a grill.

Proposition 2.2.3.4 Every ultrafilter is a grill and arbitrary unions of ultrafilters are grills.

Proof: Ultrafilters satisfy the condition $A \cup B \in \mathcal{U}$ implies $A \in \mathcal{U}$ or $B \in \mathcal{U}$. The second assertion follows from Proposition 2.2.3.3.

Proposition 2.2.3.5 The ultrafilters on X are exactly the minimal proper grills on X. Further,

 $\Gamma(X) \cap \Phi(X) = \Omega(X)$

and $c(\mathcal{G}) = \mathcal{G}$ if and only if $\mathcal{G} \in \Omega(X)$.

Proof: The first assertion follows from Proposition 2.2.3.4. A family of sets satisfying both the conditions for a grill and a filter is an ultrafilter. Hence only ultrafilters can satisfy $c(\mathcal{G}) = \mathcal{G}$; this, and the fact that they do satisfy this condition follows from Proposition 2.2.3.1.

Theorem 2.2.3.1 The mapping c is order reversing and thus $(\Gamma(X), \subset)$ is a lattice which is order isomorphic to the lattice $(\Phi(X), \supset)$.

Proof: This is a direct consequence of the definition of c. Φ is known to be a lattice, and much of its structure is known, all this information can thus be brought to bear on the lattice Γ .

Proposition 2.2.3.6 Let \mathcal{F} be a filter and \mathcal{G} a grill on X. Then $\mathcal{F} \subset \mathcal{G}$ holds if and only if there exists an ultrafilter \mathcal{B} on X such that $\mathcal{F} \subset \mathcal{B} \subset \mathcal{G}$.

Proof: $\mathcal{F} \subset \mathcal{G}$ implies that \mathcal{F} is a proper filter and \mathcal{G} a proper grill. Then $\mathcal{G} = \bigcup \{\mathcal{U}_i : i \in I\}$, where $I \neq \emptyset$. Then $c(\mathcal{G}) = \bigcap \{\mathcal{U}_i : i \in I\}$. Let us consider $F \in \mathcal{F}$. Clearly, there exists $\mathcal{U}_i \subset \mathcal{G}$ such that $F \in \mathcal{U}_i$. Let F' be an arbitrary element of $c(\mathcal{G})$. Then $F' \in \mathcal{U}_i$ and hence $F \cap F' \neq \emptyset$. It follows that $\mathcal{F} \cup c(\mathcal{G})$ is a filter subbase. Thus there exists a \mathcal{B} such that $\mathcal{F} \cup c(\mathcal{G}) \subset \mathcal{B}$. But then $c(\mathcal{G}) \subset \mathcal{B}$ so that $\mathcal{B} = \mathcal{U}_i$ for some $i \in I$.

Proposition 2.2.3.7 *If* $\mathcal{U} \subset \mathcal{G}_1 \cup \mathcal{G}_2$ *then* $\mathcal{U} \subset \mathcal{G}_1$ *or* $\mathcal{U} \subset \mathcal{G}_2$ *.*

Proof: Let us suppose the assertion is false. Then there exist $U_1 \notin \mathcal{G}_1$, $U_2 \notin \mathcal{G}_2$, $U_1, U_2 \in \mathcal{U}$. Then $U_1 \cap U_2 \in \mathcal{U}$ and hence $U_1 \cap U_2 \in \mathcal{G}_1 \cup \mathcal{G}_2$. Hence either $U_1 \cap U_2 \in \mathcal{G}_1$ or $U_1 \cap U_2 \in \mathcal{G}_2$. Clearly both alternatives lead to a contradiction.

Proposition 2.2.3.8 A relation δ on P(X) is a semi-proximity on X if and only if the following conditions are satisfied:

 $(G_1) \ \delta = \delta^{-1};$ $(G_2) \ \delta(A) \in \Gamma(X) \text{ for each } A \in P(X);$ $(G_3) \cup \{\mathcal{A} : A \in \mathcal{A}\} \subset \delta(A).$

Proof: (SP_4) and (SP_1) together are equivalent to (G_2) . The fact that grills cannot contain the empty set, proves to be convenient here. $\delta(\emptyset) = \emptyset$ need not be stated explicitly since it follows from (G_1) and (G_2) together with the fact mentioned above. (G_3) is equivalent to (SP_3) since $\cup \{\mathcal{A} : \mathcal{A} \in \mathcal{A}\}$ is exactly the set of all B satisfying $B \cap \mathcal{A} \neq \emptyset$. **Proposition 2.2.3.9** The set $\mathfrak{N}(\delta, A)$ of all δ -neighborhoods of A with respect to δ is equal to the set $c(\delta(A))$ and hence is a filter. In addition

$$\mathfrak{N}(\delta, A) = \cap \{ \mathcal{U} : \mathcal{U} \subset \delta(A) \} \subset \{ B : A \subset B \}.$$

Proof: $\delta(A)$ is a grill and hence $c(\delta(A))$ is a filter. Further, $\mathfrak{N}(\delta, A) = c(\delta(A)) = \bigcup \{\mathcal{U} : \mathcal{U} \subset \delta(A)\} = \cap \{c(\mathcal{U}) : \mathcal{U} \subset \delta(A)\} = \cap \{\mathcal{U} : \mathcal{U} \subset \delta(A)\}.$ That $\mathfrak{N}(\delta, A) \subset \{B : A \subset B\}$ follows from Proposition 2.2.3.8 (G_3).

Proposition 2.2.3.10 $B \in \delta(A)$ if and only if $B \cap N_A \neq \emptyset$ for all $N_A \in \mathfrak{N}(\delta, A)$.

Proof: $\delta(A) = c(c(\delta(A))) = d(c(\delta(A))) = d(\mathfrak{N}(\delta, A))$. Hence $B \in \delta(A)$ if and only if $B \in d(\mathfrak{N}(\delta, A))$.

Proposition 2.2.3.11 $B \in \delta(A)$ implies the existence of an ultrafilter \mathcal{U} such that

$$\mathcal{U} \subset \delta(A) \cap \delta(B)$$
 .

Proof: $B \in \delta(A)$ implies $\mathfrak{N}(\delta, B) \subset \delta(A)$ since all N_B are supersets of B. The existence of \mathcal{U} then follows from Proposition 2.2.3.6 and Proposition 2.2.3.9.

Proposition 2.2.3.12 $\delta(A \cup B) = \delta(A) \cup \delta(B)$.

Proof: $C \in \delta(A \cup B)$ if and only if $A \cup B \in \delta(C)$ if and only if $A \in \delta(C)$ or $B \in \delta(C)$ if and only if $C \in \delta(A)$ or $C \in \delta(B)$ if and only if $C \in \delta(A) \cup \delta(B)$.

Proposition 2.2.3.13 $N_A \in \mathfrak{N}(\delta, A)$ and $N_B \in \mathfrak{N}(\delta, B)$ implies that $N_A \cup N_B \in \mathfrak{N}(\delta, A \cup B)$.

Proof: If the proposition is false, there exist N_A and N_B such that $N_A \cup N_B \notin \mathfrak{N}(\delta, A \cup B)$. But then $D = (X - N_A) \cap (X - N_B) = X - (N_A \cup N_B) \in \delta(A \cup B) = \delta(A) \cup \delta(B)$. If $D \in \delta(A)$ then $X - N_A \in \delta(A)$ since $D \subset X - N_A$. This contradicts $N_A \in \mathfrak{N}(\delta, A)$. Similarly, the assumption $D \in \delta(B)$ leads to a contradiction.

Definition 2.2.3.3 For all $\mathcal{U} \in \Omega(X)$ we define

$$\delta(\mathcal{U}) = \{B : B \in \delta(U) \text{ for every } U \in \mathcal{U}\} = \cap\{\delta(U) : U \in \mathcal{U}\}.$$

Proposition 2.2.3.14 For every semi-proximity δ on X and every ultrafilter $\mathcal{U} \in \Omega(X)$ it holds:

(a) $\delta(\mathcal{U})$ is a grill; (b) $\delta(A) = \bigcup \{ \delta(\mathcal{U}) : A \in \mathcal{U} \};$ (c) $\mathcal{U} \subset \delta(\mathcal{U});$ (d) $\mathcal{B} \subset \delta(A) \Rightarrow \exists \mathcal{U}, A \in \mathcal{U}, \mathcal{B} \subset \delta(\mathcal{U});$ (e) $\mathcal{B} \subset \delta(A) \Leftrightarrow \mathcal{U} \subset \delta(\mathcal{B}).$

Proof: (a) Clearly $\delta(\mathcal{U})$ is a stack. If $A \cup B \in \delta(\mathcal{U})$, then $\mathcal{U} \subset \delta(A \cup B)$. Hence by Proposition 2.2.3.7 and Proposition 2.2.3.12 $\mathcal{U} \subset \delta(A)$ or $\mathcal{U} \subset \delta(B)$. It follows that $A \in \delta(\mathcal{U})$ or $B \in \delta(\mathcal{U})$ and hence $\delta(\mathcal{U})$ is a grill.

(b) It is an immediate consequence of Definition 2.2.3.3 that $\cup \{\delta(\mathcal{U}) : A \in \mathcal{U}\} \subset \delta(A)$. Now let $Y \in \delta(A)$ and let us assume $B \notin \delta(\mathcal{U})$ for all \mathcal{U} with $A \in \mathcal{U}$. Then for all \mathcal{U} with $A \in \mathcal{U}, \mathcal{U} \not\subset \delta(B)$ and it follows that $A \notin \delta(B)$. This contradicts $B \in \delta(A)$. This argument rests on the fact that $\delta(B)$ is a grill and thus is the union of ultrafilters. Hence A can be in $\delta(B)$ only if there exists a \mathcal{U} with $A \in \mathcal{U} \subset \delta(B)$.

(c) follows from the observation that $U \in \mathcal{U}$ implies $\mathcal{U} \subset \delta(U)$ by Proposition 2.2.3.8 (G₃). To prove (d), let us observe that $\mathcal{B} \subset \delta(A)$ implies $A \in \delta(\mathcal{B})$ which implies the existence of a \mathcal{U} with $A \in \mathcal{U} \subset \delta(\mathcal{B})$. Finally, (e) follows from the definition of $\delta(\mathcal{U})$ and the symmetry of δ .

Definition 2.2.3.4 Let δ be a semi-proximity on X. A grill \mathcal{G} on X will be called a δ -clan on X (or simply a clan) if $A \in \delta(B)$ for all $A, B \in \mathcal{G}$. If this condition holds, we also say that \mathcal{G} is δ -compatible. A δ -clan \mathcal{G} is said to be maximal if $\mathcal{G} \subset \mathcal{G}_1$, where \mathcal{G}_1 is another δ -clan, implies $\mathcal{G} = \mathcal{G}_1$.

Definition 2.2.3.5 A δ -clan \mathcal{G} on X is called δ -cluster (or simply a cluster) if the following condition is satisfied: for each $A \subset X$, $\mathcal{G} \subset \delta(A)$ implies $A \in \mathcal{G}$. A grill \mathcal{G} , which satisfies this condition, is called δ -closed.

Let us note that a cluster is exactly a grill which is both δ -compatible and δ -closed. Note also that for each $A \subset X$, $\delta(A)$ is δ -closed. Indeed, since $\delta(A) \subset \delta(B)$ implies $A \in \delta(B)$, then $B \in \delta(A)$. In general, $\delta(A)$ is not δ -clan.

The following facts are immediate: (a) if \mathcal{G}_1 and \mathcal{G}_2 are clusters from X and $\mathcal{G}_1 \subset \mathcal{G}_2$, then $\mathcal{G}_1 = \mathcal{G}_2$; (b) if $A \cap B \neq \emptyset$ for every $B \in \mathcal{G}$, where \mathcal{G} is a cluster, then $A \in \mathcal{G}$.

Proposition 2.2.3.15 For a grill \mathcal{G} the following statements are equivalent: (a) \mathcal{G} is a δ -clan;

- (b) $\mathcal{U} \subset \mathcal{G} \Rightarrow \mathcal{G} \subset \delta(\mathcal{U});$
- (c) $\mathcal{G} \subset \cap \{\delta(\mathcal{U}) : \mathcal{U} \subset \mathcal{G}\} = \cap \{\delta(A) : A \in \mathcal{G}\};$
- (d) $\mathcal{U}, \mathcal{B} \subset \mathcal{G} \Rightarrow \mathcal{U} \subset \delta(\mathcal{B}).$

Proof: $(a) \Rightarrow (b)$: If \mathcal{G} is a δ -clan and $\mathcal{U} \subset \mathcal{G}$, then $\mathcal{G} \in \delta(A)$ for all $A \in \mathcal{U}$, so that $\mathcal{G} \subset \delta(\mathcal{U})$.

 $(b) \Rightarrow (a)$: If (b) holds, let $A, B \in \mathcal{G}$. Then there exists a \mathcal{U} such that $B \in \mathcal{U} \subset \mathcal{G}$. Hence $A \in \mathcal{G} \subset \delta(\mathcal{U}) \subset \delta(B)$ so that \mathcal{G} is a δ -clan.

Clearly, (b) is equivalent to (c). Finally, (b) implies (d) and since $\mathcal{U} \subset \delta(\mathcal{B})$ holds in (d) for all $\mathcal{U} \subset \mathcal{G}$, that is for all \mathcal{G} , it follows that (d) implies (b).

Proposition 2.2.3.16 For a δ -clan the following statements are equivalent: (a) C is a cluster;

(b) $\mathcal{C} = \cap \{ \delta(\mathcal{U}) : \mathcal{U} \subset \mathcal{C} \} = \cap \{ \delta(A) : A \in \mathcal{C} \}.$

Proof: Let $C = \cap \{\delta(A) : A \in C\}$ and let us assume that $C \subset \delta(C)$; then $C \in \delta(A)$ for every $A \in C$, that is $C \in \cap \{\delta(A) : A \in C\} = C$ so that C is a cluster. If C is a cluster, let $D \in \cap \{\delta(A) : A \in C\}$; then $D \in \delta(A)$ for all $A \in C$. It follows that $C \subset \delta(D)$ and hence $D \in C$ or $\cap \{\delta(A) : A \in C\} \subset C$. That $C \subset \cap \{\delta(A) : A \in C\}$ follows from the fact that C is a δ -clan.

Proposition 2.2.3.17 If G is a cluster, then

(a) $\mathcal{G} = \cap \{\delta(A) : \delta(A) \supset \mathcal{G}\};\$

(b) $\mathcal{G} = \cap \{\delta(\mathcal{A}) : \delta(\mathcal{A}) \supset \mathcal{G}\}.$

Here $\delta(\mathcal{A}) = \{B : B \in \delta(U) \ \forall U \in \mathcal{A} \in \Omega(X)\} = \cap \{\delta(U) : U \in \mathcal{A} \in \Omega(X)\}.$ Neither of these conditions characterizes the clusters.

Proof: (a) Let us assume \mathcal{G} is a cluster. Clearly $\mathcal{G} \subset \cap \delta(A)$. Now let $D \in \delta(A)$. Then, since $\mathcal{G} \subset \delta(A)$ for all $A \in \mathcal{G}$, it follows that $A \in \delta(D)$ for all $A \in \mathcal{G}$ and thus $\mathcal{G} \subset \delta(D)$, from which $D \in \mathcal{G}$ follows. The proof of (b) is analogous. Since $\delta(A)$ is not in general a cluster but satisfies (a), and $\delta(\mathcal{A})$ is not in general a cluster but satisfies (b), neither (a) nor (b) alone can characterizes clusters.

Proposition 2.2.3.18 Let δ be a semi-proximity on X. If \mathcal{G} is a δ -clan on X, then there exists a maximal δ -clan containing \mathcal{G} . Every δ -cluster is a maximal δ -clan.

Proof: The first assertion follows from a straight forward application of Zorn's lemma. If \mathcal{G} is a cluster and \mathcal{G}^* a maximal δ -clan containing it, let $C \in \mathcal{G}^*$. Then $\mathcal{G} \subset \mathcal{G}^* \subset \delta(C)$ so that $C \in \mathcal{G}$ and hence $\mathcal{G} = \mathcal{G}^*$.

Let us note that a maximal clan need not be a cluster.

Proposition 2.2.3.19 Let δ be a semi-proximity on X and $A, B \subset X$. Then $A \in \delta(B)$ if and only if there exists a δ -clan \mathcal{G} containing sets A and B.

Proof: $A \in \delta(B)$ implies $A \in \bigcup \{\delta(\mathcal{B}) : B \in \mathcal{B}\}$, hence there exist \mathcal{A}_A and \mathcal{B}_B with $A \in \mathcal{A}_A$ and $B \in \mathcal{B}_B$ such that $\mathcal{A}_A \subset \delta(\mathcal{B}_B)$. Then $\mathcal{B}_B \subset \delta(\mathcal{A}_A)$ is also valid and hence $\mathcal{A}_A \cup \mathcal{B}_B$, which contains A and B, is also a δ -clan. If A and B belong to the same δ -clan, then $A \in \delta(B)$ by definition of a clan.

This shows that the semi-proximity δ is completely determined by knowledge of all (maximal) δ -clans.

A semi-proximity δ is said to be **cluster generated** if for each $A\delta B$ there exists a cluster \mathcal{G} such that both A and B belong to \mathcal{G} . A semi-proximity need not be cluster generated in general.

In Proposition 2.2.3.17 the intersections of the form

$$\cap \{\delta(A):\,\delta(A)\supset \mathcal{G}\} \quad ext{and} \quad \cap \{\delta(\mathcal{A}):\,\delta(\mathcal{A})\supset \mathcal{G}\}\,.$$

are encountered. This deserves further study. To facilitate this study, and since it is presently unknown under what conditions intersections of grills are grills, it is convenient to introduce the following notion.

Definition 2.2.3.6 Let \mathcal{G} be a grill on X, then

$$\mathcal{G}^{\dagger} = \left\{ \mathcal{U} : \mathcal{U} \subset \mathcal{G}
ight\}.$$

If $\mathcal{H} \subset \Omega(X)$, then

$$\mathcal{H}^{\vee} = \cup \{ \mathcal{U} : \mathcal{U} \in \mathcal{H} \}.$$

Thus for $\mathcal{G} \subset P(X)$, $\mathcal{G}^{\dagger} \subset \Omega(X)$ and for $\mathcal{H} \subset \Omega(X)$, $\mathcal{H}^{\vee} \subset P(X)$. In particular, $(\mathcal{G}^{\dagger})^{\vee} = \mathcal{G}$ for all grills. However, $(\mathcal{H}^{\vee})^{\dagger}$ is in general bigger than \mathcal{H} . On $\Omega(X)$ a topology τ can be defined by specifying that $\mathcal{H} = \{\mathcal{U}_i : i \in I\}$ is closed with respect to τ if and only if $\mathcal{B} \subset \mathcal{H}^{\vee}$ implies $\mathcal{B} \in \mathcal{H}$. The space (Ω, τ) is homeomorphic to the Czech-Stone compactification of X with the discrete topology. τ is frequently referred to as an ultrafilter topology. We conclude this observation by noticing that every \mathcal{G}^{\dagger} is a closed set in (Ω, τ) and that $(\mathcal{H}^{\vee})^{\dagger}$ is the closure of \mathcal{H} .

Definition 2.2.3.7 For every $\mathcal{U} \in \Omega(X)$ we define

$$D(\mathcal{U}) = (\cap \{\delta^{\dagger}(A) : \mathcal{U} \subset \delta(A)\})^{\vee}.$$

Here $\delta^{\dagger}(A)$ means $(\delta(A))^{\dagger}$.

Proposition 2.2.3.20 For all $\mathcal{U} \in \Omega(X)$, $\mathcal{U} \subset D(\mathcal{U}) = (\cap \{\delta^{\dagger}(A) : A \in \delta(\mathcal{U})\})^{\vee} \subset \delta(\mathcal{U})$ and $D(\mathcal{U})$ is a δ -clan.

Proof: $\mathcal{U} \subset D(\mathcal{U})$ is an immediate consequence of the definition. The new intersection formula follows from the fact that $\mathcal{U} \subset \delta(A)$ if and only if $A \in \delta(\mathcal{U})$. From this and from the fact that $\delta(\mathcal{U}) = \cap \{\delta(A) : A \in \mathcal{U}\}$, it follows that $D(\mathcal{U}) \subset \delta(\mathcal{U})$. Clearly $D(\mathcal{U})$ is a grill. New let $B, C \in D(\mathcal{U})$. Then $B \in \delta(A)$ for all $A \in \delta(\mathcal{U})$ and hence $\delta(\mathcal{U}) \subset \delta(B)$. Thus $C \in D(\mathcal{U}) \subset \delta(\mathcal{U}) \subset \delta(\mathcal{U}) \subset \delta(B)$ and $D(\mathcal{U})$ is a δ -clan.

Proposition 2.2.3.21 If C is a cluster containing U, then

$$D(\mathcal{U}) \subset \mathcal{C} \subset \delta(\mathcal{U})$$
.

Proof: $C \subset \delta(\mathcal{U})$ follows from Proposition 2.2.3.15 (b). $D(\mathcal{U}) \subset C$ follows from $C \subset \delta(\mathcal{U})$ together with Proposition 2.2.3.16 (b) and Proposition 2.2.3.20.

Proposition 2.2.3.22 If $\delta(\mathcal{U})$ is a δ -clan, then $\delta(\mathcal{U}) = D(\mathcal{U})$ and $\delta(\mathcal{U})$ is a cluster.

Proof: If $\delta(\mathcal{U})$ is a δ -clan, then by Proposition 2.2.3.15 (c) $\delta(\mathcal{U}) \subset \cap \{\delta(A) : A \in \delta(\mathcal{U})\}$, hence $\delta^{\dagger}(\mathcal{U}) \subset \cap \{\delta^{\dagger}(A) : A \in \delta(\mathcal{U})\}$ and hence $\delta(\mathcal{U}) = (\delta^{\dagger}(\mathcal{U}))^{\vee} \subset D(\mathcal{U})$. Since $D(\mathcal{U}) \subset \delta(\mathcal{U})$ there always holds that the first assertion is established. Now let C be such that $\delta(\mathcal{U}) \subset \delta(C)$. Then $\mathcal{U} \subset \delta(C)$ and hence $C \in \delta(\mathcal{U})$ so that $\delta(\mathcal{U})$ is a cluster.

Proposition 2.2.3.23 If $\mathcal{U}(x)$ is the principal ultrafilter of x, then $\delta(\mathcal{U}(x)) = \delta(\{x\})$.

Proof: $\delta(\{x\}) = \cup \{\delta(\mathcal{U}) : \{x\} \in \mathcal{U}\} = \delta(\mathcal{U}(x)).$

Proposition 2.2.3.24 The relation

 $\mathcal{B}\Delta_{\delta}\mathcal{U} \Leftrightarrow \mathcal{B} \subset \delta(\mathcal{U})$

is a reflexive and symmetric relation on $\Omega(X)$ satisfying the additional condition

(*)
$$\cap \{ \cup \{ \Delta_{\delta}(\mathcal{U}) : A \in \mathcal{U} \} : A \in \mathcal{B} \} = \Delta_{\delta}(\mathcal{B}) .$$

Proof: Since $\mathcal{U} \subset \delta(\mathcal{U})$ for all $\mathcal{U} \in \Omega(X)$, the relation Δ_{δ} is reflexive. Since, further, $\mathcal{B} \subset \delta(\mathcal{U})$ if and only if $\mathcal{U} \subset \delta(\mathcal{B})$, it follows that Δ_{δ} is symmetric. That (*) is satisfied, follows from Proposition 2.2.3.14.

Theorem 2.2.3.2 Let Δ be an arbitrary reflexive and symmetric relation on $\Omega(X)$. Let us set $\Delta(\mathcal{U}) = \bigcup \{\mathcal{B} : \mathcal{B}\Delta\mathcal{U}\}$, then the relation on P(X)defined by

$$\delta_{\Delta}(A) = \bigcup \{ \Delta(\mathcal{U}) : A \in \mathcal{U} \}$$

is a semi-proximity on X.

Proof: $\delta_{\Delta}(A)$ is a union of grills and hence is itself a grill. Further $\mathcal{U} \subset \Delta(\mathcal{U})$ leads to $\cup \{\mathcal{U} : A \in \mathcal{U}\} \subset \delta_{\Delta}(A)$. Finally, let us assume $B \in \delta_{\Delta}(A)$. Then $Y \in \mathcal{B} \subset \Delta(\mathcal{U})$, with $A \in \mathcal{U}$. Since Δ is symmetric we have $A \in \mathcal{U} \subset \Delta(\mathcal{B})$, with $B \in \mathcal{B}$, that is $A \in \delta_{\Delta}(B)$. Hence $\delta_{\Delta} = \delta_{\Delta}^{-1}$.

It is not in general true that $\Delta(\mathcal{U}) = \delta_{\Delta}(\mathcal{U})$ nor is $\Delta_{\delta_{\Delta}} = \Delta$. The reason is that Δ does not need to be satisfied (*). Thus different Δ may induce the same semi-proximity. However every semi-proximity δ is induced by at least one relation Δ on Ω . This follows from the next theorem.

Theorem 2.2.3.3 For every semi-proximity δ on X, $\delta_{\Delta_{\delta}} = \delta$.

Proof: $\delta_{\Delta_{\delta}}(A) = \bigcup \{ \Delta_{\delta}(\mathcal{U}) : A \in \mathcal{U} \} = \bigcup \{ \delta(\mathcal{U}) : A \in \mathcal{U} \} = \delta(A).$

2.2.4 Representation of semi-proximities

In 1908 F. Riesz asked to determine the class of proximity spaces (X, δ) such that there exists an extension $(\Psi, (Y, c))$ of the closure space (X, c_{δ}) satisfying the condition

 $A\delta B$ if and only if $c(\Psi(A)) \cap c(\Psi(B)) \neq \emptyset$.

In Chapter 1., Proposition 1.3.4.12 showed that each separated proximity has this property. Being motivated by this query, in this subsection there will be proved two representation theorems, one being for all separated semiproximities and the other being for all separated cluster generated semiproximities.

Theorem 2.2.4.1 Let X be a set and let δ be a relation on the power set P(X). Then the following conditions are equivalent:

I. there exists a closure space (Y, d) and a relation Ψ from X to Y such that

(a) $\Psi^{-1}[Y] = X;$ (b) $d(\Psi[x]) \cap d(\Psi[y]) = \emptyset$ for each pair of distinct points $x, y \in X;$
- (c) $\Psi[X]$ is dense in (Y, d);
- (d) $A\delta B$ if and only if $d(\Psi[A]) \cap d(\Psi[B]) \neq \emptyset$;
- II. δ is a separated semi-proximity on X.

Proof: Let us suppose that *I*. holds. Then clearly δ is a symmetric relation on the power set P(X) of *X*. Let A, B, C be the subsets of *X*. Since $d(\Psi[A]) \cap d(\Psi[\emptyset]) = \emptyset$, it follows that $A\overline{\delta}\emptyset$. Let us note that $A\delta B \cup C$ if and only if $d(\Psi[A]) \cap d(\Psi[B \cup C]) \neq \emptyset$ if and only if $d(\Psi[A]) \cap d(\Psi[B]) \neq \emptyset$ or $d(\Psi[A]) \cap d(\Psi[C]) \neq \emptyset$ if and only if $A\delta B$ or $A\delta C$. Finally, if $A \cap B \neq \emptyset$, then there exists $x \in A \cap B$ and since $\Psi^{-1}[Y] = X$ and hence, $y \in \Psi[A] \cap \Psi[B]$, consequently $A\delta B$. Also, if x, y are two distinct points of X, then $d(\Psi[x]) \cap$ $d(\Psi[y]) = \emptyset$ and hence $\{x\}\overline{\delta}\{y\}$. Thus δ is a separated semi-proximity on X. Hence *I*. implies *II*.

Conversely, let us suppose that II. holds. Let Y be the set of all maximal δ -clans. For each subset A of X let us set

$$\mathcal{A}^* = \left\{ \mathcal{G} \in Y : A \in \mathcal{G} \right\}.$$

Since maximal δ -clans are grills, it follows that if A, B are subsets of X, and \mathcal{G} is a maximal δ -clan, then $A \cup B \in \mathcal{G}$ if and only if $A \in \mathcal{G}$ or $B \in \mathcal{G}$ and hence $(A \cup B)^* = A^* \cup B^*$ for all the subsets $A, B \subset X$. Consequently, $\{A^* : A \subset X\}$ is a base for the closed sets of a topology on Y and hence $c : P(Y) \to P(Y)$ defined by

$$c(\alpha) = \cap \{A^*: \, A^* \supset \alpha\}\,, \quad \alpha \subset Y\,,$$

is a Kuratowski closure operator on Y.

Let Ψ be the relation from X to Y defined by

$${x}\Psi\mathcal{G}$$
 if and only if ${x}\in\mathcal{G}$.

Since $\{x\}\delta\{x\}$ for each $x \in X$, it follows that there exists a maximal δ -clan \mathcal{G} such that $\{x\}\in \mathcal{G}$ and hence $\{x\}\Psi\mathcal{G}$. Consequently $\Psi^{-1}[Y] = X$.

For each $\alpha \subset Y$ let us define

$$d(\alpha) = (\Psi^{-1}[\alpha])^* \cup c(\alpha) \,.$$

Since Ψ is a relation and c is a Kuratowski closure operator, it follows that d is a closure operator on Y and hence (Y, d) is a closure space. Since δ is separated and $\{x\}$ belongs to a maximal δ -clan for each $x \in X$, it follows that $\Psi^{-1}[\Psi[A]] = A$ for all $A \subset X$. Hence

$$d(\Psi[A]) = A^* \cup c(\Psi[A])$$
 for all $A \subset X$.

Let us note that if $\mathcal{G} \in \Psi[A]$, then there is an $x \in A$ such that $(x, \mathcal{G}) \in \Psi$ and hence $\{x\} \in \mathcal{G}$. But then $A \in \mathcal{G}$ and thus $\mathcal{G} \in A^*$. Therefore $\Psi[A] \subset A^*$ and hence $c(\Psi[A]) \subset A^*$. Thus $d(\Psi[A]) = A^*$ for all $A \subset X$. And in particular $d(\Psi[X]) = X^* = Y$ and hence $\Psi[X]$ is dense in (Y, d).

Let us note that $A\delta B$ if and only if there exists a maximal δ -clan which contains both A and B if and only if $A^* \cap B^* \neq \emptyset$ if and only if $d(\Psi[A]) \cap d(\Psi[B]) \neq \emptyset$.

Also if x, y are distinct points of X, then, since δ is separated, it follows that $\{x\}\overline{\delta}\{y\}$ and hence $d(\Psi[x]) \cap d(\Psi[y]) = \emptyset$. Thus *II*. implies *I*. This completes the proof. \clubsuit

Definition 2.2.4.1 Let X be a set and (Y,d) be a closure space and let Ψ be a relation from X to Y. The space X is said to be **regularly dense in** Y under the relation Ψ if the following condition holds:

given $B \subset Y$ and $y \notin d(B)$ there exists a subset A of X such that y belongs to $d(\Psi[A])$ and $d(\Psi[A]) \subset Y - d(B)$.

Theorem 2.2.4.2 Let X be a set and δ be a relation on P(X). Then the following conditions are equivalent:

I. there exist a closure space (Y, d) and a relation Ψ from X to Y such that

- (a) $\Psi^{-1}[Y] = X;$
- (b) for each pair of distinct points $x_1, x_2 \in X$ $d(\Psi[x_1]) \cap d(\Psi[x_2]) = \emptyset;$
- (c) $A\delta B$ if and only if $d(\Psi[A]) \cap d(\Psi[B]) \neq \emptyset$;
- (d) X is regularly dense in (Y, d) under Ψ .
- II. δ is a cluster generated separated proximity on X.

Proof: Let us suppose that I. holds. Then, by argument similar to the one used in the corresponding part of the previous theorem, one can show that δ is a separated semi-proximity on X. To complete the proof of the fact that I. implies II, we need to check only that δ is cluster generated.

Let $A\delta B$. Then $d(\Psi[A]) \cap d(\Psi[B]) \neq \emptyset$. Let us choose $y_0 \in d(\Psi[A]) \cap d(\Psi[B])$. Let us set $\mathcal{G}_0 = \{D \subset X : y_0 \in d(\Psi[D])\}$. Clearly \mathcal{G}_0 is a grill containing both A and B. If E and F belong to \mathcal{G}_0 , then y_0 belongs to $d(\Psi[E]) \cap d(\Psi[F])$ and hence $E\delta F$; consequently it follows that \mathcal{G}_0 is a δ -clan. Let us suppose that $H \subset X$ and $H \notin \mathcal{G}_0$. Then $y_0 \notin d(\Psi[H])$. Since X is regularly dense in (Y, d) under Ψ , it follows that there exists a subset K of X such that $y_0 \in d(\Psi[K]) \subset Y - d(\Psi[H])$ and hence $K \in \mathcal{G}_0$ and $d(\Psi[K]) \cap d(\Psi[H]) = \emptyset$, consequently $H\overline{\delta}K$. Thus \mathcal{G}_0 is a cluster containing both A and B. Hence δ is cluster generated.

Conversely, let us suppose that II. holds. Let Y be the set of all δ -clusters. For each $A \subset X$ let us define

$$A^* = \{ \mathcal{G} \in Y : A \in \mathcal{G} \}$$

Then by an argument similar to the one used in the theorem above, one can show that the function $c: P(Y) \to P(Y)$ defined by

$$c(\alpha) = \cap \{A^* : A^* \supset \alpha\} \text{ for all } \alpha \subset Y,$$

is a Kuratowski closure operator on Y. Let us define a relation Ψ from X to Y by $x\Psi \mathcal{G}$ if and only if $\{x\} \in \mathcal{G}$. Since δ is cluster generated, it follows that $\{x\}$ belongs to a cluster for each $x \in X$ and hence $\Psi^{-1}[Y] = X$. Let us define $d: P(Y) \to P(Y)$ by

$$d(\alpha) = (\Psi^{-1}[\alpha])^* \cup c(\alpha)$$
 for all $\alpha \subset Y$.

It can be verified easily that d is a closure operator on Y such that $d(\Psi[A]) = A^*$ for all $A \subset X$. Since δ is cluster generated, it follows that

 $A\delta B$ if and only if $A^* \cap B^* \neq \emptyset$ if and only if $d(\Psi[A]) \cap d(\Psi[B]) \neq \emptyset$.

Since δ is separated, it follows that for each pair of distinct points $x_1, x_2 \in X$, $d(\Psi[x_1]) \cap d(\Psi[x_2]) = \emptyset$. To complete the proof we need to check only that X is regularly dense in (Y, d) under Ψ .

Let $\mathcal{G}_0 \in Y$ and $\mathcal{G}_0 \notin d(\alpha)$ for some $\alpha \subset Y$. Hence $\mathcal{G}_0 \notin (\Psi^{-1}[\alpha])^* \cup c(\alpha)$. This means that $\Psi^{-1}[\alpha] \notin \mathcal{G}_0$ and $A \notin \mathcal{G}_0$ for some $A^* \supset \alpha$ and hence $\Psi^{-1}[\alpha] \cup A \notin \mathcal{G}_0$. Since \mathcal{G}_0 is a cluster, it follows that $\Psi^{-1}[\alpha] \cup A \notin \delta(B)$ for some $B \in \mathcal{G}_0$. Since $\delta(B)$ is a grill, then $\Psi^{-1}[\alpha] \notin \delta(B)$ and $A \notin \delta(B)$. Clearly $\mathcal{G}_0 \in B^* = d(\Psi[B])$.

Let $\mathcal{G} \in B^*$. Then $B \in \mathcal{G}$. Since \mathcal{G} is a cluster and $\Psi^{-1}[\alpha] \notin \delta(B)$, $A \notin \delta(B)$, it follows that $\Psi^{-1}[\alpha] \notin \mathcal{G}$ and $A \notin \mathcal{G}$, consequently $\mathcal{G} \notin (\Psi^{-1}[\alpha])^* \cup c(\alpha) = d(\alpha)$ and hence $\mathcal{G} \in Y - d(\alpha)$. Thus we have proved that $\mathcal{G}_0 \in d(\Psi[B]) \subset Y - d(\alpha)$. Hence X is regularly dense in (Y, d) under Ψ .

Definition 2.2.4.2 The closure space (X, c) is said to be an R_1 -closure space if for any $x \in X$ and $A \subset X$, $c(x) \cap c(A) \neq \emptyset$ implies $x \in c(A)$.

It is easily seen that a topological space is R_1 if and only if, for any points x and y, $\overline{x} \neq \overline{y}$ implies that x and y have disjoint neighborhoods. An R_1 topological space is R_0 (see Definition 2.3.2.1). Indeed, if $x \notin \overline{y}$, then the points x and y must have disjoint neighborhoods, which means $y \notin \overline{x}$. Moreover, a topological space is R_1 if and only if $\overline{x} \neq \overline{y}$ implies that \overline{x} and \overline{y} have disjoint neighborhoods. **Proposition 2.2.4.1** Let (X, τ) be a topological space. Then the following statements are equivalent:

- (a) τ is an R_1 -topology;
- (b) $\overline{\Delta} = \{(x, y) : \overline{x} = \overline{y}\} = \overline{\Delta};$
- (c) $\widetilde{\Delta}$ is closed in the product topology.

Proof: $(a) \Rightarrow (b)$: If $(x, y) \in \Delta$, then $\overline{x} = \overline{y}$. But then $(x, y) \in \overline{x} \times \overline{y} \subset G_x \times G_y$ for all the open neighborhoods G_x and G_y of x and y respectively. Thus every neighborhood of (x, y) meets Δ and $(x, y) \in \overline{\Delta}$. Conversely, let $(x, y) \in \overline{\Delta}$ and G_x , G_y be arbitrary neighborhoods of x and y respectively. Then $G_x \times G_y$ meets Δ , i.e. G_x , G_y have a common point. Since G_x , G_y are arbitrary, this means that $\overline{x} = \overline{y}$ and $(x, y) \in \overline{\Delta}$.

 $(b) \Rightarrow (c)$: Obvious.

 $(c) \Rightarrow (a)$: Let $\overline{x} \neq \overline{y}$. Then $(x, y) \in X \times X - \overline{\Delta}$ which is open. Hence there exist neighborhoods G_x , G_y of x, y respectively such that $G_x \times G_y \subset X \times X - \overline{\Delta}$. G_x , G_y cannot have common points and so (X, τ) is an R_1 space.

Proposition 2.2.4.2 If a semi-proximity δ on X satisfies the condition (RI) for each $x \in X$ and $A, B \in P(X)$, $A, B \in \delta(x)$ implies $A \in \delta(B)$, then (X, c_{δ}) is an R_1 -closure space.

Proof: On account of Proposition 2.2.1.3 c_{δ} is a closure operator. To prove that (X, c_{δ}) is an R_1 -closure space, let us suppose that $y \in c_{\delta}(x) \cap c_{\delta}(A)$. Then $y \in \delta(x)$ and $y \in \delta(A)$ which, on the other hand, implies $x \in \delta(y)$ and $A \in \delta(y)$. Since the semi-proximity δ satisfies the condition (RI), it follows that $x \in \delta(A)$.

Proposition 2.2.4.3 Let (X, c) be an R_1 -closure space and let δ_0 be a relation on P(X) defined in the following manner:

 $A\delta_0 B$ if and only if $c(A) \cap c(B) \neq \emptyset$.

Then δ_0 is a semi-proximity on X which satisfies the condition (RI) and it is compatible with the given closure, that is $c_{\delta_0} = c$.

Proof: That δ_0 is a semi-proximity on P(X) is a trivial consequence of the closure axioms. To prove that δ_0 satisfies the condition (RI), let us suppose that $A, B \in \delta_0(x)$, where $x \in X$. Then $c(A) \cap c(x) \neq \emptyset$ and $c(B) \cap c(x) \neq \emptyset$. Since c is an R_1 -closure, it follows that $x \in c(A) \cap c(B)$ and hence $A \in \delta_0(B)$. The compatibility of the semi-proximity δ_0 with the given closure c follows from the fact that the equality

$$c_{\delta_0}(A) = \{ x \in X : x \in \delta_0(A) \} = \\ = \{ x \in X : c(x) \cap c(A) \neq \emptyset \} = \{ x \in X : x \in c(A) \} = c(A)$$

holds for each $A \in P(X)$.

Definition 2.2.4.3 A semi-proximity δ on X is called **Riesz or RI-proxi**mity if it satisfies the condition (RI).

Proposition 2.2.4.4 Let (X, δ) be an RI-proximity space. If δ_0 is a relation on P(X) defined by $A\delta_0B$ if and only if $c_{\delta}(A) \cap c_{\delta}(B) \neq \emptyset$, then $A\delta_0B$ implies $A\delta B$ for all subsets A and B of X. Thus δ_0 is the smallest RI-proximity relation compatible with the closure of an R_1 -closure space.

Proof: Follows from Propositions 2.2.4.2 and 2.2.4.3.

Proposition 2.2.4.5 A semi-proximity space (X, δ) is an RI-proximity space if and only if $\delta(x)$ is a cluster for all $x \in X$.

Proof: Let us suppose that (X, δ) is a Riesz proximity space. It is evident that $\delta(x)$ is a grill for all $x \in X$. Let $A, B \in \delta(x)$. Since δ is a Riesz proximity, it follows that $A \in \delta(B)$. If $\delta(x) \subset \delta(A)$, then $x \in \delta(A)$ and hence $A \in \delta(x)$. The converse is an immediate consequence of the definition of the cluster.

Corollary 2.2.4.1 If \mathcal{G} is a cluster containing $\{x\}$, then $\mathcal{G} = \delta(x)$.

To state the representation theorem for RI-proximity spaces, we shall need the following:

Definition 2.2.4.4 A subset Y of a closure space (X, c) is regularly dense in X if for any set $F \subset X$ and $x \in X - c(F)$ there exists a subset $E \subset Y$ with the property $x \in c(E) \subset X - c(F)$.

Theorem 2.2.4.3 Let X be a set and δ a binary relation on P(X). Then the following conditions are equivalent:

(I) There exists an R_1 -closure space (Y, c) and a mapping f of X into Y such that f(X) is regularly dense in Y, where f is an isomorphism of X onto f(X) satisfying $c_{f(X)}(f(x)) = f(x)$ and

(*) $A\delta B \text{ in } X \text{ if and only if } c(f(A)) \cap c(f(B)) \neq \emptyset;$

(II) δ is a separated Riesz semi-proximity satisfying the additional condition:

if $A\delta B$ in X, then there exists a cluster \mathcal{G} to which both A and B belong.

Proof: Let us suppose that (I) holds and let us define δ on P(X) by (*). That δ is a semi-proximity follows immediately from the properties of the closure. Let us suppose that $x \in \delta(y)$. Then $c(f(x)) \cap c(f(y)) \neq \emptyset$. Since c is an R_1 -closure, it follows that $f(x) \in c(f(y))$. Thus $f(x) \in c(f(y)) \cap f(X)$, that is, $f(x) \in c_{f(X)}(f(y)) = f(y)$. Since f is an isomorphism of X onto f(X), it follows that x = y. This proves that δ is a separated semi-proximity. We shall next show that δ is a Riesz proximity. For $x \in X$, $A, B \in P(X)$, let us suppose that $A, B \in \delta(x)$. Then $c(f(x)) \cap c(f(A)) \neq \emptyset$ and $c(f(x)) \cap c(f(B)) \neq \emptyset$. That the closure operator is an R_1 implies $f(x) \in c(f(A)) \cap c(f(B))$, that is, $A \in \delta(B)$. It remains to prove that for $A \in \delta(B)$ there exists a cluster to which both A and B belong. Now $A \in \delta(B)$, which implies that there exists a $y \in c(f(A)) \cap c(f(B))$. Let us define $\tau_y = \{D \subset X : y \in c(f(D))\}$. It is obvious that $A, B \in \tau_y$. We shall omit the details of the fact that τ_y is a cluster since they are quite similar to the ones given in Lodato (see [202]).

To prove the converse, let us suppose that (II) holds. By Proposition 2.2.4.5, $\delta(x)$ is a δ -cluster for any $x \in X$. For a subset A of X, let A^* be the set of all clusters to which A belongs. We will denote the set of all clusters from X by Y. Let us observe that

$$(1) \qquad (A \cup B)^* = A^* \cup B^*,$$

since clusters are grills.

We say that a subset A of X absorbs a subset β of Y if and only if A belongs to every cluster in β , that is, $\beta \subset A^*$. For any subset β of Y, we define $c_1(\beta)$ by:

 $\mathcal{B} \in c_1(\beta)$ if and only if every subset $E \subset X$ which absorbs β is in \mathcal{B} .

It follows as in [202] that

(2)
$$c_1(\beta_1 \cup \beta_2) = c_1(\beta_1) \cup c_1(\beta_2)$$

for all subsets β_1, β_2 in P(Y) and $c_1(\mathcal{B}) = \mathcal{B}$ for every \mathcal{B} in Y.

Let f be the mapping which assigns to each $x \in X$ the cluster $\delta(x)$ determined by it. This mapping is well defined. Let us define

(3)
$$c(\beta) = (f^{-1}(\beta))^* \cup c_1(\beta).$$

Let us observe that $c(f(A)) = A^*$. By definition

$$c(f(A)) = (f^{-1}(f(A)))^* \cup c_1(f(A)) = A^* \cup c_1(f(A)) = A^*,$$

since $c_1(f(A)) \subset A^*$. The inclusion $c_1(f(A)) \subset A^*$ is a consequence of the fact that A absorbs f(A).

Now we shall show that closure properties are satisfied by the closure defined by (3).

Since $\beta \subset c_1(\beta)$, it follows that $\beta \subset c(\beta)$. The fact that $c(\emptyset) = \emptyset$ is trivial. (2) and the fact that f^{-1} distributes on unions imply the equality $c(\beta_1 \cup \beta_2) = c(\beta_1) \cup c(\beta_2)$. Thus (Y, c) is a closure space. We shall next show that (Y, c) is an R_1 -closure space. For $\mathcal{B} \in Y$, $f^{-1}(\mathcal{B})$ is either empty or equals x for some $x \in X$. If $f^{-1}(\mathcal{B}) = \emptyset$, then $c(\mathcal{B}) = c_1(\mathcal{B}) = \mathcal{B}$. On the other hand, if $f^{-1}(\mathcal{B}) = x$ for some $x \in X$, then $\mathcal{B} = \delta(x)$. Hence

$$c(\mathcal{B}) = (f^{-1}(\mathcal{B}))^* \cup c_1(\mathcal{B}) = \delta(x) \cup \delta(x) = \delta(x) = \mathcal{B}.$$

The separated character of the Riesz proximity implies that f is one-one. That f is an isomorphism will be accomplished by showing that

(i) $c_{f(X)}(f(A)) \supset f(c_{\delta}(A))$ for every A in P(X), and

(ii) $f^{-1}(c_{f(X)}(f(A))) \subset c_{\delta}(A)$ for each $A \subset X$.

For (i), let us suppose that $x \in c_{\delta}(A)$. Then $A \in \delta(x)$. Thus $\delta(x) \in A^* = c(f(A))$ which, in turn, implies $\delta(x) \in c_{f(X)}(f(A))$. In order to prove (ii), let us suppose that $\mathcal{B} \in c_{f(X)}(f(A))$. Then there exists an $x \in X$ such that $\mathcal{B} = \delta(x)$ and $\delta(x) \in c_{f(X)}(f(A)) = c(f(A)) \cap f(X)$. Thus $A \in \delta(x)$, that is, $x \in c_{\delta}(A)$.

 $A\delta B$ if and only if there exists a cluster to which both A and B belong, that is, $A^* \cap B^* \neq \emptyset$; thus $c(f(A)) \cap c(f(B)) \neq \emptyset$ if and only if $A\delta B$.

It remains to check that f(X) is regularly dense in Y. Let us suppose that $\beta \subset Y$ and $\mathcal{B}_0 \notin c(\beta) = (f^{-1}(\beta))^* \cup c_1(\beta)$. Then $f^{-1}(\beta) \notin \mathcal{B}_0$ and there exists a subset A which absorbs β and does not belong to \mathcal{B}_0 . Since \mathcal{B}_0 is, in particular, a grill, it follows that $A \cup f^{-1}(\beta) \notin \mathcal{B}_0$. Taking into account the fact that \mathcal{B}_0 is a cluster, it follows that there exists a $B \in \mathcal{B}_0$ such that $A \cup f^{-1}(\beta) \notin \delta(B)$, that is, $A \notin \delta(B)$ and $f^{-1}(\beta) \notin \delta(B)$. Let \mathcal{B} be any element of B^* . Then $B \in \mathcal{B}$ and hence $f^{-1}(\beta)$ and A do not belong to \mathcal{B} . Thus it follows that $\mathcal{B} \in Y - c(\beta)$. Clearly $\mathcal{B}_0 \in B^* = c(f(B)) \subset Y - c(\beta)$. This completes the proof. \clubsuit

2.2.5 Proximally coarse semi-unifomities

Now we shall show that every semi-proximity is induced by a semi-uniformity, and that among all uniformities inducing a given semi-proximity there exists a uniformly coarsest one which will be called the **proximally coarse** **semi-uniformity** of (X, δ) . It turns out that this semi-uniformity is a uniformity if and only if δ is induced by a uniformity.

Definition 2.2.5.1 A semi-proximity induced by a uniformity will be called *uniformizable*.

Theorem 2.2.5.1 A semi-proximity δ for a set X is uniformizable if and only if the following condition is satisfied:

 (SP_5) if $A\overline{\delta}B$, then there exists $C, D \subset X$ such that $C \cap D = \emptyset$, $A\overline{\delta}X - C$ and $X - D\overline{\delta}B$.

It is obvious that the condition (SP_5) is equivalent with the following condition:

 (SP'_5) if $A\overline{\delta}B$, then there exists δ -neighborhoods C of A and D of B such that $C \cap D = \emptyset$.

Proof: Let us suppose that δ is induced by a uniformity \mathcal{U} and $A\delta B$. By the definition of induced proximities, there exits a $U \in \mathcal{U}$ such that $U[A] \cap B = \emptyset$. Let us choose a symmetric element $V \in \mathcal{U}$ so that $V \circ V \subset U$ and let us put C = V[A] and D = V[B]. By definition, C and D are proximal neighborhoods of A and B and it remains to show that $C \cap D = \emptyset$. Assuming the contrary, we obtain $V \circ V[A] \cap B \neq \emptyset$ which implies $U[A] \cap B \neq \emptyset$, and this contradicts our assumption $U[A] \cap B = \emptyset$.

To prove the converse we must construct a uniformity inducing δ . Three lemmas will be given, concerning the construction of the uniformly coarsest semi-uniformity inducing a given semi-proximity δ which will be proved to be a uniformity if δ fulfils the condition (SP_5) .

If a semi-proximity δ for a set X is induced by a semi-uniformity \mathcal{U} and if $A\delta B$, then $U[A] \cap B \neq \emptyset$ for each $U \in \mathcal{U}$. Therefore, if we want to find a semiuniformity inducing the given semi-proximity δ , it is natural to consider the collection \mathcal{U} of all entourages U of diagonal of $X \times X$ such that $U[A] \cap B \neq \emptyset$ whenever $A\delta B$. It is easily seen that $U \in \mathcal{U}$ and $U \subset V \subset X \times X$ implies $U^{-1} \in \mathcal{U}$ and $V \in \mathcal{U}$. On the other hand, the intersection of two elements of \mathcal{U} need not belong to \mathcal{U} , and therefore \mathcal{U} need not be a semi-uniformity. It turns out that the collection \mathcal{U}' of all the elements $V \in \mathcal{U}$ of the form $\bigcup_i X_i \times X_i$, where $\{X_i\}$ is a finite cover of X, possesses the following two properties: (1) if $U \in \mathcal{U}$ and $U' \in \mathcal{U}'$, then $U \cap U' \in \mathcal{U}$, and (2) if a semiuniformity \mathcal{V} induces δ , then $\mathcal{U}' \subset \mathcal{V}$. It will follows from (1) that \mathcal{U}' is a base for a proximally continuous semi-uniformity for (X, δ) . It turns out that this semi-uniformity induces δ , and if δ fulfils (SP_5) , then this semi-uniformity is a uniformity. For convenience we shall introduce some terminology. **Definition 2.2.5.2** A finite square entourage of the diagonal of $X \times X$ is an entourage of the form $\bigcup_i X_i \times X_i$, where $\{X_i\}$ is a finite cover of X. If (X, δ) is a semi-proximity space, then a **proximal entourage of the diagonal of** $(X, \delta) \times (X, \delta)$, or δ -entourage of the diagonal of $X \times X$, is a subset U of $X \times X$ such that $A\delta B$ implies $U[A] \cap B \neq \emptyset$.

A subset U of $X \times X$ is a symmetric entourage of the diagonal of $X \times X$ if and only if U is a union of squares $A \times A$. "If" is obvious and to prove "only if", let us notice that $V = \bigcup \{((x, y) \times (x, y)) : (x, y) \in V\}$ provided that V is a symmetric entourage of the diagonal.

Every proximal entourage U of the diagonal of $(X, \delta) \times (X, \delta)$ is an entourage of the diagonal of $X \times X$. Indeed, if $x \in X$, then $x \delta x$ and hence $U[x] \cap x \neq \emptyset$, i.e. $(x, x) \in U$.

For convenience, Lemma 2.2.5.3, as the main result, will be preceded by two preparatory lemmas which are also important by themselves.

Lemma 2.2.5.1 Every finite square entourage of the diagonal of $X \times X$ is the intersection of a finite family of entourages of the form $(A \times A) \cup (B \times B)$.

Proof: Let us suppose that $U = \bigcup_{i \leq n} A_i \times A_i$, $n \in \mathbb{N}$, is an entourage of the diagonal of $X \times X$, i.e. $\{A_i\}$ is a cover of X. Assuming that $(x, y) \in X \times X - U$, let us consider the union A of all $\{A_i\}$ such that $x \in A_i$, and the union B of all the remaining sets A_i . Since $y \notin A$, it follows that $(x, y) \notin X \times X$ and since $x \notin B$, it follows that $(x, y) \notin B \times B$. Thus $U \subset ((A \times A) \cup (B \times B)) \subset (X \times X) - \{(x, y)\}$. This concludes the proof.

Lemma 2.2.5.2 Let (X, δ) be a proximity space. Each of the following two conditions is necessary and sufficient for a set $V = ((A_1 \times A_1) \cup (A_2 \times A_2) \subset X \times X$ to be a proximal entourage of the diagonal:

(a) $X - A_1 \overline{\delta} X - A_2$ (and hence $A_1 \cup A_2 = X$);

(b) if $A\delta B$, then $(A_1 \cap A)\delta(A_1 \cap B)$ or $(A_2 \cap A)\delta(A_2 \cap B)$.

Proof: Let us first notice that $X - A_1 = A_2 - A_1$ and $X - A_2 = A_1 - A_2$ if $A_1 \cup A_2 = X$.

Condition (a) is necessary because $V[A_2 - A_1] = A_2$ and $A_2 \cap (A_1 - A_2) = \emptyset$.

Condition (b) is sufficient, because $(A_i \cap A)\delta(A_i \cap B)$ implies $A_i \cap A \neq \emptyset$, $A_i \cap B \neq \emptyset$, and hence $V[A] \cap B \supset V[A_i \cap A] \cap (A_i \cap B) = A_i \cap (A_i \cap B) = A_i \cap B \neq \emptyset$. It remains to show that (a) implies (b). Assuming (a), let us suppose that $A\delta B$ and let us consider the following decompositions of A and B:

$$A = ((A_1 - A_2) \cap A) \cup ((A_1 \cap A_2) \cap A) \cup ((A_2 - A_1) \cap A),$$

$$B = ((A_1 - A_2) \cap B) \cup ((A_1 \cap A_2) \cap B) \cup ((A_2 - A_1) \cap B).$$

Since $A\delta B$, then, by Proposition 2.2.1.1, at least one of the sets of the decomposition of A must be proximal to a set from the decomposition of B. But $A_2 - A_1 \overline{\delta} A_1 - A_2$ and hence also $A \cap (A_2 - A_1) \overline{\delta} B \cap (A_1 - A_2)$, $B \cap (A_2 - A_1) \overline{\delta} A \cap (A_1 - A_2)$. It follows that both of the proximal sets in question must be contained in A_1 or in A_2 ; this concludes the proof.

Lemma 2.2.5.3 Let us suppose that (X, δ) is a semi-proximity space, \mathcal{V} being the set of all finite square proximal entourages (of the diagonal of $(X, \delta) \times (X, \delta)$) and \mathcal{W} being the set of all elements of \mathcal{V} of the form $(A \times A) \cup (B \times B)$. Obviously \mathcal{V} is a sub-base for a semi-uniformity \mathcal{U} for X. The following assertions hold:

(a) \mathcal{V} consists of finite intersections of elements of \mathcal{W} and hence \mathcal{W} is a sub-base for \mathcal{U} ;

(b) if $W \in W$ and U is any proximal entourage, then $W \cap U$ is also a proximal entourage;

(c) \mathcal{V} is multiplicative, hence a base for \mathcal{U} ; thus every element of \mathcal{U} is a proximal entourage and hence \mathcal{U} is a proximally continuous semi-uniformity for (X, δ) ;

(d) \mathcal{U} induces δ ;

(e) if a semi-uniformity \mathcal{U}_1 induces δ , then $\mathcal{U} \subset \mathcal{U}_1$;

(f) if δ fulfils the condition (SP₅), then \mathcal{U} is a uniformity.

Proof: (a) Statement follows from Lemma 2.2.5.1 and the definition of a sub-base of semi-uniformity.

(b) Let $W = (A_1 \times A_1) \cup (A_2 \times A_2) \in \mathcal{W}$, and let U be any proximal entourage. Assuming $A\delta B$, we must show that $(U \cap W)[A] \cap B \neq \emptyset$. By Lemma 2.2.5.2 we obtain that $(A_i \cap A)\delta(X_i \cap B)$ for some i = 1, 2. Since Uis a proximal entourage, it follows that $U[A_i \cap A] \cap (A_i \cap B) \neq \emptyset$. However, $(U \cap W)[A_i \cap A] = A_i \cap U[A_i \cap A]$, and consequently $(U \cap W)[A] \cap B \supset$ $(U \cap W)[A_i \cap A] \cap B \supset A_i \cap U[A_i \cap A] \cap B \neq \emptyset$.

(c) Follows immediately from (a) and (b) by induction.

(d) It remains to show that if $A\delta B$, then $U[A] \cap B = \emptyset$ for some $U \in \mathcal{U}$. Let us denote $A_1 = X - A$ and $B_1 = X - B$. It follows from Lemma 2.2.5.2 (a) that $U = (A_1 \times A_1) \cup (B_1 \times B_1)$ is a proximal entourage and hence $U \in \mathcal{W} \subset \mathcal{U}$. But, clearly, $U[A] = B_1 = X - B$. (e) Let us suppose that a semi-uniformity \mathcal{U}_1 induces δ . To prove that \mathcal{U} is contained in \mathcal{U}_1 , it is sufficient to show that the sub-base \mathcal{W} of \mathcal{U} is contained in \mathcal{U}_1 . Let $W = (A \times A) \cup (B \times B)$ be any element of \mathcal{W} . By Lemma 2.2.5.2 we obtain $X - A\overline{\delta}X - B$. By our assumption there exists a $U \in \mathcal{U}_1$ such that

(*)
$$U[X - A] \cap (X - B) = \emptyset.$$

Without any loss of generality we may assume that U is symmetric, i.e. $U = U^{-1}$. Now the proof will be accomplished if we show that $U \subset W$; and this inclusion will be derived from (*) as follows:

It is sufficient to show that $U[x] \subset W[x]$ for each $x \in X$. It follows from (*) that $U[X - A] \subset B$. But clearly W[X - A] = B and hence $U[x] \subset W[x]$ for each $x \in X - A$. Since U is symmetric, we obtain from (*) that $U[X - B] \cap (X - A) = \emptyset$ and the same argument as the one above gives $U[x] \subset W[x]$ for each $x \in X - B$. It remains to consider the case when $x \in X - ((X - A) \cup (X - B)) = A \cap B$. However, if $x \in A \cap B$, then $W[x] = A \cup B = X$ and therefore $U[x] \subset X = W[x]$.

(f) Let us suppose that δ fulfils the condition (SP_5) . To prove that \mathcal{U} is a uniformity, it is sufficient to show that for each element W of the sub-base \mathcal{W} for \mathcal{U} , there exists an element $V \in \mathcal{V}$ such that $V \circ V \subset \mathcal{W}$. Let us suppose that $W = (A \times A) \cup (B \times B) \in \mathcal{W}$. Since $X - A\overline{\delta}X - B$, there exists a proximal neighborhood B_1 of X - A and A_1 of X - B such that $A_1 \cap B_1 = \emptyset$. Let us denote that $V = (A_1 \times A_1) \cup ((A \cap B) \times (A \cap B)) \cup (B_1 \times B_1)$. Now $V \in \mathcal{V}$ because V is the intersection of two elements of \mathcal{W} , namely $(A_1 \times A_1) \cup (B \times B)$ and $(B_1 \times B_1) \cup (A \times A)$, use Lemma 2.2.5.2 (a). It will be shown that $V \circ V \subset W$. By Lemma 2.1.2.1 it follows that $V \circ V = \bigcup_{x \in X} V[x] \times V[x]$. If $x \in A_1$, then $V[x] \subset A$ and hence $V[x] \times V[x] \subset A \times A \subset W$. If $x \in X - A_1$, then $V[x] \subset B$ and hence $V[x] \times V[x] \subset B \times B \subset W$.

It is to be pointed out that Lemma 2.2.5.3 accomplishes the proof of Theorem 2.2.5.1. If δ is a semi-proximity for a set X then by Lemma 2.2.5.3 the set of all finite square δ -proximal entourages of the diagonal of $X \times X$ is a base for a semi-uniformity \mathcal{U} for X, which is the smallest semi-uniformity inducing the proximity δ . If \mathcal{U}' is any semi-uniformity inducing δ such that the set \mathcal{V}' of all finite square entourages from \mathcal{U}' is a base for \mathcal{U}' , then necessarily $\mathcal{U} \subset \mathcal{U}'$; but $\mathcal{V}' \subset \mathcal{U}$, and \mathcal{V}' being a base for \mathcal{U}' , we obtain $\mathcal{U}' \subset \mathcal{U}$ and hence $\mathcal{U}' = \mathcal{U}$. Thus we have proved the following proposition: **Proposition 2.2.5.1** Let us suppose that a semi-uniformity \mathcal{U} induces a semi-proximity δ . Then \mathcal{U} is the uniformly coarsest (i.e. smallest) semi-uniformity inducing δ if and only if the finite square elements of \mathcal{U} form a base for \mathcal{U} .

Definition 2.2.5.3 A semi-uniformity \mathcal{U} will be called **proximally coarse** if finite square elements of \mathcal{U} form a base for \mathcal{U} , i.e. by Lemma 2.2.5.3, if a semi-uniformity \mathcal{U}' induces the same proximity as \mathcal{U} , then $\mathcal{U} \subset \mathcal{U}'$, i.e. \mathcal{U} is uniformly coarser than \mathcal{U}' .

Theorem 2.2.5.2 Every semi-proximity is induced by a semi-uniformity. Among all the semi-uniformities inducing a given semi-proximity δ there exists a unique proximally coarse semi-uniformity \mathcal{U} . The set of all finite square δ -proximal entourages is a base for \mathcal{U} and \mathcal{U} is a uniformity if and only if δ is uniformizable.

Proof: Follows from Theorem 2.2.5.1, Lemma 2.2.5.3 and Proposition 2.2.5.1. ♣

Definition 2.2.5.4 A semi-uniformity \mathcal{U} for a set X is said to be **totally** bounded if for each $U \in \mathcal{U}$ there exists a finite subset $A \subset X$ such that U[A] = X.

Proposition 2.2.5.2 Every proximally coarse semi-uniformity is totally bounded and every totally bounded uniformity is proximally coarse.

Proof: Let \mathcal{U} be a proximally coarse semi-uniformity for a set X and let \mathcal{V} be the collection of all finite square elements of \mathcal{U} . Thus \mathcal{V} is a base for \mathcal{U} . If $U \in \mathcal{U}$, then $V \subset U$ for some $V = \bigcup_i A_i \times A_i \in \mathcal{V}$, where $\{A_i\}$ is a finite cover of X. Now, if A is a finite set intersecting each A_i , then clearly $V[A] = \bigcup_i A_i = X$ and hence U[A] = X.

Conversely, let us suppose that \mathcal{U} is a totally bounded uniformity for a set X and let us suppose that U is any element of \mathcal{U} . We must find a finite square element $W \in \mathcal{U}$ contained in U. Let us choose a symmetric element $V \in \mathcal{U}$ such that $V \circ V \circ V \circ V \subset U$ and a finite subset $A \subset X$ with V[A] = X, and let us put that $W = \bigcup \{ (V \circ V)[x] \times (V \circ V)[x] : x \in X \}$. Since $(V \circ V) \circ (V \circ V) \subset U$, the set W is contained in U by Lemma 2.1.2.1. To prove that $W \in \mathcal{U}$, we shall show that $W \supset V$. Given any $y \in X$, let us choose an $x \in A$ with $y \in V[x]$. It follows that $V[y] \subset V[V[x]] = (V \circ V)[x] \subset W[x]$. **Corollary 2.2.5.1** A uniformity is proximally coarse if and only if it is totally bounded.

The following example shows that a totally bounded semi-uniformity need not be proximally coarse.

Example 2.2.5.1 Let X be an infinite set, $x \in X$ and let us consider the semi-proximity δ for X such that $A\delta B$ if and only if $A \cap B \neq \emptyset$ or $A \neq \emptyset \neq B$ and $x \in A \cup B$. If u is the closure induced by δ , then $u(y) = \{x, y\}$ if $y \in X - \{x\}$ and u(x) = X. Thus X is the only neighborhood of x in (X, u) and consequently, if \mathcal{U} is a continuous semi-uniformity for X, then U[x] = X for each $U \in \mathcal{U}$. This shows that every continuous semi-uniformity for (X, u) is totally bounded. Let \mathcal{U} be the largest continuous semi-uniformity for (X, u). Clearly the set $U = \Delta_X \cup (\{x\} \times X) \cup (X \times \{x\})$ forms a base for \mathcal{U} and \mathcal{U} induces δ . On the other hand, \mathcal{U} is not proximally coarse because the set U contains no finite square element of $\mathcal{U}(X \text{ is infinite})$.

Theorem 2.2.5.3 The class of all proximally coarse semi-uniformities is hereditary and closed under arbitrary products.

Proof: If $Y \subset X$ and U is a finite square entourage of the diagonal of $X \times X$, then $(Y \times Y) \cap U$ is a finite square entourage of the diagonal of $Y \times Y$ and therefore every relativization of a proximally coarse semi-uniformity is proximally coarse.

If (X, \mathcal{U}) is the product of a family $\{(X_a, \mathcal{U}_a)\}$ and $U_a \in \mathcal{U}_a$ is a finite square, then $\{(x, y) : (pr_a x, pr_a y) \in U_a\}$ is a finite square and hence finite square elements form a sub-base for \mathcal{U} . This shows that \mathcal{U} is proximally coarse.

Theorem 2.2.5.4 Let us suppose that there exists a uniformly continuous mapping of a semi-uniform space (X, \mathcal{U}) onto a semi-uniform space (Y, \mathcal{V}) . If \mathcal{U} is a totally bounded semi-uniformity, then \mathcal{V} is also a totally bounded semi-uniformity. If \mathcal{U} is a totally bounded and proximally coarse semi-uniformity, and if \mathcal{V} is a uniformity, then \mathcal{V} is proximally coarse.

Proof: The first statement is an immediate consequence of the corresponding definition and the second one follows from the first one and Proposition 2.2.5.2. \clubsuit

It is to be noted that there exists a uniformly continuous mapping of a proximally coarse uniform space onto a semi-uniform space which is not proximally coarse. By Proposition 2.2.2.3 every uniformly continuous mapping is proximally continuous but a proximally continuous mapping for semi-uniform spaces need not be uniformly continuous (it is sufficient to take two different proximally equivalent semi-uniformities). On the other hand following holds:

Theorem 2.2.5.5 If (X, U) is a proximally coarse semi-uniform space, then every proximally continuous mapping of a semi-uniform space into X is uniformly continuous.

Proof: Let us suppose that f is a proximally continuous mapping of a semiuniform space (X_1, \mathcal{U}_1) into a proximally coarse semi-uniform space (X, \mathcal{U}) . To prove that f is uniformly continuous, it is necessary to find a sub-base \mathcal{W} for \mathcal{U} such that $f_2^{-1}(W) \in \mathcal{U}_1$ for each $W \in \mathcal{W}$. Of course, for \mathcal{W} we take the sub-base for \mathcal{U} described in Lemma 2.2.5.3, i.e. the collection of all sets W of the form $W = (A \times A) \cup (B \times B)$ such that $X - A\overline{\delta}X - B$, where δ is the semi-proximity induced by \mathcal{U} . Since f is proximally continuous, we obtain $X_1 - A_1\overline{\delta}_1X_1 - B_1$, where $A_1 = f^{-1}(A)$, $B_1 = f^{-1}(B)$ and δ_1 is the proximity induced by \mathcal{U}_1 . Thus, from Lemma 2.2.5.2, $W_1 = (A_1 \times A_1) \cup (B_1 \times B_1)$ is a δ_1 -proximal entourage of the diagonal of $X_1 \times X_1$, and consequently, by Lemma 2.2.5.3, $W_1 \in \mathcal{U}_1$. But clearly $W_1 = f_2^{-1}(W)$.

Theorem 2.2.5.6 Let us suppose that (X, \mathcal{U}) is the product of a non-void family $\{(X_a, \mathcal{U}_a) : a \in A\}$ of proximally coarse semi-uniform spaces. The proximity δ induced by \mathcal{U} is the proximally coarsest proximity for X such that all mappings $pr_a : (X, \delta) \to (X_a, \mathcal{U}_a)$ are proximally continuous.

Proof: All the mappings in question are proximally continuous because all the mappings $pr_a : (X, \mathcal{U}) \to (X_a, \mathcal{U}_a)$ are uniformly continuous and every uniformly continuous mapping is proximally continuous. Let δ_1 be any proximity for X such that all the mappings $pr_a : (X, \delta_1) \to (X_a, \mathcal{U}_a)$ are proximally continuous and let \mathcal{V} be a semi-uniformity inducing δ_1 . Since \mathcal{U}_a are proximally coarse, by Theorem 2.2.5.5 all the mappings $pr_a : (X, \mathcal{V}) \to$ (X_a, \mathcal{U}_a) are uniformly continuous and consequently, by the definition of the product semi-uniformity, $\mathcal{V} \supset \mathcal{U}$. But this implies that δ_1 is proximally finer than δ , which completes the proof. \clubsuit

Theorem 2.2.5.7 The proximally coarse semi-uniformity \mathcal{V} proximally equivalent with a given semi-uniformity \mathcal{U} for a set X is the unique semi-uniformity for X with the following property:

A mapping f of (X, \mathcal{U}) into a proximally coarse semi-uniform space (Z, \mathcal{W}) is uniformly continuous if and only if the mapping $f : (X, \mathcal{V}) \to (Z, \mathcal{W})$ is uniformly continuous.

Proof: Let f be a uniformly continuous mapping of (X, \mathcal{U}) into a proximally coarse semi-uniform space (Z, \mathcal{W}) and let \mathcal{V} be the proximally coarse semiuniformity which is proximally equivalent to \mathcal{U} . The collection \mathcal{W}' of all finite square elements of \mathcal{W} is a base for \mathcal{W} , and the set \mathcal{V}' of all $f_2^{-1}(W)$, $W \in \mathcal{W}'$, consists of finite square elements of \mathcal{U} . The finite square elements of \mathcal{U} form a base for \mathcal{V} and therefore $\mathcal{V}' \subset \mathcal{V}$. Since \mathcal{W}' is a base for \mathcal{W} , the mapping $f : (X, \mathcal{V}) \to (Z, \mathcal{W})$ is uniformly continuous. Conversely, if $f : (X, \mathcal{V}) \to (Z, \mathcal{W})$ is uniformly continuous, then $f : (X, \mathcal{U}) \to (Z, \mathcal{W})$ is uniformly continuous because \mathcal{U} is uniformly finer than \mathcal{V} . Thus \mathcal{V} fulfils the condition.

The uniqueness of \mathcal{V} is evident. \clubsuit

Proposition 2.2.5.3 A subset S of the uniform space of reals is proximally coarse if and only if S is contained in a bounded interval in \mathbb{R} .

Proof: If S is contained in no bounded interval, then one can easily construct a sequence (x_n) in S such that $|x_n - x_m| \ge 1$ for $n \ne m$. If (y_n) is a sequence in S such that $|x_n - y_n| < 1/2$, then $1 \le |x_n - x_m| \le |x_n - x_m| \le |x_n - y_n| + |y_n - y_m| + |x_m - y_m| < 1 + |y_n - y_m|$ whenever $n \ne m$, and hence $|y_n - y_m| > 0$ for $n \ne m$. But this implies that (y_n) is a one-to-one sequence. Consequently, S is not totally bounded and hence S is not proximally coarse by Proposition 2.2.5.2.

Now let S be contained in a bounded interval I = [-r, r]. According to Theorem 2.2.5.3 it is sufficient to show that I is proximally coarse. By Proposition 2.2.5.2 this will follow if the interval I is totally bounded. Given a positive s, let T be the set of all the points $s \cdot n$, $n \in \mathbb{N}$ or $-n \in \mathbb{N}$. Clearly, $T \cap [-r, r]$ is finite and if $x \in I$, then |x - y| < s for some $y \in T$. The proof is complete.

By our convention that every uniform concept applies to semi-pseudometrics, a **semi-pseudometric** is said to be **totally bounded** if the induced semi-uniformity is totally bounded. It is evident that a semi-pseudometric dfor a set X is totally bounded if and only if for each real r > 0 there exists a finite subset $A \subset X$ such that the distance from each $y \in X$ to A is smaller than r.

Theorem 2.2.5.8 Let \mathcal{U} be a semi-uniformity for a set X, δ the semiproximity induced by \mathcal{U} , and \mathcal{V} the proximally coarse semi-uniformity inducing δ (that is, \mathcal{V} is the unique proximally coarse semi-uniformity which is proximally equivalent to \mathcal{U}). Then

(a) a pseudometric d for X is a uniformly continuous pseudometric for (X, \mathcal{V}) if and only if d is a totally bounded uniformly continuous pseudometric for (X, \mathcal{U}) ;

(b) a function f on (X, \mathcal{V}) is uniformly continuous if and only if the function $f: (X, \mathcal{U}) \to \mathbb{R}$ is bounded and uniformly continuous;

(c) if a function f on (X, \mathcal{U}) is uniformly continuous, then f is proximally continuous, in symbols, $\mathbf{U}((X, \mathcal{U}), \mathbb{R}) \subset \mathbf{P}((X, \mathcal{U}), \mathbb{R});$

(d) a function f on (X, \mathcal{V}) is uniformly continuous if and only if f is a bounded proximally continuous function, in symbols $\mathbf{U}((X, \mathcal{V}), \mathbb{R}) = \mathbf{P}^*((X, \mathcal{V}), \mathbb{R})$.

Proof: (a) A totally bounded pseudo-metric is proximally coarse by Proposition 2.2.5.2, and therefore, by Theorem 2.2.5.7, a totally bounded pseudo-metric for X is uniformly continuous for (X, \mathcal{V}) if and only if it is uniformly continuous for (X, \mathcal{U}) . Thus, to prove the statement, it remains to show that every uniformly continuous pseudo-metric for a proximally coarse semi-uniform space is totally bounded, and this follows from Theorem 2.2.5.4.

(b) If f is a bounded function on X, then the subspace f(X) of \mathbb{R} is proximally coarse by Proposition 2.2.5.3 and therefore, by Theorem 2.2.5.7, the function $f : (X, \mathcal{U}) \to \mathbb{R}$ is uniformly continuous if and only if the function $f : (X, \mathcal{V}) \to \mathbb{R}$ is uniformly continuous. It remains to show that every uniformly continuous function f on a proximally coarse semi-uniform space is bounded. By Theorem 2.2.5.4, the subspace f(X) is proximally coarse and therefore, by Proposition 2.2.5.3, f(X) is a bounded subset of \mathbb{R} .

(c) This statement is a particular case of the fact that every uniformly continuous mapping is proximally continuous.

(d) The statements (b) and (c) imply the inclusion \subset . Conversely, if $f : (X, \mathcal{V}) \to \mathbb{R}$ is a bounded proximally continuous function, then f is uniformly continuous by Theorem 2.2.5.5 because f(X) is a proximally coarse subset of \mathbb{R} .

Proposition 2.2.5.4 If d_1 and d_2 are proximally continuous pseudo-metrics for a semi-proximity space (X, δ) and d_1 is totally bounded, then $d_1 + d_2$ is proximally continuous.

Proof: Let \mathcal{V} be the proximally coarse semi-uniformity which induces δ and let \mathcal{U}_i , i = 1, 2, be the uniformity induced by d_i . Since \mathcal{U}_1 is proximally coarse and proximally continuous, by Theorem 2.2.5.4, the identity mapping of (X, \mathcal{V}) into (X, \mathcal{U}) is uniformly continuous and hence $\mathcal{U}_1 \subset \mathcal{V}$. By Lemma

2.2.5.3, $\mathcal{V} \cap \mathcal{U}_2$ consists of δ -proximal entourages and hence $\mathcal{U}_1 \cap \mathcal{U}_2$ consists of δ -proximal entourages. Since $\mathcal{U}_1 \cap \mathcal{U}_2$ is a base for the uniformity induced by $d_1 + d_2$, the pseudo-metric $d_1 + d_2$ is proximally continuous for (X, δ) .

We ought to remember that the sum of two proximally continuous pseudo-metrics need not be proximally continuous and hence a uniformly finest proximally continuous uniformity for a given semi-proximity need not exist.

Theorem 2.2.5.9 Let \mathcal{U} be a uniformity and let δ be the semi-proximity induced by \mathcal{U} . Every uniformly continuous pseudo-metric for (X,\mathcal{U}) is a proximally continuous pseudo-metric for (X,δ) . If every proximally continuous pseudo-metric for (X,δ) is a uniformly continuous pseudo-metric for (X,\mathcal{U}) , then \mathcal{U} is the uniformly finest uniformity inducing δ . Finally, if \mathcal{U} is the uniformly finest uniformity inducing δ , then every proximally continuous pseudo-metric for (X,δ) is uniformly continuous for (X,\mathcal{U}) .

Proof: The first statement is a particular case of the fact that every uniformly continuous mapping is proximally continuous. If every proximally continuous pseudo-metric for (X, δ) is uniformly continuous for (X, \mathcal{U}) and \mathcal{W} is any proximally continuous uniformity for (X, δ) , then every uniformly continuous pseudo-metric for (X, \mathcal{W}) is proximally continuous for (X, δ) and hence uniformly continuous for (X, \mathcal{U}) . This implies that \mathcal{U} is uniformly finer than \mathcal{W} and establishes the second statement. The last statement follows from Proposition 2.2.5.4. Indeed, if d is any proximally continuous pseudo-metric for (X, δ) together with d generate proximally continuous uniformity \mathcal{W} for (X, δ) by Proposition 2.2.5.4 which evidently induces δ , and hence $\mathcal{W} \subset \mathcal{U}$. Thus d is a uniformly continuous pseudo-metric for (X, δ) .

Proposition 2.2.5.5 Let δ be a semi-proximity for a set X induced by a pseudo-metric d and let \mathcal{U} be the uniformity induced by d. Then \mathcal{U} is the uniformly finest uniformity which induces δ .

Proof: If D is a proximally continuous pseudo-metric for (X, δ) , then the mapping $J : (X, d) \to (X, D)$ is proximally continuous and hence, by Theorem 2.2.2.1, uniformly continuous. Thus every proximally continuous pseudo-metric is a uniformly continuous pseudo-metric for (X, \mathcal{U}) . By the preceding theorem \mathcal{U} has the property in question. **Corollary 2.2.5.2** If d is a totally bounded pseudo-metric, then the uniformity \mathcal{U} induced by d is the unique uniformity inducing the same semiproximity δ as d.

Proof: Since \mathcal{U} is proximally coarse, \mathcal{U} is the smallest uniformity among all the uniformities inducing δ . By Proposition 2.2.5.5, \mathcal{U} is the largest among these uniformities.

2.2.6 Uniformizable proximities

By Definition 2.2.5.1 a semi-proximity is uniformizable if it is induced by a uniformity, and by Theorem 2.2.5.1 a semi-proximity is uniformizable if and only if it fulfils the condition (SP_5) , i.e. if it is a proximity. Here we shall describe uniformizable semi-proximities by means of proximally continuous pseudo-metrics and functions.

Theorem 2.2.6.1 Each of the following three conditions is necessary and sufficient for a semi-proximity space (X, δ) to be uniformizable:

(a) $A\delta B$ provided that $A, B \subset X$ and the distance from A to B is zero for each totally bounded proximally continuous pseudo-metric for (X, δ) ;

(b) $A\delta B$ provided that $A, B \subset X$ and the distance from A to B is zero for each proximally continuous pseudo-metric for (X, δ) ;

(c) if $A\delta B$ then there exists a bounded proximally continuous function f on (X, δ) which is 0 on A and 1 on B.

Proof: I. First we shall show that the conditions (a), (b) and (c) are equivalent to each other. It is sufficient to prove $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$. Clearly $(a) \Rightarrow (b)$, and to prove $(b) \Rightarrow (c)$ let us assume (b) and let $A\overline{\delta}B$. By (b) we can take a proximally continuous pseudo-metric d for (X, δ) such that the distance from A to B in (X, d) is positive, let us say r. Now let us consider the function g(x) = d(x, A) on (X, δ) and let us put $f(x) = \min(1, g(x)/r)$. Clearly $0 \leq f \leq 1$ and f is 0 on A and 1 on B. Next, $g: (X, d) \to \mathbb{R}$ is a Lipschitz mapping, hence uniformly continuous and thus proximally continuous. Since g is proximally continuous. It remains to show that $(c) \Rightarrow (a)$. Assuming (c), let $A\overline{\delta}B$. We must find a proximally continuous, totally bounded pseudo-metric d for (X, δ) such that the distance from A to B in (X, d) is positive. Let us take a bounded proximally continuous function f on (X, δ) which is 0 on A and 1 on B, and let us consider the pseudo-metric d for (X, δ) such that the distance from A to B in (X, d) is positive. Let us take a bounded proximally continuous function f on (X, δ) which is 0 on A and 1 on B, and let us consider the pseudo-metric

 $d = \{(x, y) \to |f(x) - f(y)| : (x, y) \in X \times X\}$. Evidently d is totally bounded and the distance from A to B in (X, d) is 1. It remains to show that d is a proximally continuous pseudo-metric for (X, δ) . This follows immediately from the fact that, denoting by \mathcal{U} the proximally coarse semi-uniformity of (X, δ) , the function $d_1 = d : (X, \mathcal{U}) \times (X, \mathcal{U}) \to \mathbb{R}$ is uniformly continuous since it is the composition of two uniformly continuous mappings. Namely, $d_1 = (\{(r, s) \to |r - s|\} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}) \circ (f \times f : (X, \mathcal{U}) \times (X, \mathcal{U}) \to \mathbb{R} \times \mathbb{R})$ and this shows that d is a uniformly continuous pseudo-metric for (X, \mathcal{U}) and hence a proximally continuous pseudo-metric for (X, δ) .

It is to be noted that the proximal continuity of d can be proved directly: if $A\delta B$, then the distance from f(A) to f(B) is zero in \mathbb{R} and therefore, the distance from A to B in (X, d) is zero. This establishes the proximal continuity of d.

II. Condition (c) is sufficient. Assuming (c) we shall prove that condition (SP_5) is fulfilled. If $A\overline{\delta}B$ and f is a proximally continuous function on (X, δ) which is 0 on A and 1 on B, then the sets $U = \{x : f(x) < 1/2\} = f^{-1}((\leftarrow, 1/2))$ and $V = \{x : f(x) > 1/2\} = f^{-1}((1/2, \rightarrow))$ are disjoint proximal neighborhoods of A and B in (X, δ) .

III. Condition (b) is necessary. Let (X, δ) be uniformizable and let \mathcal{U} be a uniformity which induces δ . If $A\overline{\delta}B$, then $U[A] \cap B = \emptyset$ for some $U \in \mathcal{U}$, and \mathcal{U} being a uniformity, we can choose a uniformly continuous pseudo-metric d for (X,\mathcal{U}) such that d(x,y) < 1 implies $(x,y) \in U$. Clearly, the distance from A to B in (X,d) is at least 1. Since d is a uniformly continuous pseudo-metric for (X,\mathcal{U}) , d is a proximally continuous pseudo-metric for (X,δ) .

Corollary 2.2.6.1 If δ_1 and δ_2 are uniformizable proximities for a set X, then δ_1 is proximally coarser than δ_2 if and only if, for each bounded proximally continuous function f on (X, δ_1) , the function $f : (X, \delta_2) \to \mathbb{R}$ is proximally continuous.

Roughly speaking, a uniformizable proximity space is uniquely determined by the collection of all bounded proximally continuous functions.

2.2.7 Proximally continuous functions

In this section our purpose is to prove that, for each semi-proximity space (X, δ) , the set of all bounded proximally continuous functions on X, denoted by $\mathbf{P}^*(X, \mathbb{R})$, is a closed sub-lattice-algebra of the topological lattice-algebra unif $\mathbf{F}^*(X, \mathbb{R})$ of all bounded mapping of X into \mathbb{R} . The symbol $\mathbf{F}^*(X, \mathbb{R})$ denotes the normed lattice-algebra of all bounded function of X into \mathbb{R} .

Proposition 2.2.7.1 Let (X, δ) be a semi-proximity space. The sum of two proximally continuous functions on X of which one is bounded, is a proximally continuous function. The product of two bounded proximally continuous functions on X is a proximally continuous function.

Proof: I. We shall need the following property of bounded proximally continuous functions: if f is a proximally continuous function on (X, δ) , r is a positive real number and $A\delta B$, then there exist $A' \subset A$ and $B' \subset B$ such that $A'\delta B'$ and the diameters of the sets f(A') and f(B') are at the most r. As the set f(X) is contained in a bounded interval, we can choose a finite family $\{I_k\}$ of intervals which covers f(X) and such that the length of each I_k is r. Thus $\{f^{-1}(I_k)\}$ is a finite cover of X and the diameter of each set $f(f^{-1}(I_k)) \subset I_k$ is at the most r. Now if $A\delta B$, then, by Proposition 2.2.1.6, for some i and j, $(A \cap f^{-1}(I_i))\delta(B \cap f^{-1}(I_j))$ and the diameters of the sets $f(A \cap f^{-1}(I_i))$ and $f(B \cap f^{-1}(I_k))$ are at the most r.

II. Now let f and g be two proximally continuous functions, f bounded and h = f + g. Let us suppose that $A\delta B$. To prove that the distance from h(A) to h(B) is zero, it is sufficient to show that the distance from h(A) to h(B) is at the most 3r for each positive real number r. Let r > 0. Let us choose $A' \subset A$ and $B' \subset B$ such that $A'\delta B'$ and the diameters of the sets f(A') and f(B') are at the most r which is possible by I. Now if $x \in A'$ and $y \in B'$, then the distance from f(x) to f(y) is at the most 2rbecause the distance of the set f(A') from f(B') is zero (f is proximally continuous) and their diameters are at the most r. Since g is proximally continuous, the distance from g(A') to g(B') is zero and therefore we can choose $x \in A'$ and $y \in B'$ so that |g(x) - g(y)| < r. Now $|h(x) - h(y)| \leq$ $|f(x) - f(y)| + |g(x) - g(y)| \leq 2r + r = 3r$, which shows that the distance from h(A) to h(B) is at the most 3r.

III. Let us suppose that f and g are bounded proximally continuous functions, $|f(x)| \leq K$ and $|g(x)| \leq K$ for each $x \in X$, where K > 0, $h = f \cdot g$, and $A\delta B$. To prove that the distance from h(A) to h(B) is zero, it is sufficient to show that, for each r > 0, the distance from h(A) to h(B)is at the most 3Kr. Let r > 0. By I. we can choose $A' \subset A$ and $B' \subset B$ so that $A'\delta B'$ and the diameters of the sets f(A') and f(B') are at the most r. Since the distance from g(A') to g(B') is zero, we can choose $x \in A'$ and $y \in B'$ such that |g(x) - g(y)| < r, since the distance from f(A') to f(B') is zero and the diameters of these sets are at the most r, we obtain $|f(x)g(x) - f(y)g(y)| \leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \leq K \cdot 3r$, and consequently the distance from h(A) to h(B) is at the most 3rK. This concludes the proof. \clubsuit The sum of two unbounded proximally continuous functions need not be proximally continuous. The product of two proximally continuous functions need not be proximally continuous. The product of two proximally continuous functions need not be proximally continuous even if one of the functions is bounded.

Proposition 2.2.7.2 The uniform limit of proximally continuous functions is a proximally continuous function. In other words, $\mathbf{P}(X, \mathbb{R})$ is closed in unif $\mathbf{F}(X, \mathbb{R})$ for each proximity space X.

Proof: Let us suppose that a net $\{f_a\}$ of proximally continuous functions on a proximity space (X, δ) converges uniformly to f, i.e. $\{f_a\}$ converges to f in unif $\mathbf{F}(X, \mathbb{R})$. Let $A\delta B$ and r be a positive real number. We shall prove that the distance from f(A) to f(B) is at the most 3r. Since $\{f_a\}$ converges to funiformly, there exists an index a so that $|f_a(x) - f(x)| \leq r$ for each $x \in X$. Since f_a is proximally continuous, the distance from $f_a(A)$ to $f_a(B)$ is zero and therefore we can choose an $x \in A$ and a $y \in B$ so that $|f_a(x) - f_a(y)| < r$. So, $|f(x) - f(y)| \leq |f(x) - f_a(x)| + |f_a(x) - f_a(y)| + |f_a(y) - f(y)| < 3r$.

Now we can prove the main result of this section.

Theorem 2.2.7.1 The set $\mathbf{P}^*(X, \mathbb{R})$ of all bounded proximally continuous functions on a proximity space X is a closed sub-lattice-algebra of the normed lattice-algebra $\mathbf{F}^*(X, \mathbb{R})$ of all bounded mappings of X into \mathbb{R} .

Proof: Clearly, every constant function on X is proximally continuous. Further, if f is a proximally continuous function, then |f| is also proximally continuous because $d(|f|(A), |f|(B)) \leq d(f(A), f(B))$ for each $A, B \subset X$. This inequality follows from the inequality $||x| - |y|| \leq |x - y|$ which holds for all real numbers x and y. Clearly, if f is proximally continuous and r is a real number, then $r \cdot f$ is also proximally continuous. It remains to show that f + g and $f \cdot g$ are proximally continuous whenever f and g are bounded proximally continuous functions, and that, if a net $\{f_a\}$ of proximally continuous functions converges to f in unif $\mathbf{F}(X, \mathbb{R})$, then f is proximally continuous. Indeed, the proximal continuity of the functions $\sup(f, g)$ and $\inf(f, g)$, where f and g are bounded proximally continuous functions, follows from the following obvious equalities:

$$\begin{aligned} \sup(f,g) &= f + \sup(0,g-f) = f + (|g-f| + (g-f))/2 = \\ &= (|g-f| + (f-g))/2, \\ \inf(f,g) &= -\sup(-f,-g). \end{aligned}$$

The remaining statements are particular cases of Proposition 2.2.7.1 and Proposition 2.2.7.2. \clubsuit

Theorem 2.2.7.2 (a) If (X, U) is a semi-uniform space and (G, u; +) commutative topological group, then $\mathbf{U}(X, G)$ is a closed subgroup of the group unif $\mathbf{F}(X, G)$ and contains all constant mappings.

(b) If (X, \mathcal{U}) is a semi-uniform space and $(R, +, \cdot; \| \|)$ is a normed ring, then the set $\mathbf{U}^*(X, R)$ of all bounded uniformly continuous mappings of Xinto R is an ring; if $R = \mathbb{R}$, then $\mathbf{U}^*(X, \mathbb{R})$ contains, with each f, the function |f|.

Proof: (a) It is a well known fact that the set $\mathbf{U}(X,G)$ is closed in unif $\mathbf{F}(X,G)$. Since (G,u;+) is commutative, the mapping h(x,y) = x - y from $G \times G$ into G is uniformly continuous. Now, if f and g are uniformly continuous mappings, then $f - g = h \circ (f \times g)$, and consequently f - g is uniformly continuous as the composition of two uniformly continuous mappings. Hence $\mathbf{U}(X,G)$ is a subgroup.

(b) Let us suppose that d is the pseudo-metric corresponding to the norm of R, i.e. d(x, y) = ||x - y||, and let d_1 be the pseudo-metric for $R \times R$ such that $d_1((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d(y_1, y_2)$. It is easily seen that the mapping $\{(x, y) \to x \cdot y\} = (R \times R, d_1) \to (R, d)$ is Lipschitz continuous and hence uniformly continuous on each set $A \times A$, where A is a bounded subset of R. Now, as in (a), we find that $f \cdot g$ is uniformly continuous whenever fand g are bounded uniformly continuous mappings. Finally, if $R = \mathbb{R}$, then, evidently, $h = \{x \to |x|\} : \mathbb{R} \to \mathbb{R}$ is uniformly continuous and hence, if f is a uniformly continuous mapping into \mathbb{R} , then |f| is uniformly continuous as the composite of f and h.

2.2.8 Stone-Weierstrass theorem

By the Weierstrass theorem, for each bounded continuous function f on a bounded closed interval I of reals and for each positive real r there exists a polynomial function $g(x) = \sum_{i \leq n} a_i x^i$ such that |f(x) - g(x)| < r for each $x \in I$. In other words, if \mathcal{F} is the set of all polynomial functions on I, then \mathcal{F} is dense in the normed algebra $\mathbf{C}^*(I, \mathbb{R})$ of all bounded continuous functions on I. Let us notice that \mathcal{F} is the smallest subalgebra of $\mathbf{C}^*(I, \mathbb{R})$ containing the functions $\{x \to 1\} : I \to \mathbb{R}$ and $J = \{x \to x\} : I \to \mathbb{R}$. Thus the Weierstrass theorem can be stated as follows: the smallest subalgebra of $\mathbf{F}^*(I, \mathbb{R})$ containing the constant function $\{x \to 1\}$ and the identity function $J : I \to \mathbb{R}$ is dense in $\mathbf{C}^*(I, \mathbb{R})$. Further, clearly, the proximity of I is the proximally coarsest proximity for I such that $J: I \to \mathbb{R}$ is a proximally continuous function, and it turns out that $\mathbf{C}^*(I, \mathbb{R}) = \mathbf{P}^*(I, \mathbb{R})$. This follows from the compactness of I. Thus $J: I \to \mathbb{R}$ entirely determines the proximity of I, and the smallest subalgebra of $\mathbf{P}^*(I, \mathbb{R})$ containing $J: I \to \mathbb{R}$, and the constant function $\{x \to 1\}$ is dense in the normed algebra $\mathbf{P}^*(I, \mathbb{R})$. It turns out that this is true in general, for an appropriate definition of "entirely determines".

Definition 2.2.8.1 We shall say that a collection \mathcal{M} of mappings of a semiproximity space (X, δ) into a semi-proximity (Y, δ_1) projectively generates the semi-proximity of X if δ is the proximally coarsest semi-proximity for X such that all mappings $f \in \mathcal{M}$ are proximally continuous.

We will begin with a consideration of the proximity space projectively generated by a family of mapping into proximity spaces.

Proposition 2.2.8.1 Let \mathcal{F} be a collection of bounded functions on a set X. There exists a unique semi-proximity δ for X such that (X, δ) is projectively generated by the collection of all functions $f : (X, \delta) \to \mathbb{R}$, $f \in \mathcal{F}$. The set \mathcal{D} of all pseudo-metrics $d_f = \{(x, y) \to |f(x) - f(y)| : (x, y) \in X \times X\}$, $f \in \mathcal{F}$, generates a proximally coarse semi-uniformity of (X, δ) . If \mathcal{D}' is the smallest set containing \mathcal{D} and such that $d_1, d_2 \in \mathcal{D}' \Rightarrow d_1 + d_2 \in \mathcal{D}'$, then $A\delta B$ if and only if the distance in (X, d) from A to B is zero for each $d \in \mathcal{D}'$.

Proof: I. Let \mathcal{U} be the semi-uniformity generated by the collection \mathcal{D} of pseudo-metrics. Then the sets of the form $\{(x, y) : d(x, y) < r\}, d \in \mathcal{D}, r > 0$, form a sub-base for \mathcal{U} , and \mathcal{U} is a uniformity. Clearly, each $d \in \mathcal{D}$ is totally bounded and hence \mathcal{U} is totally bounded. \mathcal{U} , being a uniformity, is proximally coarse by Proposition 2.2.5.2.

II. Let δ be the proximity induced by \mathcal{U} . Clearly, the last statement of this proposition holds. Hence every $f : (X, \delta) \to \mathbb{R}, f \in \mathcal{F}$, is proximally continuous.

III. It remains to prove that δ is the proximally coarsest proximity for X such that all the functions $f: (X, \delta) \to \mathbb{R}, f \in \mathcal{F}$, are proximally continuous. Let δ_1 be any proximity for X such that all the functions $f: (X, \delta) \to \mathbb{R}$, $f \in \mathcal{F}$, are proximally continuous. We shall show that δ_1 is proximally finer than δ . Since each $f: (X, \delta_1) \to \mathbb{R}, f \in \mathcal{F}$, is proximally continuous, each $d_f, f \in \mathcal{F}$, is a proximally continuous pseudo-metric for (X, δ) . Each d_f being totally bounded and all the elements of \mathcal{D}' are proximally continuous pseudo-metrics for (X, δ) by Proposition 2.2.5.4; hence, \mathcal{U} is a proximally continuous uniformity for (X, δ_1) . Thus δ is proximally coarser than δ_1 .

Let us assume that a proximity space (X, δ) is projectively generated by a collection \mathcal{F} of bounded functions, and for each $f \in \mathcal{F}$ let d_f be the pseudometric defined in the formulation of Proposition 2.2.8.1. If $A\delta B$, then the distance from A to B is zero in each (X, d_f) . It is easy to find an example such that A and B are distant in (X, δ) but not proximal in each (X, d_f) . If the set of all d_f is addition-stable, then by Proposition 2.2.8.1 $A\overline{\delta}B$ implies that A and B are distant in some (X, d_f) . Similarly, if $A\delta B$ then f(A) is proximal to f(B) in \mathbb{R} for each $f \in \mathcal{F}$, but the converse is not true. This follows from the similar results for d_f . It is interesting to show that the converse is not true even if \mathcal{F} is a linear space. We shall only construct such an \mathcal{F} with the following algebraic property: $f_1, f_2 \in \mathcal{F} \Rightarrow f_1 + f_2 \in \mathcal{F}$. Using this example the reader may construct without difficulty such a linear space \mathcal{F} .

Example 2.2.8.1 Let (X, δ) be a subspace of \mathbb{R} , $X = I_1 \cup I_2 \cup I_3$, $I_1 = [0, 1]$, $I_2 = [2, 3]$, $I_3 = [4, 5]$, and let us consider the following two functions f and g on (X, δ) : f(x) = g(x) = x for $x \in I_1$, f(x) = x - 2 and g(x) = x for $x \in I_2$ and finally, f(x) = x - 2 and g(x) = x - 4 for $x \in I_3$.

It is easily seen that the collection $\{f, g\}$ projectively generates (X, δ) . Let \mathcal{F} be the set of all linear combinations $\lambda f + \mu g$ with non-negative λ and μ . We shall show that $h(I_1) \cap h(I_2 \cup I_3) \neq \emptyset$ for each $h \in \mathcal{F}$ (on the other hand, I_1 and $I_2 \cup I_3$ are far in (X, δ)). Let $h = \lambda f + \mu g$, $\lambda \ge 0$, $\mu \ge 0$. It is easily seen that $h(I_1) = [0, \lambda + \mu]$, $h(I_2) = [2\mu, 3\mu + \lambda]$, $h(I_3) = [2\lambda, 3\lambda + \mu]$. It is clear that $t = \min(2\lambda + 2\mu) \le \lambda + \mu$ and hence $t \in h(I_1) \cap h(I_2 \cup I_3)$.

Let us suppose that a proximity space X is generated by a collection \mathcal{F} of bounded proximally continuous functions. By the preceding example it is not true that, if A and B are distant in X, then f(A) and f(B) are distant in \mathbb{R} for some $f \in \mathcal{F}$. On the other hand, one has the following, essentially weaker, result:

Proposition 2.2.8.2 Let us suppose that a semi-proximity space (X, δ) is projectively generated by a collection \mathcal{F} of bounded functions. Then $A\delta B$ if and only if the following condition is fulfilled: if A is the union of a finite family $\{A_i\}$ and B is the union of a finite family $\{B_j\}$, then there exist indices i and j such that $f(A_i)$ is proximal to $f(B_j)$ for each $f \in \mathcal{F}$.

Proof: For each $f \in \mathcal{F}$ let us put that $\delta_f = \{(A, B) : f(A) \text{ is proximal to } f(B) \text{ in } \mathbb{R}\}$. It is easy to verify that each δ_f is a proximity for X and δ is the

proximally coarsest proximity for X proximally finer than each δ_f , $f \in \mathcal{F}$. Now, the statement is implied by the following proposition:

Proposition 2.2.8.3 Let X be a set and let $\{\delta_i : i \in I\}$ be a family of semi-proximity relations for X. There exists a proximally coarsest semi-proximity δ for X proximally finer than each δ_i , $i \in I$. If $I \neq \emptyset$, then $A\delta B$ if and only if $A, B \subset X$ and the following condition is fulfilled:

If $\{A_j\}$ is a finite cover of A, and $\{B_k\}$ is a finite cover of B, then there exist indices j and k such that $A_j\delta_i B_k$ for each $i \in I$.

Proof: See the proof of Theorem 1.1.4.1.

Proposition 2.2.8.4 Let us suppose that \mathcal{F} is a collection of functions on a semi-proximity space (X, δ) satisfying the following condition:

if $A\overline{\delta}B$ and if r is a positive real number, then there exists an $f \in \mathcal{F}$ such that $0 \leq f(x) \leq r$ for each $x \in X$, f(A) = 0, f(B) = r.

Then for each non-negative bounded proximally continuous function g on X and each positive real r there exists a finite family $\{f_i\}$ in \mathcal{F} such that $|g(x) - \sum_i f_i(x)| \leq r$ for each $x \in X$.

Proof: Let g be a non-negative bounded proximally continuous function on X and let r > 0. Let k be the smallest positive integer such that $g(x) \leq kr$ for each $x \in X$. For each $i \leq k$ let $A_i = \{x : g(x) \leq ir\}$. If $1 \leq i \leq k$, then the sets A_{i-1} and $X - A_i$ are distant in X and therefore we can choose an $f_i \in \mathcal{F}$ such that $0 \leq f_i(x) \leq r$ for each $x \in X$ and $f_i(A_{i-1}) = 0$, $f_i(X - A_i) = r$. It is easy to verify that $|g(x) - \sum_{1 \leq i \leq k} f_i(x)| \leq r$ for each $x \in X$.

Corollary 2.2.8.1 If a linear subspace \mathcal{F} of $\mathbf{P}^*(X, \mathbb{R})$ fulfils the above condition, then \mathcal{F} is dense in the normed space $\mathbf{P}^*(X, \mathbb{R})$.

Proof: To prove the corollary, it is sufficient to notice that \mathcal{F} contains all constant functions. Given an r > 0, there exists an $f \in \mathcal{F}$ such that $f(\emptyset) = 0, f(X) = r$ (because $\emptyset \overline{\delta} X$), and hence f(x) = r for each $x \in X$.

Lemma 2.2.8.4 Let us suppose that (X, δ) is a semi-proximity space and let \mathcal{F} be a sublattice-module of $\mathbf{P}^*(X, \mathbb{R})$ containing all constant functions, and projectively generating X. Then for each $A\overline{\delta}B$ and each positive real number r there exist an $f \in \mathcal{F}$ so that f is 0 on A, r on B and $0 \leq f(x) \leq r$ for each $x \in X$. **Proof:** I. It will suffice to prove that, given $A\overline{\delta}B$ and r > 0, there exist finite families $\{A_i\}$ and $\{B_j\}$ such that $A = \bigcup_i A_i, B = \bigcup_j B_j$, and for each of the indices i and j there exists a required function f_{ij} for A_i and B_j , i.e. f_{ij} is 0 on A_i , r on B_j and $0 \leq f_{ij}(x) \leq r$ for each $x \in X$. Indeed, $f = \inf_i \sup_j \{f_{ij}\}$ is then a required function for A and B.

II. Let us suppose $A\overline{\delta}B$, $A \neq \emptyset \neq B$ and let f be an element of \mathcal{F} such that the distance from f(A) to f(B) is positive, let us say r (such an element need not exist). Let us choose a finite decomposition $\{A_i\}$ of A and $\{B_j\}$ of B such that the diameter of each set $f(A_i)$ as well as each $f(B_j)$ is smaller than r/2. This is possible because f is bounded. We may and shall assume that $A_i \neq \emptyset \neq B_j$ for each i and j. If $x' \in A_i$, $y' \in B_j$ and f(x') < f(y'), then f(x) < f(y) for each $x \in A_i$ and $y \in B_j$. Indeed, since the distance from $f(A_i)$ to $f(B_j)$ is at least the one from f(A) to f(B), i.e. r, and |f(x) - f(x')| < r/2, |f(y) - f(y')| < r/2, we obtain $f(x) < f(x') + r/2 \leq f(y') - r/2 < f(y)$. Similarly, if f(x') > f(y') for some $x' \in A_i$, $y' \in B_j$, then f(x) > f(y) for each $x \in A_i$ and each $y \in B_j$. If f(x) < f(y) for each $x \in A_i$ and $y \in B_j$.

$$h_{ij} = \{z \to \min(f(z), \inf f(B_j))\} : X \to \mathbb{R}, g_{ij} = \{z \to \max(h_{ij}(z), \sup f(A_i))\} : X \to \mathbb{R}, f_{ij} = \{z \to (g_{ij}(z) - \sup f(A_i))\} : X \to \mathbb{R}.$$

Clearly, the function h_{ij} , and hence g_{ij} , and finally f_{ij} all belong to \mathcal{F} , f_{ij} being zero on A_i and $d(f(A_i), f(B_j)) \ge r$. Now, given a positive real s, for an appropriate real t, $t \cdot f_{ij}$ is s on B_j and zero on A_i . Similarly, if f(x) > f(y) for $x \in A_i$ and $y \in B_j$, then the same construction leads to a function $f \in \mathcal{F}$ which is zero on B_j and s on A_i .

III. Now let us suppose that $A\delta B$. Since \mathcal{F} generates δ , by Proposition 2.2.8.2 there exist finite decompositions $\{A_i\}$ of A and $\{B_j\}$ of B such that for each i and j there exists an $f \in \mathcal{F}$ so that the distance from $f(A_i)$ to $f(B_j)$ is positive. Applying II. to each pair A_i , B_j we obtain finite decompositions $\{C_k\}$ of A and $\{D_l\}$ of B such that for each k and l there exists a function in \mathcal{F} which is zero on C_k , s on D_l and its range is contained in the interval [0, s]. The proof is complete.

Theorem 2.2.8.1 Stone-Weierstrass Theorem (for semi-proximity spaces). Let (X, δ) be a semi-proximity space projectively generated by a collection \mathcal{M} of bounded functions, and let \mathcal{F} be the smallest subalgebra of $\mathbf{F}^*(X, \mathbb{R})$ containing \mathcal{M} and the constant function $\{x \to 1\} : X \to \mathbb{R}$. Then the closure of \mathcal{F} in $\mathbf{F}^*(X, \mathbb{R})$ is $\mathbf{P}^*(X, \mathbb{R})$. In other words, a bounded function-

tion f on X is proximally continuous if and only if the following condition is fulfilled:

For each positive real r there exists a polynomial function

$$P = \{(z_0, z_1, \dots, z_n) \to \sum_{i_j \leq k} a_{i_0 \dots i_n} z_0^{i_0} \dots z_n^{i_n} : z_j \in \mathbb{R}\} : \mathbb{R}^n \to \mathbb{R}$$

and functions $f_0, \ldots, f_n \in \mathcal{M}$ such that $|f(x) - P(f_0(x), \ldots, f_n(x))| \leq r$ for each $x \in X$.

Proof: Let us suppose that a semi-proximity space (X, δ) is projectively generated by a collection \mathcal{M} of bounded functions, and let \mathcal{F} be the smallest algebra containing \mathcal{M} and the constant function $\{x \to 1\} : X \to \mathbb{R}$, and hence all constant functions. Let us consider the closure \mathcal{G} of \mathcal{F} in $\mathbf{F}^*(X, \mathbb{R})$. Since $\mathcal{M} \subset \mathbf{P}^*(X, \mathbb{R})$, $\{\{x \to 1\} : X \to \mathbb{R}\} \in \mathbf{P}^*(X, \mathbb{R})$ and $\mathbf{P}^*(X, \mathbb{R})$ is a closed subalgebra of $\mathbf{F}^*(X, \mathbb{R})$ by Theorem 2.2.7.1, $\mathcal{G} \subset \mathbf{P}^*(X, \mathbb{R})$ holds. Clearly, \mathcal{G} is closed in $\mathbf{F}^*(X, \mathbb{R})$ (the closure structure of $\mathbf{F}^*(X, \mathbb{R})$ is topological) and \mathcal{G} is an algebra because it is the closure of an algebra, namely of \mathcal{F} . Since \mathcal{G} is a closed algebra, \mathcal{G} is a lattice. Since $\mathbf{P}^*(X, \mathbb{R}) \supset \mathcal{G} \supset \mathcal{M}$ and \mathcal{M} projectively generates (X, δ) , \mathcal{G} also projectively generates (X, δ) , and therefore by Lemma 2.2.8.4, \mathcal{G} is dense in $\mathbf{P}^*(X, \mathbb{R})$. Since \mathcal{G} is closed, $\mathcal{G} = \mathbf{P}^*(X, \mathbb{R})$.

The concluding theorems are intended to clarify the relations between proximities and sets of bounded functions. We shall need the following description of the proximity of bounded subsets of \mathbb{R} .

Proposition 2.2.8.5 *A* bounded subset *A* of \mathbb{R} is proximal to a subset *B* of \mathbb{R} if and only if $\overline{A} \cap \overline{B} \neq \emptyset$.

Proof: If $\overline{A} \cap \overline{B} \neq \emptyset$, then the distance from A to B is zero and hence the sets A and B are proximal (without any supposition on A). Conversely, assuming that a bounded set A is proximal to a set B, i.e. the distance from A to B is zero, we can take sequences $\{x_n\}$ in A and $\{y_n\}$ in B such that the sequence $\{|x_n - y_n|\}$ converges to zero. Since A is bounded, some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges to a point x. Clearly, $x \in A$. Since $|x - y_{n_i}| \leq |x - x_{n_i}| + |x_{n_i} - y_{n_i}|$, the sequence $\{y_{n_i}\}$ also converges to x. Thus $x \in \overline{B}$ and hence $x \in \overline{A} \cap \overline{B}$.

Theorem 2.2.8.2 Let (X, δ) be a uniformizable semi-proximity space and let H be a closed linear subspace of $\mathbf{F}^*(X, \mathbb{R})$ containing the constant function $\{x \to 1\}: X \to \mathbb{R}$. The following statements are equivalent: (a) $H = \mathbf{P}^*(X, \mathbb{R});$

(b) *H* is a subalgebra of $\mathbf{F}^*(X, \mathbb{R})$ (i.e. $g_1, g_2 \in H \Rightarrow g_1 \cdot g_2 \in H$), if $A\delta B$ and $f \in H$, then $\overline{f(A)} \cap \overline{f(B)} \neq \emptyset$, and if $A\overline{\delta}B$ then there exists an $f \in H$, $0 \leq f \leq 1$, which is 0 on A and 1 on B;

(b') H is a sublattice of $\mathbf{F}^*(X, \mathbb{R})$ (i.e. $g \in H \Rightarrow |g| \in H$, or equivalently, $\underline{g_1, g_2} \in H \Rightarrow \sup(g_1, g_2) \in H$, $\inf(g_1, g_2) \in H$), if $A\delta B$ and $f \in H$, then $\overline{f(A)} \cap \overline{f(B)} \neq \emptyset$, and if $A\overline{\delta}B$, then there exists an $f \in H$, $0 \leq f \leq 1$, which is 0 on A and 1 on B;

(c) *H* is a subalgebra of $\mathbf{F}^*(X, \mathbb{R})$ and $A\delta B$ if and only if $\overline{f(A)} \cap \overline{f(B)} \neq \emptyset$ for each $f \in H$;

(c') *H* is a sublattice of $\mathbf{F}^*(X, \mathbb{R})$ and $A\delta B$ if and only if $\overline{f(A)} \cap \overline{f(B)} \neq \emptyset$ for each $f \in H$;

(d) H is a subalgebra of $\mathbf{F}^*(X, \mathbb{R})$ and projectively generates δ ;

(d') H is a sublattice of $\mathbf{F}^*(X, \mathbb{R})$ and projectively generates δ .

Proof: Evidently (b) implies (c), and (b') implies (c'). By Proposition 2.2.8.5 (c) implies (d), and (c') implies (d'). Every closed subalgebra is a sublattice and therefore(d) implies (d'). By the proof of Theorem 2.2.8.1, (d') implies (a). It remains to show that (a) implies both (b) and (b'). This follows from Theorem 2.2.6.1 and Theorem 2.2.7.1.

Historical and bibliographic notes

The concept of the semi-proximity spaces was introduced by M. Hushek in 1964 in paper [146] (see also [147]). In the revised edition of book "Topological Spaces" from E. Czech (the first edition was published by Czech in 1959) Z. Frolik and M. Katetov in 1966 [63] gave the most complete exposition of the theory of semi-proximity spaces. The results in subsections 2.1. and 2.2. were proved in that book. Filters were introduced by H. Cartan [48], grills by G. Choquet [57] and stacks by G. Grimeisen [125]. Grills are also mentioned by G. Nöbeling [245] (see also [281]). Examples of grills in theory of semi-proximity spaces, without awareness of the general concept, go back to F. Riesz [273], who was dealing with certain grills which, in our present terminology, are maximal δ -clans. S. Leader [184] introduced clusters which are also maximal δ -clans. He further [187] pointed out the duality between maximal round filters (the ends in Ju. M. Smirnoff's terminology) and clusters. M. W. Lodato [201], [202] introduced bunches. The proofs of assertions explained in subsection are given by W. J. Thron [320]. Theorems 2.2.4.1 and 2.2.4.2 were proved by K. C. Chattopadhyay in 1985 in his paper [51]. Propositions 2.2.4.1-2.2.4.5 and Theorem 2.2.4.3 were proved by Chattopadhyay in [53]. The Stone-Weierstrass Theorem for proximity spaces was proved by Smirnoff in 1960 [304]. Smirnoff devoted that paper to the memory on E. Czech. In the revised edition of Czech's book "Topological Spaces" Frolik and Katetov proved the Stone-Weierstrass theorem for semi-proximity spaces.

2.3 LO- and S-proximity spaces

2.3.1 The notion and basic properties of LO-proximities

Definition 2.3.1.1 Let X be a set. A semi-proximity relation δ defined on the power set of X is called a **Lodato** or **LO-proximity** if it satisfies the following condition:

(LO) $A\delta B$ and $b\delta C$ for each $b \in B$ together imply $A\delta C$. An LO-proximity δ is **separated** if it is a separated semi-proximity. The pair (X, δ) , where δ is a (separated) LO-proximity, is referred to as a (separated) Lodato- or (separated) LO-proximity space.

It is easy to show that the conditions (LO)- and

$$(LO')$$
 $c_{\delta}(A) \in \delta(B) \Rightarrow A \in \delta(B),$

where $c_{\delta}(A) = \{x \in X : x \in \delta(A)\}$, are equivalent. Indeed, if $c_{\delta}(A)\delta B$ holds, then $x\delta A$ for each $x \in c_{\delta}(A)$ holds. Therefore $A\delta B$ by (LO). To prove the converse, let us suppose that the condition (LO) holds. Since $b\delta C$ for each $b \in B$, $b \in c_{\delta}(C)$ for each $b \in B$ holds and therefore $B \subset c_{\delta}(C)$. But then $A\delta c_{\delta}(C)$ holds, from which, by (LO'), it follows that $A\delta C$.

It is evident that (by (LO')) in every LO-proximity space $A\delta B$ is true if and only if $c_{\delta}(A)\delta c_{\delta}(B)$.

It is obvious that every LO-proximity is an RI-proximity.

Proposition 2.3.1.1 Every proximity space (X, δ) is an LO-proximity space.

Proof: It is sufficient to show that (B_5) implies (LO). Let us suppose $A\delta B$ and $b\delta C$ for all $b \in B$, but $A\overline{\delta}C$. Then, by Proposition 1.1.1.3, there exist E and F such that $A\overline{\delta}X - F$, $C\overline{\delta}X - E$ and $E \cap F = \emptyset$. Let us suppose that $A\delta E$. Since $E \subset X - F$, by Proposition 2.2.1.1 $A\delta X - F$ holds, which is a contradiction. Hence $A\overline{\delta}E$. Now $B \cap (X - E) = \emptyset$ holds. Indeed, if $b \in B \cap (X - E)$, then, by Proposition 2.2.1.1, $b\delta C$ would imply $C\delta X - E$ which is also a contradiction. Hence $B \subset E$. But then, since $A\delta B$, by Proposition 2.2.1.1, $A\delta E$ holds, and this is a contradiction. \clubsuit **Definition 2.3.1.2** Let (X, δ) be an LO-proximity space. The set $B \in P(X)$ is a δ -neighborhood of a set $A \in P(X)$, in the notation $A \ll B$, if $A\overline{\delta}X - B$.

Proposition 2.3.1.2 The relation \ll in an LO-proximity space (X, δ) fulfills the following conditions:

 $(NL_1) X \ll X;$

 $(NL_2) A \ll B \text{ implies } A \subset B;$

 $(NL_3) A \subset B \ll C \subset D$ implies $A \ll D$;

 (NL_4) $A \ll B_k$, k = 1, 2, if and only if $A \ll B_1 \cap B_2$;

 $(NL_5) A \ll B \text{ implies } X - B \ll X - A;$

 (NL_6) $A \ll B$ implies that, for all C, $A \ll C$ or there exists $x \in X - C$ such that $x \ll B$.

If δ is a separated relation, then

 $(NL_7) x \ll X - y$ if and only if $x \neq y$.

Conversely, if \ll is a binary relation on the power set of X fulfilling $(NL_1) - (NL_6)$ and δ is defined by

 $A\overline{\delta}B$ if and only if $A \ll X - B$,

then δ is an LO-proximity on X. Furthermore, B is a δ -neighborhood of A with respect to δ if and only if $A \ll B$. Moreover, if \ll also fulfills (NL_7) , then δ is a separated LO-proximity.

Proof: The proof of $(NL_1) - (NL_5)$ is straightforward and is left to the reader.

 (NL_6) Let us suppose that $A \ll B$ and $A \not\ll C$. Furthermore, let us suppose that $x \not\ll B$ for every $x \in X - C$. Then $A\delta X - C$ and for each $x \in X - C \ x\delta X - B$ holds. But then $A\delta X - B$ holds according to (LO), which is a contradiction.

 (NL_7) Let us suppose that $x \ll X - y$. Then $x\overline{\delta}y$, so that by (SP_3) $x \neq y$. Conversely, let us suppose that $x \neq y$. Then, by (SP_5) , $x\overline{\delta}y$ follows. Consequently, $x\overline{\delta}(X - (X - y))$, that is $x \ll X - y$.

To prove the converse of the theorem, let us suppose that the relation \ll satisfies the conditions $(NL_1) - (NL_6)$.

 (SP_2) Let us suppose that $A\overline{\delta}B$. Then $A \ll X - B$ and therefore, by $(NL_5), B \ll X - A$ holds. But then $B\overline{\delta}A$ holds.

 (SP_1) Since, by (NL_1) , $X \ll X$, it is $X\overline{\delta}\emptyset$. But then $\emptyset\overline{\delta}X$ holds according to (SP_2) .

 $(SP_3) \ A\overline{\delta}B$ implies $A \ll X - B$. Consequently, by $(NL_2), A \subset X - B$ holds, so that $A \cap B = \emptyset$.

 (SP_4) Let us suppose that $A\overline{\delta}C$ and $B\overline{\delta}C$, that is $A \ll X - C$ and $B \ll X - C$. Then $C \ll X - A$ and $C \ll X - B$ by (NL_5) and hence by $(NL_4), C \ll (X - A) \cap (X - B)$; so that $C \ll X - (A \cup B)$. Now, by (NL_5) , we have that $A \cup B \ll X - C$, and consequently, $A \cup B\overline{\delta}C$. It is obvious that the converse also holds.

(LO) To prove this conditions, it is sufficient to show that $A\overline{\delta}B_1$ implies $A\overline{\delta}C_1$ or there exists $x \in C_1$ such that $x\overline{\delta}B_1$. Let $B_1 = X - B$ and $C_1 = X - C$. $A\overline{\delta}X - B$ implies, by definition, that $A \ll B$. Thus, by (NL_6) , we have that $A \ll C$ or there exists an $x \in X - C$ such that $x \ll B$. Hence $A\overline{\delta}X - C$ or there exists an $x \in X - C$ such that $x\overline{\delta}X - B$.

Hence δ is an LO-proximity.

Let us suppose that B is a δ -neighborhood of A with respect to δ . Then $A\overline{\delta}X - B$, that is $A \ll B$.

Conversely, let us suppose that $A \ll B$. Then $A\overline{\delta}X - B$. But then B is a δ -neighborhood of A with respect to δ .

Let us suppose that \ll satisfied (NL_7) , and $x\delta y$. Then $x \ll X - y$ which implies $x \neq y$.

Conversely, let us suppose that $x \neq y$. Then $x \ll X - y$ so that $x\overline{\delta}y$. This completes the proof of the proposition.

2.3.2 The compatibility of LO-proximity with topology

Theorem 2.3.2.1 Let (X, δ) be an LO-proximity space. The function $A \to c_{\delta}(A)$, where $c_{\delta}(A) = \{x \in X : x \delta A\}$, is a Kuratowski closure function.

Proof: According to Proposition 2.2.1.3, it is sufficient to show that inclusion $c_{\delta}(c_{\delta}(A)) \subset c_{\delta}(A)$ holds for any subset A of X. Let us suppose that $x \in c_{\delta}(c_{\delta}(A))$. Then $x\delta c_{\delta}(A)$. Since $a\delta A$ for all $a \in c_{\delta}(A)$, by (LO) we have that $x\delta A$. Therefore $x \in c_{\delta}(A)$ is true.

Definition 2.3.2.1 A topological space (X, τ) is symmetric or R_0 -space if one of the following equivalent conditions is satisfied:

(a) for every $x, y \in X$, $x \in \overline{y}$ implies $y \in \overline{x}$;

(b) for every open set $G \ x \in G$ implies $\overline{x} \subset G$;

(c) for every closed set F and any point $x \in X - F$, there exists a neighborhood of F not containing x;

(d) $x \neq y$ implies either $\overline{x} = \overline{y}$ or $\overline{x} \cap \overline{y} = \emptyset$.

Proposition 2.3.2.1 Let (X, τ) be a R_0 -topological space. The relation δ_0 on P(X), defined by $A\delta_0B$ if and only if $\overline{A}^{\tau} \cap \overline{B}^{\tau} \neq \emptyset$, is an LO-proximity on

X for which $\tau_{\delta_0} = \tau$ holds. Furthermore, if (X, δ) is an LO-proximity space such that $\tau_{\delta} = \tau$, then $\delta_0 \subset \delta$, i.e. δ_0 is the largest compatible LO-proximity.

Proof: It is obvious that the relation δ_0 is a semi-proximity on X. We must show that δ_0 fulfills the condition (LO). Let us suppose that for some point b and a set C we have $\overline{b}^{\tau} \cap \overline{C}^{\tau} \neq \emptyset$. Then there exists a point $c \in \overline{C}^{\tau}$ such that $c \in \overline{b}^{\tau}$. Since τ is R_0 -topology on X, we have that $b \in \overline{c}^{\tau} \subset \overline{C}^{\tau}$. Hence, if $\overline{A}^{\tau} \cap \overline{B}^{\tau} \neq \emptyset$ and $\overline{b}^{\tau} \cap \overline{C}^{\tau} \neq \emptyset$ for every $b \in B$, then $\overline{B}^{\tau} \subset \overline{C}^{\tau}$ so that $\overline{A}^{\tau} \cap \overline{C}^{\tau} \neq \emptyset$. Consequently, δ_0 satisfies (LO).

To show that $\tau_{\delta_0} = \tau$, it is sufficient to show that $x\delta_0 B$ if and only if $x \in \overline{B}^{\tau}$. Clearly, $x \in \overline{B}^{\tau}$ implies $\overline{x}^{\tau} \cap \overline{B}^{\tau} \neq \emptyset$. Hence $x\delta_0 B$.

Conversely, let us suppose that $x\delta_0 B$. Then, for some $y, y \in \overline{x}^{\tau} \cap \overline{B}^{\tau}$ holds. Hence $\overline{y}^{\tau} \subset \overline{B}^{\tau}$ and $y \in \overline{x}^{\tau}$. But since τ is a R_0 -topology, $y \in \overline{x}^{\tau}$ implies $x \in \overline{y}^{\tau}$. Hence $x \in \overline{B}^{\tau}$.

The proof of the last part of the theorem is straightforward.

Corollary 2.3.2.1 A topology τ on X is generated by some LO-proximity on X if and only if τ is a R_0 -topology.

Proof: Let us suppose there exists an LO-proximity δ on X such that $\tau_{\delta} = \tau$. Let $x \in \overline{y}^{\tau} = \overline{y}^{\tau_{\delta}}$. Then $x \delta y$ so that $y \delta x$ and $y \in \overline{x}^{\tau_{\delta}} = \overline{x}^{\tau}$. The converse is an immediate consequence of Proposition 2.3.2.1.

Proposition 2.3.2.2 Let (X, δ) be an LO-proximity space. Let τ_{δ} be the topology on X. Then for all $A, B \in P(X)$ there follows:

- (a) $A\delta B$ if and only if $A\delta B$;
- (b) $A \ll B$ implies $\overline{A} \ll B$;
- (c) $A \ll B$ implies $A \ll int B$.

Proof: (a) Let us suppose that $\overline{A}\delta\overline{B}$. By definition for all $b \in \overline{B}$ we have that $b\delta B$. Hence by (*LO*), $\overline{A}\delta B$ and therefore $B\delta\overline{A}$ holds by (*SP*₂). But for all $a \in \overline{A}$ we have that $a\delta A$, so that by (*LO*), $B\delta A$. Hence $A\delta B$. The converse is a consequence of Proposition 2.2.1.1.

The proofs of the last two statements are similar to the proofs of the corresponding statements for proximity spaces and are therefore left to the reader. \clubsuit

Proposition 2.3.2.3 Let δ be an LO-proximity on X and $A \subset X$. Then Int $A = \{x : x \ll A\}$.

Proof: It is easily shown that $C \in \tau_{\delta}$ if and only if $x \ll C$ for every $x \in C$. Let $B = \{x : x \ll A\}$. It is clear that $int A \subset B \subset A$. Consequently, it is sufficient to show that if $x \in B$, then $x \ll B$. Let $x \in B$. Then $x \ll A$; hence by $(NL_6) x \ll B$ or there exists $y \in X - B$ such that $y \ll A$. But if $y \ll A$, then $y \in B$; hence $x \ll B$.

Proposition 2.3.2.4 Let (X, δ_X) and (Y, δ_Y) be LO-proximity spaces. If $f : (X, \delta_X) \to (Y, \delta_Y)$ is δ -continuous, then $f : (X, \tau_{\delta_X}) \to (Y, \tau_{\delta_Y})$ is continuous. The converse is not true in general but it is true if $\delta_X = \delta_0$, where δ_0 is LO-proximity defined in Proposition 2.3.2.1.

Proof: It is sufficient to prove that $f(\overline{A}) \subset \overline{f(A)}$ for each $A \subset X$. Let $x \in \overline{A}$, that is $x\delta_X A$. Since f is δ -continuous, we have that $f(x)\delta_Y f(A)$. Hence $f(x) \in \overline{f(A)}$.

That in general the converse is not true, one can be see by taking $X = Y = \mathbb{R}$, where δ_X is the usual metric proximity, and $\delta_Y = \delta_0$.

Finally, let us suppose that f is a continuous mapping and $\delta_X = \delta_0$. To show that f is a δ -continuous mapping, we must show that $A\delta_X B$ implies $f(A)\delta_Y f(B)$. Since $\delta_X = \delta_0$, $A\delta_X B$ implies $\overline{A} \cap \overline{B} \neq \emptyset$; so that $f(\overline{A}) \cap f(\overline{B}) \neq \emptyset$. This shows that $f(\overline{A})\delta_Y f(\overline{B})$. Since f is continuous, it follows that $f(\overline{A}) \subset \overline{f(A)}$ and $f(\overline{B}) \subset \overline{f(B)}$, and so, by Proposition 2.2.1.1, $\overline{f(A)}\delta_Y \overline{f(B)}$. Finally, by Proposition 2.3.2.2, it follows that $f(A)\delta_Y f(B)$.

2.3.3 Extensions of LO-proximity spaces

Proposition 2.3.3.1 Let (X, δ) be an LO-proximity space. Then the class π_x of all subsets of X which are close to the point $x \in X$ is a cluster from X.

Proof: Let us first suppose that $A \in \pi_x$ and $A \subset B$. Then $A\delta x$ and therefore, by Proposition 2.2.1.1, $B\delta x$. Hence $B \in \pi_x$.

Let us suppose that $A, B \in \pi_x$. Then $A\delta x$ and $B\delta x$ so that, by (LO), $A\delta B$.

Now let us suppose that $A \cup B \in \pi_x$. Then $(A \cup B)\delta x$ and, by (SP_4) , this means that either $A\delta x$ or $B\delta x$, that is, either $A \in \pi_x$ or $B \in \pi_x$ holds.

Finally let us suppose that $A\delta C$ for every $C \in \pi_x$. Since, by (SP_3) , $\{x\} \in \pi_x$, it follows that $A\delta x$. Therefore $A \in \pi_x$.

Proposition 2.3.3.2 Let us suppose that π is a cluster from an LO-proximity space (X, δ) . Then

- (a) if $A \in \pi$ and $a\delta B$ for every $a \in A$, then $B \in \pi$;
- (b) if $\{x\} \in \pi$, then $\pi = \pi_x$;
- (c) if $A \subset X$, then either $A \in \pi$ or $X A \in \pi$;
- (d) if A is a subset of X which meets every member of π , then $A \in \pi$.

Proof: (a) Let $C \in \pi$. Then, $C\delta A$ by the definition of a cluster. Since $a\delta B$ for every $a \in A$, it follows that $C\delta B$. But then, $B \in \pi$ by the definition of a cluster.

(b) If $A \in \pi$, then $A\delta x$ by the definition of a cluster.

(c) Since $A \cup (X - A) = X \in \pi$, then, either $A \in \pi$ or $X - A \in \pi$ holds by the definition of a cluster.

(d) If A is a subset of X which meets every member of π , then $A\delta B$ for any member B of π . But then, $A \in \pi$ by the definition of a cluster.

Definition 2.3.3.1 A subset X of a topological space Y is regularly dense in Y if, given U open in Y and p being a point in U, there exists a subset E of X with $p \in \overline{E} \subset U$, the closure being taken in Y.

Proposition 2.3.3.3 If X is regularly dense in Y, then X is dense in Y. If Y is regular and X is dense in Y, then X is regularly dense in Y.

Proof: Y is open in Y, hence, for any point $p \in Y$, there exists a subset E of X such that $p \in \overline{E} \subset \overline{X} \subset Y$. Since this is true for any $p \in Y$, it follows that $Y \subset \overline{X} \subset Y$.

For Y regular, $y \in Y$ and U an open set of Y containing y there follows the existence of an open set V of Y containing y such that $\overline{V} \subset U$. Now $E = V \cap X$ is a subset of X and $\overline{E} = \overline{V \cap X} = \overline{V} \subset U$, with the second equality following from the density of X in Y. Thus, $y \in \overline{E} \subset U$.

Theorem 2.3.3.1 Given a set X and some binary relation δ on the power set of X, the following conditions are equivalent:

(I) there exists a T_1 -topological space Y and a mapping $f: X \to Y$ such that f(X) is regularly dense in Y and

(1) $A\delta B \text{ in } X \text{ if and only if } \overline{f(A)} \cap \overline{f(B)} \neq \emptyset \text{ in } Y;$

(II) δ is an LO-proximity satisfying the additional condition:

(a) if $A\delta B$, then there exists a cluster π to which both A and B belong.

Proof: Let us suppose that (I) holds. That (X, δ) is semi-proximity space is a trivial consequence of the properties of the closure. To prove the condition

(LO), let us suppose that $A\delta B$ and $b\delta C$ for all $b \in B$. Then $f(A) \cap f(B) \neq \emptyset$ and $\overline{f(b)} \cap \overline{f(C)} \neq \emptyset$ for all $b \in B$, which, since Y is T_1 , implies that $\underline{f(b)} \in \overline{f(C)}$ for all $b \in B$. Thus $f(B) \subset \overline{f(C)}$ or $\overline{f(B)} \subset \overline{f(C)}$ so that $\overline{f(A)} \cap \overline{f(C)} \neq \emptyset$ showing that $A\delta C$. Let us prove that δ satisfies the condition (a). Since $\overline{f(A)} \cap \overline{f(B)} \neq \emptyset$, let $c \in \overline{f(A)} \cap \overline{f(B)}$. Let us define π to be the class of all subsets S of X such that $c \in \overline{f(S)}$. Clearly A and Bare in π . It is obvious that π is a clan. Let us suppose that $\overline{f(D)} \cap \overline{f(C)} \neq \emptyset$ for every $C \in \pi$ but that $D \notin \pi$, i.e. $c \notin \overline{f(D)}$. Therefore $c \in Y - \overline{f(D)}$ and, since f(X) is regularly dense in Y, there exists a subset E of X such that $c \in \overline{f(E)} \subset Y - \overline{f(D)}$. In other words, there exists an E in π such that $\overline{f(D)} \cap \overline{f(C)} = \emptyset$. This contradicts the hypothesis that $\overline{f(D)} \cap \overline{f(C)} \neq \emptyset$ for every $C \in \pi$. Thus (II) is satisfied.

For the converse let us suppose that (II) holds. For any subset A of X, let \mathcal{A} be the set of all the clusters π_a determined by the points a in A. Let $\overline{\mathcal{A}}$ be the set of all the clusters to which A belongs. It is evident that $A \in \pi_a$ for each $a \in A$ and so $\mathcal{A} \subset \overline{\mathcal{A}}$. We will denote $\overline{\mathcal{X}}$, the set of all the clusters from X, by Y.

A subset A of X **absorbs** a subset β of Y if and only if A belongs to every cluster in β , that is, if and only if $\overline{\mathcal{A}}$ contains β . For any subset β of Y we define the closure $cl(\beta)$ of β by

(2)
$$\pi \in cl(\beta) \quad \text{if and only if every subset } E \text{ of } X$$
which absorbs β is in π .

We will next show that

(3)
$$cl(\mathcal{A}) = \overline{\mathcal{A}}$$

Let us suppose that $\pi \in cl(\mathcal{A})$. Since A absorbs $\mathcal{A}, A \in \pi$, so that $\pi \in \overline{\mathcal{A}}$. On the other hand, if $\pi \in \overline{\mathcal{A}}$, then $A \in \pi$. Now let P be in every π_a in \mathcal{A} , i.e. $P\delta a$ for every $a \in A$ and hence $A \subset P^{\delta} = \{x : x\delta P\}$. Thus, by Proposition 2.3.3.2 (a), $P \in \pi$ so that $\pi \in cl(\mathcal{A})$.

We will now show that the Kuratowski closure axioms are satisfied by the closure defined by (2).

 (K_1) Let us suppose $\pi \in cl(\emptyset)$. Since it is obviously true that every subset of X absorbs \emptyset , then every subset of X is in π . Thus, $\emptyset \delta X$, which contradicts (SP_1) .

 (K_2) It is evident that $\beta \subset cl(\beta)$. Indeed, if *E* absorbs β , then $E \in \pi$ for every $\pi \in \beta$.

 (K_3) Let us suppose that $\pi \in cl(\beta \cup \gamma)$, that B absorbs β and that C absorbs γ . Then $B \cup C$ absorbs $\beta \cup \gamma$ so that $B \cup C \in \pi$. But, by

the definition of the cluster, this means that either $B \in \pi$ or $C \in \pi$, that is $\pi \in cl(\beta)$ or $\pi \in cl(\gamma)$. Thus $\pi \in cl(\beta) \cup cl(\gamma)$ and it follows that $cl(\beta \cup \gamma) \subset cl(\beta) \cup cl(\gamma)$. On the other hand, $\pi \in cl(\beta) \cup cl(\gamma)$ implies that either $\pi \in cl(\beta)$ or $\pi \in cl(\gamma)$. Now, if *E* absorbs $\beta \cup \gamma$, then *E* absorbs β and it also absorbs γ . Hence, $E \in \pi$ showing that $\pi \in cl(\beta \cup \gamma)$.

 (K_4) Let us suppose that $\pi \in cl(cl(\beta))$ and that E absorbs β . By (2), E absorbing β implies that E absorbs $cl(\beta)$. Hence $E \in \pi$ showing that $\pi \in cl(\beta)$.

To show that the topology is T_1 , let us suppose $\pi' \in cl(\pi)$, where π and π' are clusters from X. This means that every set in π is also in π' . Thus, $\pi \subset \pi'$ and therefore $\pi = \pi'$. Hence, $cl(\pi) = \pi$ for every point π in the space Y.

The correspondence which, to each point $x \in X$, assigns the cluster π_x determined by it, is a well-defined transformation mapping X into Y which we will denote by f. Let us note that $f(A) = \mathcal{A}$ for every subset A of X, so in order to show that (2) holds, we must show that, using (3),

(4) $A\delta B$ in X if and only if $\overline{\mathcal{A}} \cap \overline{\mathcal{B}} \neq \emptyset$ in Y.

If $A\delta B$, then, by (a), there exists a cluster π to which both A and B belong. Thus, by the definition of $\overline{\mathcal{A}}$, it follows that $\pi \in \overline{\mathcal{A}} \cap \overline{\mathcal{B}}$. On the other hand, if $\pi \in \overline{\mathcal{A}} \cap \overline{\mathcal{B}}$, then A and B are in π so that $A\delta B$.

To show that $f(X) = \mathcal{X}$ is regularly dense in Y let us suppose that α is an open subset of Y and that $\pi \in \alpha$. We thus have $\pi \notin Y - \alpha = cl(Y - \alpha)$. This means, by (2), that there exists some subset E of X such that E is in every cluster of $Y - \alpha$ but that E is not in π . Hence, there is a $C \in \pi$ such that $E\overline{\delta}C$.

Since \overline{C} is the set of all clusters to which C belongs, it follows that $\pi \in \overline{C}$. And since E belongs to every cluster in $Y - \alpha$ and $E\overline{\delta}C$, then C cannot belong to any cluster in $Y - \alpha$ by the definition of the cluster. Hence \overline{C} is contained in α and we have shown that \mathcal{X} is regularly dense in Y. The proof is now complete.

Theorem 2.3.3.2 Given a set X and some binary relation δ on the power set of X, the following conditions are equivalent:

(I) there exists a T_1 topological space (Y, τ) in which X can be topologically embedded as a regularly dense subset so that

 $A\delta B$ in X if and only if $\overline{A}^{\tau} \cap \overline{B}^{\tau} \neq \emptyset$ in Y;

(II) δ is a separated LO-proximity satisfying the condition (a) of Theorem 2.3.3.1.
Proof: The proof is similar to the one of Theorem 2.3.3.1. To see that LOproximity δ is separated, let us note that $\overline{x} \cap \overline{y} \neq \emptyset$ implies that $x \cap y \neq \emptyset$, or x = y.

To show that our embedding is topological, let us note first that the correspondence between X and \mathcal{X} induced by the identification of x with the cluster π_x determined by it is one-to-one. To see that the mapping is δ -homeomorphism, we must show that, if A is a subset of X, $x \in A^{\delta}$ if and only if $\pi_x \in cl_Y(\mathcal{A})$, where $cl_Y(\mathcal{A})$ is the closure of \mathcal{A} relative to the space Y.

So, let us suppose $x \in A^{\delta}$ and that P absorbs \mathcal{A} . Then P is a member of every π_a in \mathcal{A} and it follows that $a\delta P$ for every $a \in A$. Thus, $A \subset P^{\delta}$ and since $A \in \pi_x$ it follows, from Proposition 2.3.3.2 (a), that $P \in \pi_x$. Thus, $\pi_x \in cl_Y(\mathcal{A})$.

On the other hand, let us suppose that $\pi_x \in cl_Y(\mathcal{A})$. Then since A absorbs \mathcal{A} , it follows that $A \in \pi_x$, i.e. $A\delta x$ and hence $x \in A^{\delta}$. This completes the proof. \clubsuit

Definition 2.3.3.2 A non-empty collection σ of an LO-proximity space (X, δ) is called a **bunch** if the following conditions are satisfied:

- (B₁) if $A, B \in \sigma$, then $A\delta B$;
- (B₂) if $A \cup B \in \sigma$, then $A \in \sigma$ or $B \in \sigma$;
- (B₃) if $A \in \sigma$ and $a\delta B$ for every $a \in A$, then $B \in \sigma$.

Proposition 2.3.3.4 A non-empty collection σ of an LO-proximity space (X, δ) is a bunch if and only if the following conditions are satisfied:

- (B'_1) if $A, B \in \sigma$, then $A\delta B$;
- $(B'_2) A \cup B \in \sigma$ if and only if $A \in \sigma$ or $B \in \sigma$;
- (B'_3) $A \in \sigma$ if and only if $\overline{A} \in \sigma$.

Proof: Let us note that any bunch is closed under the operation of superset. The collection which satisfies the condition $(B'_1) - (B'_3)$ also has this property. Therefore, to prove that σ is a bunch, we must show only that the conditions (B_3) and (B'_3) are equivalent.

Let us suppose that (B_3) is true and let $\overline{A} \in \sigma$. Since $a\delta A$ for every $a \in \overline{A}$, by (B_3) it follows that $A \in \sigma$. The converse obviously holds.

Let us suppose now that the condition (B'_3) is true. Let us suppose that $A \in \sigma$ and $a\delta B$ for every $a \in A$. Since $a\delta B$ for every $a \in A$, it follows that $a \in \overline{B}$ for every $a \in A$. Therefore the inclusion $A \subset \overline{B}$ holds which implies that $\overline{B} \in \sigma$. But then $B \in \sigma$ according to the condition (B'_3) .

Proposition 2.3.3.5 In every LO-proximity space any cluster is a bunch, but the converse need not be true.

Proof: Let σ be a cluster in an LO-proximity space (X, δ) . To prove that σ is a bunch, it is sufficient to prove that the condition (B_3) is satisfied. Let C be any member of a cluster σ . Then $C\delta A$ by the definition of the cluster. Since $a\delta B$ for each $a \in A$, by (LO), $C\delta B$ is true. But then $B \in \sigma$ by the definition of the cluster.

If (X, δ) is a non discrete proximity space and $x \in X$, then $\{A \subset X : A \neq \{x\} \text{ and } A\delta x\}$ is a bunch which is not a cluster.

If σ is a bunch and $\{x\} \in \sigma$, then it is easy to show that $\sigma = \sigma_x$.

Proposition 2.3.3.6 If \mathcal{L} is an ultrafilter in an LO-proximity space (X, δ) , then $b(\mathcal{L}) = \{A \subset X : \overline{A} \in \mathcal{L}\}$ is a bunch called the **bunch generated by** ultrafilter \mathcal{L} .

Proof: (B'_1) : If $A, B \in b(\mathcal{L})$, then $\overline{A}, \overline{B} \in \mathcal{L}$, so that $\overline{A} \cap \overline{B} \neq \emptyset$. Thus $\overline{A}\delta\overline{B}$, and so $A\delta B$.

 $\begin{array}{l} (B_2')\colon A\cup B\in b(\mathcal{L})\Leftrightarrow \overline{A\cup B}\in \mathcal{L}\Leftrightarrow \overline{A}\cup \overline{B}\in \mathcal{L}\Leftrightarrow \overline{A}\in \mathcal{L} \text{ or } \overline{B}\in \mathcal{L}\Leftrightarrow \mathcal{A}\in b(\mathcal{L}) \text{ or } B\in b(\mathcal{L}).\\ (B_3')\colon A\in b(\mathcal{L})\Leftrightarrow \overline{A}\in \mathcal{L}\Leftrightarrow \overline{(\overline{A})}\in \mathcal{L}\Leftrightarrow \overline{A}\in b(\mathcal{L}). \end{array}$

Proposition 2.3.3.7 A separated LO-proximity space X is compact if and only if every bunch $b(\mathcal{L})$ generated by a closed ultrafilter \mathcal{L} in X is a point cluster.

Proof: Let \mathcal{L} be a closed ultrafilter in X. Then $b(\mathcal{L}) = \sigma_{x_0}$ for some $x_0 \in X$ implies $\{x_0\} \in b(\mathcal{L})$. This shows that $x_0 \in L$ for every $L \in \mathcal{L}$ (since each L is closed) and so $\{x_0\} \in \mathcal{L}$, since \mathcal{L} is maximal. Conversely, if X is compact and \mathcal{L} is a closed ultrafilter, then \mathcal{L} has a cluster point x. Since \mathcal{L} is maximal, $\{x\} \in \mathcal{L} \subset b(\mathcal{L})$ and so $b(\mathcal{L}) = \sigma_x$.

In [238] it is shown that a nonempty family π of the subsets of proximity space (X, δ) is a cluster if and only if there exists an ultrafilter \mathcal{L} in X such that $\pi = \pi(\mathcal{L}) = \{A \subset X : A\delta L \text{ for every } L \in \mathcal{L}\}$. $\pi(\mathcal{L})$ is called **the cluster generated by** \mathcal{L} .

Proposition 2.3.3.8 Let \mathcal{I} be a ring of subsets of X, i.e. let \mathcal{I} be closed under finite unions and finite intersections. Let us suppose \mathcal{P} is a subset of \mathcal{I} such that

(a) $\emptyset \notin \mathcal{P}$,

(b) for $A, B \in \mathcal{I}, A \cup B \in \mathcal{P} \Leftrightarrow A \in \mathcal{P}$ or $B \in \mathcal{P}$,

(c) $A \in \mathcal{P}$ and $A \subset B \in \mathcal{I}$ implies $B \in \mathcal{P}$.

Then for given $A_0 \in \mathcal{P}$, there exists a prime \mathcal{I} -filter \mathcal{L} such that $A_0 \in \mathcal{L} \subset \mathcal{P}$. (Let us recall that \mathcal{L} is a prime \mathcal{I} -filter means that \mathcal{L} is a filter of subsets of \mathcal{I} satisfying the additional condition: for $A, B \in \mathcal{I}, A \cup B \in \mathcal{L}$ implies that $A \in \mathcal{L}$ or $B \in \mathcal{L}$). If $\mathcal{I} = P(X)$, then \mathcal{L} is an ultrafilter.

Proof: By Zorn's lemma, there exists a maximal collection $\mathcal{L} \subset \mathcal{P}$ satisfying (a) $A_0 \in \mathcal{L}$ and (b) $A_i \in \mathcal{L}, 1 \leq i \leq n$, implies $\cap_i A_i \in \mathcal{P}$. Clearly, $\emptyset \notin \mathcal{L}$ and $\mathcal{L} \neq \emptyset$. If $A, B \in \mathcal{L}$, then $A \cap B \in \mathcal{P}$, and since \mathcal{L} is maximal, $A \cap B \in \mathcal{L}$. Similarly, if $A \in \mathcal{L}$ and $A \subset B \in \mathcal{I}$, then $B \in \mathcal{L}$. Thus \mathcal{L} is an \mathcal{I} -filter. To show that \mathcal{L} is prime, let us suppose that $A, B \in \mathcal{I} - \mathcal{L}$. Then there exist $A_1, B_1 \in \mathcal{L}$ such that $A \cap A_1, B \cap B_1 \notin \mathcal{P}$. Setting $E = A_1 \cap B_1 \in \mathcal{L}$ we find that $(A \cup B) \cap E \notin \mathcal{P}$ (by (b)), i.e. $A \cup B \notin \mathcal{L}$. The last part is obvious.

Proposition 2.3.3.9 In a proximity space (X, δ) , a nonempty family π of subsets of X is a filter if and only if it is a maximal bunch.

Proof: Let us suppose that π is a cluster in X contained in a bunch σ . If $A \in \sigma$ and $B \in \pi$, then $A, B \in \sigma$, and so $A\delta B$. By the definition of the cluster, $A \in \pi$ and therefore $\pi = \sigma$.

Conversely, let us suppose that σ is a maximal bunch. By Proposition 2.3.3.8 there exists an ultrafilter $\mathcal{L} \subset \sigma$. Clearly, $\sigma \subset \sigma(\mathcal{L}) = \{A \subset X : A\delta L \text{ for every } L \in \mathcal{L}\}$ which is a cluster. Since σ is maximal, $\sigma = \sigma(\mathcal{L})$ is a cluster.

Proposition 2.3.3.10 In a proximity space (X, δ) every bunch is contained in a unique cluster.

Proof: If σ is a bunch in X, then by Zorn's lemma, σ is contained in a maximal bunch which is a cluster by Proposition 2.3.3.9. To show uniqueness, let us suppose, on the contrary, that σ is contained in two different clusters: π_1 and π_2 . Then there exist sets $A_i \in \pi_i$, i = 1, 2, such that $A_1 \overline{\delta} A_2$. According to Proposition 1.1.1.3 there exist $E_i \in P(X)$ such that $A_i \overline{\delta} X - E_i$, $i = 1, 2, E_1 \overline{\delta} E_2$. Since $A_i \overline{\delta} X - E_i$ and $A_i \in \pi_i$ it follows that $X - E_i \notin \sigma$. This implies that $E_i \in \sigma$, i.e. $E_1 \delta E_2$, which is a contradiction.

Definition 2.3.3.3 *Let* (X, δ) *be an LO-space and let* Σ_X *be the family of all bunches in* X*. A set* $A \in P(X)$ *is said to* **absorb** $A \subset \Sigma_X$ *if* $A \in \sigma$ *for every* $\sigma \in A$ *.*

Lemma 2.3.3.1 Let (X, δ) be an LO-proximity space and let Σ_X be the set of all bunches in X. The operator cl defined on $P(\Sigma), \Sigma \subset \Sigma_X$, by

$$cl(\mathcal{A}) = \{ \sigma \in \Sigma : A \text{ absorb } \mathcal{A} \text{ implies } A \in \sigma \}$$

for each $\mathcal{A} \in P(\Sigma)$, is a Kuratowski closure operator on Σ . The resulting topology on Σ is called the **absorbtion** or **A-topology**.

Proof: (K_1) : It is obvious $cl(\emptyset) = \emptyset$. (K_2) : Clearly, $\mathcal{A} \subset cl \mathcal{A}$ for every $\mathcal{A} \in P(\Sigma)$. (K_3) : Let us suppose that $\sigma \in cl (\mathcal{A} \cup \mathcal{B})$, that \mathcal{A} absorbs \mathcal{A} and that \mathcal{B} absorbs \mathcal{B} . Then $\mathcal{A} \cup \mathcal{B}$ absorbs $\mathcal{A} \cup \mathcal{B}$ and so $\mathcal{A} \cup \mathcal{B} \in \sigma$. By (\mathcal{B}'_2) $\mathcal{A} \in \sigma$ or $\mathcal{B} \in \sigma$, i.e. $\sigma \in cl\mathcal{A}$ or $\sigma \in cl\mathcal{B}$. Thus $cl(\mathcal{A} \cup \mathcal{B}) \subset cl\mathcal{A} \cup cl\mathcal{B}$. To show the converse, let us suppose that $\sigma \in cl\mathcal{A} \cup cl\mathcal{B}$. If C absorbs $\mathcal{A} \cup \mathcal{B}$, then $C \in \sigma$, i.e. $\sigma \in cl(\mathcal{A} \cup \mathcal{B})$. (K_4) : Let us suppose that $\sigma \in cl(cl\mathcal{A})$ and \mathcal{A} absorbs \mathcal{A} . Then \mathcal{A} absorbs $cl\mathcal{A}$, and so $\mathcal{A} \in \sigma$, i.e. $\sigma \in cl\mathcal{A}$. Thus $cl(cl\mathcal{A}) \subset cl\mathcal{A}$ and, using (K_2) , the equality follows.

Lemma 2.3.3.2 Let (X, δ) be an LO-proximity space and let $\Sigma \subset \Sigma_X$. The A-topology on Σ is T_1 if and only if $\sigma_1, \sigma_2 \in \Sigma$, $\sigma_1 \neq \sigma_2$ implies $\sigma_1 \not\subset \sigma_2$ and $\sigma_2 \not\subset \sigma_1$.

Proof: If Σ is a T_1 -space, $\sigma_1, \sigma_2 \in \Sigma, \sigma_1 \neq \sigma_2$, then $\sigma_1 \notin cl \sigma_2$, and so there exists an $A \subset X$ such that $A \in (\sigma_2 - \sigma_1)$. The converse follows from the fact that under the hypothesis $cl \sigma = \{\sigma\}$ for each $\sigma \in \Sigma$.

Theorem 2.3.3.3 Let (X, δ) be an LO-proximity space, and let $\Phi : X \to \Sigma_X$ be the mapping defined by $\Phi(x) = \sigma_x$, where σ_x is a point cluster generated by x. Then Φ is continuous and closed. $\Phi(X)$ is dense in Σ_X . If δ is a separated LO-proximity, then X is homeomorphic to $\Phi(X)$.

Proof: That Φ is continuous and closed follows from the fact that $x\delta A$ if and only if $A \in \sigma_x$, i.e. $x \in \overline{A}$ if and only if $\sigma_x \in cl \Phi(A)$. $cl \Phi(X) = \{\sigma \in \Sigma_X : X \in \sigma\} = \Sigma_X$. Finally, if δ is a separated LO-proximity, then $x \neq y$ implies $\sigma_x \neq \sigma_y$, and so Φ is one-to-one.

Corollary 2.3.3.1 If (a) $\Phi(X) \subset \Sigma \subset \Sigma_X$, (b) $A\delta B$ implies there exists a $\sigma \in \Sigma$ containing both A and B, (c) the A-topology on Σ is T_1 , then Φ is a δ -isomorphism between X and $\Phi(X)$, the latter having the subspace LO-proximity induced by δ_0 on Σ .

Proof: $A\delta B$ if and only if there exists an $\sigma \in \Sigma$ such that $A, B \in \sigma$ if and only if there exists an $\sigma \in cl \Phi(A) \cap cl \Phi(B)$ if and only if $\Phi(A)\delta_0\Phi(B)$.

Theorem 2.3.3.4 Let (X, δ_X) , (Y, δ_Y) be LO-proximity spaces and let f: $(X, \delta_X) \to (Y, \delta_Y)$ be δ -continuous. Then there exists an associated mapping $f_{\Sigma} : \Sigma_X \to \Sigma_Y$ defined by $f_{\Sigma}(\sigma) = \{E \subset Y : f^{-1}(\overline{E}) \in \sigma\}$ which is continuous with respect to the A-topologies on Σ_X and Σ_Y and for which $f_{\Sigma}(\sigma_x) = \sigma_{f(x)}$ holds for each $x \in X$. If δ_X and δ_Y are separated, then f_{Σ} may be considered a continuous extension of f as follows:

$$\begin{array}{cccc} X & \stackrel{f}{\longrightarrow} & Y \\ \Phi_X & & & \downarrow \Phi_Y \\ \Sigma_X & \stackrel{f_\Sigma}{\longrightarrow} & \Sigma_Y \end{array}$$

Proof: We will first show that if $\sigma \in \Sigma_X$, then $f_{\Sigma}(\sigma) \in \Sigma_Y$ by verifying $(B'_1) - (B'_3)$. (B'_1) : If $A, B \in f_{\Sigma}(\sigma)$, then $f^{-1}(A), f^{-1}(B) \in \sigma$ which implies that $f^{-1}(\overline{A})\delta_X f^{-1}(\overline{B})$. Since f is a δ -continuous mapping, $\overline{A}\delta_Y\overline{B}$ and by Proposition 2.3.2.2 $A\delta_YB$. $(B'_2) A \cup B \in f_{\Sigma}(\sigma) \Leftrightarrow f^{-1}(\overline{A}\cup\overline{B}) \in \sigma \Leftrightarrow f^{-1}(\overline{A}) \cup f^{-1}(\overline{B}) \in \sigma \Leftrightarrow f^{-1}(\overline{A}) \in \sigma$ or $f^{-1}(\overline{B}) \in \sigma \Leftrightarrow A \in f_{\Sigma}(\sigma)$ or $B \in f_{\Sigma}(\sigma)$. (B'_3) : $\overline{A} \in f_{\Sigma}(\sigma) \Leftrightarrow f^{-1}(\overline{A}) \in \sigma \Leftrightarrow A \in f_{\Sigma}(\sigma)$.

Next we will show that the mapping f_{Σ} is continuous with respect to the A topologies on Σ_X and Σ_Y , i.e. if a $\sigma \in cl \mathcal{A}$, where $\mathcal{A} \subset \Sigma_X$, then $f_{\Sigma}(\sigma) \in cl(f_{\Sigma}(\mathcal{A}))$. If this is not true, then there exists a set $E \subset Y$ which absorbs $f_{\Sigma}(\mathcal{A})$ but $E \notin f_{\Sigma}(\sigma)$. Then $f^{-1}(\overline{E})$ absorbs \mathcal{A} , but $f^{-1}(\overline{E}) \notin \sigma$, i.e. $\sigma \notin cl \mathcal{A}$, which is a contradiction. Finally, for $x \in X$, $f_{\Sigma}(\sigma_x) = \{A \subset Y : f^{-1}(\overline{A}) \in \sigma_x\} = \{A \subset Y : x \delta_X f^{-1}(\overline{A})\} = \{A \subset Y : x \in f^{-1}(\overline{A})\} = \{A \subset Y : f(x) \in \overline{A}\} = \sigma_{f(x)}$.

Identifying X with $\Phi_X(X)$ and Y with $\Phi_Y(Y)$, we have the following fundamental extension theorem.

Theorem 2.3.3.5 Let (X, δ_X) and (Y, δ_Y) be separated LO-proximity spaces. Then every δ -continuous mapping $f : X \to Y$ has a continuous extension $f_{\Sigma} : \Sigma_X \to \Sigma_Y$.

Theorem 2.3.3.6 Let X be a dense, separated subspace of an LO-proximity space (T, δ) (δ need not be separated). Then the mapping $\Psi_T : T \to \Sigma_X$ defined by $\Psi_T(t) = \sigma^t = \{E \subset X : t\delta E\}$ is continuous. If $t \in X$, then $\sigma^t = \sigma_t$, the point cluster being generated by t, i.e. $\Psi_T | X = \Phi_X$. If further T is T₃-space, then Ψ_T is a homeomorphism of T into Σ_X .

Proof: We will first verify that $\sigma^t \in \Sigma_X$. Since $\overline{X} = T$, it follows that $X \in \sigma^t$ and so $\sigma^t \neq \emptyset$. (B'_1) : If $A, B \in \sigma^t$, then $t\delta A, t\delta B$ and hence $A\delta B$.

 $\begin{array}{l} (B_2'): \ A \cup B \in \sigma^t \Leftrightarrow t\delta A \cup B \Leftrightarrow t\delta A \text{ or } t\delta B \Leftrightarrow A \in \sigma^t \text{ or } B \in \sigma^t. \\ A \in \sigma^t \Leftrightarrow t\delta A \Leftrightarrow t\delta \overline{A} \Leftrightarrow \overline{A} \in \sigma^t. \end{array}$

Further, we will show that Ψ_T is continuous by proving that $x \in \overline{E}$, $E \subset T$, implies $\Psi_T(x) \in cl(\Psi_T(E))$. If, on the contrary, $\Psi_T(x) \notin cl(\Psi_T(E))$, then there exists an $A \subset X$ which absorbs $\Psi_T(E)$. But $A \notin \Psi_T(x) = \sigma^x$. This implies that $\overline{E} \subset \overline{A}$ and $x \notin \overline{A}$, which is a contradiction.

It is obvious $\Psi_T|X = \Phi_X$. Finally, let us suppose that T is a T_3 -space. If $t_1, t_2 \in T$, $t_1 \neq t_2$, then t_1 and t_2 have disjoint neighborhoods N_1 and N_2 respectively. Clearly, $N_1 \cap X \in \sigma^{t_1} - \sigma^{t_2}$ and $N_2 \cap X \in \sigma^{t_2} - \sigma^{t_1}$, i.e. $\sigma^{t_1} \neq \sigma^{t_2}$, showing that Ψ_T is an one-to-one mapping. To prove that Ψ_T is a homeomorphism, it is sufficient to show that Ψ_T is closed. Let us suppose that $E \subset T$ and $x \notin \overline{E}$. Since T is a T_3 -space, there exist disjoint neighborhoods N_x and N_E of x and E respectively. Since $\overline{X} = T$, $N_E \cap X$ absorbs $\Psi_T(E)$ but does not belong to σ^x . Hence $\Psi_T(x) = \sigma^x \notin cl(\Psi_T(E))$, i.e. Ψ_T is closed.

Theorem 2.3.3.7 Let (X, δ) be a separated proximity space and let X^* be its Smirnoff compactification (i.e. the space of all the clusters in X with the A-topology). The mapping $\Theta = \Theta_X : \Sigma_X \to X^*$ given by $\Theta(\pi) = \pi_{\Theta}$, which is the unique cluster containing π , is continuous. Moreover, $\Theta(\pi_X) = \pi_X$.

Proof: To show that Θ is continuous, we must prove that if $\pi \in cl\mathcal{A}$, $\mathcal{A} \subset \Sigma_X$, then $\Theta(\pi) \in cl(\Theta(\mathcal{A}))$. If, on the contrary, $\Theta(\pi) \notin cl(\Theta(\mathcal{A}))$, then, since X^* is T_3 , there exist disjoint neighborhoods U_1 and U_2 of $\Theta(\pi)$ and $\Theta(\mathcal{A})$ respectively. Clearly, $U_2 \cap X$ absorbs \mathcal{A} but does not belong to π , i.e. $\pi \notin cl(\mathcal{A})$, which is a contradiction. It is clear that $\Theta(\pi_x) = \pi_x$ is clear from the fact that π_x is a cluster.

Definition 2.3.3.4 If σ is a bunch in an LO-proximity space (X, δ) , then σ converges to x if the neighborhood filter \mathcal{N}_x of x is a subclass of σ .

Theorem 2.3.3.8 Let (X, δ_0) be a T_3 LO-proximity space and let $\Sigma \subset \Sigma_X$ be such that each $\sigma \in \Sigma$ converges to a (unique) $x_{\sigma} \in X$. Then the mapping $\Theta = \Theta_X : \Sigma \to X$ given by $\Theta(\sigma) = x_{\sigma}$ is continuous.

Proof: Similar to the one of Theorem 2.3.3.7.

Theorem 2.3.3.9 Let (X, δ) be a separated LO-proximity space and let X^* be the family of all maximal bunches in X with the A-topology. Then X^* is a compact T_1 -space containing a dense homeomorphic copy of X.

Proof: By Theorem 2.3.3.3 and Lemma 2.3.3.2 we need only prove that X^* is compact. Since $\{A'_{\alpha} : A_{\alpha} \text{ is closed in } X\}$, where $A'_{\alpha} = \{\sigma \in X^* : A_{\alpha} \in \sigma\}$, is a base for closed sets in X^* , it is sufficient to show that if $\{A'_{\alpha} : \alpha \in \Lambda\}$ has the finite intersection property, then $\cap \{A'_{\alpha} : \alpha \in \Lambda\} \neq \emptyset$. Clearly, if $\{A'_{\alpha} : \alpha \in \Lambda\}$ has the finite intersection property, then $\mathcal{I} = \{A_{\alpha} : \alpha \in \Lambda\}$ is a family of closed subsets of X with the property: every finite subfamily of \mathcal{I} is a subclass of some $\sigma \in X^*$. Let Δ be the family of all collections \mathcal{J} of closed subsets of X such that $(a) \ \mathcal{I} \subset \mathcal{J}$ and $(b) \ \{G_i : 1 \leq i \leq n\} \subset \mathcal{J}$ implies there exists a $\sigma \in X^*$ such that $G_i \in \sigma, 1 \leq i \leq n$. By Zorn's lemma, Δ has a maximal element \mathcal{M} . It is easily verified that $b(\mathcal{M}) = \{E \subset X : \overline{E} \in \mathcal{M}\}$ is a bunch in X. By Zorn's lemma $b(\mathcal{M})$ is contained in a $\sigma_0 \in X^*$. Clearly, $\sigma_0 \in \cap \{A'_{\alpha} : \alpha \in \Lambda\}$, and so X^* is compact.

Theorem 2.3.3.10 Let (X, δ) be a separated LO-proximity space such that if $A\delta B$, then there exists a bunch in X containing both A and B. Then there exists a compact T_1 -space X^* containing a dense homeomorphic copy $\Phi(X)$ of X and such that $A\delta B$ if and only if $cl \Phi(A) \cap cl \Phi(B) \neq \emptyset$ in X^* . Furthermore, every δ -continuous mapping f from one separated LOproximity space to another (Y, δ_Y) has a continuous extension f_{Σ} from X^* to Σ_Y .

Proof: Let X^* be the family of all maximal bunches in X with the A-topology. Then by Theorem 2.3.3.9 X^* is a compact, T_1 -space containing a dense homeomorphic copy of X. That $A\delta B$ if and only if $cl \Phi(A) \cap cl \Phi(B) \neq \emptyset$ follows as in Corollary 2.3.3.1 by noting that every bunch is contained in a maximal bunch. The last part has been established in Theorem 2.3.3.4.

Theorem 2.3.3.10 generalized Smirnoff's theorem, for, in every proximity space (X, δ) , $A\delta B$ implies there exists a cluster π in X which contains both A and B. Also, Proposition 2.3.3.9 shows that X^* is the Smirnoff compactification of X. Finally, let us suppose that δ_X and δ_Y are separated proximities and that $f : (X, \delta_X) \to (Y, \delta_Y)$ is δ -continuous. By Theorem 2.3.3.7 the mapping $\Theta_Y : \Sigma_Y \to Y^*$, which assigns to each bunch in Ythe unique cluster containing it, is continuous. Hence f has an extension $\overline{f}: X^* \to Y^*$ given by $\overline{f} = \Theta_Y \circ f_{\Sigma}$.

Theorem 2.3.3.11 Let X be a separated dense subspace of an LO-proximity space (T, δ_0) . Let (Y, δ) be a separated proximity space and let Y^* be its Smirnoff compactification. Then a continuous mapping $f : X \to Y$ has a continuous extension $\overline{f} : T \to Y^*$ if and only if f is δ -continuous. **Proof:** If f has an extension \overline{f} , then by Theorem 2.3.2.4, $\overline{f} : T \to Y^*$ is δ -continuous and so is its restriction $f = \overline{f}|X$. To prove the sufficiency, let us suppose that f is δ -continuous and let us consider the following diagram:

 $\overline{f} = \Theta_Y \circ f_\Sigma \circ \Psi_T$ is a continuous extension from T to Y^* .

If Y is a compact T_2 -space, then Y is homeomorphic to Y^* , and we may consider \overline{f} to be a mapping from T to Y. Thus we get a result of Taimanov (see [316]): A necessary and sufficient condition that a continuous function $f: X \to Y$, where X is dense in a T_1 -space T and Y is a compact T_2 -space, has a continuous extension $\overline{f}: T \to Y$ is that for every pair of disjoint closed sets F_1, F_2 in Y, $cl_T f^{-1}(F_1) \cap cl_T f^{-1}(F_2) = \emptyset$.

2.3.4 The notion and basic properties of S-proximity spaces

Definition 2.3.4.1 A semi-proximity relation δ defined on the power set of X is called an **S-proximity** if it satisfies the following condition:

(S) $x\delta B \neq \emptyset$ and $b\delta C$ for every $b \in B$ implies $x\delta C$.

An S-proximity δ is **separated** if it is a separated semi-proximity. The pair (X, δ) , where δ is a (separated) S-proximity, is called a **(separated)** S-proximity space.

Every S-proximity δ on X induces a T_1 -topology τ_{δ} in the following manner: $G \in \tau_{\delta}$ if and only if, for each $x \in G$, $x\overline{\delta}X - G$. Conversely, every T_1 -space (X, τ) has a compatible S-proximity δ^* defined by

 $A\delta^*B$ if and only if $(\overline{A} \cap B) \cup (A \cap \overline{B}) \neq \emptyset$.

The S-proximity δ^* is the largest compatible S-proximity on X, that is, if δ is any other compatible S-proximity on X, then $A\delta^*B$ implies $A\delta B$.

Every (separated) proximity is a (separated) LO-proximity which, in turn, is an S-proximity. If δ is an S-proximity, then, as in the proximity or LO-proximity, $A\delta B$, $A \subset C$, $B \subset D$ implies $C\delta D$. But $\overline{A}\delta \overline{B}$ does not imply $A\delta B$. For example, if X is the reals with the S-proximity δ^* , then $(0,1)\overline{\delta}^*(1,2)$ although $[0,1]\delta^*[1,2]$.

A function f from one S-proximity space (X, δ_X) to another (Y, δ_Y) is called S-proximally continuous if $A\delta_X B$ implies $f(A)\delta_Y f(B)$.

Proposition 2.3.4.1 Every S-proximally continuous function is continuous. The converse is not true in general, but it holds in the case that $\delta_X = \delta_s$.

Proof: The first part is obvious. If $X = Y = \mathbb{R}$, $\delta_X = \delta_w$ and $\delta_Y = \delta_s$, then the identity mapping is continuous, but not S-proximally continuous. Finally, let us suppose that f is continuous and $\delta_X = \delta_s$. If $A\delta_s B$ then $(\overline{A} \cap B) \cup (A \cap \overline{B}) \neq \emptyset$. This implies that $(f(\overline{A}) \cap f(B)) \cup (f(\underline{A}) \cap f(\overline{B})) \neq \emptyset$. Since f is continuous, $f(\overline{A}) \subset \overline{f(A)}$, $f(\overline{B}) \subset \overline{f(B)}$ and so $(\overline{f(A)} \cap f(B)) \cup (f(A) \cap \overline{f(B)}) \neq \emptyset$, showing that $f(A)\delta_Y f(B)$.

Proposition 2.3.4.2 Let (Y, δ_Y) be an S-proximity space and let $f : X \to Y$ be a one-to-one mapping. Then δ_X , defined by $A\delta_X B$ if $f(A)\delta_Y f(B)$, is the smallest S-proximity on X which makes f S-proximally continuous.

Proof: We need verify only (S) as the other properties of an S-proximity follow easily. Let us suppose that $x\delta_X B$ and $b\delta_X C$ for each $b \in B$. Then $f(x)\delta_Y f(B)$ and $f(b)\delta_Y f(C)$ for every $f(b) \in f(B)$. Since δ_Y is an Sproximity, $f(x)\delta_Y f(C)$ showing that $x\delta_X C$. If δ is any S-proximity on Xsuch that f is S-proximally continuous, then $A\delta B$ implies $f(A)\delta_Y f(B)$ and this, in turn, implies that $A\delta_X B$ showing that $\delta > \delta_X$.

Now we introduce the concept of a band which is analogous to that of a cluster or a bunch.

Definition 2.3.4.2 A non-empty family σ of subsets of an S-proximity space (X, δ) is a **band** if:

- (a) $A, B \in \sigma$ implies $\overline{A\delta B}$;
- (b) $A \cup B \in \sigma$ implies $A \in \sigma$ or $B \in \sigma$;
- (c) $A \in \sigma$ and $a\delta B$ for every $a \in A$ implies $B \in \sigma$.

In an S-proximity space, clearly, every cluster is a bunch and every bunch is a band. In an LO-proximity space every band is a bunch. However, in the S-proximity space (\mathbb{R}, δ_s) , $\sigma_1 = \{A \subset \mathbb{R} : \{1\}\delta_s A\}$ is a band but not a bunch; this can be seen from the fact that (0, 1) and (1, 2) are both in σ_1 , but $(0, 1)\overline{\delta}_s(1, 2)$.

The following results follow easily from the definitions.

Proposition 2.3.4.3 Let (X, δ) be an S-proximity space and let σ be a band over X. Then

- (a) $A \in \sigma$, $A \subset B$ implies $B \in \sigma$, and hence $X \in \sigma$;
- (b) $A \subset X$ implies $A \in \sigma$ or $X A \in \sigma$;
- (c) $A \in \sigma$ if and only if $A \in \sigma$;
- (d) for each $x \in X$, $\sigma_x = \{A \subset X : x \delta A\}$ is a band, called a point band;
- (e) if $\{x\} \in \sigma$, then $\sigma = \sigma_x$; consequently, if $\sigma_x \subset \sigma$, then $\sigma = \sigma_x$;
- (f) $x \neq y$ implies $\sigma_x \neq \sigma_y$;

(g) if \mathcal{L} is a closed ultrafilter in X, then $b(\mathcal{L}) = \{E \subset X : \overline{E} \in \mathcal{L}\}$ is a band, called a band generated by \mathcal{L} .

The proof of the following theorem is essentially the same as the one given in Proposition 2.3.3.7.

Theorem 2.3.4.1 An S-proximity space is compact if and only if every band $b(\mathcal{L})$, generated by a closed ultrafilter \mathcal{L} on X, is a point band.

Proposition 2.3.4.4 Let Y be a T_1 -space and let $f : X \to Y$ be a one-toone mapping. Let δ_X be the S-proximity as defined in Proposition 2.3.4.2 corresponding to $\delta_Y = \delta_s$ on Y. Then for each $y \in f(X)$, $\sigma^y = \{A \subset Y : y \in f(\overline{A})\} = \sigma_x$ holds, where $x = f^{-1}(y)$.

Proof: $A \in \sigma^y$ if and only if $y \in f(\overline{A})$ if and only if $x \in \overline{A}$ if and only if $x \delta A$ if and only if $A \in \sigma_x$.

2.3.5 Embedding of an S-proximity spaces. Extension of continuous function

In this subsection we will prove that every S-proximity space (X, δ) , satisfying condition (II) of Theorem 2.3.5.1 below, can be S-proximally embedded in a Hausdorff space Y with the S-proximity δ_s . This result generalizes the results of Smirnoff [294] and Lodato [201].

Theorem 2.3.5.1 Let X be a non-empty set and let δ be a binary relation on the power set of X. For each $A \subset X$ let $A^* = \{x \in X : x\delta A\}$. Then the following statements are equivalent:

(I) There exists a Hausdorff space Y and a one-to-one mapping $f: X \to Y$ such that

(a)
$$\overline{f(X)} = Y;$$

(b) $f(A^*) = \overline{f(A)} \cap f(X);$

(c) $A\delta B$ if and only if $f(A)\delta_s f(B)$ in Y.

(II) δ is a separated S-proximity on X satisfying the following condition: there exists a family Σ of bands over X such that

(a) $A\delta B$ implies that there exists a $\sigma \in \Sigma$ such that $A, B \in \sigma$ and $\sigma = \sigma_x$ for some $x \in A \cup B$;

(b) if $\sigma, \sigma^1 \in \Sigma$ and if either $A \in \sigma$ or $B \in \sigma^1$ for all subsets A, B of X such that $A \cup B = X$, then $\sigma = \sigma^1$.

Proof: $(I) \Rightarrow (II)$: Proposition 2.3.4.2 and condition (I)(c) show that δ is an S-proximity on X. Also f is S-proximally continuous and so X is a Hausdorff space. Let us note that $A^* = \overline{A}$. Let $\Sigma = \{\sigma^y : y \in f(X)\}$ which, by Proposition 2.3.4.4, is a family of bands. We will now show that the conditions (II)(a) and (II)(b) are satisfied. $A\delta B$ implies $(\overline{f(A)} \cap f(B)) \cup (f(A) \cap \overline{f(B)}) \neq \emptyset$. Without loss of generality, let us suppose $y \in f(A) \cap \overline{f(B)}$. Then $\sigma^y \in \Sigma$ and clearly $A \in \sigma^y$. From $(I)(b), y \in f(\overline{B})$ i.e. $B \in \sigma^y$. By Proposition 2.3.4.4, $\sigma^y = \sigma_x$ for some $x \in A$. To prove (II)(b), let us suppose that $\sigma^{y_1}, \sigma^{y_2} \in \Sigma, y_1 \neq y_2$. Since Y is Hausdorff, there exist disjoint neighborhoods V_1, V_2 of y_1, y_2 respectively. Let us set that $A = f^{-1}(f(X) - V_2), B = f^{-1}(f(X) - V_1)$. Then $A \notin \sigma^{y_1}, B \notin \sigma^{y_2}$ and $f(A \cup B) = f(X)$ implies $A \cup B = X$.

 $(II) \Rightarrow (I)$: Since $x\delta x$, by (II) (a) there exists a $\sigma \in \Sigma$ such that $\{x\} \in \sigma$ and by Proposition 2.3.4.3 (e), $\sigma = \sigma_x$. Let us set that $Y = \Sigma$ and let us define $f: X \to Y$ by $f(x) = \sigma_x \in \Sigma$. Clearly f is one-to-one. Let us define a closure operator cl on Σ by $\sigma \in cl(\mathcal{A}), \mathcal{A} \subset \Sigma$ if and only if every subset E of X which absorbs \mathcal{A} belongs to σ . It is easily verified that cl is a Kuratovski closure operator and that (II)(b) implies that Y is Hausdorff (see the proof of Theorem 2.3.3.1). We need to verify only (I)(c). If $A\delta B$, then there exists a $\sigma \in \Sigma$ such that $A, B \in \sigma = \sigma_x$ for some $x \in A \cup B$. Without loss of generality, let us suppose that $x \in A$; then $\sigma = \sigma_x \in f(A)$ and $B \in \sigma$ implies $\sigma \in cl(f(B))$ i.e. $f(A) \cap cl(f(B)) \neq \emptyset$. Clearly, $f(A)\delta_s f(B)$. Conversely, if $y \in (cl(f(A)) \cap f(B)) \cup (f(A) \cap cl(f(B)))$, then $y = \sigma_x$ for some $x \in A \cup B$. Again, let us suppose that $x \in A$; then $\sigma_x \in cl(f(B))$ implies $B \in \sigma_x$. This shows that $x\delta B$ i.e. $x \in A \cap \overline{B}$ i.e. $A\delta_s B$ in X, which, in turn, implies $A\delta B$.

We will now consider the problem of extending a continuous function from a dense subspace of an S-proximity space and obtain several generalizations of the known results in this area.

Let (X, δ) be an S-proximity space and let Σ_X be the family of all bands over X. The mapping $\varphi = \varphi_x : X \to \Sigma_X$ defined by $\varphi(x) = \sigma_x$, where σ_x is the point band, can be shown to be a homeomorphism of X onto a dense subspace of Σ_X (with the A-topology) as in Theorem 2.3.3.3. The proof of the following result is similar to the one of Theorems 2.3.3.4 and 2.3.3.5.

Theorem 2.3.5.2 Let (X, δ_X) and (Y, δ_Y) be S-proximity spaces and let $f: X \to Y$ be an S-proximally continuous mapping. Then there exists an associated mapping $f_{\Sigma}: \Sigma_X \to \Sigma_Y$ defined by $f_{\Sigma}(\sigma) = \{E \subset Y: f^{-1}(\overline{E}) \in \sigma\}$. The mapping f_{Σ} is continuous and $f_{\Sigma}(\sigma_x) = \sigma_{f(x)}$. Hence, identifying X with $\varphi_X(X)$ and Y with $\varphi_Y(Y)$, f_{Σ} is a continuous extension of f.

The following is an improved version of Theorem 2.3.3.11.

Theorem 2.3.5.3 Let X be a dense subspace of an S-proximity space $(\alpha X, \delta_s)$, let (Y, δ) be a separated proximity space and let Y^* be its Smirnoff compactification. Then a continuous mapping f from X to Y has a continuous extension \overline{f} from αX to Y^* if and only if f is an S-proximally continuous mapping.

Historical and bibliographic notes

M. W. Lodato introduced and developed the LO-proximity relation in papers [201] and [202]. The notion of a regular dense subset of topological space was introduced by Lodato in [201]. In that paper Lodato proved Theorems 2.3.3.1 and 2.3.3.2. The notion of a bunch in an LO-proximity space was introduced by Lodato in paper [202]. Theorems 2.3.3.3-2.3.3.9 were proved by M. S. Gagrat and S. A. Naimpally in 1971 in paper [115]. Theorem 2.3.3.10 was first proved by Lodato [202]. We have shown the short proof of this theorem which was given by Gagrat and Naimpally in [115]. Theorem 2.3.3.11 as a generalization of A. D. Taimanov's Theorem [316] was also proved in [115] by Gagrat and Naimpally (see also [224]). The concept of S-proximity spaces were defined independently by S. B. Krishna Murti [169], P. Szymanski [315] and A. D. Wallace [329] (see also [330]). Proposition 2.3.4.3, Theorems 2.3.4.1, 2.3.5.1, 2.3.5.2 and 2.3.5.3 were given in [116] by Gagrat and Naimpally.

2.4 M-uniform spaces

2.4.1 The notion and basic properties of M-uniform spaces

Let \mathcal{U} be a non empty subset of $P(X \times X)$, where X is a non empty set. Let us consider the following properties of \mathcal{U} : $(M_1) \Delta \subset U$ for every $U \in \mathcal{U}$;

 $(M_2) \bigcap \{ U : U \in \mathcal{U} \} = \Delta;$

 $(M_3) U = U^{-1}$ for every $U \in \mathcal{U}$;

 (M_4) for every $A \in P(X)$ and every $U, V \in \mathcal{U}$ there exists a $W \in \mathcal{U}$ such that $W[A] \subset U[A] \cap V[A]$;

 $(M_5) \ U \cap V \in \mathcal{U}$ for every $U, V \in \mathcal{U}$;

 (M_6) for each $A, B \in P(X)$ and $U \in \mathcal{U}$, if $V[A] \cap B \neq \emptyset$ for all $V \in \mathcal{U}$, then there exists an $x \in B$ and there exists a $W \in \mathcal{U}$ such that $W[x] \subset U[A]$;

 (M_7) for every $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ such that $V \circ V \subset U$;

 (M_8) if $U \in \mathcal{U}$ and $U \subset V = V^{-1} \subset X \times X$, then $V \in \mathcal{U}$.

It is obvious that (M_5) implies (M_4) . Also (M_7) implies (M_6) . Indeed, let $A, B \in P(X)$ and let $U \in \mathcal{U}$. By (M_7) there exists a $V \in \mathcal{U}$ such that $V \circ V \subset U$. By hypothesis, there exists an $x \in V[A] \cap B$. Hence there exists a $z \in A$ such that $(z, x) \in V$. Let $p \in V[x]$. Then $(x, p) \in V$; hence $(z, p) \in U$. Thus $p \in U[A]$ and $V[x] \subset U[A]$.

Definition 2.4.1.1 Let \mathcal{U} be a non empty subset of $P(X \times X)$. \mathcal{U} is an M-uniformity or symmetric generalized uniformity on X if \mathcal{U} satisfies the conditions (M_1) , (M_3) , (M_4) , (M_6) and (M_8) . \mathcal{U} is a correct uniformity on X if \mathcal{U} satisfies (M_1) , (M_3) , (M_4) , (M_7) and (M_8) . \mathcal{U} is a symmetric uniformity if it satisfies (M_1) , (M_3) , (M_5) , (M_7) and (M_8) . \mathcal{U} is a symmetric uniformity if it satisfies (M_1) , (M_3) , (M_5) , (M_7) and (M_8) . A pair (X,\mathcal{U}) is called an M-uniform space or symmetric generalized uniform space if \mathcal{U} is an M-uniformity on X. Similarly, we define correct uniform spaces and symmetric uniform spaces.

Lemma 2.4.1.1 Let (X, \mathcal{U}) be an M-uniform space. If $(x, y) \in V$ for every $V \in \mathcal{U}$ and $(y, z) \in V$ for every $V \in \mathcal{U}$, then $(x, z) \in V$ for every $V \in \mathcal{U}$.

Proof: Let $U \in \mathcal{U}$. By hypothesis, $V[x] \cap \{y\} \neq \emptyset$ for every $V \in \mathcal{U}$. Hence by (M_6) there exists $W_1 \in \mathcal{U}$ such that $W_1[y] \subset U[x]$. But since $V[y] \cap \{z\} \neq \emptyset$ for every $V \in \mathcal{U}$, there exists $W_2 \in \mathcal{U}$ such that $W_2[z] \subset W_1[y]$. Hence $z \in W_1[y] \subset U[x]$, so that $(x, z) \in U$.

Theorem 2.4.1.1 Let (X, U) be an *M*-uniform space. If U has the smallest element with respect to the set inclusion, then (X, U) is a symmetric uniform space.

Proof: By the hypothesis \mathcal{U} immediately satisfies (M_5) . We will now show that \mathcal{U} satisfies (M_7) . Let $U \in \mathcal{U}$ and let V be the smallest element in \mathcal{U} . Let us suppose that $(x, y) \in V$ and $(y, z) \in V$. Then by Lemma 2.4.1.1 $(x, z) \in V$ holds, so that $V \circ V \subset U$.

Proposition 2.4.1.1 Let (X, U) be an M-uniform space. Then the function $g: P(X) \to P(X)$, defined by $x \in g(A)$ if $U[x] \cap A \neq \emptyset$ for all $U \in U$, is a Kuratowski closure function.

Proof: (K_1) : Let us suppose that there exists a point $x \in g(\emptyset)$. Then $U[x] \cap \emptyset \neq \emptyset$ for every $U \in \mathcal{U}$ which, of course, is impossible.

 (K_2) : Let $x \in A$. Then $U[x] \cap A \neq \emptyset$ for all $U \in \mathcal{U}$, so that $x \in g(A)$. Thus $A \subset g(A)$ holds for every $A \in P(X)$.

 (K_3) : Let us suppose that $x \notin g(A) \cup g(B)$. Then there exists $U_1, U_2 \in \mathcal{U}$ such that $U_1[x] \cap A = \emptyset$ and $U_2[x] \cap B = \emptyset$. Then by (M_4) there exists a $W \in \mathcal{U}$ such that $W[x] \subset U_1[x] \cap U_2[x]$. But then $W[x] \cap (A \cup B) \subset$ $(U_1[x] \cap A) \cup (U_2[x] \cap B) = \emptyset$ which implies $x \notin g(A \cup B)$. It is clear that the converse inclusion holds.

 (K_4) : Let us suppose $x \in g(g(A))$. Let $U \in \mathcal{U}$. Then $V[x] \cap g(A) \neq \emptyset$ for every $V \in \mathcal{U}$. But by (M_6) there exists an $x_0 \in g(A)$, and there exists a $W \in \mathcal{U}$ such that $W[x_0] \subset U[x]$. Since $W[x_0] \cap A \neq \emptyset$, $U[x] \cap A \neq \emptyset$ holds. Consequently $x \in g(A)$. The converse is obviously true.

Definition 2.4.1.2 The topology induced on X by the Kuratowski closure function g in the above theorem is called the **uniform topology on X** induced by \mathcal{U} , and is denoted with $\tau_{\mathcal{U}}$.

Proposition 2.4.1.2 Let (X, U) be an M-uniform space. Then $A \in \tau_{\mathcal{U}}$ if and only if for every $x \in A$ there exists a $U \in \mathcal{U}$ such that $U[x] \subset A$.

Proof: Let us suppose that $A \in \tau_{\mathcal{U}}$. Then X - A is closed. Let $x \in A$. Since $x \notin \overline{X - A}$, there exists a $U \in \mathcal{U}$ such that $U[x] \cap (X - A) = \emptyset$. Thus we have that $U[x] \subset A$.

Conversely, let us suppose that $x \in A$ and that there exists a $U \in \mathcal{U}$ such that $U[x] \subset A$, i.e. $U[x] \cap (X - A) = \emptyset$. Then $x \notin \overline{X - A}$, so that X - A contains all its accumulation points. Hence X - A is closed and therefore A is open.

The following theorem and corollary are very important for the development of the theory of M-uniform spaces.

Theorem 2.4.1.2 Let (X, U) be an M-uniform space. Then $Int A = \{x : U[x] \subset A \text{ for some } U \in U\}$ holds for every $A \in P(X)$.

Proof: Let $B = \{x : U[x] \subset A \text{ for some } U \in \mathcal{U}\}$. It is clear that $Int A \subset B \subset A$. Consequently, it is sufficient to show that X - B is closed. Let us suppose that $y \in \overline{X - B}$. Then $V[y] \cap (X - B) \neq \emptyset$ for every $V \in \mathcal{U}$. Let

us suppose that $y \in B$. Then there exists $U_1 \in \mathcal{U}$ such that $U_1[y] \subset A$. But then, by (M_6) , there exists an $x \in X - B$ and there exists a $W \in \mathcal{U}$ such that $W[x] \subset U_1[y] \subset A$. So, $x \in B$ which is a contradiction. Consequently, $y \in X - B$ and X - B is closed.

Corollary 2.4.1.1 For every $x \in X$ the family $\{U[x] : U \in U\}$ is a base for the neighborhood system of x.

Proof: Let M be an open set that contains a point $x \in X$. There exists, by Proposition 2.4.1.2, a $U \in \mathcal{U}$ such that $U[x] \subset M$. But by Theorem 2.4.1.2 $x \in Int(U[x])$ holds. Hence U[x] is a neighborhood of x.

Theorem 2.4.1.3 Let (X, \mathcal{U}) be an M-uniform space. Then for every $A \subset X \ \overline{A} = \bigcap \{ U[A] : U \in \mathcal{U} \}$ holds.

Proof: Let $x \in \overline{A}$. Then $U[x] \cap A \neq \emptyset$ for all $U \in \mathcal{U}$, so that $x \in U^{-1}[A]$ for all $U \in \mathcal{U}$. But since $U = U^{-1}$, this implies $x \in U[A]$ for all $U \in \mathcal{U}$.

Conversely, let us suppose that $x \in U[A]$ for all $U \in \mathcal{U}$. Then $x \in U^{-1}[A]$ for all $U \in \mathcal{U}$, so that $U[x] \cap A \neq \emptyset$ for all $U \in \mathcal{U}$. Hence $x \in \overline{A}$.

Proposition 2.4.1.3 If (X, U) is an M-uniform space, then the following statements are equivalent:

- (a) $\tau_{\mathcal{U}}$ is a T_0 -topology;
- $(b) \cap \{U : U \in \mathcal{U}\} = \Delta;$
- (c) $\tau_{\mathcal{U}}$ is a T_1 -topology.

Proof: $(a) \Rightarrow (b)$: Let us suppose that $\tau_{\mathcal{U}}$ is a T_0 -topology and $x \neq y$. Let us suppose that there exists an open set M such that $y \in M$ and $x \notin M$. Then by Corollary 2.4.1.1 there exists a $U \in \mathcal{U}$ such that $U[y] \subset M$. Consequently, $x \notin U[y]$; so that $(x, y) \notin U$. Hence $\bigcap \{U : U \in \mathcal{U}\} = \Delta$.

 $(b) \Rightarrow (c)$: Let us assume that $\bigcap \{U : U \in \mathcal{U}\} = \Delta$ and let us suppose that $x \neq y$. Then $(x, y) \notin U_1$ and $(y, x) \notin U_1$ for some $U_1 \in \mathcal{U}$. Hence $y \notin U_1[x]$ and $x \notin U_1[y]$; so, by Corollary 2.4.1.1, we have that $\tau_{\mathcal{U}}$ is T_1 .

Definition 2.4.1.3 A decomposition of a set X is a disjoint family \mathcal{D} of the subsets of X whose union is X. A decomposition \mathcal{D} of a topological space (X, τ) is **upper semi-continuous** if for each $D \in \mathcal{D}$ and each open set A containing D, there exists an open set B such that $D \subset B \subset A$, and B is the union of members of \mathcal{D} .

Proposition 2.4.1.4 Let (X, U) be an M-uniform space. Then $R = \bigcap \{U : U \in U\}$ is an equivalence relation on X, and X/R is an upper semicontinuous decomposition of $(X, \tau_{\mathcal{U}})$.

Proof: Clearly, R is reflexive and symmetric, and by Lemma 2.4.1.1, R is transitive. Let $x \in A \in \tau_{\mathcal{U}}$. Then there exists $U \in \mathcal{U}$ such that $U[x] \subset A$. Since $R \subset U$ for every $U \in \mathcal{U}$, we have that $R[x] \subset A$ for every $x \in A$. Hence $A = \bigcup \{R[x] : x \in A\}$. But $R[x] \in X/R$ for every $x \in A$. Therefore X/R is an upper semi-continuous decomposition of $(X, \tau_{\mathcal{U}})$.

Definition 2.4.1.4 \mathcal{B} is a base for an M-uniformity \mathcal{U} on X if:

(a) $\mathcal{B} \subset \mathcal{U};$

(b) for every $U \in \mathcal{U}$ there exists a $V \in \mathcal{B}$ such that $V \subset U$.

 \mathcal{B} is called an **open base** if each element of \mathcal{B} is open with respect to the product topology on $X \times X$. Similarly we define a **closed base**.

Definition 2.4.1.5 S is a subbase for an *M*-proximity U on X if the set B of all finite intersections of the elements of S is a base for U.

Lemma 2.4.1.2 Let (X, \mathcal{U}) be an M-uniform space. If V is a closed set in $X \times X$ with the product topology of $\tau_{\mathcal{U}}$, then, for each $x \in X$, the set V[x] is closed with respect to $\tau_{\mathcal{U}}$.

Proof: Let $x_0 \in X$ and let $\{y_n : n \in D\}$ be a net in $V[x_0]$. Then $\{(x_0, y_n) : n \in D\}$ is a net in V. Let us suppose that (y_n) converges to b. We know that the constant net (x_0) converges to x_0 . Hence $\{(x_0, y_n) : n \in D\}$ converges to $(x_0, b) \in V$; so that $b \in V[x]$ and $V[x_0]$ is closed.

Proposition 2.4.1.5 Let (X, U) be an M-uniform space. If U has a close base, then $\tau_{\mathcal{U}}$ is regular.

Proof: This is an immediate consequence of Lemma 2.4.1.2 and Corollary 2.4.1.1.

Theorem 2.4.1.4 A subset \mathcal{B} of $P(X \times X)$ is a base for some M-uniformity on X if and only if \mathcal{B} satisfies (M_1) , (M_3) (M_4) and (M_6) .

Proof: Clearly, if \mathcal{B} is a base for some M-uniformity on X, then \mathcal{B} satisfies $(M_1), (M_3), (M_4)$ and (M_6) .

Conversely, let $\mathcal{U} = \{U : U = U^{-1} \text{ and } V \subset U \text{ for some } V \in \mathcal{B}\}.$ Clearly, \mathcal{U} satisfies $(M_1), (M_3)$ and (M_8) . We will now show that \mathcal{U} satisfies (M_4) . Let $A \in P(X)$ and let $U, V \in \mathcal{U}$. There exists $U_1, V_1 \in \mathcal{B}$ such that $U_1 \subset U$ and $V_1 \subset V$. But since \mathcal{B} satisfies (M_4) , there exists $W \in \mathcal{B}$ such that $W[A] \subset U_1[A] \cap V_1[A]$. Since $U_1[A] \cap V_1[A] \subset U[A] \cap V[A]$, \mathcal{U} satisfies (M_4) . We will now show \mathcal{U} satisfies (M_6) . Let $A, B \in P(X)$ and let $U \in \mathcal{U}$, and let us suppose that $V[A] \cap B \neq \emptyset$ for all $V \in \mathcal{U}$. Then $V[A] \cap B \neq \emptyset$ for all $V \in \mathcal{B}$. But there exists $U_1 \in \mathcal{B}$ such that $U_1 \subset U$. But since \mathcal{B} satisfies (M_6) , there exists an $x \in B$ and there exists a $W \in \mathcal{B}$ such that $W[x] \subset U_1[A]$. But $U_1[A] \subset U[A]$. Consequently, \mathcal{U} satisfies (M_6) .

2.4.2 LO-proximity induced by an M-uniformity

Theorem 2.4.2.1 Let \mathcal{U} be a subset of $P(X \times X)$ with the property that for all $U \in \mathcal{U}$, U^{-1} contains a member of \mathcal{U} . Let us define a relation $\delta_{\mathcal{U}}$ on P(X) by

 $A\delta_{\mathcal{U}}B$ if $U[A] \cap B \neq \emptyset$ for all $U \in \mathcal{U}$.

Then $\delta_{\mathcal{U}}$ is an LO-proximity on X if and only if \mathcal{U} satisfies (M_1) , (M_4) and (M_6) .

Proof: Let us suppose that \mathcal{U} satisfies (M_1) , (M_4) and (M_6) . We will show that $\delta_{\mathcal{U}}$ satisfies the conditions $(SP_1) - (SP_4)$ and (LO). To simplify the notation we will write δ in place of $\delta_{\mathcal{U}}$.

 (SP_1) : Holds immediately from the definition of δ and the fact that the members of \mathcal{U} are non empty by (M_1) .

 (SP_2) : Let us suppose that $A\overline{\delta}B$. There exists, by hypothesis, a $U \in \mathcal{U}$ such that $U[A] \cap B = \emptyset$. Let us suppose that $U^{-1}[B] \cap A \neq \emptyset$. Let $x_0 \in U^{-1}[B] \cap A$. Then $x_0 \in U^{-1}[B]$ and therefore there exists $y_0 \in B$ such that $(y_0, x_0) \in U^{-1}$, and consequently, $(x_0, y_0) \in U$. But this means that $y_0 \in U[A] \cap B$ which is a contradiction. Hence $U^{-1}[B] \cap A = \emptyset$. But, by hypothesis, $V \subset U^{-1}$, where $V \in \mathcal{U}$ so that $V[B] \cap A = \emptyset$. Hence $B\overline{\delta}A$.

 (SP_3) : Let us suppose that $A \cap B \neq \emptyset$. By $(M_1) \ U[A] \cap B \neq \emptyset$ for all $U \in \mathcal{U}$. Therefore $A\delta B$ is true.

 (SP_4) : Let us suppose that $C\delta A$ and $C\delta B$. Then there exist $U, V \in \mathcal{U}$ such that $U[C] \cap A = \emptyset$ and $V[C] \cap B = \emptyset$. But, by (M_4) , there exists a $W \in \mathcal{U}$ such that $W[C] \subset U[C] \cap V[C]$. Consequently, $W[C] \cap (A \cup B) = \emptyset$. Thus we have that $C\overline{\delta}(A \cup B)$.

(*LO*): Let us suppose that $A\delta B$ and $b\delta C$ for all $b \in B$, but $A\delta C$. Then there exists a $U \in \mathcal{U}$ such that $U[A] \cap C = \emptyset$. Since $A\delta B$, $V[A] \cap B \neq \emptyset$ for all $V \in \mathcal{U}$ holds, so that by (M_6), there exists $x_0 \in B$ and there exists $W \in \mathcal{U}$ such that $W[x_0] \subset U[A] \subset X - C$. This implies that $W[x_0] \cap C = \emptyset$, so that $x_0 \overline{\delta}C$ which is a contradiction since $x_0 \in B$.

Conversely, let us suppose that δ is an LO-proximity on X and let us show that \mathcal{U} satisfies (M_1) , (M_4) and (M_6) .

 (M_1) : Let $U \in \mathcal{U}$. If $x \in X$, then $\{x\} \cap \{x\} \neq \emptyset$ implies by (SP_3) that $x\delta x$. Consequently, $U[x] \cap x \neq \emptyset$ so that $(x, x) \in U$. Hence $\Delta \subset U$.

 (M_4) : Suppose that this is not true. Then there exists $A \in P(X)$ and $U, V \in \mathcal{U}$ such that for every $W \in \mathcal{U}$ there exists $x \in W[A]$ such that $x \notin U[A] \cap V[A]$. For each $W \in \mathcal{U}$ let $B(W) = \{x : x \in W[A] \text{ and } x \notin U[A] \cap V[A]\}$. Let $B = \cup \{B(W) : W \in \mathcal{U}\}$. Let us suppose that there exists $U_a \in \mathcal{U}$ such that $U_a[A] \cap B = \emptyset$. Then, since $B(U_a) \subset B$, $U_a[A] \subset U[A] \cap V[A]$ holds, but, by assumption, this is not possible. Hence $M[A] \cap B \neq \emptyset$ for all $M \in \mathcal{U}$, so that $A\delta B$. Let $B_1 = B - U[A]$ and $B_2 = B - V[A]$. Clearly, $U[A] \cap B_1 = \emptyset$ and $V[A] \cap B_2 = \emptyset$, so that $A\overline{\delta}B_1$ and $A\overline{\delta}B_2$. Consequently, by $(SP_4) A\overline{\delta}(B_1 \cup B_2)$. By the definition of $B, B = B_1 \cup B_2$ holds. Hence $A\overline{\delta}B$ which is a contradiction.

 (M_6) : Let us suppose that (M_6) is not true. Then there exist $A, B \in P(X)$ and $U \in \mathcal{U}$ such that $V[A] \cap B \neq \emptyset$ for all $V \in \mathcal{U}$ and every $b \in B$ and for every $W \in \mathcal{U}$ we have that $W[b] \cap (X - U[A]) \neq \emptyset$. Consequently, $A\delta B$ and $b\delta(X - U[A])$ for every $b \in B$, so that by (LO), $A\delta(X - U[A])$. But $U[A] \cap (X - U[A]) = \emptyset$, so that $A\overline{\delta}(X - U[A])$. Hence our assumption leads to a contradiction.

Definition 2.4.2.1 The LO-proximity induced on X in the above theorem is called the **uniform LO-proximity** and is denoted by $\delta_{\mathcal{U}}$.

Corollary 2.4.2.1 If (X, U) is an M-uniform space, then $\tau_{\mathcal{U}} = \tau_{\delta_{\mathcal{U}}}$.

Proposition 2.4.2.1 If (X, U) is an M-uniformity on X, then:

(a) $A\delta_{\mathcal{U}}B$ if and only if $(A \times B) \cap U \neq \emptyset$ for every $U \in \mathcal{U}$;

(b) $A \ll B$ if and only if there exists a $U \in \mathcal{U}$ such that $U[A] \subset B$.

Proof: (a) Let us suppose that $(A \times B) \cap U \neq \emptyset$ for every $U \in \mathcal{U}$. Then $U[A] \cap B \neq \emptyset$ for every $U \in \mathcal{U}$, so that $A\delta B$.

Conversely, let us suppose that $A\delta_{\mathcal{U}}B$ and $U \in \mathcal{U}$. Then, since $U[A] \cap B \neq \emptyset$, there exists a $b \in U[A] \cap B$. Hence there exists $a \in A$ for which $(a, b) \in U$ so that $(A \times B) \cap U \neq \emptyset$.

(b) Let us suppose that $A \ll B$. Then $A\overline{\delta}X - B$ so that there exists a $U \in \mathcal{U}$ such that $U[A] \cap (X - B) = \emptyset$. Hence $U[A] \subset B$.

Conversely, let us suppose that there exists an $U \in \mathcal{U}$ such that $U[A] \subset B$. Then $U[A] \cap (X - B) = \emptyset$. Hence $A\overline{\delta}_{\mathcal{U}}X - B$ so that $A \ll B$.

Theorem 2.4.2.2 Let (X, δ) be an LO-proximity space. Then there exists an M-uniformity $\mathcal{U}_{1\delta}$ on X such that $\delta_{\mathcal{U}_{1\delta}} = \delta$.

Proof: For every $A, B \in P(X)$ let $U_{A,B} = X \times X - ((A \times B) \cup (B \times A))$. Let $\mathcal{V} = \{U_{A,B} : A\delta B\}$. It is clear that \mathcal{V} satisfies (M_1) and (M_3) . We will now show that $A\overline{\delta}B$ if and only if for some $C, D, C\overline{\delta}D$ and $U_{C,D}[A] \cap B = \emptyset$. Let us suppose that $A\delta B$ and that there exists $t \in U_{A,B}[A] \cap B$. Then there exists $s \in A$ such that $(s,t) \in U_{A,B}$. But this is a contradiction since $(s,t) \in A \times B$. Hence $U_{A,B}[A] \cap B = \emptyset$. Consequently, let us suppose that there exist C, D such that $C\overline{\delta}D$ and $U_{C,D}[A] \cap B = \emptyset$. We will first assume that $A \subset C \cup D$; for, if $t \in A - (C \cup D)$, then $U_{C,D}[t] = X$ and so $U_{C,D}[A] = X$, which is also a contradiction. Next, we will show that $A \subset C$ or $A \subset D$. Let us suppose that there exist $t_1, t_2 \in A$ such that $t_1 \in C$ and $t_2 \in D$. Then $U_{C,D}[t_1] = X - D$ and $U_{C,D}[t_2] = X - C$. But since $C\overline{\delta}D$, we know by (SP_3) that $(X - C) \cup (X - D) = X$. Hence $U_{C,D}[t_1] \cup U_{C,D}[t_2] = X$ so that $U_{C,D}[A] = X$ which is a contradiction. Consequently, $A \subset C$ or $A \subset D$. Let us suppose that the first case is true. Then $U_{C,D}[A] = X - D$ so that $B \subset D$, and by Proposition 2.2.1.1, $A\delta B$. The proof in the second case is similar.

By the above argument and Theorem 2.4.2.1 \mathcal{V} also satisfies (M_4) and (M_6) . Consequently, by Theorem 2.4.1.4 $\mathcal{U}_{1\delta} = \{U : U = U^{-1} \text{ and } V \subset U \text{ for some } V \in \mathcal{V}\}$ is an M-uniformity on X. It is clear that $\delta_{\mathcal{U}_{1\delta}} = \delta$.

Corollary 2.4.2.2 A topology τ on X is the uniform topology for some M-uniformity on X if and only if τ is a R_0 -topology.

Proof: This is an immediate consequence of Corollary 2.3.2.1 and Theorem 2.4.2.2. ♣

Corollary 2.4.2.3 An M-uniformity $U_{1\delta}$ constructed in Theorem 2.4.2.2 is totally bounded.

Proof: If $U_{A,B} \in \mathcal{U}_{1\delta}$ and if (x_a, y_b) is any element of $A \times B$, then $U_{A,B}[x_a] = X - B$ and $U_{A,B}[x_b] = X - A$, so that, since $A \cap B = \emptyset$, $U_{A,B}[x_a] \cup U_{A,B}[x_b] = X$.

Example 2.4.2.1 There exists an M-uniform space that does not satisfy (M_5) .

Let δ be the usual proximity for the reals \mathbb{R} . Let $\mathcal{U}_{1\delta}$ be the M-uniformity on \mathbb{R} as constructed in Theorem 2.4.2.2. Let $A = [1, 2], B = [2, 3], A_1 = [3, 4]$ and $B_1 = [4, 5]$. Clearly $A\overline{\delta}A_1$ and $B\overline{\delta}B_1$. We will show that there does not exist P, Q such that $P\overline{\delta}Q$ and $U_{P,Q} \subset U_{A,A_1} \cap U_{B,B_1}$. For let us suppose that there does exist such a P and Q. Then $E_1 = (A \times A_1) \cup (B \times B_1) \subset P \times Q$ and $E_2 = (A_1 \times A) \cup (B_1 \times B) \subset Q \times P$. But $(3,5) \in B \times B_1$ implies that $(3,5) \in E_1$ so that $(3,5) \in P \times Q$ and hence $3 \in P$. Also, $(3,1) \in A_1 \times A$, so that $(3,1) \in E_2$ and hence $(3,1) \in Q \times P$. This means that $3 \in Q$. Hence, $P \cap Q \neq \emptyset$, so that, by $(SP_3), P\delta Q$ which is a contradiction.

2.4.3 Proximity class of M-uniformities

Definition 2.4.3.1 If δ is an LO-proximity on X, then the class of all *M*-uniformities \mathcal{U} on X such that $\delta = \delta_{\mathcal{U}}$ is called a **proximity class of** *M*-uniformities on X and is denoted by $\pi(\delta)$.

Theorem 2.4.3.1 Let (X, δ) be an LO-proximity space. Then $\mathcal{U}_{1\delta}$ constructed in Theorem 2.4.2.2 is the smallest element of $\pi(\delta)$, where the partial order on $\pi(\delta)$ is the set inclusion.

Proof: Let $\mathcal{U} \in \pi(\delta)$. If $U_{A,B} \in \mathcal{U}_{1\delta}$, then $A\overline{\delta}B$. According to Proposition 2.4.2.1 (a), there exists a $V \in \mathcal{U}$ such that $(A \times B) \cap V = \emptyset$. But since $V = V^{-1}$ we have that $(B \times A) \cap V = \emptyset$. Hence $V \subset U_{A,B}$, so that $U_{A,B} \in \mathcal{U}$.

Theorem 2.4.3.2 Let (X, δ) be an LO-proximity space. The union \mathcal{B} of an arbitrary family of members of $\pi(\delta)$ is a base for an M-uniformity in $\pi(\delta)$.

Proof: It is clear that \mathcal{B} satisfies (M_1) and (M_3) . By the definition of $\pi(\delta)$, $A\delta B$ holds if and only if $U[A] \cap B \neq \emptyset$ for every $U \in \mathcal{B}$. Hence, by Theorem 2.4.2.1, \mathcal{B} satisfies (M_4) and (M_6) . Consequently, by Theorem 2.4.1.4, \mathcal{B} is a base for an M-uniformity on X which is clearly in $\pi(\delta)$.

Corollary 2.4.3.1 Let (X, δ) be an LO-proximity. Then $\pi(\delta)$ has the biggest element.

Proof: It is an immediate consequence of Theorem 2.4.3.2.

Definition 2.4.3.2 Let (X, U) be an *M*-uniform space. (X, U) is δ -correct if there exists an *LO*-proximity δ on *X* such that the family $S = \{U_{A,B} : A\overline{\delta}B\}$ is a subbase for U. δ is called the generator proximity for U. **Lemma 2.4.3.3** Let (A_1, \ldots, A_n) and (B_1, \ldots, B_n) be n-tuples of non-void subsets of a set X. Let $U = U_{A_1,B_1} \cap \ldots \cap U_{A_n,B_n}$ and let $I_1 = \{k_1, \ldots, k_p\}$ and $I_2 = \{j_1, \ldots, j_q\}$ be subsets of $\{1, 2, \ldots, n\}$. Let us suppose that $x_0 \in$ $(A_{k_1} \cap \ldots \cap A_{k_p} \cap B_{j_1} \cap \ldots \cap B_{j_q})$ and $x_0 \notin A_i$ if $i \notin I_1$ and $x_0 \notin B_i$ if $i \notin I_2$. Then $U[x_0] = E$, where E is equal to

$$(X - B_{k_1}) \cap \ldots \cap (X - B_{k_p}) \cap (X - A_{j_1}) \cap \ldots \cap (X - A_{j_q}).$$

Proof: To simplify the language we will abbreviate the hypothesis of the lemma as follows: "Let us suppose that $x_0 \in (A_{k_1} \cap \ldots \cap A_{k_p} \cap B_{j_1} \cap \ldots \cap B_{j_q})$ and let x_0 be in no other A_i or B_i ." By De Morgan's law $U = X \times X - \bigcup_{i=1}^{n} [A_i \times B_i) \cup (B_i \times A_i)]$. Let us suppose that $t \in U[x_0]$. Then $(x_0, t) \in U$, so that, since $x_0 \in (A_{k_1} \cap \ldots \cap A_{k_p} \cap B_{j_1} \cap \ldots \cap B_{j_q})$ we have that $t \notin B_{k_i}$, $i = 1, \ldots, p$, and $t \notin A_{j_i}$, $i = 1, \ldots, q$. Consequently, $t \in E$ and $U[x_0] \subset E$. To show the reverse inclusion, let us suppose that there exists $t_1 \in E - U[x_0]$. Then $(x_0, t_1) \notin U$, so that (x_0, t_1) is an element of $\bigcup_{i=1}^{n} [(A_i \times B_i) \cup (B_i \times A_i)]$. Let us suppose that $m \neq k_i$ for $i = 1, \ldots, p$, so that $x_0 \in A_m$ and $m \notin I_i$, which is a contradiction. Let us suppose that $(x_0, t_1) \in B_m \times A_m$, where $1 \leq m \leq n$. Then since $t_1 \in E$, we have that $m \neq j_i$ for $i = 1, \ldots, q$, so that $x_0 \in B_m$ and $m \notin I_2$, which is a contradiction. Hence $E = U[x_0]$.

Definition 2.4.3.3 Let (A_1, \ldots, A_n) and (B_1, \ldots, B_n) be n-tuples of nonvoid subsets of a set X. Let $I_1 = \{k_1, \ldots, k_p\}$ and $I_2 = \{j_1, \ldots, j_q\}$ be any two subsets of $\{1, \ldots, n\}$ and let

$$E = \{x : x \in A_i \Leftrightarrow i \in I_1 \text{ and } x \in B_i \Leftrightarrow i \in I_2\}$$

If $E \neq \emptyset$, we will call that the set E a **residual intersection** of the sets A_i and B_i .

It is clear that residual intersections are mutually disjoint so that the family \mathcal{B} of all residual intersections of A_i and B_i provides a decomposition of $\bigcup \{A_i \cup B_i : i = 1, ..., n\}$ into mutually disjoint sets.

Theorem 2.4.3.3 Let (X, U) be a δ -correct M-uniform space. Then (X, U) is totally bounded.

Proof: Let $U \in \mathcal{U}$ and let δ be a generating proximity for \mathcal{U} . Then there exists a finite family of sets A_1, \ldots, A_n and B_1, \ldots, B_n such that $A_i \overline{\delta} B_i$ for $i = 1, \ldots, n$ and $U_{A_1, B_1} \cap \ldots \cap U_{A_n, B_n} = V \subset U$. Now, if $\bigcup \{A_i \cup B_i : i = I\}$

 $1, \ldots, n\} \neq X$, then, for any $x_0 \in X - \bigcup \{A_i \cup B_i : i = 1, \ldots, n\}$ we have that $V[x_0] = X$, and the theorem follows; so we assume that $\bigcup \{A_i \cup B_i : i = 1, \ldots, n\} = X$. Let \mathcal{Q} be the family of all residual intersections of A_i and B_i . From each $R \in \mathcal{Q}$ let us choose one and only one point and let us denote that point as x_R . Let $S = \{x_R : R \in \mathcal{Q}\}$. Clearly, since \mathcal{Q} is finite, S is also finite. We will now show that V[S] = X. Let $z \in X$. Since we assume that $\bigcup \{A_i \cup B_i : i = 1, \ldots, n\} = X$, we have that $z \in R$ for some $R \in \mathcal{Q}$. Consequently, for some k_1, \ldots, k_p and $j_1, \ldots, j_q, z \in A_{k_1} \cap \ldots \cap A_{k_p} \cap B_{j_1} \cap \ldots \cap B_{j_q}$ and z is in no other A_i or B_i . But by definition of S there exists $x_R \in S$ such that $x_R \in A_{k_1} \cap \ldots \cap A_{k_p} \cap B_{j_1} \cap \ldots \cap B_{j_q}$ and x_R is in no other A_i or B_i . By Lemma 2.4.3.3 we have that $V[x_R]$ is equal to $(X - B_{k_1}) \cap \ldots \cap (X - B_{k_p}) \cap (X - A_{j_1}) \cap \ldots \cap (X - A_{j_q})$. But since $A_i \overline{\delta} B_i$ for all i, we have that $z \notin B_{k_i}$ for $i = 1, \ldots, p$ and $z \notin A_{j_i}$ for $i = 1, \ldots, q$. Consequently, $z \in V[x_R]$. But z is an arbitrary point in X. Hence V[S] = X, so that U[S] = X.

Theorem 2.4.3.4 Let (X, U) be a δ -correct M-uniform space. Then (X, U) has an open base.

Proof: Let $U \in \mathcal{U}$. Then there exists a finite family of sets A_1, \ldots, A_n and B_1, \ldots, B_n such that $A_i \delta B_i$ for $i = 1, \ldots, n$ and $V = U_{A_1,B_1} \cap \ldots \cap U_{A_n,B_n} \subset U$. But for each $i, 1 \leq i \leq n, A_i \subset \overline{A_i}$ and $B_i \subset \overline{B_i}$ so that $U_{\overline{A_i},\overline{B_i}} \subset U_{A_i,B_i}$. But by Theorem 2.3.2.2 (a), $\overline{A_i} \delta \overline{B_i}$ for $i = 1, \ldots, n$, so that $U_{\overline{A_i},\overline{B_i}} \in \mathcal{U}$ for $1 \leq i \leq n$. But it is easily shown that $U_{\overline{A_i},\overline{B_i}}$ is open for $i = 1, \ldots, n$. Hence V is open.

It is clear that $\mathcal{U}_{1\delta}$ as constructed in Theorem 2.4.2.2 has an open base; for, if $U_{A,B}$ is an element of $\mathcal{U}_{1\delta}$, then by the same argument that is given above, we have that $U_{\overline{A},\overline{B}} \subset U_{A,B}$; $U_{\overline{A},\overline{B}} \in \mathcal{U}_{1\delta}$ and $U_{\overline{A},\overline{B}}$ is open.

Lemma 2.4.3.4 Let us suppose that $\{A_i\}$ and $\{B_i\}$, i = 1, ..., n, are finite sequences of non empty subsets of a set X such that $B_i \subset A_i$ for all i and $\cup \{B_i : i = 1, ..., n\} = X$. Then we have that

$$F = X \times X - \bigcup_{i=1}^{n} [[(X - A_i) \times B_i] \cup [B_i \times (X - A_i)]] \subset \bigcup_{i=1}^{n} A_i \times A_i.$$

Proof: Let $(x, y) \in F$. Since $\cup \{B_i : i = 1, ..., n\} = X$, we have that $(x, y) \in B_{k_1} \times B_{k_2}$ where $1 \leq k_1, k_2 \leq n$. It is clear that $(x, y) \notin (X - A_{k_2}) \times B_{k_2}$, so that $x \in A_{k_2}$, since $y \in B_{k_2}$. But $B_{k_2} \subset A_{k_2}$ and therefore we have that $(x, y) \in A_{k_2} \times A_{k_2}$.

Lemma 2.4.3.5 Let (X, δ) be a proximity space. Let \mathcal{U} be a totally bounded M-uniformity on X that is in a proximity class $\pi^*(\delta)$ of symmetric uniformities on X. Then for every $U \in \mathcal{U}$ there exist sets A_1, \ldots, A_n and B_1, \ldots, B_n such that $U_{A_1,B_1} \cap \ldots \cap U_{A_n,B_n} \subset U$ and $A_i \overline{\delta} B_i$ for $i = 1, \ldots, n$.

Proof: Let $U \in \mathcal{U}$. We know that there exists $V \in \mathcal{U}$ such that $V = V^{-1}$ and $V \circ V \circ V \subset U$. Since (X, \mathcal{U}) is totally bounded, there exist the sets B_1, \ldots, B_n such that $\bigcup_{i=1}^n B_i = X$ and $\bigcup_{i=1}^n B_i \times B_i \subset V$. Let $A_i = V[B_i]$. Since $V[B_i] \cap (X - V[B_i]) = \emptyset$, $i = 1, \ldots, n$, we have that $B_i \ll A_i$ for $i = 1, \ldots, n$. Also, by a straightforward calculation, we can show that $A_i \times A_i \subset V \circ V \circ V$ for $i = 1, \ldots, n$. Hence we have that $\bigcup_{i=1}^n A_i \times A_i \subset U$. By Lemma 2.4.3.4

$$X \times X - \bigcup_{i=1}^{n} [[(X - A_i) \times B_i] \cup [B_i \times (X - A_i)]] \subset \bigcup_{i=1}^{n} A_i \times A_i,$$

so that $U_{B_1, X-A_1} \cap \ldots \cap U_{B_n, X-A_n} \subset U$, and $B_i \overline{\delta} X - A_i$ for $i = 1, \ldots, n$.

Theorem 2.4.3.5 A symmetric uniform space (X, U) is totally bounded if and only if the family $S = \{U_{A,B} : A\overline{\delta}B\}$ is a subbase for (X, U) for some proximity δ on X.

Proof: Let us suppose that $S = \{U_{A,B} : A\overline{\delta}B\}$ is a subbase for \mathcal{U} for some proximity δ on X. Then \mathcal{U} is a δ -correct symmetric uniformity on X and, hence, by Theorem 2.4.3.3, \mathcal{U} is totally bounded.

Conversely, let us suppose that \mathcal{U} is totally bounded. It is known (see [317]) that for some proximity δ , $\mathcal{U} \in \pi^*(\delta)$. Let us suppose that $A_i \overline{\delta} B_i$ for $i = 1, \ldots, n$. For each $i = 1, \ldots, n$ there exists a symmetric $V_i \in \mathcal{U}$ such that $(A_i \times B_i) \cap V_i = \emptyset$ and hence such, that $V_i \subset U_{A_i,B_i}$. Consequently, we have that $V_1 \cap \ldots \cap V_n \subset U_{A_1,B_1} \cap \ldots \cap U_{A_n,B_n} = U$, so that $U \in \mathcal{U}$. By this fact and Lemma 2.4.3.5 there follows that the family $\mathcal{S} = \{U_{A,B} : A\overline{\delta}B\}$ is a subbase for \mathcal{U} .

Lemma 2.4.3.6 Let (X, δ) be an LO-proximity space. Let (C_1, \ldots, C_n) and (D_1, \ldots, D_n) be n-tuples of non empty subsets of X such that $C_i \overline{\delta} D_i$ for $i = 1, \ldots, n$. Then $(C_1 \cap \ldots \cap C_n) \overline{\delta} (D_1 \cup \ldots \cup D_n)$.

Proof: Let us suppose that $(C_1 \cap \ldots \cap C_n)\delta(D_1 \cup \ldots \cup D_n)$. Then, by (SP_4) there holds $(C_1 \cap \ldots \cap C_n)\delta D_k$ for some $k, 1 \leq k \leq n$. But $(C_1 \cap \ldots \cap C_n) \subset C_k$, so that by Proposition 2.2.1.1 $C_k\delta D_k$ which is a contradiction.

Lemma 2.4.3.7 Let (X, δ) be an LO-proximity space. Then $P\overline{\delta}Q$ if and only if there exist n-tuples (A_1, \ldots, A_n) and (B_1, \ldots, B_n) of subsets of X such that $(U_{A_1,B_1} \cap \ldots \cap U_{A_n,B_n})[P] \cap Q = \emptyset$ and $A_i\overline{\delta}B_i$ for $i = 1, \ldots, n$.

Proof: If $P\overline{\delta}Q$, then, by the same argument that is given at the beginning of the proof of Theorem 2.4.2.2, $U_{P,Q}[P] \cap Q = \emptyset$.

Conversely, let $V = U_{A_1,B_1} \cap \ldots \cap U_{A_n,B_n}$. Since $V[P] \cap Q = \emptyset$, $P \subset$ $\cup \{A_i \cup B_i : i = 1, \dots, n\}$ holds. Let $\mathcal{E} = \{E_1, \dots, E_m\}$ be the pairwise disjoint family of all residual intersections of the sets A_i and B_i that have a non empty intersection with P. Clearly, $P \subset M = \bigcup \{E_c : c = 1, \dots, m\}$. By Lemma 2.4.3.3, since \mathcal{E} is a pairwise disjoint family, if $t_1 \in P \cap E_c$ and $t_2 \in P \cap E_c$ where $1 \leq c \leq m$, then $V[t_1] = V[t_2]$. Let $F_c = V[t_c]$ for $c = 1, \ldots, m$, where t_c is a fixed point in E_c . Then, $V[P] = \bigcup \{F_c : c = 0\}$ $1, \ldots, m$ holds. But, since $V[P] \cap Q = \emptyset$, we have that $Q \subset X - V[P]$, so that by De Morgan's law, $Q \subset N$ where $N = \cap \{X - F_c : c = 1, \dots, m\}$. Let $E_c \in \mathcal{E}$ where $1 \leq c \leq m$. We may assume that $E_c \subset E_c^* = A_{k_1} \cap \ldots \cap$ $A_{k_p} \cap B_{j_1} \cap \ldots \cap B_{j_q}$ for some $k_1, \ldots, k_p, j_1, \ldots, j_q$ and that E_c intersects no other A_i or B_i . Consequently, by Lemma 2.4.3.3 and De Morgan's law, $X - F_c = B_{k_1} \cup \ldots \cup B_{k_p} \cup A_{j_1} \cup \ldots \cup A_{j_q}$. Hence, by Lemma 2.4.3.6, $E_c^* \overline{\delta} X - F_c$ where $1 \leq c \leq m$, so that by Proposition 2.2.1.1, $E_c \overline{\delta} X - F_c$ where $1 \leq c \leq m$. Hence, again by Lemma 2.4.3.6, $M\delta N$, so that, by Proposition 2.2.1.1, $P\delta Q$.

Lemma 2.4.3.8 Let (X, \mathcal{U}) be a δ -correct *M*-uniform space with a generating proximity δ . Then $\delta_{\mathcal{U}} = \delta$.

Proof: Let us suppose that $P\overline{\delta}Q$. Then, by Lemma 2.4.3.7, there exists $U \in \mathcal{U}$ such that $U[P] \cap Q = \emptyset$, so $P\overline{\delta}_{\mathcal{U}}Q$.

Conversely, let us suppose that $P\overline{\delta}_{\mathcal{U}}Q$. Then there exists $V \in \mathcal{U}$ such that $V[P] \cap Q = \emptyset$ so that, by Lemma 2.4.3.7, $P\overline{\delta}Q$.

Theorem 2.4.3.6 Let (X, δ) be an LO-proximity space. In $\pi(\delta)$ there exists one and only one δ -correct M-uniformity $\mathcal{U}_{2\delta}$ on X.

Proof: Let $S = \{U_{A,B} : A\bar{\delta}B\}$ and let $\mathcal{B} = \{\text{all finite intersectins of mem$ $bers of <math>S\}$. It is clear that \mathcal{B} satisfies (M_1) and (M_3) . By Lemma 2.4.3.7 and Theorem 2.4.2.1 we have that \mathcal{B} also satisfies (M_4) and (M_6) . Consequently, by Theorem 2.4.1.4, we have that $\mathcal{U}_{2\delta} = \{U : U = U^{-1} \text{ and } U \subset V \text{ for some } V \in \mathcal{B}\}$ is an M-uniformity on X. It is clear that $\mathcal{U}_{2\delta}$ is δ -correct, and that $\mathcal{U}_{2\delta} \in \pi(\delta)$ by Lemma 2.4.3.8. We will now show that $\mathcal{U}_{2\delta}$ is the only δ -correct M-uniformity on X that is in $\pi(\delta)$. For this, let us suppose that $\mathcal{V} \in \pi(\delta)$ and that (X, \mathcal{V}) is δ -correct uniformity with generating proximity δ_1 . Clearly, $\delta_1 \neq \delta$ if $\mathcal{U}_{2\delta} \neq \mathcal{V}$. But by Lemma 2.4.3.8 we have that $\delta_{\mathcal{V}} = \delta_1$ which is a contradiction, since we assume that $\mathcal{V} \in \pi(\delta)$. Hence $\mathcal{V} = \mathcal{U}_{2\delta}$.

Let us note that, if $U, V \in \mathcal{U}_{2\delta}$ (as constructed in proof of above theorem), then $U \cap V \in \mathcal{U}_{2\delta}$. Hence if δ is the usual proximity of the reals \mathbb{R} , then $\mathcal{U}_{1\delta}$ (as constructed in the proof of Theorem 2.4.2.2) is properly contained in $\mathcal{U}_{2\delta}$. Hence we can see that a proximity class of M-uniformities may contain two distinct, totally bounded uniformities. It can easily be shown that a proximity class may contain more than two distinct, totally bounded uniformities.

Corollary 2.4.3.2 Let (X, δ) be a proximity space. There exists in $\pi(\delta)$ one and only one totally bounded symmetric uniformity on X.

Proof: By Theorem 2.4.3.5 and Theorem 2.4.3.6 it is sufficient to show that $\mathcal{U}_{2\delta}$ satisfies (M_7) . Let us note that if $V_i \circ V_i \subset U_i$ for $i = 1, \ldots, n$, then $(V_1 \cap \ldots V_n) \circ (V_1 \cap \ldots \cap V_n) \subset U_1 \cap \ldots \cap U_n$, where V_i and U_i are subsets of $X \times X$ for $i = 1, 2, \ldots, n$. Consequently, it is sufficient to show that for each $U_{A,B} \in \mathcal{U}_{2\delta}$ there exists a $V \in \mathcal{U}_{2\delta}$ such that $V \circ V \subset U_{A,B}$. We will now show the existence of such a V. By Proposition 1.1.1.3 there exist sets C and D such that $C \cap D = \emptyset$ and $A \ll C$ and $B \ll D$. Let $V = U_{A,X-C} \cap U_{B,X-D}$. We will show that $V \circ V \subset U_{A,B}$. Let us suppose that $(x, y) \in V$ and $(y, z) \in V$. We must show that $(x, z) \in U_{A,B}$ or, equivalently, that $(x, z) \notin (A \times B) \cup (B \times A)$. Clearly, if $x \notin A \cup B$, then for every $t \in X$, $(x, t) \in U_{A,B}$ holds. Hence we may assume that $x \in A \cup B$. Two cases now occur. Case 1: $x \in A$ and case 2: $x \in B$. These are the only possibilities for x since $A \cap B = \emptyset$.

If $x \in A$, then $z \notin B$. Let us suppose that $z \in B$. Then $(y, z) \in C \times B$. But since $C \cap D = \emptyset$, i.e. $C \subset X - D$, we have that $C \times B \subset (X - D) \times B$. Hence $(y, z) \notin V$ which is a contradiction. Therefore, if $x \in A$, then $(x, z) \notin A \times B$, so that $(x, z) \in U_{A,B}$.

By a similar argument we get that $x \in B$ implies $z \notin A$. Now from $x \in B$, there follows $(x, z) \notin B \times A$, so that $(x, z) \in U_{A,B}$.

The uniformity $\mathcal{U}_{2\delta}$ satisfies (M.5), but might fail to satisfy (M_7) . For, let (X, τ) be any R_0 -topological space which is not completely regular. Let us define the relation δ_0 on P(X) by $A\delta_0 B$ if and only if $\overline{A} \cap \overline{B} \neq \emptyset$, so that $\tau_{\delta_0} = \tau$. Then $\mathcal{U}_{2\delta_0}$ cannot satisfy (M_7) : for, if so, then $\mathcal{U}_{2\delta_0}$ would be a symmetric uniformity and hence τ would be a completely regular topology.

2.4.4 Complete M-uniform spaces

Definition 2.4.4.1 Let (X, U) be an M-uniform space. A filter \mathcal{F} on X is weakly Cauchy with respect to the uniformity \mathcal{U} if for every $U \in \mathcal{U}$ there exists $x \in X$ such that $U[x] \in \mathcal{F}$. \mathcal{F} is Cauchy filter with respect to \mathcal{U} if for every $U \in \mathcal{U}$ there exists $A \in \mathcal{F}$ such that $A \times A \subset U$.

Definition 2.4.4.2 A Cauchy filter in (X, U) is an *infrafilter* if it does not properly contain a Cauchy filter.

Definition 2.4.4.3 An *M*-uniform space (X, U) (or *M*-uniformity U) is complete if every weakly Cauchy filter on X has a cluster point in X. (X, U) is Δ -complete if, whenever (X, U) is uniformly isomorphic to a dense subspace (X_a, U_a) of (X_b, U_b) , then $X_a = X_b$.

In a similar way we define a Δ -complete (separated) correct uniform space by taking \mathcal{U}_a and \mathcal{U}_b to be (separated) correct uniformities.

Definition 2.4.4. An *M*-uniform space (X_b, U_b) is a **completion** of the *M*-uniform space (X, U) if (X_b, U_b) is complete and (X, U) is uniformly isomorphic to a dense subspace (X_a, U_a) of (X_b, U_b) .

Proposition 2.4.4.1 Every Cauchy filter on (X, U) is weakly Cauchy filter.

Proof: Let $U \in \mathcal{U}$. There exists $F \in \mathcal{F}$ such that $F \times F \subset U$. Let $x_0 \in F$. Then $F \subset U[x_0]$ so that $U[x_0] \in \mathcal{F}$.

The following theorem points out that it is not reasonable to require every weakly Cauchy filter in an M-uniform space to converge in order for the space to be "complete".

Proposition 2.4.4.2 If (X, τ) is a connected R_0 -topological space, then there exists a totally bounded M-uniformity \mathcal{U} on X such that $\tau_{\mathcal{U}} = \tau$ and every filter in X is weakly Cauchy with respect to \mathcal{U} .

Proof: We know by Corollary 2.3.2.1 that there exists an LO-proximity δ on X such that $\tau_{\delta} = \tau$. Let $\mathcal{U}_{1\delta}$ be the uniformity on X that we constructed in the proof of Theorem 2.4.2.2. $\mathcal{U}_{1\delta} \in \pi(\delta)$, so that $\tau_{\mathcal{U}_{1\delta}} = \tau$. Let $U \in \mathcal{U}_{1\delta}$. Then there exist sets $A \subset X$ and $B \subset X$ such that $U_{\overline{A},\overline{B}} \subset U_{A,B} \subset U$. But since τ is connected, there exists $x_0 \in X - (\overline{A} \cup \overline{B})$, so that $U_{\overline{A},\overline{B}}[x_0] = X$. Hence every filter on X is weakly Cauchy with respect to \mathcal{U} .

Example 2.4.4.1 There exists an M-uniform space (X, \mathcal{U}) and there exists a filter \mathcal{F} on X such that \mathcal{F} is weakly Cauchy with respect to \mathcal{U} , but \mathcal{F} is not Cauchy with respect to \mathcal{U} .

Indeed, let (X, τ) be any connected T_1 -topological space with at least two distinct points. By Corollary 2.3.2.1 there exists an LO-proximity δ on X such that $\tau_{\delta} = \tau$. Since τ is T_1 , it follows by Proposition 2.4.1.3 that δ is separated. Let $\mathcal{U}_{1\delta}$ be the uniformity on X constructed in Theorem 2.4.2.2. Let us consider the filter $\mathcal{F} = \{X\}$ on X. As it was shown in the proof of Proposition 2.4.4.2, \mathcal{F} is weakly Cauchy with respect to $\mathcal{U}_{1\delta}$. Let x_1 and x_2 be any two distinct points in X. Let us consider U_{x_1,x_2} . Since δ is separated, $x_1\overline{\delta}x_2$, so that $U_{x_1,x_2} \in \mathcal{U}_{1\delta}$. Hence \mathcal{F} is not Cauchy filter with respect to $\mathcal{U}_{1\delta}$.

Proposition 2.4.4.3 If an M-uniform space (X, U) has an open base, then every convergent filter on X relative to τ_U is a Cauchy filter.

Proof: Let $U \in \mathcal{U}$. Since \mathcal{U} has an open base, there exists $U_1 \in \mathcal{U}$ such that $U_1 \subset U$ and U_1 is open in the product topology on $X \times X$. Let us suppose that \mathcal{F} is a filter on X which converges to x_0 . Since U_1 is open, there exists an open set $A \in \mathcal{N}_{x_0}$, where \mathcal{N}_{x_0} is a neighborhood system of the point x_0 , such that $A \times A \subset U_1$. But $A \in \mathcal{F}$. Hence \mathcal{F} is Cauchy with respect to \mathcal{U} .

Proposition 2.4.4.4 Every convergent filter on an M-uniform space (X, U) is a weakly Cauchy filter.

Proof: Let \mathcal{F} be a filter on X which converges to $x_0 \in X$ relative to $\tau_{\mathcal{U}}$. If $U \in \mathcal{U}$, then, by Corollary 2.4.1.1, $U[x_0]$ is an element of the neighborhood system of the point x_0 . Hence $U[x_0] \in \tau$.

Proposition 2.4.4.5 Let (X, U) be a correct uniform space. A filter \mathcal{F} on X is Cauchy with respect to U if it is weakly Cauchy with respect to U.

Proof: Let us suppose that \mathcal{F} is a weakly Cauchy filter with respect to \mathcal{U} . If $U \in \mathcal{U}$, then there exists a $V \in \mathcal{U}$ such that $V \circ V \subset U$; there also exists $x_0 \in X$ such that $V[x_0] \in \mathcal{F}$. Let $(a, b) \in V[x_0] \times V[x_0]$. Then $a \in V[x_0]$ and $b \in V[x_0]$. Consequently, $(a, b) \in V \circ V \subset U$.

The following facts about infrafilters are easily established.

Proposition 2.4.4.6 Let \mathcal{F} be an infrafilter and let \mathcal{F}_1 be a Cauchy filter in a correct uniform space (X, \mathcal{U}) . Let $\mathcal{U}(\mathcal{F}_1) = \{U[F] : F \in \mathcal{F}_1 \text{ and } U \in \mathcal{U}\}$. Then

(a) $\mathcal{U}(\mathcal{F}_1)$ is an infrafilter contained in \mathcal{F}_1 ;

(b) \mathcal{F}_1 is an infrafilter if and only if, for every $A \in \mathcal{F}_1$, there exists a $B \in \mathcal{F}_1$ and a $U \in \mathcal{U}$ such that $U[B] \subset A$;

(c) the neighborhood system \mathcal{N}_x of the point x is an infrafilter in (X, \mathcal{U}) ; (d) \mathcal{F} has an open base;

(e) for every $U, V \in \mathcal{U}$ there exists a $W \in \mathcal{U}$ such that, if $F \in \mathcal{F}$ and $F \times F \subset W$, then $F \times F \subset U \cap V$;

(f) if (X_a, \mathcal{U}_a) is a dense subspace of a correct uniform space (X_b, \mathcal{U}_b) and if \mathcal{F}_0 is an infrafilter in (X_b, \mathcal{U}_b) , then $\mathcal{B} = \{U[F] \cap X_a : F \in \mathcal{F}_0 \text{ and } U \in \mathcal{U}_b\}$ is a base for an infrafilter \mathcal{F}_0^* in (X_a, \mathcal{U}_a) .

Let us note that in the proof of Proposition 2.4.1.1 we actually only used the following weak form of (M_4) :

 (M_4^*) for every $x \in X$ and $U, V \in \mathcal{U}$ there exists a $W \in \mathcal{U}$ such that $W[x] \subset U[x] \cap V[x]$.

Definition 2.4.4.5 Let X be a non empty set. A non empty subset \mathcal{U} of $P(X \times X)$ is a semi-correct uniformity on X if \mathcal{U} satisfies (M_2) , (M_3) , (M_4^*) , (M_7) and (M_8) .

By the above statement, if (X, \mathcal{U}) is a semi-correct uniform space, then the function $g: P(X) \to P(X)$ defined by $x \in g(A)$ if and only if $U[x] \cap A \neq \emptyset$ for all $U \in \mathcal{U}$, is a Kuratowski closure function. By a straightforward computation it is possible to show that if (X_a, \mathcal{U}_a) is a dense subspace of the semi-correct uniform space (X_b, \mathcal{U}_b) , then (X_a, \mathcal{U}_a) is a separated correct uniform space if and only if (X_b, \mathcal{U}_b) is a separated correct uniform space. Also, it is easy to show that a subset \mathcal{B} of $P(X \times X)$ is a base for some semi-correct uniformity on X if and only if \mathcal{B} satisfies $(M_2), (M_3), (M_4^*)$ and (M_7) .

Theorem 2.4.4.1 Let (X, U) be a separated, correct uniform space. Then the following statements are equivalent:

- (a) (X, \mathcal{U}) is Δ -complete;
- (b) every infrafilter is a neighborhood system of some point;
- (c) every Cauchy filter on (X, \mathcal{U}) converges;
- (d) (X, \mathcal{U}) is complete.

Proof: $(a) \Rightarrow (b)$: The neighborhood system of the point x will be denoted by \mathcal{N}_x . Let us suppose that there exists at least one infrafilter on X which is not a neighborhood system of a point in X. Let X_b be the family of all infrafilters on X. Let X_a be the family of all neighborhood systems of the points in X. It is clear that $X_a \subset X_b$. For each $U \in \mathcal{U}$ let $\overline{U} = \{(P_1, P_2) : F \times F \subset U \text{ for some } F \in P_1 \cap P_2\}$. Let $\mathcal{B} = \{\overline{U} : U \in \mathcal{U}\}$. To show that \mathcal{B} is a base for a semi-correct uniformity \mathcal{U}_b on X_b , it is sufficient to prove that \mathcal{B} satisfies $(M_2), (M_3), (M_4^*)$ and (M_7) .

 (M_2) : Let us suppose that $P_1, P_2 \in X_b$, $P_1 \neq P_2$ and $(P_1, P_2) \in \overline{U}$ for every $\overline{U} \in \mathcal{B}$. Then, for every $U \in \mathcal{U}$ there exists $F \in P_1 \cap P_2$ such that $F \times F \subset U$. Hence $P_3 = P_1 \cap P_2$ is a Cauchy filter, so that, since $P_3 \subset P_1$ and $P_3 \subset P_2$ we have by Definition 2.4.4.1 that $P_1 = P_2 = P_3$ which is a contradiction.

 (M_3) Since $U = U^{-1}$ for every $U \in \mathcal{U}$, it is clear that $\overline{U}^{-1} = \overline{U}$ for every $\overline{U} \in \mathcal{B}$.

 (M_4^*) : Let $P \in X$ and let $\overline{U}, \overline{V} \in \mathcal{B}$. By Proposition 2.4.4.6 (e), there exists $W \in \mathcal{B}$ such that for all $F \in P$ if $F \times F \subset W$ then $F \times F \subset U \cap V$. We claim that $\overline{W}[P] \subset \overline{U}[P] \cap \overline{V}[P]$. For, let us suppose that $P_1 \in \overline{W}[P]$. Then $(P, P_1) \in \overline{W}$, so that there exists $F \in P \cap P_1$ such that $F \times F \subset W$ and hence $F \times F \subset U \cap V$. Consequently, $P_1 \in \overline{U}[P] \cap \overline{V}[P]$.

 (M_7) : Let us suppose that $\overline{U} \in \mathcal{B}$. There exists $V \in \mathcal{U}$ such that $V \subset U$ and $V \circ V \subset U$. We claim that $\overline{V} \circ \overline{V} \subset \overline{U}$. Let us suppose that $(P_1, P_2) \in \overline{V}$ and let $(P_2, P_3) \in \overline{V}$. Then there exists $F \in P_1 \cap P_2$ such that $F \times F \subset V$ and there exists $G \in P_2 \cap P_3$ such that $G \times G \subset V$. But this implies that $E \times E \subset U$ for some $E \in P_1 \cap P_3$. Hence $(P_1, P_3) \in \overline{U}$. (Let $E = G \cup F$.)

Consequently, $\mathcal{U}_b = \{\overline{U} : \overline{U} = \overline{U}^{-1} \text{ and } \overline{V} \subset \overline{U} \text{ for some } \overline{V} \in \mathcal{B}\}$ is a semi-correct uniformity on X_b .

Let us consider the mapping $h: X \to X_b$ defined by $h(x) = \mathcal{N}_x$. Since (X,\mathcal{U}) is separated, $\tau_{\mathcal{U}}$ is T_0 , so that h is 1-1. Clearly, h is onto X_a . Let $\overline{U} \in \mathcal{B}$. There exists an open set $V \in \mathcal{U}$ such that $V \subset U$ and $V \circ V \subset U$. Let us suppose that $(x,y) \in V$. Then, by a straightforward calculation, it can be shown that if $F = V[x] \cap V[y]$, then $F \times F \subset U$ and $F \in \mathcal{N}_x \cap \mathcal{N}_y$, so that $(\mathcal{N}_x, \mathcal{N}_y) \in \overline{U}$. Conversely, let us suppose that $(\mathcal{N}_x, \mathcal{N}_y) \in \overline{U}$. Then it immediately holds that $(x, y) \in U$. Hence we have that (X, \mathcal{U}) is uniformly isomorphic to (X_a, \mathcal{U}_a) where \mathcal{U}_a is the relativization of \mathcal{U}_b to X_a .

Let us suppose that P_1 is any point in X_b . Let \overline{U} be any element of \mathcal{B} . $(P_1, P_2) \in \overline{U}$, so that, by Proposition 2.4.4.6 (d), there exists an open set $F \in P_1$ such that $F \times F \subset U$. Let $x_0 \in F$. Then $F \in \mathcal{N}_{x_0}$, so that $(P_1, \mathcal{N}_{x_0}) \in \overline{U}$. Hence X_a is dense in X_b . Consequently, (X_b, \mathcal{U}_b) is a separated correct uniform space.

Thus we can see that if there exists at least one infrafilter which is not the neighborhood system of some point in X, then it is possible to construct a separated correct uniform space (X_b, \mathcal{U}_b) such that (X, \mathcal{U}) is uniformly isomorphic to a dense subspace (X_a, \mathcal{U}_a) of (X_b, \mathcal{U}_b) and $X_a \neq X_b$. Consequently, (X, \mathcal{U}) is not Δ -complete.

 $(b) \Rightarrow (a)$: Let us suppose that (X, \mathcal{U}) is not Δ -complete. Then (X, \mathcal{U}) is uniformly isomorphic to a dense subspace (X_a, \mathcal{U}_a) of (X_b, \mathcal{U}_b) and $X_a \neq X_b$. Let us suppose that $P \in X_b - X_a$. Let $\mathcal{F} = \mathcal{N}_P$. Since \mathcal{F} is an infrafilter, by Proposition 2.4.4.6, it induces in (X_a, \mathcal{U}_a) an infrafilter \mathcal{F}^* . But, by hypothesis, $\mathcal{F}^* = \mathcal{N}_{P_1}$ for some point $P_1 \in X_a$. Hence $P \in \cap \{F : F \in \mathcal{F}\}$ and $P_1 \in \cap \{F : F \in \mathcal{F}\}$. But since \mathcal{F} is a Cauchy filter, this means that $(P, P_1) \in \overline{U}$ for every $\overline{U} \in \mathcal{U}_b$, and since (X_b, \mathcal{U}_b) is separated, this is a contradiction.

 $(b) \Rightarrow (c)$: Let \mathcal{F} be a Cauchy filter in (X, \mathcal{U}) . By Proposition 2.4.4.6 (a), \mathcal{F} contains an infrafilter \mathcal{F}_1 in (X, \mathcal{U}) . But, by hypothesis, $\mathcal{F}_1 = \mathcal{N}_{x_0}$ for some $x_0 \in X$. Hence \mathcal{F} converges to x_0 .

 $(c) \Rightarrow (b)$: Let \mathcal{F} be an infrafilter in (X, \mathcal{U}) . By hypothesis, $\mathcal{N}_{x_0} \subset \mathcal{F}$ for some $x_0 \in X$. But \mathcal{N}_{x_0} is a Cauchy filter in (X, \mathcal{U}) . Hence $\mathcal{F} = \mathcal{N}_{x_0}$.

 $(c) \Rightarrow (d)$: Let \mathcal{F} be a weakly Cauchy filter with respect to \mathcal{U} . By Proposition 2.4.4.5 \mathcal{F} is a Cauchy filter with respect to \mathcal{U} . But then \mathcal{F} is convergent and hence has a cluster point.

 $(d) \Rightarrow (c)$: Let \mathcal{F} be a Cauchy filter with respect to \mathcal{U} . By Proposition 2.4.4.1 \mathcal{F} is a weakly Cauchy filter with respect to \mathcal{U} and hence has a cluster point. By Proposition 2.4.4.6 (g), \mathcal{F} is convergent.

Proposition 2.4.4.7 If (X, U) is a totally bounded M-uniform space, then every ultrafilter on X is a weakly Cauchy filter.

Proof: Let \mathcal{F} be an ultrafilter in (X, \mathcal{U}) and let $V \in \mathcal{U}$. There exist x_1, x_2, \ldots, x_n in X such that $X = V[x_1] \cup \ldots \cup V[x_n]$. But then, since $X \in \mathcal{F}$, there holds by Proposition 2.4.4.6 (c), that for some m, where $1 \leq m \leq n, V[x_m] \in \mathcal{F}$.

Theorem 2.4.4.2 An M-uniform space (X, U) is complete and totally bounded if and only if (X, τ_U) is compact.

Proof: Let us assume that $(X, \tau_{\mathcal{U}})$ is compact and let $U \in \mathcal{U}$. Let us consider the family $\{U[x] : x \in X\}$. By Corollary 2.4.1.1, for each $x \in X$, $Int U[x] \neq \emptyset$ holds. Therefore for each $x \in X$ there exists an open set O_x such that $x \in O_x \subset U[x]$. Hence, since $\tau_{\mathcal{U}}$ is compact, there exist x_1, \ldots, x_n such that $X = U[x_1] \cup \ldots \cup U[x_n]$, so that (X, \mathcal{U}) is totally bounded. Let \mathcal{F} be a weakly Cauchy filter. Since $\tau_{\mathcal{U}}$ is compact, \mathcal{F} has a cluster point, so that (X, \mathcal{U}) is complete.

Conversely, let \mathcal{F} be an ultrafilter on X. Since (X, \mathcal{U}) is totally bounded, \mathcal{F} is a weakly Cauchy filter by Proposition 2.4.4.7. But since (X, \mathcal{U}) is complete, \mathcal{F} has a cluster point, so that by Proposition 2.4.4.6 (d), \mathcal{F} is convergent. Consequently, by Proposition 2.4.4.6 (f), $\tau_{\mathcal{U}}$ is compact.

Corollary 2.4.4.1 Let (X, U) be a separated, correct uniform space. Then $(X, \tau_{\mathcal{U}})$ is compact if and only if (X, U) is totally bounded, and every infrafilter on X is a neighborhood system of some point in X.

Proof: This is an immediate consequence of Theorem 2.4.4.1 and Theorem 2.4.4.2. \clubsuit

Corollary 2.4.4.2 Every closed subspace (Y, \mathcal{V}) of a complete M-uniform space (X, \mathcal{U}) is a complete space.

Proof: Let (Y, \mathcal{V}) be a closed subspace of (X, \mathcal{U}) . Let \mathcal{F}_1 be any weakly Cauchy filter on Y relative to \mathcal{V} . \mathcal{F}_1 can be considered as a filter base for a filter \mathcal{F}_1^* on X. It is clear that \mathcal{F}_1^* is a weakly Cauchy filter on X, relative to \mathcal{U} and hence has a cluster point $x_0 \in X$. But then x_0 is a cluster point of \mathcal{F}_1 , so that x_0 is an accumulation point of Y. Since Y is closed, $x_0 \in Y$. Hence (Y, \mathcal{V}) is complete. \clubsuit

Definition 2.4.4.6 Let (X, τ) be a topological space. Let \mathcal{U} be any structure on X which generates a topology $\tau_{\mathcal{U}}$ on X. Then \mathcal{U} is compatible with the topology τ if $\tau_{\mathcal{U}} = \tau$.

Theorem 2.4.4.3 A R_0 topological space (X, τ) is compact if and only if it is complete with respect to every compatible M-uniformity \mathcal{U} on X.

Proof: Let (X, \mathcal{U}) be compatible with (X, τ) . By Theorem 2.4.4.2, (X, \mathcal{U}) is complete.

Conversely, we know by Corollary 2.3.2.1 that there exists a an LO-proximity δ on X such that $\tau_{\delta} = \tau$. Let $\mathcal{U}_{1\delta}$ be the M-uniformity on X constructed in Theorem 2.4.2.2. We know that $\tau_{\mathcal{U}_{1\delta}} = \tau$, so that, by hypothesis, $\mathcal{U}_{1\delta}$ is complete. But, by Corollary 2.4.2.3 $\mathcal{U}_{1\delta}$ is totally bounded. Hence by Theorem 2.4.4.2, τ is compact.

Let us note the analogy between the above theorem and the theorem of Niemytzki and Tychonoff who states that a metrizable topological space is compact if and only if it is complete in every compatible metric (see [241]). Also, let us recall the theorem of Doss which states that a completely regular space (X, τ) is compact if and only if it is complete with respect to every compatible uniformity \mathcal{U} on X (see [89]). **Proposition 2.4.4.8** An M-uniform space (X, U) is totally bounded if and only if every filter on X is contained in a weakly Cauchy filter.

Proof: Let us suppose that (X, \mathcal{U}) is totally bounded and let \mathcal{F} be a filter on X. By Proposition 2.4.4.6 (b), \mathcal{F} is contained in an ultrafilter \mathcal{F}_1 which, by Proposition 2.4.4.7, is a weakly Cauchy filter.

Conversely, let us suppose that every filter on X is contained in a weakly Cauchy filter. Let $U \in \mathcal{U}$. For every finite subset $E \subset X$ let us assume that $U[E] \neq X$, so that $X - U[E] \neq \emptyset$. The family $\{X - U[E] :$ E is a finite subset of X $\}$ is easily shown to be a base for a filter, which is, by hypothesis, contained in a weakly Cauchy filter \mathcal{F} . For some point $x_0 \in X \ U[x_0] \in \mathcal{F}$ is true. On the other hand, since $\{x_0\}$ is a finite set, $X - U[x_0] \in \mathcal{F}$. But since $U[x_0] \cap (X - U[x_0]) = \emptyset$, we have that $\emptyset \in \mathcal{F}$ which is a contradiction.

Proposition 2.4.4.9 Let an M-uniform space (X, U) be a totally bounded, dense subspace of M-uniform space (X_a, U_a) . If every element of every weakly Cauchy filter on X_a has a non empty interior (relative to τ_{U_a}), and if every weakly Cauchy filter (relative to U) on X has a cluster point in X_a , then (X_a, U_a) is complete.

Proof: Let \mathcal{F} be a weakly Cauchy filter on X_a such that $Int F \neq \emptyset$ for every $F \in \mathcal{F}$. Since X is dense in X_a , $F \cap X \neq \emptyset$ for every $F \in \mathcal{F}$. Let $\mathcal{B} = \{F \cap X : F \in \mathcal{F}\}$. Clearly, \mathcal{B} is a base for a filter \mathcal{F}_1 on X which is, by Proposition 2.4.4.8, contained in a weakly Cauchy filter \mathcal{F}_2 on X. But, by hypothesis, \mathcal{F}_2 has a cluster point $x_0 \in X_a$. Let $U \in \mathcal{U}_a$ and let $F \in \mathcal{F}$. Then $U[x_0] \cap (F \cap X) \neq \emptyset$, so that $U[x_0] \cap F \neq \emptyset$. Hence x_0 is a cluster point for \mathcal{F} and (X_a, \mathcal{U}_a) is complete. \clubsuit

Theorem 2.4.4.4 If an M-uniform space (X, \mathcal{U}) is separated and Δ -complete, then every weakly Cauchy filter on X is the neighborhood system of some point in X.

Proof: The neighborhood system of the point x will be denoted by \mathcal{N}_x . Let us suppose that there exists at least one weakly Cauchy filter on X which is not the neighborhood system of a point in X. Let X_b be the family of all weakly Cauchy filters on X. Let X_a be the family of all neighborhood systems of the points in X. It is clear that $X_a \subset X_b$. To construct the uniformity \mathcal{U}_b on X_b in the proper way, let us assign to each filter P in the set X_b a point $x_P \in X$ in the following way: $x_P = x_1$ if $P = \mathcal{N}_{x_1}$, and x_P is any point in X if $P \neq \mathcal{N}_x$ for every $x \in X$. For each $U \in \mathcal{U}$ let $\overline{U} = \{(P_1, P_2) : (x_{P_1}, x_{P_2}) \in U\}$. Let \mathcal{B} be equal to $\{\overline{U} : U \in \mathcal{U}\}$. We will show that \mathcal{B} is a base for an M-uniformity \mathcal{U}_b on X_b . By Theorem 2.4.1.4 it is sufficient to show that \mathcal{B} satisfies $(M_1), (M_3), (M_4)$ and (M_6) .

 (M_1) : Let $\overline{U} \in \mathcal{B}$. Since $(x_P, x_P) \in U$ for every $P \in X_b$ we have that $(P, P) \in \overline{U}$ for every $P \in X_b$.

 (M_3) : Since $U = U^{-1}$ for every $U \in \mathcal{U}$, we have that $\overline{U}^{-1} = \overline{U}$ for every $\overline{U} \in \mathcal{B}$.

 (M_4) : Let $A^* \subset X_b$ and let $\overline{U}, \overline{V} \in \mathcal{B}$. Let $A = \{x_P : P \in A^*\}$. There exists, by (M_4) , a $W \in \mathcal{U}$ such that $W[A] \subset U[A] \cap V[A]$. Let $P_1 \in \overline{W}[A^*]$. Then $(P_a, P_1) \in \overline{W}$ for some $P_a \in A^*$, so that $(x_{P_a}, x_{P_1}) \in W$. Consequently, $x_{P_1} \in W[A]$, so that $x_{P_1} \in U[A] \cap V[A]$. But this means that there exists $x_{P_r} \in A$ and $x_{P_s} \in A$ such that $(x_{P_r}, x_{P_1}) \in U$ and $(x_{P_s}, x_{P_1}) \in V$, so that $P_1 \in \overline{U}[A^*] \cap \overline{V}[A^*]$. Hence, there exists a $\overline{W} \in \mathcal{B}$ such that $\overline{W}[A^*] \subset \overline{U}[A^*] \cap \overline{V}[A^*]$.

 $(B_6): \text{ Let } A^*, B^* \subset X_b \text{ and let } \overline{U}, \overline{V} \in \mathcal{B}. \text{ Let us suppose that } \overline{V}[A^*] \cap B^* \neq \emptyset. \text{ Let } A = \{x_P : P \in A^*\}, B = \{x_P : P \in B^*\} \text{ and let } P_c \in \overline{V}[A^*] \cap B^*. \text{ Then } P_c \in \overline{V}[A^*] \text{ and } P_c \in B^*, \text{ so that for some } P_a \in A^* \text{ we have that } (P_a, P_c) \in \overline{V} \text{ and hence } (x_{P_a}, x_{P_c}) \in V. \text{ Consequently, since } \overline{V} \text{ is any element in } \mathcal{B}, V[A] \cap B \neq \emptyset \text{ for all } V \in \mathcal{U}. \text{ But by } (M_6) \text{ there exists a } W \in \mathcal{U} \text{ and an element } x_{P_b} \in B \text{ such that } W[x_{P_b}] \subset U[A]. \text{ Let } P_1 \in \overline{W}[P_b]. \text{ Then } (x_{P_b}, x_{P_1}) \in W \text{ and } x_{P_1} \in W[x_{P_b}], \text{ so that there exists an } x_{P_d} \in A \text{ such that } (x_{P_d}, x_{P_1}) \in U \text{ or equivalently, } (P_d, P_1) \in \overline{U} \text{ and } P_1 \in \overline{U}[A^*]. \text{ Hence } \overline{W}[P_b] \subset \overline{U}[A^*]. \text{ Consequently, } \mathcal{U}_b = \{\overline{U} : \overline{U} = \overline{U}^{-1} \text{ and } \overline{V} \subset \overline{U} \text{ for some } \overline{V} \in \mathcal{B}\} \text{ is an M-uniformity on } X_b.$

Let us consider the mapping $h: X \to X_b$ defined by $h(x) = \mathcal{N}_x$. Since (X, \mathcal{U}) is separated, $\tau_{\mathcal{U}}$ is T_0 , so that h is 1-1. Clearly h is onto X_a . Let $U \in \mathcal{U}$ and let $(x, y) \in U$. Then $(\mathcal{N}_x, \mathcal{N}_y) \in \overline{U}$. Conversely, let us suppose that $(\mathcal{N}_x, \mathcal{N}_y) \in \overline{U}$. Then $(x, y) \in U$. Hence (X, \mathcal{U}) is uniformly isomorphic to (X_a, \mathcal{U}_a) where \mathcal{U}_a is the relativization of \mathcal{U}_b to X_a .

Let us suppose that P_1 is any point in X_b . Let \overline{U} be any element of \mathcal{B} . $(P_1, P_1) \in \overline{U}$, so that $(x_{P_1}, x_{P_1}) \in U$. Hence $(P_1, \mathcal{N}_x) \in \overline{U}$, so that X_a is dense in X_b .

Thus we can see that if there exists at least one weakly Cauchy filter which is not the neighborhood system of some point in X, then it is possible to construct an M-uniform space (X_b, \mathcal{U}_b) such that (X, \mathcal{U}) is uniformly isomorphic to a dense subspace (X_a, \mathcal{U}_a) of (X_b, \mathcal{U}_b) and $X_a \neq X_b$. Consequently, (X, \mathcal{U}) is not Δ -complete.

If (X_b, \mathcal{U}_b) , as constructed in the proof of Theorem 2.4.4.4, is complete, then (X, \mathcal{U}) is complete. To prove this fact, let us suppose that \mathcal{F} is a weakly Cauchy filter on X and let $\mathcal{F}^* = \{h(F) : F \in \mathcal{F}\}$. It is obvious that \mathcal{F}^* is a base for a filter $\mathcal{F}_1^* \in X_b$, where \mathcal{F}_1^* is a weakly Cauchy filter with respect to \mathcal{U}_b . Thus \mathcal{F}_1^* has a cluster point $P_1 \in X_b$. P_1 is also a cluster point for \mathcal{F}^* . Let $F \in \mathcal{F}$ and $F^* = h(F)$. Let $U \in \mathcal{U}$. Then there exists $\mathcal{N}_{x_1} \in \overline{U}[P_1] \cap F^*$, so that $x_1 \in U[x_{P_1}] \cap F$. Consequently, x_{P_1} is a cluster point for \mathcal{F}_1 and (X, \mathcal{U}) is complete. Thus we see that the construction used in the proof of Theorem 2.4.4.4 does not yield a completion for (X, \mathcal{U}) .

Theorem 2.4.4.5 Every separated correct uniform space has a unique completion.

Proof: We will show that (X_b, \mathcal{U}_b) , as constructed in the proof of Theorem 2.4.4.1, is complete. Let $\overline{U} \in \mathcal{B}$ and let $\overline{\mathcal{F}}$ be an infrafilter in (X_b, \mathcal{U}_b) . By Proposition 2.4.4.6 (f), $\overline{\mathcal{F}}$ induces in X_a the infrafilter \mathcal{F}^* in (X_a, \mathcal{U}_a) which is the natural image under the mapping h (as defined in the proof of Theorem 2.4.4.1) of the filter \mathcal{F} in (X, \mathcal{U}) . We will now show that \mathcal{F} , which, of course, is an element of X_b , is a cluster point for $\overline{\mathcal{F}}$. By Proposition 2.4.4.6 (d) there exists an open $\overline{G} \in \overline{\mathcal{F}}$ such that $\overline{G} \times \overline{G} \subset \overline{U}$. Let $G^* = \overline{G} \cap X$. Let $G = h^{-1}(G^*)$. It is clear that G is open in $X, G \in \mathcal{F}$ and $G \times G \subset U$. Hence for every $x \in G$ we have that $\mathcal{N}_x \in \overline{U}[\mathcal{F}]$, so that $G^* \subset \overline{U}[\mathcal{F}]$. But by Proposition 2.4.4.6 $(g), \mathcal{N}_{\mathcal{F}} \subset \overline{\mathcal{F}}$, so that, since $\overline{\mathcal{F}}$ is an infrafilter, $\overline{\mathcal{F}} = \mathcal{N}_{\mathcal{F}}$. Consequently, by Theorem 2.4.4.1, (X_b, \mathcal{U}_b) is complete.

That the completion is unique is shown in a straightforward manner. \clubsuit

The existence of a completion for more general types of M-uniform spaces is an open question.

Historical and bibliographic notes

The results of this section are based on papers [201] and [202] by M. W. Lodato, the paper [217] by Mordkovich and paper [94] by V. A. Efremovich, A. G. Mordkovich and V. Ju. Sandberg. In papers [219] by C. J. Mozzochi the notion of an M-uniform space is generalized in such a way that every uniformity of that kind generates an LO-proximity in a natural way. It is then shown that the classical theorem, which states that every proximity class of M-uniformities contains one and only one totally bounded uniformity, can be generalized to these M-uniform and LO-proximity spaces in such a way that the classical theorem follows as an immediate corollary. Generalizations and partial generalizations are also obtained for many other classical theorems concerning a uniform continuity, uniform convergence, convergence in proximity, completeness and compactness.

The correct spaces of Efremovich, Mordkovich and Sandberg are a special subclass of M-uniform spaces. The axioms for the correct spaces are almost as strong as those for an M-uniform space.

All the results of this section were proved by C. J. Mozzoochi in [219] (see also [220], [222], [223] and [224]).

2.5 R- and RC-proximity spaces

2.5.1 The notion and basic properties of R-proximities

Definition 2.5.1.1 A semi-proximity relation δ defined on the power set of X is called an *R*-proximity if it satisfies the following condition:

(R) (Axiom of regularity) if $\{x\}\overline{\delta}A$, then there is $B \subset X$ such that $\{x\}\overline{\delta}X - B$ and $B\overline{\delta}X - A$.

An *R*-proximity δ is **separated** if it is a separated semi-proximity. The pair (X, δ) , where δ is a (separated) *R*-proximity, is referred to as a (**separated**) *R***-proximity space**.

Clearly the concept of an R-proximity is a generalization of the Efremovich proximity.

Proposition 2.5.1.1 If (X, δ) is an *R*-proximity space, then it is an *S*-proximity space.

Proof: The proof is essentially the same as the one given in Proposition 2.3.1.1. \clubsuit

Definition 2.5.1.2 A δ -neighborhood of a set $A \subset X$ in an R-proximity space (X, δ) is a set B such that $A\overline{\delta}X - B$.

One can prove the following proposition in a manner similar to the proof of Proposition 2.3.1.2.

Proposition 2.5.1.2 Let \ll be a relation on P(X) such that the following conditions are satisfied:

(a) $\emptyset \ll A$ for each $A \subset X$;

(b) if $A \ll B$, then $A \subset B$;

- (c) if $A \subset A_1 \ll B_1 \subset B$, then $A \ll B$;
- (d) if $A \ll B_1$, i = 1, 2, then $A \ll B_1 \cap B_2$;
- (e) if $A \ll B$, then $X B \ll X A$;
- (f) if $x \ll A$, then there exists a set $B \subset X$ such that $x \ll B \ll A$.

Then there exists a unique R-proximity δ for X such that $A \ll B$ if and only if $A\overline{\delta}X - B$, that is, the set B is a δ -neighborhood of the set A.

Proposition 2.5.1.3 If A and B are subsets of an R-proximity space (X, δ) , then $A \ll B$ implies that

(a)
$$A \subset cA \subset B$$
, and (b) $A \subset X - c(X - B) \subset B$,

where $cA = \{x : x\delta A\}$.

Proof: The proof of this inclusions is the same as the one given in Proposition 2.3.1.1.

Proposition 2.5.1.4 Let (X, δ) be a separated *R*-proximity space. The function $A \to cA$, where $cA = \{x \in X : x\delta A\}$, is a Kuratovski closure function. The topology induced by the proximity δ is regular and c is the closure operator induced by the topology.

Proof: It is sufficient to show that ccA = cA for every $A \subset X$. To verify that ccA = cA, we need only to show that $ccA \subset cA$. Now if $x \notin cA$, we have $x\overline{\delta}A$, that is, $x \ll X - A$, so by (P) there exists a set $B \subset X$ such that $x \ll B \ll X - A$. According to Proposition 2.5.1.3 we have that $B \subset X - cA \subset X - A$, and hence, from Proposition 2.5.1.2, we have that $x \ll X - cA$, which is equivalent to $x \notin ccA$.

We have now shown that c is the closure operator of the topology that it induces: the closed sets are precisely the sets of the form cA for some $A \subset X$. Now this fact, along with Proposition 2.5.1.3, shows that the proximal neighborhood filter of each point of X is a **regular filter** (that is, a filter with a base of open sets and a base of closed sets). In particular, the proximal neighborhood filter of each point is contained in the neighborhood filter of the point. Since, by the definition of the topology, the converse inclusion also holds, equality of the two filters holds. It therefore holds that the neighborhood filter of each point of the space is regular, that is, the topology is regular.

The next result includes a generalization of the converse of Proposition 2.5.1.4.
Proposition 2.5.1.5 Let us suppose that Z is a regular topological space and that X is a dense subspace of Z. Let us define a relation between subsets of X by setting $A\delta B$ if $cl_Z A \cap cl_Z B \neq \emptyset$.

(a) The relation δ is a separated R-proximity on X.

(b) A filter on X is round if and only if it is the trace of a filter that is regular on Z.

Proof: (a) $(SP_1) - (SP_3)$ and (SP_5) are immediate, and (SP_4) follows from the distributivity of the closure with respect to finite unions. To show (R), we observe that if V is a neighborhood (in X) of $x \in X$, there exists a closed neighborhood (in Z) B of $x \in Z$ and an open neighborhood (in Z) W of $x \in Z$ such that $W \cap X = V$ and $B \subset W$. Setting $A = B \cap X$, we find that $x \ll A \ll V$.

(b) Let us suppose that γ is a round filter on X, and let ζ be the filter on Z generated by $\{cl_ZF : F \in \gamma\}$. Then ζ certainly has a base of closed sets. Now if $F \in \gamma$ and $G \in \gamma$ with $G \ll F$, then $cl_ZG \cap cl_Z(X - F) = \emptyset$. Since X is dense in Z, it also holds that $cl_ZF \cup cl_Z(X - F) = Z$. It follows that $cl_ZG \subset Z - cl_Z(X - F) \subset cl_ZF$, and we have thus shown that ζ also has a base of open sets. Thus ζ is a regular filter, and it clearly induces γ on X.

Conversely, let us suppose that ζ is a regular filter on Z. Since X is dense in Z and ζ has a base of open sets, every member of ζ intersects X and so the trace γ on X of ζ exists. If $V \in \gamma$, so that $V = W \cap X$ for some open set $W \in \zeta$, let P be any member of ζ such that $cl_Z P \subset W$. Then if $Q = P \cap X$ there follows that $cl_Z Q \cap cl_Z (X - V) = \emptyset$, so $Q \ll V$ and $Q \in \gamma$. Thus we have shown that γ is a round filter. \clubsuit

The proximity defined on a regular space by declaring sets to be near if their closures intersect is, according to Proposition 2.5.1.5, an R-proximity that induces the topology of the space. We can state the following:

Theorem 2.5.1.1 A topology is regular if and only if it is the topology induced by a separated R-proximity.

There may, of course, be many R-proximities that induce a given regular topology.

2.5.2 **R**-proximities and LO-proximities

There are three semi-proximities that can be defined on any T_1 space and that will be useful in the examples below. These semi-proximities are con-

sidered in Example 2.2.2.1. It is appropriate here to observe that the semiproximities considered in Example 2.2.2.1 are more general than those that we consider, since the property (SP_5) need not be satisfied by the semiproximity of Example 2.2.2.1.

The proximities considered below do satisfy (SP_5) ; however, since the associated topologies are T_1 , the following results are readily established from the definitions.

Proposition 2.5.2.1 If X is a T_1 space, then $A\delta_c B$ if and only if $A\delta_w B$ or both A and B are infinite.

Proposition 2.5.2.2 If X is a T_1 space, then the proximity δ_w is the finest LO-proximity that induces the topology of X.

If X is regular, then δ_w is an **LR-proximity**, that is, the separated semi-proximity that is simultaneously an LO-proximity and an R-proximity.

Proposition 2.5.2.3 Let X be a T_1 space with no isolated point. Then the LO-proximity δ_c induces the topology of X and is not an R-proximity.

Proof: Let us suppose that δ_c is an R-proximity. Then the topology of X is regular. If X has the cofinite topology, then it is finite and so every point is isolated. If X does not have the cofinite topology, then there is $x \in X$ and a neighborhood V of x such that X - V is infinite. Now there is a neighborhood W of x such that $W \ll V$, and it follows that W is finite, and therefore x is an isolated point.

Corollary 2.5.2.1 There exists a compact Hausdorff space X such that δ_c is not an R-proximity and thus $\delta_c \neq \delta_w$.

The interest of the corollary lies in the fact that it has shown that we can have two distinct LO-proximities inducing the topology of a compact Hausdorff space, although, according to Proposition 2.5.2.4 below, δ_w is the unique LR-proximity that induces the topology.

Lemma 2.5.2.1 Any proximity finer than an *R*-proximity and inducing the same topology is also an *R*-proximity.

Proof: Let δ be an R-proximity and let us suppose that δ_1 is a finer proximity giving the same topology, and let us write \gg , \geq for the corresponding proximal neighborhood relations. Now, if $V \geq x$, then, since both topologies are the same, there also follows $x \ll V$, and since δ is an R-proximity, there is a W with $x \ll W \ll V$. It now follows, since δ_1 is finer than δ , that $x \leqslant W \leqslant V$.

Corollary 2.5.2.2 The proximity δ_s on a regular T_1 space is an *R*-proximity.

Proof: It is finer than the R-proximity δ_w , and induces the same topology.

Corollary 2.5.2.3 The proximity δ_s on a normal but not hereditarily normal (=completely normal) Hausdorff space is an R-proximity that is not an LO-proximity.

An example of a compact Hausdorff space for which the three proximities δ_c , δ_w and δ_s are all distinct, is provided by an uncountable product of unit intervals, since such a space is not hereditarily normal and has no isolated points. This also shows that two distinct R-proximities can induce the topology of a compact Hausdorff space. The following result is now quite interesting; although LO-proximities and R-proximities need not be unique on a compact Hausdorff space, the combined property is unique.

Proposition 2.5.2.4 The proximity δ_w is the only LR-proximity on a compact Hausdorff space.

Proof: By Proposition 2.5.2.2 such a proximity is certainly coarser than δ_w , so we need only to show that it is also finer than δ_w . This can be shown by a device similar to the one usually used to show that a compact Hausdorff space is normal. This is a generalization of the usual theorem that a compact Hausdorff space has only one completely regular proximity.

2.5.3 RC-proximities

First we will establish some properties of round filters with respect to an R-proximity that will be needed in this subsection.

Proposition 2.5.3.1 Let (X, δ) be an *R*-proximity space. Then

- (a) every round filter is a regular filter;
- (b) every neighborhood filter is a maximal round filter;
- (c) every round filter is contained in a maximal round filter;
- (d) distinct maximal round filters contain disjoint open members.

Proof: Property (a) follows from Proposition 2.3.1.1 and Proposition 2.5.1.3. Property (b) follows from (a) together with the facts that neighborhood filters are round and maximal regular in a regular space. Property (c) is established in the usual manner using Zorn's lemma. To show (d), let us observe that by (a), round filters are open filters. Also, if the sup of two round filters is a filter, then by Proposition 2.5.1.2, this sup is a round filter. Thus, if two round filters do not contain disjoint open sets, then their sup is a round filter containing each, and this establishes (d).

Definition 2.5.3.1 A topological space is said to be **regular-closed** if it is regular, and cannot be nontrivially densely embedded in a regular space.

The term regular as used herein includes T_1 separation.

Since every compact space is regular-closed, then any completely regular space can be embedded in a regular-closed space, namely, any compactification of it. It is known that a regular-closed space need not be compact (see [26]); also, it is known that there exists a regular space that cannot be densely embedded in a regular-closed space (see [141]).

Definition 2.5.3.2 A topological space is said to be an **RC-regular space** if it can be densely embedded in a regular-closed space.

It follows from the above remarks that the class of RC-regular spaces lies properly between the class of regular spaces and the class of completely regular spaces.

We shall now give the axiom that is used for the connection with the regular-closed spaces. It deals with a different type of neighborhood relation between subsets of an R-proximity space X.

Definition 2.5.3.3 A subset B of X surrounds the subset A if every maximal round filter that intersects A (that is, every member of the filter intersects A) contains B.

Definition 2.5.3.4 A separated *R*-proximity that satisfies the condition:

(RC) (axiom of RC-regularity) the subset B surrounds the subset A if and only if $A \ll B$,

is said to be an **RC-proximity**.

Proposition 2.5.3.2 Let Z be a regular-closed topological space, and let X be a dense subspace of Z. If δ is a separated R-proximity induced on X by Z by the method described in Proposition 2.5.1.5, then

(a) the relation δ is an RC-proximity on X;

(b) the maximal round filters on X are precisely the traces on X of the neighborhood filters of the points of Z.

Proof: We will show (b) first. Since Z is regular, by Proposition 2.5.1.5 (b) the trace γ of the neighborhood filter ζ of a point $z \in Z$ is a round filter. If η is a round filter and $\gamma \subset \eta$, by Proposition 2.5.1.5 (b) there exists a regular filter ν on Z whose trace on X is η . Since ζ is a maximal regular filter, we must have $\nu \subset \zeta$ and thus $\eta \subset \gamma$. Conversely, if γ is a maximal round filter, it is the trace on X of a regular filter on Z, and since Z is regular-closed, this regular filter has a cluster point. The trace on X of the neighborhood filter of this cluster point must be the given maximal round filter.

To show (a), let us suppose that A and B are the subsets of X and $B \ll A$. By definition of the proximity this is equivalent to $cl_Z(X-A) \cap cl_Z B = \emptyset$. Now if γ is a maximal round filter on X, then by (b) we know that γ is the trace on X of the neighborhood filter of some point $z \in Z$. If γ intersects B, then $z \in cl_Z B$, and so there is a neighborhood V of z disjoint from X - A, from which we find that $A \in \gamma$. We have thus shown that if $B \ll A$, then A surrounds B.

Conversely, let us suppose that A and B are subsets of X and that A surrounds B. Let $z \in cl_Z B$ and let γ be the trace on X of the neighborhood filter of z. Then by $(b) \gamma$ is a maximal round filter. Since γ intersects B we must have $A \in \gamma$, from which it follows that $z \notin cl_Z(X - A)$. Thus $B \ll A$.

According to this proposition, we can now state:

Theorem 2.5.3.1 The topology of every RC-regular space is induced by an RC-proximity. \clubsuit

2.5.4 Absolutely closed RC-proximities

We now introduce a completness condition on RC-proximities that is a generalization of a condition given by Smirnoff in [294]. It will prove to be characteristic for a regular-closed space in the same way that Smirnoff's condition is characteristic for compact spaces. We will also show that a regular-closed space has the topology induced by precisely one RC-proximity, just as a compact space has the topology induced by precisely one completely regular proximity.

Definition 2.5.4.1 An RC-proximity is **absolutely closed** if every maximal round filter is the proximal neighborhood filter of some point of the space (that is, converges in the topology induced by the proximity).

It will later become apparent that this is equivalent to stating that there is no proper dense embedding (in either the proximal or topological space) of the space into an RC-proximity space, which is the condition corresponding to Smirnoff's definition.

Theorem 2.5.4.1 If an RC-proximity space is absolutely closed, then its induced topology is a regular-closed topology, and the proximity is given by: A and B are far if and only if they have disjoint closures.

Proof: We will establish the second statement first. Let us suppose that A and B are subsets of an RC-proximity space X, and that $A\overline{\delta}B$, that is, $A \ll X - B$. Since the proximity satisfies (RC) it follows that X - B surrounds A, and so every maximal round filter that intersects A contains X - B. Now, by Proposition 2.5.3.1 (a), neighborhood filters are maximal round, and thus we can see that any neighborhood filter that intersects A fails to intersect B, that is, A and B have disjoint closures.

Conversely, let us suppose that $A\delta B$, that is, $A \not\ll X - B$. Then, by (RC), X-B does not surround A, so there is some maximal round filter that intersects A and intersects B. Since we are assuming that the proximity is absolutely closed, this maximal round filter must be the neighborhood filter of some point of the space, and this point is in the closure of both A and B.

Having characterized the proximity, we will establish that the induced topology is regular-closed. According to Proposition 2.5.3.1 (a) every round filter is a regular filter. Observing that every open set containing a closed set is a round neighborhood of the closed set, by the above characterization of the proximity, we can see that every regular filter is a round filter, thus every maximal regular filter converges and the topology is regular-closed.

The following theorem is a generalization of Theorem 8. in [294].

Theorem 2.5.4.2 An RC-proximity space is absolutely closed if and only if the induced topology is regular-closed.

Proof: That an absolutely closed RC-proximity induces a regular-closed topology is a part of Theorem 2.5.4.1. To show the converse, that an RC-proximity space whose induced topology is regular-closed is absolutely closed, we can see that a maximal round filter (being a regular filter by Proposition 2.5.3.1 (*a*)) must have a cluster point, to which it must then converge (since neighborhood filters are round by Proposition 2.5.3.1 (*b*)).

Using Theorems 2.5.3.1, 2.5.4.1 and 2.5.4.2, we can establish the following two results:

Theorem 2.5.4.3 A topological space is regular-closed if and only if it has the topology induced by an absolutely closed RC-proximity.

Theorem 2.5.4.4 There is precisely one RC-proximity that induces the topology of a regular-closed space. \clubsuit

2.5.5 The ideal space of an RC-proximities

The final link in chain connecting RC-proximities and RC-regular spaces is to show that a space having topology induced by an RC-proximity is an RC-regular space, and it is this problem that we will pay our attention to.

Let δ be an RC-proximity on X. We shall construct a set rX and an absolutely closed RC-proximity π on rX such that X is naturally embedded in rX as a dense subspace both in the topological and the proximal sense.

Let rX be the disjoint union of X with an index set for the family of nonconvergent maximal round filters on X. For $p \in rX$, let us define O^p as follows: if $p \in X$ then O^p is the filter of proximal neighborhoods of p, and if $p \in rX - X$, then O^p is the nonconvergent maximal round filter for which p is the index.

Let us define a relation π on subsets of rX by $P\pi Q$ if there is $p \in rX$ such that for each $V \in O^p$ there is $(a, b) \in P \times Q$ with $V \in O^a$ and $V \in O^b$. We shall show that π is an absolutely closed RC-proximity on rX, that it induces the proximity δ on the subset X, and that every point of rX is related to X under π . An immediate consequence will be that the topology induced on X by δ is RC-regular.

Properties (SP_1) and (SP_3) are clear, and (SP_4) is readily shown. Property (SP_5) follows from Proposition 2.5.3.1 (d). Since the relation δ satisfies (RC), it is easy to see that for the subsets A and B of X, it follows that $A\delta B$ if and only if $A\pi B$; thus the relation π does indeed induce the relation δ on the subset X. To show that every point of rX is related to X under π , we can merely see that if $p \in rX$ and $V \in O^p$, there is $x \in X$ with $V \in O^x$.

We will now introduce some useful notation. For $A \subset X$, let $A' = \{p \in rX : A \in O^p\}$, and let $A^* = A \cup A'$. Also let $A^\circ = \{x \in X : x \ll A\}$. It is easy to see that $(A^\circ)' = (A^\circ)^* = A'$. Given a filter γ on X, let γ^* be the filter on rX generated by $\{F^* : F \in \gamma\}$. Finally, we will note that $P \leq Q$ for $P\overline{\pi}(rX - Q)$. The following lemma is useful in proving that π has the properties (R) and (RC).

Lemma 2.5.5.2 (a) $p \leq R$ if and only if there is $V \in O^p$ with $V' \subset R$.

(b) For $A, B \subset X$, $A^* \leq B^*$ if and only if $A \ll B$.

(c) If ζ is a round filter on rX, then the trace γ of ζ on X exists and $\zeta = \gamma^*$.

(d) γ^* is a (maximal) round filter on rX if and only if γ is a (maximal) round filter on X.

Proof: (a) If $\{p\}\overline{\pi}(rX - R)$, then, since for each $V \in O^p$ there is $p \in \{p\}$ with $V \in O^p$, we must have some $V \in O^p$ such that $V \notin O^b$ for any $b \in rX - R$; equivalently, $V' \subset R$. Conversely, if $\{p\}\pi(rX - R)$, it can easily be seen that we must have $V' \notin R$ for each $V \in O^p$.

(b) Let us suppose that $A\delta(X - B)$. Then, using (RC) we can see that there is O^p such that O^p intersects A and O^p intersects X - B. Since $A \subset A^*$ and $X - B \subset rX - B^*$, then $A^*\pi(rX - B^*)$. Conversely, let us suppose that $A^*\pi(rX - B^*)$. Then there is O^p such that for each $V \in O^p$ there is $(a,b) \in A^* \times (rX - B^*)$ with $V \in O^a$ and $V \in O^b$. Now, if $a \in A$, then $a \in A \cap V \neq \emptyset$; if $a \in A^* - A$ then $a \in A'$, so $A \in O^a$, and since $V \in O^a$ as well, it also follows that $A \cap V \neq \emptyset$. If $b \in (rX - B^*) \cap X$, then $b \in (X - B) \cap V$; if $b \in (rX - B^*) - X$, then $B \notin O^b$, thus $V \not\subset B$, and so $V \cap (X - B) \neq \emptyset$. Therefore the set B does not surround A and so, by (RC) we have $A \not\ll B$, that is, $A\delta(X - B)$.

(c) Let $A \in \zeta$. Since ζ is a round filter, there exists $B \in \zeta$ with $B \leq A$. Since $B \neq \emptyset$, there exists $p \in B$, and thus $p \leq A$. It follows immediately from (a) that $A \cap X \neq \emptyset$. We have thus shown that every member of ζ intersects X, and so the trace γ of ζ on X exists.

We shall now show that $(B \cap X)^* \subset A$, which will establish that $\zeta \subset \gamma^*$. It certainly holds that $B \cap X \subset A \cap X$; now if $p \in (B \cap X)^* - (B \cap X)$, then $B \cap X \in O^p$, and so, for each $V \in O^p$, there is $b \in B \cap X$ with $B \cap X \in O^b$, thus, since $B\overline{\pi}(rX - A)$, there must be $W \in O^p$ with $W' \subset A$, and $p \in W' \subset A$ follows.

To show the converse that $\gamma^* \subset \zeta$ we shall show that $B \subset (A \cap X)^*$. If $p \in B$, then $p \leq A$, and so, by using $(a), (A \cap X) \in O^p$ holds and $p \in (A \cap X)^*$ follows.

(d) This follows immediately from (b) and (c). \clubsuit

Lema 2.5.5.2 (c) and (d) establishes a one-to-one correspondence from the maximal round filters on X onto the maximal round filters on rX. They show in particular that the maximal round filters on rX are precisely the filters $(O^p)^*$ for some $p \in rX$.

It is immediate from Lema 2.5.5.2 (a) and (b) that π satisfies (R). We shall now demonstrate that π also satisfies (RC). Let us suppose that $P, Q \subset rX$ and $P\pi Q$. Then there exists a maximal round filter O^p on X such that for each $V \in O^p$ there exists $(p,q) \in P \times Q$ with $V \in O^p$ and $V \in O^q$; then the maximal round filter $(O^p)^*$ on rX intersects P and does not contain rX - Q, so rX - Q does not surround P.

Conversely, let us suppose that rX - Q does not surround P. Then there is a maximal round filter on rX that intersects P and does not contain rX - Q, that is, it intersects P and Q. Letting O^p be the trace of this filter on X, it follows that for each $V \in O^p$ there is $(a, b) \in P \times Q$ such that $V \in O^a$ and $V \in O^b$, and therefore $P\pi Q$.

We have now shown that π is an RC-proximity on rX, that π induces δ on its subspace X and that X is proximally dense in rX. This establishes in particular that the space X with the topology induced by δ is densely embedded in the space rX with the topology induced by π . If we show that rX with this topology is regular-closed, then we will have shown that X with topology induced by δ is RC-regular. To show that rX is regular-closed, we shall show that π is absolutely closed and shall apply Theorem 2.5.4.1.

The proximity π is absolutely closed if every maximal round filter on rX converges in the topology of the proximity. Now one need only observe that a maximal round filter on rX is of the form $(O^p)^*$ for some $p \in rX$, and that the following sets are all bases for $(O^p)^*$: $\{V^* : V \in O^p\}$; $\{V' : V \in O^p\}$ and $\{A \subset rX : p \leq A\}$. Thus, $(O^p)^*$ converges to $p \in rX$.

Definition 2.5.5.1 The set rX with the proximity π is called the ideal space of the proximity δ .

Summing up the preceding conclusions, we have the following results:

Theorem 2.5.5.1 The ideal space of an RC-proximity is an absolutely closed RC-proximity space, and its induced topology is regular-closed. The given space is a dense subspace of its ideal space, in both the topological and the proximal sense. \clubsuit

Corollary 2.5.5.1 Every RC-proximity space is RC-regular.

The following theorem, which is a generalization of the corresponding result for completely regular proximities, is an immediate consequence of the preceding results.

Theorem 2.5.5.2 There is a one-to-one correspondence from the collection of RC-proximities for an RC-regular space onto the collection of regularclosed embeddings of the spaces, given by letting an RC-proximity correspond to its ideal space.

We will order regular-closed embeddings of an RC-regular space by stating Z > X if there is a mapping h from the regular-closed embedding Z(necessarily) onto the regular-closed embedding Y that reduces to the identity on the subspace X. Then it is not difficult to show that the corresponding proximities are comparable in the sense that sets which are far in the proximity of Y are far in the proximity of Z when Z > Y.

Historical and bibliographic notes

The notions of R-proximities, RC-proximities and absolutely closed RCproximities were introduced in 1975 by D. Harris in paper [131]. The Rproximities are the generalization of V. A. Efremovich's proximities in the class of regular spaces. The RC-proximities were introduced by Harris to characterize the space that can be embedded in a regular-closed space. All the results of this section, except the results of subsection 5.2., were proved by Haris in that paper. In paper [132] he introduced the notion of an LRproximity space and proved Propositions 2.5.2.1-2.5.2.4 The notion of base and subbase of an R-proximity was introduced by the author in 1987 [76]. In a very specific manner V. Fedorchuk introduced θ -proximities in regular topological spaces [97]. In 1989 G. Di Maio and S. A. Naimpally in their paper [68] introduced the concept of a \mathcal{D} -proximity (\mathcal{D} standing for D, D^* , d, d^*) which is distinct from the known proximities such as EF, LO, R and S. Each of these \mathcal{D} -proximities, besides satisfying the axioms of semi-proximity, fulfils a condition which is weaker than the one of an EF or R-proximity and stronger than the one of an LO or S-proximity. In paper [69] Di Maio and Naimpally introduced the class of a **D**-proximity (**D** standing for G_m, G_m^*) g_m, g_m^*, g_m^* and p_m^* , where m is an infinite cardinal) (see also [67] and [71]).

2.6 Proximity approach to semi-metric and developable spaces

2.6.1 The introductory notions

In this section we will study semi-metric and developable spaces via generalized proximities and uniformities. We assume that all the proximities considered in this section are separated.

Definition 2.6.1.1 A semi-pseudo-metizable space (X, d) is a T_1 - space together with a real-valued function d on $X \times X$ such that

 $\begin{array}{l} (a) \ d(x,x) = 0 \ for \ each \ x \in X; \\ (b) \ d(x,y) = d(y,x) \geqslant 0 \ for \ all \ x,y \in X; \\ (c) \ clA = \{x \in X: \ d(x,A) = 0\} \ for \ each \ A \subset X. \\ If \\ (d) \ d(x,y) = 0 \ implies \ x = y, \end{array}$

then X is called a semi-metrizable space.

Let (X, d) be a semi-metrizable space. It can easily be verified that the relation δ_d defined on the set X by

 $A\delta_d B$ if and only if d(A, B) = 0

is an S-proximity on X. For $\varepsilon > 0$ we will set

$$V_{\varepsilon} = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}.$$

Clearly $V_{\varepsilon}[x] = S(x, \varepsilon)$, the sphere with center x and radius ε . We will set $\mathcal{U}_d = \{U = U^{-1} \subset X \times X : V_{1/n} \subset U \text{ for some } n \in \mathbb{N}\}$. For $\mathcal{U} \subset P(X \times X)$ let us define a relation $\delta_{\mathcal{U}}$ on P(X) by

 $A\delta_{\mathcal{U}}B$ if and only if $(A \times B) \cap U \neq \emptyset$ for each $U \in \mathcal{U}$.

Clearly, if d is a semi-metric on X, then $\delta_d = \delta_{\mathcal{U}_d}$.

Definition 2.6.1.2 A refining family Σ on a topological space (X, τ) is a family $\{\alpha_i : i \in I\}$ of open covers of X such that for each $x \in G \in \tau$, there exists an $i \in I$ such that $st(x, \alpha_i) \subset G$. In the case when $I = \mathbb{N}$, Σ is called a **development** on X and the pair (X, Σ) is called a **developable** space. In this case, it is well known that Σ may be replaced by another development $\Sigma' = \{\beta_j : j \in \mathbb{N}\}$ such that if j < k then $\beta_k \subset \beta_j$; we will assume that Σ already satisfies this condition. We will also assume that the developable spaces are T_1 .

Lemma 2.6.1.1 Every refining family $\Sigma = \{\alpha_i : i \in I\}$ on a T_1 space (X, τ) induces a compatible LO-proximity δ_{Σ} on X, where δ_{Σ} is defined by

$$A\delta_{\Sigma}B$$
 if and only if $st(A, \alpha_i) \cap B \neq \emptyset$ for each $i \in I$.

Proof: It is obvious that δ_{Σ} is a separated semi-proximity compatible with the topology τ . We will now show that δ_{Σ} satisfies the condition (LO). Let us suppose that $A\delta_{\Sigma}B$ and $b\delta_{\Sigma}C$ for each $b \in B$. Then $A\delta_{\Sigma}B$ implies that for each $i \in I$, there exists a $b \in B$ such that $b \in st(A, \alpha_i)$. Since $st(A, \alpha_i)$ is open, there is a $j \in I$ such that $b \in st(b, \alpha_j) \subset st(A, \alpha_i)$. Since $b\delta_{\Sigma}C$, $C \cap st(b, \alpha_j) \neq \emptyset$ and this, in turn, implies that $C \cap st(A, \alpha_i) \neq \emptyset$, i.e. $A\delta_{\Sigma}C$.

For a developable space $(X, \Sigma), \Sigma = \{\lambda_n : n \in \mathbb{N}\}$, we will define d_{Σ} by

$$d_{\Sigma}(x,y) = \inf \left\{ \frac{1}{n+1} : y \in st(x,\lambda_n) \right\}.$$

It can easily be seen that d_{Σ} is a compatible semi-metric on X and that $\delta_d = \delta_{\Sigma}$. For each $n \in \mathbb{N}$, let us set that

$$B_n = \bigcup \{ G \times G : G \in \lambda_n \}.$$

Lemma 2.6.1.2 $B_n = V_{1/n}$.

Proof: $(x, y) \in V_{1/n}$ if and only if $d(x, y) \leq 1/(n+1) < 1/n$ if and only if $y \in st(x, \lambda_n)$ if and only if $(x, y) \in B_n$.

Definition 2.6.1.3 An *S*-uniformity base \mathcal{B} on X is a family of subsets of $X \times X$ such that

 $(S_1) \cap \{U : U \in \mathcal{B}\} \supseteq \Delta;$

 (S_2) $U = U^{-1}$ for each $U \in \mathcal{B}$;

 (S_3) for each $A \subset X$ and $U, V \in \mathcal{B}$, there exists a $W \in \mathcal{B}$ such that $W[A] \subset U[A] \cap V[A];$

 (S_4) for each $p \in X$, $B \subset X$ and $U \in \mathcal{B}$, if $V[p] \cap B \neq \emptyset$ for each $V \in \mathcal{B}$, then there exists an $x \in B$ and a $W \in \mathcal{B}$ such that $W[x] \subset U[p]$;

(S₅) for each $U \in \mathcal{B}$, $U \subset V = V^{-1} \subset X \times X$ implies $V \in \mathcal{B}$.

An S-proximity on X is separated if

 $(S_6) \cap \{U : U \in \mathcal{B}\} = \Delta.$

 $\mathcal{U} \subset X \times X$ is said to be an S-uniformity (separated S-uniformity) in X if there exists a family $\mathcal{B} \subset \mathcal{U}$ satisfying conditions $(S_1) - (S_5) ((S_1) - (S_6))$ above and for each $U \in \mathcal{U}$ there exists a $B \in \mathcal{B}$ such that $B \subset U$.

If \mathcal{U} is an S-uniformity, then $\tau_{\mathcal{U}}$ is defined as usual: $G \in \tau_{\mathcal{U}}$ if and only if for each $x \in G$, there exists a $U \in \mathcal{U}$ such that $U[x] \subset G$. If $\tau = \tau_{\mathcal{U}}$, we will say that τ and \mathcal{U} are **compatible**.

Theorem 2.6.1.1 If \mathcal{U} consists of symmetric subsets of $X \times X$, then \mathcal{U} is an *M*-uniformity base (resp. *S*-uniformity base) if and only if $\delta_{\mathcal{U}}$ is a LO proximity (resp. an *S*-proximity).

Let (X, d) be a semi-metrizable space and let us set that $V_{1/n} = \{(x, y) : d(x, y) < 1/n\}$. Then $\{V_{1/n} : n \in \mathbb{N}\}$ is a countable base for a compatible S-uniformity.

2.6.2 Semi-metrizable spaces

In this subsection we will suppose that (X, d) is a semi-metrizable space and consider the effects of the various forms of continuity properties of d on the topology of X and on the proximity δ_d .

Lemma 2.6.2.3 In the following consideration (a) and (b) are equivalent and each implies (c):

- (a) semi-metric d is separately upper semi-continuous;
- (b) for each $\varepsilon > 0$ and $x \in X$, $S(x, \varepsilon)$ is open;
- (c) δ_d is a LO-proximity on X.

Proof: $(a) \Rightarrow (b)$: Let $y \in S(x, \varepsilon)$, i.e. $d(x, y) < \varepsilon$. Since d is upper semi-continuous in y, for every $\eta > 0$ there is a neighborhood N_y of y such that $d(x, z) < d(x, y) + \eta$ for each $z \in N_y$. Let us choose $\eta < \varepsilon - d(x, y)$. Then clearly $N_y \subset S(x, \varepsilon)$, showing thereby that $S(x, \varepsilon)$ is open.

 $(b) \Rightarrow (a)$: Let us suppose that d(x, y) = r and $\varepsilon > 0$. Clearly $y \in S(x, r + \varepsilon)$, which is open, and hence there exists a neighborhood N_y of y such that $N_y \subset S(x, r + \varepsilon)$. But this means that for each $z \in N_y$, $d(x, z) < d(x, y) + \varepsilon$, i.e. d is separately upper semi-continuous.

 $(a) \Rightarrow (c)$: Let us suppose that $A\delta_d B$ and $b\delta_d C$ for each $b \in B$. Then for each $\varepsilon > 0$ there exists an $a \in A$ and a $b \in B$ such that $d(a, b) < \varepsilon$. Since d is upper semi-continuous at b, there exists a neighborhood N_b of b such that $d(a, x) < \varepsilon$ for each $x \in N_b$. Also, $b\delta_d C$ implies the existence of a point $c \in C \cap N_b$ and hence, $d(a, c) < \varepsilon$, i.e. $A\delta_d C$.

Corollary 2.6.2.1 If a semi-metric d is separately upper semi-continuous, then U_d is an M-uniformity (obviously with a countable base).

Lemma 2.6.2.4 If semi-metric d is separately lower semi-continuous, then X is regular.

Proof: Let A be a closed set in X and $p \in X - A$. Then d(p, A) = r > 0and hence for each $a \in A$, $d(p, a) \ge r$. Since d is lower semi-continuous at a, there exists a neighborhood N_a of a such that for each $x \in N_a$, d(p, x) > r/2. Let us set that $N_A = \bigcup \{N_a : a \in A\}$. Then N_A is a neighborhood of A and $N_A \cap S(p, r/2) = \emptyset$, thereby showing that X is regular.

Theorem 2.6.2.1 If semi-metric d is separately continuous, then X is Tychonoff space.

Proof: Let us suppose that A is a closed subset of X and that $x \in X - A$. We may assume that d(x, A) = 1. As in the proof of Urisohn's lemma, we will now show how to construct, for each positive rational $r \in [0, 1]$, an open set V_r such that $x \in V_r$ for each such r, $\overline{V_r} \subset V_s$ whenever r < s and each $V_r \subset X - A$. We set $V_r = \{y \in X : d(x, y) < r\}$ which is open by Lemma 2.6.2.3. Further, the separate lower semi-continuity of d is equivalent to the set $\{y \in X : d(x, y) \leq r\}$, being closed for each r.

The following theorem is an analogue of the result: A T_1 -space is uniformizable with a countable base if and only if it is metrizable (and hence has a metric d such that δ_d is a proximity).

Theorem 2.6.2.2 A T_1 -space is M-uniformizable with a countable base if and only if it has a compatible semi-metric d such that δ_d is an LO-proximity.

Proof: Sufficiency is evident from Theorem 2.6.1.1 and necessity follows from Theorem 2.6.1.1 and the result of C. M. Pareec, making use of the remarks just preceding Definition 2.6.1.2. \clubsuit

The following analogue of the above result is proved in a similar way.

Theorem 2.6.2.3 A T_1 -space is S-uniformizable with a countable base if and only if it has a compatible semi-metric d (obviously δ_d is an S-proximity).

2.6.3 Developable spaces

In this subsection we will suppose that (X, Σ) is a developable space with $\Sigma = \{\lambda_n : n \in N\}$ where each λ_n is an open cover of X and $\lambda_{n+1} \subset \lambda_n$. Let d_{Σ} be the induced semi-metric on X. Then $\delta_d = \delta_{\Sigma}$ is an LO-proximity on X.

Lemma 2.6.3.5 Semi-metric d is upper semi-continuous.

Proof: If $p, q \in X$, we have to consider two cases: (i) d(p,q) = 0 and (ii) d(p,q) = 1/(m+1) for some $m \in \mathbb{N}$. Let $\varepsilon > 0$ be arbitrary. In case (i), δ_d is an LO-proximity by Lemma 2.6.2.3 and p = q. Now, let us choose $n \in \mathbb{N}$ such that $1/n < \varepsilon$; then $p, q \in G$ for some $G \in \lambda_n$. For all $(x, y) \in G \times G$, $d(x, y) \leq 1/(n+1) < \varepsilon$. In case (ii), $p, q \in G$ for some $G \in \lambda_m$. Then for all $(x, y) \in G \times G$, $d(x, y) \leq 1/(m+1) < 1/(m+1) + \varepsilon$. Thus the result is proved.

The above result (in conjunction with Lemma 2.6.2.3) provides an alternate proof of the fact that δ_{Σ} is a compatible LO-proximity and also shows that \mathcal{U}_d is compatible M-uniformity with a countable open base $\{B_n : n \in \mathbb{N}\}$. This provides a motivation for the next result.

Theorem 2.6.3.1 A T_1 -space is developable if and only if it is M-uniformizable with a countable open base.

Proof: Necessity follows the remarks preceding this theorem and sufficiency from Brown's result (see [36]) that a space is developable if and only if it has a compatible semi-metric d for which every convergent sequence is Cauchy, using Proposition 2.4.4.3.

In [56] H. Cook states that if a compatible semi-metric d on X is continuous, then X is developable. The following characterization of developable spaces is an improvement of this result.

Theorem 2.6.3.2 A T_1 -space X is developable if and only if it has a compatible upper semi-continuous semi-metric.

Proof: Since the necessity has been proved in Lemma 2.6.3.5, we need prove only the sufficiency. Assuming d to be an upper semi-continuous semi-metric, we prove that $V_{1/n}$ is open for each $n \in \mathbb{N}$ and then the result will follow from Theorem 2.6.3.1. If $(p,q) \in V_{1/n}$, then d(p,q) < 1/n. Let

us choose $\varepsilon > 0$ such that $\varepsilon < 1/n - d(p,q)$. Then from the upper semicontinuity of d, there exist neighborhoods N_p , N_q of p, q respectively, such that $d(x,y) < d(p,q) + \varepsilon < 1/n$ for each $(x,y) \in N_p \times N_q$. This shows that $N_p \times N_q \subset V_{1/n}$, showing thereby that $V_{1/n}$ is open.

Definition 2.6.3.1 A proximity base \mathcal{B} for a proximity space (X, δ) is a family of the subsets of X such that if $A\overline{\delta}B$, then there exist $C, D \in \mathcal{B}$ such that $A \subset C, B \subset D$ and $C\overline{\delta}D$.

In an LO-space, we may assume that the members of \mathcal{B} are closed. The following is an improvement of Theorem 2.4.2.2.

Theorem 2.6.3.3 If \mathcal{B} is a proximity base for an LO-proximity space (X, δ) , then $\{U_{A,B} : A, B \in \mathcal{B} \text{ and } A\overline{\delta}B\}$, where $U_{A,B} = X \times X - [(A \times B) \cup (B \times A)]$, is a base for an M-uniformity \mathcal{U}_1 on X.

A developable space (X, Σ) is said to be **totally bounded** if for each $n \in \mathbb{N}$, there exists a finite set $F \subset X$ such that $st(F, \lambda_n) = X$. It is known that a proximity space (X, δ) has a countable base if and only if X has a compatible totally bounded metric (see [238]). The following is a partial generalization of the this result.

Theorem 2.6.3.4 If an LO-proximity space (X, δ) has a countable closed base \mathcal{B} , then X is a totally bounded developable space.

Proof: $\{U_{A,B} : A, B \in \mathcal{B}, A\delta B\}$ is a countable open base for a compatible M-uniformity, which is also totally bounded. The result then follows from Theorem 2.6.3.1.

2.6.4 Metrizable spaces

A. Arhangel'skii (see [21]) proved that a semi-metrizable space (X, d) is metrizable if δ_d is a proximity. A glance at his proof shows that the following, improved version is true.

Theorem 2.6.4.1 A T_1 -space is uniformizable with a countable base if and only if it has a compatible semi-metric d such that δ_d is an R-proximity if and only if it is metrizable.

S. I. Nedev (see [239]) proved that a semi-metric space (X, d) is metrizable if d(A, x) is a continuous function of x for each $A \subset X$. The following is an improvement. **Theorem 2.6.4.2** A semi-metric space (X, d) is metrizable if and only if for all closed subsets A of X, d(A, x) is lower semi-continuous.

Proof: Let us suppose that A is closed, that B is compact and that $A \cap B = \emptyset$. Since the function d(A, x) is lower semi-continuous and B is compact, it follows that d(A, B) = d(A, x) for some $x \in B$. This implies that X is metrizable (see [21]).

We now give a table which shows the relationship of exposed results with the classical Alexandroff-Urysohn uniform metrization theorem. Let us suppose that $\{W_n : n \in \mathbb{N}\}$ is a countable family of symmetric subsets of $X \times X$ satisfying (a) $\bigcap_{n=1}^{\infty} W_n = \Delta$ and (b) for each $x \in X$, $\{W_n[x] : n \in \mathbb{N}\}$ forms a neighborhood base at x. Let d be the semi-metric by $\{W_n\}$, namely

$$d(x,y) = \inf\left\{\frac{1}{n+1} : y \in W_n[x]\right\}$$

We will assume without any loss of generality that $W_{n+1} \subset W_n$. Finally let us set that $\mathcal{U} = \{U \subset X \times X : W_n \subset U \text{ for some } n \in \mathbb{N}\}.$

| \mathcal{U} is an | \mathcal{U} is an | \mathcal{U} is an M-uniformity. | \mathcal{U} is a unifor- |
|---------------------|----------------------|-----------------------------------|----------------------------|
| S-proximity | M-uniformity | Each W_n is open | mity |
| δ_d is an S- | δ_d is an LO- | d is u.s.c. | δ_d is a prox- |
| proximity | proximity | δ_d is an LO- | imity |
| | | proximity | |
| X is semi- | X is semi- | X is | X is |
| metrizable | metrizable | developable | metrizable |

2.6.5 Metric spaces, developable spaces, semi-metric spaces and mappings connected with them

In this subsection $f : X \to Y$ will denote a mapping from a proximity space (X, δ) onto a T_1 space Y. When X is developable, semi-metrizable or metrizable, δ will denote the corresponding naturally induced proximity relation as defined in subsection 1. of this section. Several kinds of mappings, which have been defined for the case in which X is metrizable, can be redefined more generally when X is a proximity space; this is done by replacing the condition d(A, B) > 0 by $A\overline{\delta}B$. These mappings have been systematically discussed in the metric case by Arkhangel'skii [21]. Although we consider their generalizations, for the sake of simplicity we will keep the same terminology and attach δ before each term.

The strategy consists in defining a binary relation δ' on the power set of Y as follows

(*)
$$E\overline{\delta}'F$$
 if and only if $f^{-1}(E)\overline{\delta}f^{-1}(B)$.

It is easily verified that δ' is a semi-proximity and so δ' is almost a quotient semi-proximity. To let δ' be a semi-proximity, naturally we will have to put some additional conditions on f; also it is clear that if δ' satisfies stronger proximity conditions, so must δ .

Our first task is to find conditions on f which make δ' an S-proximity compatible with the topology of Y.

Definition 2.6.5.1 The mapping f is called δ -**pseudo-open** if for each $y \in Y$ and $A \subset X$, if $f^{-1}(y)\overline{\delta}(X - A)$ then $y \in int f(A)$.

It is easily verified that if f is open or closed, then it is also δ -pseudo-open.

Lemma 2.6.5.6 If δ is an S-proximity on X and f is δ -pseudo-open, then $y\overline{\delta}'E$ implies $y \notin \overline{E}$.

Proof: $y\overline{\delta}'E$ implies $f^{-1}(y)\overline{\delta}f^{-1}(E)$, and since f is δ -pseudo-open, it follows that $y \in \operatorname{int} f(X - f^{-1}(E))$. But $E \cap f(X - f^{-1}(E)) = \emptyset$ and hence $y \notin \overline{E}$.

Corollary 2.6.5.1 If δ is an S-proximity and f is either open or closed, then $y\overline{\delta}'E$ implies $y \notin \overline{E}$.

Definition 2.6.5.2 The mapping f is called a δ -uniform if for each $y \in Y$ and each neighborhood N_y of y, $f^{-1}(y)\overline{\delta}(X - f^{-1}(N_y))$.

Lemma 2.6.5.7 If δ is an S-proximity, then f is a δ -uniform if and only if $y \notin \overline{E}$ implies $y\overline{\delta}'E$.

Proof: If f is a δ -uniform and $y \notin \overline{E}$, then $f^{-1}(y)\overline{\delta}f^{-1}(E)$ and by (*), $y\overline{\delta}'E$. Conversely, if N_y is a neighborhood of y, then $y \notin \overline{Y - N_y}$ implies $y\overline{\delta}'(Y - N_y)$ and this is equivalent to $f^{-1}(y)\overline{\delta}(X - f^{-1}(N_y))$. Thus f is a δ -uniform mapping.

Lemma 2.6.5.8 If δ is an *R*-proximity and *f* is continuous and compact, then $y \notin \overline{E}$ implies $y\overline{\delta}'E$.

Proof: If $y \notin \overline{E}$ then $f^{-1}(y) \cap f^{-1}(\overline{E}) = \emptyset$. Since f is compact, $f^{-1}(y)$ is compact and since f is continuous, $f^{-1}(\overline{E})$ is closed. Finally, δ being an R-proximity on X, then $f^{-1}(y)\overline{\delta}f^{-1}(E)$, which in turn implies $y\overline{\delta}'E$.

The following result is now obvious.

Theorem 2.6.5.1 The relation δ' is a compatible S-proximity on Y in the following cases:

- (a) δ is an S-proximity, f is δ -pseudo-open and δ -uniform;
- (b) δ is an R-proximity, f is δ -pseudo-open, continuous and compact.

Next we will find out when δ' is an LO-proximity.

Theorem 2.6.5.2 If δ is an LO-proximity on X and f is a δ -uniform and open mapping, then δ' is a compatible LO-proximity on Y.

Proof: By Theorem 2.6.5.1 (b), δ' is a compatible S-proximity and since f is a δ -uniform mapping, it follows that f is continuous. This, together with the openness of f, implies that for each $B \subset Y$, $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$. Now $A\delta'B$ if and only if $f^{-1}(A)\delta f^{-1}(B)$ if and only if $f^{-1}(A)\delta f^{-1}(\overline{B})$ if and only if $f^{-1}(\overline{A})\delta f^{-1}(\overline{B})$ if and only if $\overline{A}\delta'\overline{B}$, thereby showing that δ' is an LO-proximity.

To investigate when δ' is an *R*-proximity, we will introduce a stronger type of mapping.

Definition 2.6.5.3 The mapping f is called a δ -completely uniform if for each neighborhood N_y of y in Y, there exists a neighborhood N'_y of y such that $f^{-1}(N'_y)\overline{\delta}(X - f^{-1}(N_y))$.

Since the identity mapping on X is not a δ -completely uniform unless δ is an *R*-proximity, it is clear that in order to have a meaningful discussion, we must have δ as an *R*-proximity.

Theorem 2.6.5.3 If δ is an *R*-proximity on *X* and if *f* is an open and δ uniform mapping, then *f* is a δ -completely uniform mapping if and only if δ' is a compatible *R*-proximity. **Proof:** To prove the necessity, we will first note that Theorem 2.6.5.2 shows that δ' is a compatible LO-proximity on Y. To see that δ' is an R-proximity, let us note that $y\overline{\delta}'E$ implies $y \in Y - \overline{E}$ which is open. Since f is δ -completely uniform, there is a neighborhood N_y of y such that $f^{-1}(N_y)\overline{\delta}f^{-1}(E)$, i.e. $N_y\overline{\delta}'E$. Since trivially $y\overline{\delta}(Y - N_y)$, it follows that δ' is an R-proximity.

Conversely, let us suppose that f is an open δ -uniform mapping and that δ' is an R-proximity. If N_y is a neighborhood of $y \in Y$, then $y\overline{\delta}(Y-N_y)$ and, since δ' is an R-proximity, there exists a set $A \subset Y$ such that $y\overline{\delta}'(Y-A)$ and $A\overline{\delta}'(Y-N_y)$, i.e. $y \in int(A)$ and $f^{-1}(A)\overline{\delta}f^{-1}(Y-N_y) = X - f^{-1}(N_y)$. This proves that f is δ -completely uniform.

In view of Theorem 2.6.4.1, it is not necessary to investigate when δ' is a proximity and we now pay our attention to the case when (X, d) is a semi-metric space. In this case we suppose that $\delta = \delta_d$.

Definition 2.6.5.4 The mapping f is called a T_1 -mapping if for every pair of distinct points y_1 , y_2 of Y, $f^{-1}(y_1)\overline{\delta}f^{-1}(y_2)$ is true; equivalently, $d(f^{-1}(y_1), f^{-1}(y_2)) > 0$.

Lemma 2.6.5.9 If (X, d) is a semi-metric space and f is a T_1 mapping, then

$$\rho(y_1, y_2) = d(f^{-1}(y_1), f^{-1}(y_2))$$

is a semi-metric on Y (not necessarily compatible with the topology of Y). \clubsuit

Corollary 2.6.5.2 If either d is a metric and f is compact or if d is a semi-metric and f is δ -uniform, then ρ is a semi-metric on Y.

Lemma 2.6.5.10 Under the conditions of Lemma 2.6.5.9, for subsets A, B of Y,

$$\rho(A, B) = d(f^{-1}(A), f^{-1}(B))$$

Proof: $\rho(A, B) = \inf \{ d(f^{-1}(a), f^{-1}(b)) : a \in A, b \in B \}.$ Now

$$d(f^{-1}(A), f^{-1}(B)) \leqslant d(f^{-1}(a), f^{-1}(b))$$

for each $a \in A$, $b \in B$ and this implies that $d(f^{-1}(A), f^{-1}(B)) \leq \rho(A, B)$. Conversely, for every $\varepsilon > 0$, there exists $x \in f^{-1}(A)$, $y \in f^{-1}(B)$ such that $d(x, y) < d(f^{-1}(A), f^{-1}(B)) + \varepsilon$, and hence

$$\rho(A,B) \leqslant d(x,y) < d(f^{-1}(A), f^{-1}(B)) + \varepsilon \,,$$

i.e. $\rho(A,B) \leqslant d(f^{-1}(A),f^{-1}(B))$, thus proving the result.

Corollary 2.6.5.3 Under the conditions of Lemma 2.6.5.10, $A\delta_{\rho}B$ if and only if $f^{-1}(A)\delta f^{-1}(B)$.

Lemma 2.6.5.11 If d is (resp. separately) upper semi-continuous and if f is open δ -uniform, then ρ is (resp. separately) upper semi-continuous.

Proof: Let $\rho(y_1, y_2) = r$ and let $\varepsilon > 0$. Then there are $x_i \in f^{-1}(y_i)$, i = 1, 2 such that $d(x_1, x_2) < r + \varepsilon/2$. Since d is upper semi-continuous, there exist neighborhoods N_{x_i} of x_i , i = 1, 2, such that for all $z_i \in N_{x_i}$, i = 1, 2, $d(z_1, z_2) < r + \varepsilon$. But this shows that $\rho(p_1, p_2) < r + \varepsilon$ for $p_i \in f(N_{x_i})$, i = 1, 2, and the result follows since f is open. The case of the separate upper semi-continuity is similarly handled.

It is well known that if X is metrizable and f is open δ -uniform, then Y is a developable space. The following theorem gives an improvement and a more "symmetric" result from Theorem 2.6.3.2 and Lemma 2.6.5.11.

Theorem 2.6.5.4 If X is developable space and if f is open δ -uniform mapping with respect to the proximity δ induced by the development on X, then Y is also a developable space.

Theorem 2.6.5.5 If d is a metric on X and if f is a continuous, open and compact mapping, then Y is a developable space.

Proof: By setting $\delta = \delta_d$, $\delta' = \delta_\rho$, the result follows from the continuity of d, Theorem 2.6.5.1, Lemma 2.6.5.11 and Theorem 2.6.3.2.

Theorem 2.6.5.6 If d is a metric on X and if f is open and δ -completely uniform mapping, then Y is a metrizable space.

Proof: Let us set that $\delta = \delta_d$ and $\delta' = \delta_{\rho}$. It follows from Theorem 2.6.5.3 that δ' is a compatible *R*-proximity on *Y*. Also from Corollary 2.6.5.2, ρ is a semi-metric on *Y* and the compatibility of δ' implies the compatibility of ρ . Hence by Theorem 2.6.4.1, *Y* is a metrizable space.

Historical and bibliographic notes

All the results in this section were proved by M. Gagrat and S. A. Naimpally in 1973 in paper [117]. (see also [71])

2.7 Unified approach to symmetric proximities

2.7.1 Unified approach to symmetric proximities

In this subsection we will develop a classification scheme under which many of the known types of proximities can be subsumed. The scheme depends on certain functions f which we shall now define.

Definition 2.7.1.1 Let \mathfrak{A} be a family of ordered pairs (δ, \mathcal{G}) , where δ is a semi-proximity on some set X and \mathcal{G} is a grill on the same set X. By (a grill operator) f we shall denote functions defined on \mathfrak{A} and satisfying

 $f(\delta, \mathcal{G}) \in \Gamma(X(\delta))$

and

$$\mathcal{G} \subset f(\delta, \mathcal{G})$$
 for all $(\delta, \mathcal{G}) \in \mathfrak{A}$

A semi-proximity δ on X is called an **f-proximity** if

$$f(\delta, \delta(A)) = \delta(A)$$

holds for all $A \in P(X)$.

The family of all f-proximities will be denoted by \mathfrak{R}_f . The subfamily of \mathfrak{R}_f consisting of all $\delta \in \mathfrak{R}_f$ with $c_{\delta} = c$, and hence also with the same reference set X, shall be denoted by $\mathfrak{M}_f(X, c)$.

Definition 2.7.1.2 A function f of the type defined above will be said to be in the class A_0 if for all semi-proximities δ

 $\mathcal{G} \subset \mathcal{G}' \Rightarrow f(\delta, \mathcal{G}) \subset f(\delta, \mathcal{G}'), \quad \forall \mathcal{G}, \mathcal{G}' \in \Gamma(X(\delta)).$

The class A_1 consists of all functions f for which

$$f(\delta, \mathcal{G}_1 \cup \mathcal{G}_2) = f(\delta, \mathcal{G}_1) \cup f(\delta, \mathcal{G}_2)$$

is valid for all semi-proximities δ and arbitrary grills \mathcal{G}_1 , \mathcal{G}_2 on $X(\delta)$. The function f belongs to the **class** $\mathbf{A_2}$ if

$$f(\delta, \cup \mathcal{G}_i) = \cup f(\delta, \mathcal{G}_i)$$

for arbitrary unions of grills on the same set X and an arbitrary semi-proximity δ on X. If

$$f(\delta, \mathcal{G}) = f(\delta^*, \mathcal{G})$$

for all δ , δ^* with $c_{\delta} = c_{\delta^*}$ and all grills \mathcal{G} on X, then f is said to belong to the **class I**. A function f satisfying

$$f(\delta, \mathcal{G}) \subset f(\delta^*, \mathcal{G}),$$

for all $\mathcal{G} \in \Gamma(X)$ and for all proximities δ and δ^* on the same set satisfying $\delta \subset \delta^*$ is said to belong to the **class M**. Finally, **F** is the class of functions for which

$$f(\delta, f(\delta, \mathcal{G})) = f(\delta, \mathcal{G})$$

holds for all $\delta \in \mathfrak{N}_f$ and all \mathcal{G} on $X(\delta)$.

Let us note that in order to belong to the classes A_0 , A_1 , A_2 , I and M the function f is required to satisfy certain conditions for all $(\delta, \mathcal{G}) \in \mathfrak{A}$, while, in order to belong to F, it is required to satisfy the condition in question only for $\delta \in \mathfrak{R}_f$.

Let us further observe that if (δ, \mathcal{G}) and $(\delta^*, \mathcal{G}^*)$ are elements of \mathfrak{A} , the domain of definition of all functions f, then δ and δ^* may be proximities on different sets X and X^* , but \mathcal{G} must be in $\Gamma(X)$ and $\mathcal{G}^* \in \Gamma(X^*)$. Finally, recall that $\delta \in \mathfrak{R}_f$ if and only if it satisfies $f(\delta, \delta(A)) = \delta(A)$ for all $A \in P(X)$. The "variable" δ enters each of these equations (one for each A) in three places, and the f-proximities are exactly those δ for which all of the equations are valid for fixed f.

From now on, all f will be assumed to satisfy the two conditions imposed in Definition 2.7.1.1. The conditions listed in Definition 2.7.1.2 will be assumed as needed, and will be listed in the statement of each theorem in which they are used.

Definition 2.7.1.3 A grill \mathcal{G} satisfying

$$f(\delta, \mathcal{G}) = \mathcal{G}$$

will be called an *f*-grill with respect to δ . For $\delta \in \mathfrak{R}_f$ a δ -clan \mathcal{G} will be called an *f*-bunch with respect to δ if

$$f(\delta, \mathcal{G}) = \mathcal{G}.$$

Since a semi-proximity δ may be in \Re_f for different f, it is possible for a δ -clan to be an f-bunch for different f. The possibility of different combinations is increased by the fact that \mathcal{G} may be a clan for different semi-proximities δ . Similarly, \mathcal{G} may be an f-grill for different f and with respect to a variety of δ . **Proposition 2.7.1.1** For $\delta \in \mathfrak{R}_f$, $f \in A_0$, $\delta(\mathcal{U})$ is an f-grill with respect to δ .

Proof: $\delta(\mathcal{U}) = \cap \{\delta(A) : A \in \mathcal{U}\}$. Thus $f(\delta, \delta(\mathcal{U})) \subset f(\delta, \delta(A)) = \delta(A)$. Hence $\delta(\mathcal{U}) \subset f(\delta, \delta(\mathcal{U})) \subset \cap \{\delta(A) : A \in \mathcal{U}\} = \delta(\mathcal{U})$. Clearly, $\delta(A)$ is an *f*-grill with respect to δ for all *f*-proximities δ .

Proposition 2.7.1.2 Let $\delta \in \mathfrak{R}_f$, $f \in A_0 \cap F$.

- (a) If \mathcal{G} is a δ -clan, then $f(\delta, \mathcal{G})$ is an f-bunch.
- (b) Every maximal δ -clan is a maximal f-bunch.
- (c) Every cluster is a maximal f-bunch.

(d) $f(\delta, \mathcal{U})$ is an f-bunch for every ultrafilter \mathcal{U} , $f(\delta, \mathcal{U})$ is the smallest f-bunch containing \mathcal{U} . Every minimal f-bunch is of the form $f(\delta, \mathcal{U})$.

Proof: (a) The key of the proposition is in part (a). Since \mathcal{G} is a δ -clan, there follows $f(\delta, \mathcal{G}) \subset f(\delta, \delta(A)) = \delta(A)$ for all $A \in \mathcal{G}$. By symmetry $\mathcal{G} \subset \delta(B)$ for all $B \in f(\delta, \mathcal{G})$. Thus, finally $f(\delta, \mathcal{G}) \subset \delta(B)$ for all $B \in f(\delta, \mathcal{G})$ and it follows that $f(\delta, \mathcal{G})$ is a δ -clan. Since $f \in F$, $f(\delta, \mathcal{G})$ is an f-bunch.

(b) If \mathcal{G} is a maximal δ -clan, then, since $f(\delta, \mathcal{G})$ is a δ -clan, we must have $\mathcal{G} = f(\delta, \mathcal{G})$ and hence \mathcal{G} is an f-bunch. Clearly, \mathcal{G} is a maximal f-bunch since every f-bunch is a δ -clan.

In (d) let us note that we are not asserting that every $f(\delta, \mathcal{U})$ is a minimal f-bunch.

Proposition 2.7.1.3 If $\delta \in \mathfrak{R}_f$ and $f \in A_0$, then $D(\mathcal{U})$ is an f-bunch and $f(\delta, \mathcal{U}) \subset D(\mathcal{U})$.

Proof: $D^{\dagger}(\mathcal{U}) \subset (f(\delta, D(\mathcal{U})))^{\dagger} \subset \delta^{\dagger}(A)$ for all $A \in \delta(\mathcal{U})$ implies $D(\mathcal{U}) = f(\delta, D(\mathcal{U}))$. By Proposition 2.2.3.20 $D(\mathcal{U})$ is a δ -clan, thus $D(\mathcal{U})$ is an f-bunch. Since $\mathcal{U} \subset D(\mathcal{U}), f(\delta, \mathcal{U}) \subset D(\mathcal{U})$ holds.

Definition 2.7.1.4 *For all* $(\delta, \mathcal{G}) \in \mathfrak{A}$ *we will define*

$$\begin{split} i(\delta, \mathcal{G}) &= \mathcal{G} \,;\\ s(\delta, \mathcal{G}) &= \mathcal{G} \cup \{A : \, (\exists x) \, c_{\delta}(A) \in \delta(\{x\}) \subset \mathcal{G}\} \,;\\ r(\delta, \mathcal{G}) &= \mathcal{G} \cup \{A : \, (\exists x) \, \{x\} \in \delta(A) \cap \mathcal{G}\} \,;\\ b(\delta, \mathcal{G}) &= \{A : \, c_{\delta}(A) \in \mathcal{G}\} \,;\\ h(\delta, \mathcal{G}) &= \mathcal{G} \cup \{A : \, (\exists a) \, a \in A \,, \, \mathfrak{N}(\delta, \{a\}) \subset \mathcal{G}\} \,;\\ e(\delta, \mathcal{G}) &= \{A : \, \mathfrak{N}(\delta, A) \subset \mathcal{G}\} \,. \end{split}$$

Proposition 2.7.1.4 The functions *i*, *s*, *r*, *b*, *h* and *e* satisfy the conditions of Definition 2.7.1.1.

Proof: The second condition is trivially satisfied by i, s, r, h. For b we will note that $A \in \mathcal{G}$ implies $c_{\delta}(A) \in \mathcal{G}$ and hence $A \in b(\mathcal{G})$. The function e satisfies the condition because $A \in \mathcal{G}$ implies $\mathfrak{N}(\delta, A) \subset \mathcal{G}$.

We will now turn to the first condition. Trivially $i(\delta, \mathcal{G})$ is a grill on X for all $\mathcal{G} \in \Gamma(X)$. To see that $s(\delta, \mathcal{G})$ is a grill, it suffices to show that $\{A : c_{\delta}(A) \in \delta(\{x\}) \subset \mathcal{G}\} = \mathfrak{S}$ is a grill. We have $\emptyset \notin \mathfrak{S}$ and $B \supset A \in \mathfrak{S}$ implies $B \in \mathfrak{S}$ since $c_{\delta}(B) \supset c_{\delta}(A)$. Finally, since $c_{\delta}(A \cup B) = c_{\delta}(A) \cup c_{\delta}(B)$ and $\delta(\{x\})$ is a grill, $A \cup B \in \mathfrak{S}$ also implies $A \in \mathfrak{S}$ or $B \in \mathfrak{S}$.

Similarly, to show that $r(\delta, \mathcal{G})$ is a grill, it suffices to consider $\mathfrak{T} = \{A : \exists x, \{x\} \in \delta(A) \cap \mathcal{G}\}$. Clearly $\emptyset \notin \mathfrak{T}$ since $\delta(\emptyset) = \emptyset$. Further $A \subset B$ implies $\delta(A) \subset \delta(B)$ and hence $B \supset A \in \mathfrak{T}$ implies $B \in \mathfrak{T}$. Since $\delta(A \cup B) = \delta(A) \cup \delta(B)$ by Proposition 2.2.3.12, it follows that $A \cup B \in \mathfrak{T}$ implies $A \in \mathfrak{T}$ or $B \in \mathfrak{T}$.

Very similar arguments, employing among others Proposition 2.2.3.13 and the fact that $\mathfrak{N}(\delta, \emptyset) = P(X)$, which cannot be contained in any grill, can be used to show that b, h and e also map into $\Gamma(X)$.

Proposition 2.7.1.5 $i, r, b \in A_2 \subset A_1 \subset A_0, h, e \in A_1 \subset A_0 and s \in A_0.$

Proof: That *i* and *b* are totally additive is immediate. Also $\{x\} \in \delta(A) \cap (\cup \mathcal{G}_i)$ clearly implies $\{x\} \in \delta(A) \cap \mathcal{G}_i$ for at least one *i* and hence $r \in A_2$.

It follows from Proposition 2.2.3.6 and Proposition 2.2.3.7 that $\mathfrak{N}(\delta, A) \subset \mathcal{G}_1 \cup \mathcal{G}_2$ implies $\mathfrak{N}(\delta, A) \subset \mathcal{G}_1$ or \mathcal{G}_1 and hence h and e are additive.

Finally, it is easy to see that $\mathcal{G}_1 \subset \mathcal{G}_2$ implies $s(\delta, \mathcal{G}_1) \subset s(\delta, \mathcal{G}_2)$.

Proposition 2.7.1.6 *i*, *s*, *r*, *b*, $h \in I$.

Proof: By Proposition 2.2.1.4 $\delta(\{x\})$ and hence $\mathfrak{N}(\delta, \{x\})$ depends only on $\mathfrak{M}_f(X, c)$. This establishes the result for s and h. For the remaining functions it follows directly from their definitions.

Proposition 2.7.1.7 *i*, *s*, *r*, *b*, *h*, $e \in M$.

Proof: There holds that $c_{\delta^*}(A) \supset c_{\delta}(A)$ and $\mathfrak{N}(\delta^*, A) \subset \mathfrak{N}(\delta, A)$ as well as $\delta^*(\{x\}) \supset \delta(\{x\})$ for $\delta^* \supset \delta$. The proposition follows easily from these observations. For example, if $A \in s(\delta, \mathcal{G}), A \notin \mathcal{G}$, then there exists an xsuch that $c_{\delta}(A) \in \delta(\{x\})$. Hence $c_{\delta^*}(A) \in \delta(\{x\})$ and $c_{\delta^*}(A) \in \delta^*(\{x\})$. It follows that $A \in s(\delta^*, \mathcal{G})$.

Proposition 2.7.1.8 $\delta \in \mathfrak{R}_s$ if and only if $\{x\} \in \delta(c_{\delta}(A))$ implies $\{x\} \in \delta(A)$. c_{δ} is a Kuratowski closure operator for $\delta \in \mathfrak{R}_s$.

Proof: $\{x\} \in \delta(c_{\delta}(A))$ if and only if $c_{\delta}(A) \in \delta(\{x\})$. Hence $A \in s(\delta, \delta(\{x\}))$. Since $\delta \in \mathfrak{R}_s$, it follows that $A \in \delta(\{x\})$ and thus $\{x\} \in \delta(A)$.

If the condition is satisfied, then $s(\delta, \delta(A)) = \delta(A) \cup \{B : c_{\delta}(B) \in \delta(\{x\}) \subset \delta(A)\} \subset \delta(A) \cup \{B : B \in \delta(\{x\}) \subset \delta(A)\} = \delta(A)$ and hence $\delta \in \mathfrak{R}_s$. It now follows that $c_{\delta}(c_{\delta}(A)) = c_{\delta}(A)$ and therefore c_{δ} is a Kuratowski closure operator.

Proposition 2.7.1.9 If $\delta \in \mathfrak{R}_r$, then $\{z\} \in \delta(\{x\}) \cap \delta(\{y\})$ implies $\delta(\{x\}) = \delta(\{y\})$.

Proof: We will show that $\{x\} \in \delta(\{y\})$ implies $\delta(\{x\}) \subset \delta(\{y\})$. The proposition then follows easily since $\delta(\{z\}) \subset \delta(\{x\})$ implies $\{z\} \in \delta(\{x\})$ and hence $\{x\} \in \delta(\{z\})$.

If $\{x\} \in \delta(\{y\})$ then $\{x\} \in \delta(B) \cap \delta(\{y\})$ for all $B \in \delta(\{x\})$. Hence $\delta(\{x\}) \subset r(\delta, \delta(\{y\})) = \delta(\{y\})$.

Proposition 2.7.1.10 If $\delta \in \mathfrak{R}_b$, then c_{δ} is a Kuratowski closure operator.

Proof: $x \in c_{\delta}(c_{\delta}(A))$ if and only if $c_{\delta}(A) \in \delta(\{x\})$ if and only if $A \in b(\delta, \delta(\{x\})) = \delta(\{x\})$ if and only if $x \in c_{\delta}(A)$.

Proposition 2.7.1.11 $\delta \in \mathfrak{R}_b$ if and only if $c_{\delta}(A) \subset \delta(B)$ implies $A \in \delta(B)$.

Proof: $\delta \in \mathfrak{R}_b$ if and only if $b(\delta, \delta(B)) = \delta(B)$ if and only if $c_{\delta}(A) \in \delta(B)$ implies $A \in \delta(B)$.

Proposition 2.7.1.12 $\delta \in \mathfrak{R}_h$ if and only if $A \in \mathfrak{N}(\delta, \{x\})$ implies there exists a $B \in \mathfrak{N}(\delta, \{x\})$ such that $A \in \mathfrak{N}(\delta, B)$.

Proof: The contrapositive of the statement is: if for all $B \in \mathfrak{N}(\delta, \{x\})$ $X - A \in \delta(B)$ then $X - A \in \delta(\{x\})$. Setting X - A = C and using the symmetry of δ , this can be reworded: $\mathfrak{N}(\delta, \{x\}) \subset \delta(C)$ implies $\{x\} \in \delta(C)$. This is exactly the statement $h(\delta, \delta(C)) \subset \delta(C)$.

Proposition 2.7.1.13 $\delta \in \mathfrak{R}_e$ if and only if $A \in \mathfrak{N}(\delta, B)$ implies the existence of $C \in \mathfrak{N}(\delta, B)$ such that $A \in \mathfrak{N}(\delta, C)$.

Proof: The argument is analogous to the one given in the proof of the preceding theorem. \clubsuit

Theorem 2.7.1.1 The following equivalences hold:

- (a) $\delta \in \mathfrak{R}_s$ if and only if δ is an S-proximity;
- (b) $\delta \in \mathfrak{R}_r$ if and only if δ is an RI-proximity;
- (c) $\delta \in \mathfrak{R}_b$ if and only if δ is an LO-proximity;
- (d) $\delta \in \mathfrak{R}_h$ if and only if δ is an *R*-proximity;
- (e) $\delta \in \mathfrak{R}_e$ if and only if δ is a proximity.

Proof: The result follows from Propositions 2.7.1.8, 2.7.1.9, 2.7.1.11, 2.7.1.12 and 2.7.1.13. ♣

Proposition 2.7.1.14 *i*, *s*, *r*, *b*, *h*, $e \in F$.

Proof: The proposition is trivial for f = i. For $\delta \in \mathfrak{R}_s$ it follows from the proof of Proposition 2.7.1.8 that $s(\delta, \mathcal{G}) = \mathcal{G}$.

 $\{y\} \in r(\delta, \mathcal{G})$ holds if and only if there exists an $\{x\} \in \mathcal{G}$ such that $\{x\} \in \delta(\{y\})$, if and only if $\{y\} \in \delta(\{x\})$ if and only if $\delta(\{x\}) = \delta(\{y\})$. Hence $r(\delta, \mathcal{G})$ contains all singletons $\{y\} \in \delta(\{x\})$ for all $\{x\} \in \mathcal{G}$. For $B \in r(\delta, r(\delta, \mathcal{G})) - r(\delta, \mathcal{G})$ we must have $\{y\} \in \delta(B)$ or $B \in \delta(\{y\}) = \delta(\{x\})$. Hence $\{x\} \in \delta(B)$ and $B \in r(\delta, \mathcal{G})$.

 $A \in b(\delta, b(\delta, \mathcal{G}))$ implies $c_{\delta}(A) \in b(\delta, \mathcal{G})$ which in turn implies $c_{\delta}(c_{\delta}(A)) \in \mathcal{G}$. Since $\delta \in \mathfrak{R}_b$, we have from Proposition 2.7.1.10 $c_{\delta}(c_{\delta}(A)) = c_{\delta}(A)$ and hence $A \in b(\delta, \mathcal{G})$.

If $\{a\} \in h(\delta, h(\delta, \mathcal{G}))$ then either $\{a\} \in h(\delta, \mathcal{G})$ or $\mathfrak{N}(\delta, \{a\}) \subset h(\delta, \mathcal{G})$. Now either $\mathfrak{N}(\delta, \{a\}) \subset \mathcal{G}$ or there exists $N_a \in \mathfrak{N}(\delta, \{a\}), N_a \notin \mathcal{G}$. Let C consist of all c for which $\mathfrak{N}(\delta, \{c\}) \subset \mathcal{G}$. It follows from Theorem 2.7.1.1 and the known properties of R-proximities that h-proximities induce a Kuratowski closure operator and hence the N_a are ordinary topological neighborhoods of a and satisfy the usual neighborhood axioms. It follows that there exists an $N_a^* \subset N_a$ such that $N_a^* \subset \mathfrak{N}(\delta, \{d\})$ for all $d \in N_a^*$. Then $N_a \notin \mathcal{G}$ implies $N_a^* \notin \mathcal{G}$ and $\mathfrak{N}(\delta, \{a\}) \subset h(\delta, \mathcal{G})$. Hence in particular $N_a^* \in h(\delta, \mathcal{G})$. It follows that $N_a^* \cap C \neq \emptyset$. Thus $N_a^* = N_c$ for some $c \in C$ and we arrive at the contradiction $N_a^* \in \mathcal{G}$.

If $A \in e(\delta, e(\delta, \mathcal{G}))$ then $\mathfrak{N}(\delta, A) \subset e(\delta, \mathcal{G})$ which implies $\mathfrak{N}(\delta, N_A) \subset \mathcal{G}$ for all $N_A \in \mathfrak{N}(\delta, A)$. By Proposition 2.7.1.13 for every N_A there exists N'_a such that $N_A \in \mathfrak{N}(\delta, N'_A)$. Hence $N_A \in \mathcal{G}$ for all $N_A \in \mathfrak{N}(\delta, A)$. It follows that $\mathfrak{N}(\delta, A) \subset \mathcal{G}$ and $A \in e(\delta, \mathcal{G})$.

Theorem 2.7.1.2 For all semi-proximities δ

$$e(\delta, \mathcal{U}) = \delta(\mathcal{U})$$

Proof: $e(\delta, \mathcal{U}) = \{B : \mathfrak{N}(\delta, B) \subset \mathcal{U}\} = \{B : \mathcal{U} \subset \delta(B)\} = \{B : B \in \delta(\mathcal{U})\} = \delta(\mathcal{U}).$

Proposition 2.7.1.15 For $\delta \in \mathfrak{R}_r$

$$r(\delta, \mathcal{U}(x)) = \delta(\mathcal{U}(x)) = \delta(\{x\}).$$

$$\begin{aligned} \mathbf{Proof:} \ \delta(\mathcal{U}(x)) &= \delta(\{x\}) = r(\delta, \delta(\{x\})) \supset r(\delta, \mathcal{U}(x)) = \\ &= \mathcal{U}(x) \cup \{A : \ \exists \{y\} \in \delta(A) \cap \mathcal{U}(x)\} = \\ &= \mathcal{U}(x) \cup \{A : \ \{x\} \in \delta(A)\} = \delta(\{x\}). \end{aligned}$$

Proposition 2.7.1.16 $\delta \in \mathfrak{R}_r$ if and only if $\delta(\{x\})$ is a cluster for all $x \in X(\delta)$.

Proof: Let $A, B \in \delta(\{x\})$ then $A \in r(\delta, \delta(B)) = \delta(B)$ hence $\delta(\{x\})$ is a cluster, since all $\delta(A)$ are δ -closed. If $\delta(\{x\})$ is a cluster, then $A, B \in \delta(\{x\})$ implies that $A \in \delta(\{B\})$ and hence $r(\delta, \delta(B)) = \delta(B)$, that is $\delta \in \mathfrak{R}_r$.

Proposition 2.7.1.17 $\delta \in \mathfrak{R}_h$ if and only if $\delta^{\dagger}(A) \cap \delta^{\dagger}(\{x\}) \neq \emptyset \Rightarrow A \in \delta(\{x\}).$

Proof: The condition $\delta^{\dagger}(A) \cap \delta^{\dagger}(\{x\}) \neq \emptyset$ is equivalent to $\mathfrak{N}(\delta, \{x\}) \subset \mathcal{B} \subset \delta(A)$. Hence the assumption of the proposition is equivalent to $h(\delta, \delta(A)) = \delta(A)$.

Proposition 2.7.1.18 $\delta(\{x\})$ satisfies $\mathcal{B} \subset \delta(\{x\}) \Rightarrow \delta(\mathcal{B}) \subset \delta(\{x\})$ if and only if $\delta \in \mathfrak{R}_h$.

Proof: If $\delta \in \mathfrak{R}_h$ then $\mathcal{B} \subset \delta(\{x\})$ and $A \in \mathcal{B}$ imply $\mathcal{B} \in \delta^{\dagger}(\{x\}) \cap \delta^{\dagger}(A)$ so that $A \in \delta(\{x\})$ and hence $\delta(\mathcal{B}) \subset \delta(\{x\})$. The converse also easily follows by Proposition 2.7.1.17.

The significance of Propositions 2.7.1.16 and 2.7.1.18 is that while the Riesz condition is strong enough to make all $\delta(\{x\}) = \delta(\mathcal{U}(x))$ clusters, it is not strong enough to make $\mathfrak{N}(\delta, \{x\})$ round filters.

Proposition 2.7.1.19 If for $\delta \in \mathfrak{R}_q$, $f(\delta, \mathcal{G}) \subset g(\delta, \mathcal{G})$ then $\mathfrak{R}_q \subset \mathfrak{R}_f$.

Proof: $\delta(A) = g(\delta, \delta(A)) \supset f(\delta, \delta(A)) \supset \delta(A)$ implies $f(\delta, \delta(A)) = \delta(A)$ for $\delta \in \mathfrak{R}_g$.

Theorem 2.7.1.3 $\mathfrak{R}_s \supset \mathfrak{R}_b, \mathfrak{R}_r \supset \mathfrak{R}_b, \mathfrak{R}_h \supset \mathfrak{R}_e, \mathfrak{R}_b \supset \mathfrak{R}_e.$

Proof: The proof easily follows by Proposition 2.7.1.19 together with $s \subset b$, $r \subset b$, $h \subset e$, $b \subset e$ which are easily established.

Theorem 2.7.1.4 $\Re_s \supset \Re_h$.

Proof: This was proved in Proposition 2.5.1.1. In this framework a proof could be based on Proposition 2.7.1.12. \clubsuit

Historical and bibliographic notes

All results of this section were proved by W. J. Thron in 1973 in paper [320]. Many other results in connection with the exposed problems in this section can be found in papers [247], [248], [50] and [54].

Chapter 3

Quasi-uniform spaces and quasi-proximity spaces

3.1 Quasi-uniform space

3.1.1 Elementary properties of quasi-uniformities

Definition 3.1.1.1 A quasi-uniformity for a set X is a filter \mathcal{U} on $X \times X$ satisfying the following two conditions:

 $(QU_1) \Delta_X \subseteq U$ for each $U \in \mathcal{U}$;

 (QU_2) for each $U \in \mathcal{U}$, there is a $V \in \mathcal{U}$ such that $V \circ V \subset U$.

A quasi-uniform space is a pair (X, \mathcal{U}) , where X is a set and \mathcal{U} is a quasi-uniformity on X. The members of a quasi-uniform structure \mathcal{U} are called entourages.

If \mathcal{U} is a quasi-uniformity on X, then $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ is also a quasi-uniformity on X and is called the **conjugate quasi-uniformity** of \mathcal{U} or **conjugate** of \mathcal{U} .

Definition 3.1.1.2 Let \mathcal{U} be a quasi-uniformity on X. A subfamily \mathcal{B} of \mathcal{U} is said to be a **quasi-uniform base** or **base** for \mathcal{U} if every entourage of \mathcal{U} contains some member of \mathcal{B} .

Clearly \mathcal{B} is a quasi-uniform base of a quasi-uniformity \mathcal{U} if and only if it is a filter-base for \mathcal{U} . If \mathcal{B} is a base for \mathcal{U} , then \mathcal{B}^{-1} is clearly a base for \mathcal{U}^{-1} .

A simple proof of the following proposition is omitted.

Proposition 3.1.1.1 Let \mathcal{B} be a family of subsets of $X \times X$ such that

(a) $\Delta \subset B$, for each $B \in \mathcal{B}$;

- (b) for B_1 , $B_2 \in \mathcal{B}$, there exists $B_3 \in \mathcal{B}$ such that $B_3 \subset B_1 \cap B_2$;
- (c) for each $B \in \mathcal{B}$, there exists an $A \in \mathcal{B}$ such that $A \circ A \subset B$.

Then there exists a unique quasi-uniformity \mathcal{U} on X for which \mathcal{B} is a base. \mathcal{U} is said to be generated by \mathcal{B} and may be defined as the family $\{U : B \subset U \text{ for some } B \in \mathcal{B}\}$.

Example 3.1.1.1 In the following examples X is a set linearly ordered by the relation <.

(a) If W denotes the "upper triangle" $\{(x, y) : x \leq y\}$, then $\{W\}$ is a quasi-uniform base.

(b) If $W_a = \{(x, y) : x = y \text{ or } a < x < y\}$, where a is some fixed element of X, then $\{W_a\}$ is a quasi-uniform base.

(c) For some fixed elements $a, b \in X$, the "vertical strip" $V_{a,b} = \{(x, y) : x = y \text{ or } a \leq x \leq b\}$ constitutes a quasi-uniform base.

(d) $\{W \cap V_{a,b}\}$ is a quasi-uniform base.

(e) If $T_{a,b} = \{(x,y) : a \leq x \leq y \leq b\} \cup \Delta$, then $\{T_{a,b}\}$ is a quasi-uniform base.

(f) If $H_a = \{(x, y) : x = y \text{ or } a \leq x\}, \{H_a\}$ is a quasi-uniform base.

(g) If $L_a = \{(x, y) : x \leq a \leq y\} \cup \Delta$, then $\{L_a\}$ is a quasi-uniform base.

(h) Each of the following is a quasi-uniform base:

$$\{W_a : a \in X\}, \{V_{a,b} : a, b \in X\}, \{H_a : a \in X\}.$$

(i) Let \mathcal{B} consist of all the sets B which properly contain Δ so that $(x, y) \in B$ implies $y \ge x$. Then \mathcal{B} is a quasi-uniform base.

(j) Let \mathbb{R} be the set of real numbers with the usual order, and let $W_{\varepsilon} = \{(x,y) : y - x < \varepsilon\}$. Then $\{W_{\varepsilon} : \varepsilon > 0\}$ is a quasi-uniform base for quasi-uniformity \mathcal{W} .

(k) Let \mathbb{R} be the set of all real numbers. For each $\varepsilon > 0$ let us define the relation $V_{\varepsilon} = \{(x, y) : x \leq y < x + \varepsilon\}$. Then $\{V_{\varepsilon} : \varepsilon > 0\}$ is a quasi-uniform base.

Proposition 3.1.1.2 Let n be any natural number and let \mathcal{B} be a base for \mathcal{U} . Then $\{B^n : B \in \mathcal{B}\}$ is also a base for \mathcal{U} .

Proof: Easily follows by using condition (c) of Proposition 3.1.1.1 and the induction on n.

Definition 3.1.1.3 If \mathcal{B}_1 and \mathcal{B}_2 are bases for quasi-uniformities on a set X, \mathcal{B}_1 is **finer** then \mathcal{B}_2 (and \mathcal{B}_2 is **coarser** than \mathcal{B}_1) and denotes $\mathcal{B}_1 < \mathcal{B}_2$, whenever this relation holds for the filters \mathcal{B}_1 and \mathcal{B}_2 . Two bases that determine the same quasi-uniformity are said to be **equivalent**.

Proposition 3.1.1.3 Two quasi-uniform bases \mathcal{B}_1 and \mathcal{B}_2 are equivalent if and only if $\mathcal{B}_1 < \mathcal{B}_2$ and $\mathcal{B}_2 < \mathcal{B}_1$.

If \mathcal{U} is a quasi-proximity on X, then the family $\{U \cap U^{-1} : U \in \mathcal{U}\}$ is a base for a quasi-uniformity \mathcal{U}^* , which is the coarsest uniformity containing \mathcal{U} .

In practice, a quasi-uniformity is often most easily described by defining a base for it. So all the quasi-uniformities given in Example 3.1.1.1 were defined in terms of bases. It is possible, however, for two equivalent bases to appear dissimilar. For example, if for each $n \in \mathbb{N}$ we set $U_n = \{(x, y) :$ $-2/n \leq x - y \leq 1/n\}$, then $\{U_n; n \in \mathbb{N}\}$ and $\{U_{\varepsilon} : \varepsilon > 0\}$, where $U_{\varepsilon} =$ $\{(x, y) : |x - y| < \varepsilon\}$, are equivalent bases by Proposition 3.1.1.3.

Definition 3.1.1.4 A subfamily S of a quasi-uniformity U is a **quasi-uniform subbase** or **subbase** for U if the family B of finite intersections of members of S is a base for U.

If \mathcal{S} is a subbase for \mathcal{U} , then \mathcal{S}^{-1} is a subbase for \mathcal{U}^{-1} .

We now give sufficient conditions for a family to be a subbase for a quasi-uniformity. Once again it should be noted that the conditions in the following proposition are only sufficient but not necessary, for S to be a quasi-uniform base.

Proposition 3.1.1.4 *If* $S \subset P(X \times X)$ *satisfies*

(a) $\Delta \subset S$ for each $S \in S$;

(b) for each $S \in S$, there exists a $T \in S$ such that $T \circ T \subset S$,

then S is a quasi-uniform subbase.

Proof: If \mathcal{B} is the family of all finite intersections of members of \mathcal{S} , then \mathcal{B} is obviously closed under finite (non-empty) intersections. Also, if $B \in \mathcal{B}$, then $B = \bigcap_{i=1}^{n} S_i$, $S_i \in \mathcal{S}$. For each $1 \leq i \leq n$, there exists a $T_i \in \mathcal{S}$ such that $T_i \circ T_i \subset S_i$. Let $A = \bigcap_{i=1}^{n} T_i \in \mathcal{B}$. Then $A \circ A \subset B$. By Proposition 3.1.1.1, \mathcal{B} is a quasi-uniform base.

Proposition 3.1.1.5 If \mathcal{U} is a quasi-uniformity on X, then the family $\mathcal{U}[x] = \{U[x] : U \in \mathcal{U}\}\$ is a neighborhood system at x, for each $x \in X$.

If \mathcal{B} is a base (resp. subbase) for \mathcal{U} , then $\mathcal{B}[x] = \{B[x] : B \in \mathcal{B}\}$ is the base (resp. subbase) for the neighborhood system $\mathcal{U}[x]$.

Proof: (a) For each $U \in \mathcal{U}$, $\Delta \subset U$ and so for each $x \in X$, $x \in \Delta[x] \subset U[x]$.

(b) Since $U \cap V \in \mathcal{U}$ whenever $U, V \in \mathcal{U}$, and $U[x] \cap V[x] = (U \cap V)[x]$, it follows that $U[x] \cap V[x] \in \mathcal{U}[x]$.

(c) Let U[x] be given and $U[x] \subset A$. Then $V = U \cup (A \times A) \in \mathcal{U}$ and V[x] = A which, therefore, belongs to $\mathcal{U}[x]$.

(d) Given U[x], we will show that there exists a V[x] such that $U[x] \in \mathcal{U}[y]$ for all $y \in V[x]$. We choose an entourage V such that $V \circ V \subset U$. Let $y \in V[x]$ and $z \in V[y]$. Then $z \in (V \circ V)[x] \subset U[x]$ as claimed.

Thus $\mathcal{U}[x]$ is a neighborhood system at x, for each $x \in X$.

In the case of base \mathcal{B} , (c) is omitted and in (b) equality is replaced by inclusion.

Definition 3.1.1.5 If \mathcal{U} is a quasi-uniformity on X, then the topology defined by the neighborhood system $\{\mathcal{U}[x] : x \in X\}$ (or, equivalently, by $\{\mathcal{B}[x] : x \in X\}$, where \mathcal{B} base or subbase for \mathcal{U}) is called the **topology** generated by \mathcal{U} or briefly the **topology of** \mathcal{U} and is denoted by $\tau(\mathcal{U})$ or $\tau_{\mathcal{U}}$.

Referring to Example 3.1.1.1, the quasi-uniformity defined in (a) induces the topology generated by the rays $[a, \rightarrow)$; the one in (j) induces the topology of the rays (\leftarrow, a) . All the quasi-uniformities in (h), though different from each other, induce the discrete topology.

It is clear from the above examples that different quasi-uniformities on X may induce the same topology. Two quasi-uniformities (or bases or subbases) on X are said to be **compatible** if they induce the same topology. This relation is obviously an equivalence relation. Also, for \mathcal{U}_1 and \mathcal{U}_2 to be compatible, it is necessary and sufficient that the families $\mathcal{U}_1[x]$ and $\mathcal{U}_2[x]$ are identical for every $x \in X$.

It may be expected that if two quasi-uniformities are compatible, then so are their conjugates. This, however, is not true. In fact, most of the interesting situations in the following sections come from this fact.

Topologies induced by conjugate quasi-uniformities are called **conjugate topologies**. From the above remarks, it is apparent that the conjugate of a topology is not necessarily unique. There are generally several conjugate topologies for a given τ . A topology is said to be **self-conjugate** if it coincides with at least one of its conjugates. For example, a uniform topology is self-conjugate.

Definition 3.1.1.6 A property \mathcal{P} is said to be **conjugate invariant** if, whenever \mathcal{P} holds in a topology τ , it holds in every conjugate of τ .

Because of the multiplicity of conjugate topologies in general, the conjugate invariance is a very demanding requirement and we will show that most topological properties are not conjugate invariants.

Definition 3.1.1.7 If (X, τ) is a topological space and W is a relation on X such that for each $x \in X$, W[x] is a neighborhood of x, then W is called a **neighbornet** of (X, τ) . If neighbornet W of (X, τ) is a symmetric (transitive) relation on X, then W is a **symmetric (transitive) neighbornet**. If for each $x \in X$, W[x] is an open (closed) set, then W is an **open (closed) neighbornet**. If W[x] = W[y] for each $y \in W[x] \cap W^{-1}[x]$, then W is an **unsymmetric neighbornet**.

Definition 3.1.1.8 A sequence (U_n) of neighbornets of a space (X, τ) is a normal sequence of neighbornets if $U_{n+1}^2 \subset U_n$ for each $n \in \mathbb{N}$, and a neighbornet U of (X, τ) is a normal neighbornet provided that U is a member of some normal sequence of neighbornets of (X, τ) .

Every transitive neighbornet is a normal neighbornet. If (U_n) is a normal sequence of neighbornets, then for each $k \in \mathbb{N}$, $U_1 \circ U_2 \circ \ldots \circ U_k \subset U_1^2$ and $\bigcap_{n=1}^{\infty} U_n$ is a transitive relation. For any neighbornet U the set $U^{\infty} = \bigcup \{U^n : n \in \mathbb{N}\}$ is always a transitive neighbornet. If (X, \mathcal{U}) is a quasi-uniform space and $U \in \mathcal{U}$, then U is a normal neighbornet of $(X, \tau_{\mathcal{U}})$. If for each neighbornet U of X there exists a neighbornet V of X such that $V^{2n} \subset U^n$, then for each neighbornet W of X, W^n is a normal neighbornet.

Proposition 3.1.1.6 Let (X, τ) be a topological space and let V be a neighbornet of (X, τ) . Then for each $A \subset X$, $\overline{A} \subset V^{-1}[A]$, and the following statements are equivalent:

- (a) V^{-1} is a neighbornet;
- (b) V contains a symmetric neighbornet;
- (c) for each subset A of X, $\overline{A} \subset V[A]$.

Proof: Let A be a subset of X. If $p \in \overline{A}$, then $V[p] \cap A \neq \emptyset$ so that $p \in V^{-1}[A]$. Thus $\overline{A} \subset V^{-1}[A]$. The implications $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ are evident. To see that $(c) \Rightarrow (a)$, let us suppose that for each subset A of $X, \overline{A} \subset V[A]$ and let $x \in X$. Evidently $x \notin V[X - V^{-1}[x]]$ and so $x \notin \overline{X - V^{-1}[x]}$. Hence $V^{-1}[x]$ is a neighborhood of x.

Proposition 3.1.1.7 Let (X, U) be a quasi-uniform space and let $G \subset X$. Then G is $\tau_{\mathcal{U}}$ -open if and only if for each $x \in G$, there exists an entourage $U \in \mathcal{U}$ such that $U[x] \subset G$. **Proof:** The condition is clearly equivalent to saying that G is a $\tau_{\mathcal{U}}$ -neighborhood of each of its points.

By definition, for each $x \in X$ and each $U \in \mathcal{U}$, U[x] is a $\tau_{\mathcal{U}}$ -neighborhood of the point x. Although U[x] need not be a $\tau_{\mathcal{U}}$ -open set, there is always a base \mathcal{B} for \mathcal{U} such that for each $B \in \mathcal{B}$ and $x \in X$, B[x] is $\tau_{\mathcal{U}}$ -open.

Proposition 3.1.1.8 If \mathcal{B} is a base (subbase) for a quasi-uniformity \mathcal{U} and $\beta = \{B : \text{ for each } x \in B, \text{ there exists } U \in \mathcal{B} \text{ such that } U[x] \subset B\}, \text{ then } \beta$ is a base (subbase) for $\tau_{\mathcal{U}}$.

Proposition 3.1.1.9 Let \mathcal{B} be a quasi-uniform base on X and $A \subset X$. Then

$$\overline{A} = \bigcap_{U \in \mathcal{B}} U^{-1}[A] \,.$$

Proof: By Proposition 3.1.1.8, $x \in \overline{A}$ if and only if $U[x] \cap A \neq \emptyset$ for each $U \in \mathcal{B}$, if and only if there exists an $a \in A$ such that $a \in U[x]$ for every $U \in \mathcal{B}$, if and only if $x \in U^{-1}[a]$ for $a \in A$ and each $U \in \mathcal{B}$, if and only if $x \in U^{-1}[a]$ for $a \in A$ and each $U \in \mathcal{B}$, if and only if $x \in U^{-1}[A]$ for each $U \in \mathcal{B}$.

As a special case, we get the following familiar result for uniform spaces. If \mathcal{U} is a uniformity on X and $A \subset X$, then

$$(*) \qquad \qquad \overline{A} = \bigcap_{U \in \mathcal{U}} U[A]$$

holds.

Definition 3.1.1.9 A topological space (X, τ) is said to be **quasi-unifor**mizable if there exists a quasi-uniformity \mathcal{U} on X which induces τ . Then \mathcal{U} is said to be compatible with τ .

It is well known that a topological space is uniformizable if and only if it is completely regular. Now we will show that every topological space is quasi-uniformizable.

Lemma 3.1.1.1 For each $A \subset X$, let us define that $S_A = (A \times A) \cup ((X - A) \times X)$. Then the sets S_A , as A runs through any family $\Sigma \subset P(X)$, forms a quasi-uniform subbase. Further, the quasi-uniformity generated in such a way induces the topology generated by the subbase Σ .

Proof: By virtue of Proposition 3.2.1.1, the first part is proved if we verify that for every $A \in \Sigma$, $(a) \ \Delta \subset A$ and $(b) \ S_A \circ S_A \subset S_A$. (a) is obvious. To prove (b), let us suppose that (x, y), $(y, z) \in S_A$. If $x \in A$, then $y \in A$ which in turn implies $z \in A$. On the other hand, if $x \in X - A$, $(x, z) \in (X - A) \times X$. In either case $(x, z) \in S_A$. Now by Proposition 3.1.1.5, the sets $S_A[x]$ form a local subbase at x. Since

$$S_A[x] = \begin{cases} A & \text{if } x \in A, \\ X & \text{if } x \notin A, \end{cases}$$

these sets generate the same topology as Σ .

Theorem 3.1.1.1 Every topological space (X, τ) is quasi-uniformizable.

Proof: By Lemma 3.1.1.1, the family $\{S_G : G \in \tau\}$ generates a quasiuniformity which is compatible with τ .

Theorem 3.1.1.1 was first proved by A. Császár (see [61]) in terms of syntopogenic structures. The proof is far from being simple. Subsequently W. J. Pervin [255] gave a more direct and simpler proof, which is presented here. For this reason, the quasi-uniformity constructed in the above proof will be referred to as the **Pervin quasi-uniformity** of τ and denoted by **P**.

Let $\{\mathcal{U}_i : i \in i\}$ be a family of quasi-uniformities on a set X. The **supremum** of $\{\mathcal{U}_i : i \in I\}$ is the smallest (in the set-inclusion sense) quasiuniformity on X that is finer than every \mathcal{U}_i . For any family $\{\mathcal{U}_i : i \in I\}$ the supremum always exists and is generated by the subbase $\{U_i : U_i \in \mathcal{U}_i\}$. This supremum coincides with the coarsest filter finer than every \mathcal{U}_i . The supremum of a family \mathcal{U}_i of quasi-uniformities is denoted by $\vee_i \mathcal{U}_i$. The **infimum** of a family $\{\mathcal{U}_i : i \in I\}$ is similarly defined and denoted by $\wedge_i \mathcal{U}_i$. The infimum always exists; it is the supremum of the family of all quasiuniformities that are coarser than every \mathcal{U}_i .

3.1.2 Quasi-uniformly continuous mappings

If $f: X \to Y$ is a mapping and \mathcal{V} is a quasi-uniformity or quasi-uniform base on Y, then $f_2^{-1}(\mathcal{U}) = \{f_2^{-1}(V) : V \in \mathcal{V}\}$ is a quasi-uniform base on X. Indeed, (a) the family $f_2^{-1}(\mathcal{U})$ is a filter base, (b) Δ is contained in every element $f_2^{-1}(V)$ and (c) for any $f_2^{-1}(V)$, $f_2^{-1}(W) \circ f_2^{-1}(W) \subset f_2^{-1}(W \circ W) \subset$ $f_2^{-1}(V)$, where $W \in \mathcal{V}$ such that $W \circ W \subset V$. The quasi-uniformity so generated is called **the pre-image** of \mathcal{V} and is denoted by $f_2^{-1}(\mathcal{V})$.
If $f: X \to Y$ is a mapping, then the image $f(\mathcal{U}) = \{f(U) : U \in \mathcal{U}\}$ of a quasi-uniformity \mathcal{U} on X need not be a quasi-uniform base on Y.

Definition 3.1.2.1 Let (X, U) and (Y, V) be quasi-uniform spaces. A mapping $f : X \to Y$ is said to be **quasi-uniformly continuous** if one of the following equivalent conditions is satisfied:

(a) $V \in \mathcal{V}$ implies $f_2^{-1}(V) \in \mathcal{U}$;

(b) there exists a base or a subbase \mathcal{B} of \mathcal{V} such that $B \in \mathcal{B}$ implies $f_2^{-1}(B) \in \mathcal{U}$;

(c) for each $V \in \mathcal{V}$, there exists a $U \in \mathcal{U}$ such that $(x, y) \in U$ implies $(f(x), f(y)) \in V$, i.e. $f_2(U) \subset V$;

(d) \mathcal{U} is finer than $f_2^{-1}(\mathcal{V})$.

Simple proof of the equivalence of conditions (a) - (d) is omitted.

Every constant mapping is trivially quasi-uniformly continuous, and so is every mapping from a discrete quasi-uniform structure and every mapping into an indiscrete quasi-uniform structure. The identity mapping i: $(X, \mathcal{U}) \to (X, \mathcal{V})$ is quasi-uniformly continuous if and only if $\mathcal{V} \subset \mathcal{U}$. Also, a mapping remains quasi-uniformly continuous if we replace \mathcal{U} by a finer quasi-uniformity and (or) \mathcal{V} by a coarser quasi-uniformity. The composition of two quasi-uniformly continuous mappings is quasi-uniformly continuous.

For each $\varepsilon > 0$ and $f : X \to \mathbb{R}$, we will let $U_{(\varepsilon,f)} = f_2^{-1}(W_{\varepsilon}) = \{(x,y) : f(x) - f(y) < \varepsilon\}$. Let \mathcal{U} be a quasi-uniformity on X. Then $f : (X,\mathcal{U}) \to (\mathbb{R},\mathcal{W})$ is quasi-uniformly continuous if and only if for each $\varepsilon > 0$, $U_{(\varepsilon,f)} \in \mathcal{U}$. Then $Q(\mathcal{U})$ ($QB(\mathcal{U})$) denotes the set of all (bounded) quasi-uniformly continuous functions from (X,\mathcal{U}) to (\mathbb{R},\mathcal{W}) .

Proposition 3.1.2.1 Every quasi-uniformly continuous mapping is continuous.

Proof: Let $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be a quasi-uniformly continuous mapping and let V[f(x)] be a neighborhood of f(x), $V \in \mathcal{V}$. By hypothesis, there exists a $U \in \mathcal{U}$ such that $f_2(U) \subset V$. So, if $y \in U[x]$, then $f(y) \in V[f(x)]$ and f is continuous.

It is well known that a continuous mapping need not be uniformly continuous. It follows that a continuous mapping need not be quasi-uniformly continuous.

Proposition 3.1.2.2 If $f : (X, U) \to (Y, V)$ is a quasi-uniformly continuous mapping, then $f : (X, U^{-1}) \to (Y, V^{-1})$ is as well.

Proof: This follows from the fact that $f_2^{-1}(V^{-1}) = (f_2^{-1}(V))^{-1}$.

Proposition 3.1.2.3 Let (X, Q) and (Y, \mathcal{R}) be quasi-uniform spaces, let $\mathcal{U} = \sup\{\mathcal{Q}, Q^{-1}\}$ and let $\mathcal{V} = \sup\{\mathcal{R}, \mathcal{R}^{-1}\}$. If $f : (X, Q) \to (Y, \mathcal{R})$ is quasi-uniformly continuous, then $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is uniformly continuous.

Proof: By Proposition 3.1.2.2, $f: (X, \mathcal{Q}^{-1}) \to (Y, \mathcal{R}^{-1})$ is quasi-uniformly continuous. Since $\{R \cap R^{-1} : R \in \mathcal{R}\}$ is a base for \mathcal{V} , it suffices to show that $f_2^{-1}(R \cap R^{-1}) \in \mathcal{U}$. But this is an easy consequence of $f_2^{-1}(R \cap R^{-1}) = f_2^{-1}(R) \cap f_2^{-1}(R^{-1}), f_2^{-1}(R) \in \mathcal{Q}$ and $f_2^{-1}(R^{-1}) \in \mathcal{Q}^{-1}$.

Corollary 3.1.2.1 Let (X, U) be a uniform space and (Y, \mathcal{R}) a quasi-uniform space with $\mathcal{V} = \sup\{\mathcal{R}, \mathcal{R}^{-1}\}$. Then $f : (X, U) \to (Y, \mathcal{R})$ is quasiuniformly continuous if and only if $f : (X, U) \to (Y, \mathcal{V})$ is uniformly continuous.

Proof: " \Rightarrow " part follows from Proposition 3.1.2.3. For the converse, if $R \in \mathcal{R}$, then trivially $f_2^{-1}(R) \in \mathcal{U}$.

Definition 3.1.2.2 A bijection $f : (X, U) \to (Y, V)$ is called a **quasi-unimorphism** if f and f^{-1} are quasi-uniformly continuous. Two quasi-uniform spaces are called quasi-unimorphic if there exists a quasi-unimorphism between them.

The identity mapping of a quasi-uniform structure, inverse of a quasiunimorphism and the composition of quasi-unimorphisms are again quasiunimorphisms. Thus, quasi-uniform isomorphism is an equivalence relation between quasi-uniform spaces. A property which is invariant under quasiunimorphism is called a quasi-uniformly invariant. From Proposition 3.1.2.1 and the subsequent remark, it follows that every quasi-unimorphism is a homeomorphism, but not conversely. Consequently, every topological property is also a quasi-uniform invariant, but not the other way round. On the other hand, every quasi-uniform invariant is also a uniform invariant, but not the other way round.

Lemma 3.1.2.2 Let (X, \mathcal{U}) be a quasi-uniform space and $Y \subset X$. Let $\mathcal{U}|Y$ denote the trace of \mathcal{U} by $Y \times Y$, i.e. $\mathcal{U}|Y = \{U \cap (Y \times Y) : U \in \mathcal{U}\}$. Then $\mathcal{U}|Y$ is a quasi-uniformity on Y.

Proof: (a) Clearly $\Delta \subset A$ for each $A \in \mathcal{U}|Y$.

(b) If $A, B \in \mathcal{U}|Y$, then $A = U \cap (Y \times Y)$, $B = V \cap (Y \times Y)$ for some $U, V \in \mathcal{U}$. Hence $A \cap B = (U \cap V) \cap (Y \times Y) \in \mathcal{U}|Y$ since $U \cap V \in \mathcal{U}$.

(c) If $A \in \mathcal{U}|Y$, then $A = U \cap (Y \times Y)$ for some $U \in \mathcal{U}$. But there exists a $V \in \mathcal{U}$ such that $V \circ V \subset U$. If $B = V \cap (Y \times Y)$, then $B \in \mathcal{U}|Y$ and $B \circ B \subset A$.

(d) Let $A \in \mathcal{U}|Y$ and let $A \subset B$. Then $A = U \cap (Y \times Y)$ for some $U \in \mathcal{U}$. By setting $V = U \cup B \in \mathcal{U}$ and so $B = V \cap (Y \times Y) \in \mathcal{U}|Y$.

This proves that $\mathcal{U}|Y$ is a quasi-uniformity on Y. \clubsuit

In the same way, we can prove that if \mathcal{B} is a base (subbase) for \mathcal{U} , then $\mathcal{B}|Y$ is a base (subbase) for $\mathcal{U}|Y$.

In the notation of Lemma 3.1.2.2 $\mathcal{U}|Y$ is called the **relative quasi**uniformity on Y, and $(Y, \mathcal{U}|Y)$ is called a **quasi-uniform subspace**.

If \mathcal{U} induces the topology τ on X, then the topology induced by $\mathcal{U}|Y$ on Y is precisely the subspace topology of τ . Also, if f is quasi-uniformly continuous on (X, \mathcal{U}) , then the restriction mapping f|Y is quasi-uniformly continuous on $(Y, \mathcal{U}|Y)$.

Definition 3.1.2.3 Let (X_i, \mathcal{U}_i) be a family of quasi-uniform spaces, indexed by I, and let $X = \prod_{i \in I} X_i$. The **product quasi-uniformity** \mathcal{U} is the smallest quasi-uniformity on X which makes every projection quasiuniformly continuous. Then (X, \mathcal{U}) is called the **product quasi-uniform space**. \mathcal{U} is denoted by $\prod_{i \in I} \mathcal{U}_i$.

Let $S(i, U_i) = (p_i)_2^{-1}(U_i) = \{(x, y) : (p_i(x), p_i(y)) \in U_i\}$ for any $U_i \subset X_i \times X_i$. Then for each $i \in I$ and each $U_i \in \mathcal{U}_i$, $S(i, U_i)$ is an entourage in the product quasi-uniformity. Hence, the product quasi-uniformity is the smallest one that contains all sets $S(i, U_i), i \in I, U_i \in \mathcal{U}_i$. It is clear that (a) $\Delta \subset S(i, U_i)$ for all $i \in I, U_i \in \mathcal{U}_i$; (b) $S(i, V_i) \circ S(i, V_i) \subset S(i, U_i)$ provided $V_i \circ V_i \subset U_i$.

By Proposition 3.2.1.1, it follows that the family $S = \{S(i, U_i) : i \in U_i \in U_i\}$ is a quasi-uniform subbase. From the foregoing remarks it is obvious that the quasi-uniformity generated by S on X is precisely the product quasi-uniformity. This gives us an alternative way of defining the product quasi-uniformity.

We will note that the conclusions of the last paragraph still hold if \mathcal{U}_i , $i \in I$, are taken to be quasi-uniform bases on X_i .

Proposition 3.1.2.4 Let (X_i, \mathcal{U}_i) , $i \in I$, be a family of quasi-uniform spaces and let (X, \mathcal{U}) be their product. Then the topology of \mathcal{U} is the product of the topologies of \mathcal{U}_i , $i \in I$.

Proof: We have already seen that the family of all sets of the form $S(i, U_i)$, $i \in I$, is a quasi-uniform subbase for \mathcal{U} . If $x \in X$, then by Proposition 3.1.1.5 the family of sets of the form $\{y : (x, y) \in S(i, U_i)\}$ is a subbase for the neighborhood system of x in the product topology induced by \mathcal{U} . But this is precisely the subbase in the Tychonoff product topology induced by the topologies \mathcal{U}_i .

Proposition 3.1.2.5 Let $f : (X, U) \to (Y, V)$ be a mapping, where (Y, V) is the product of family (Y_i, V_i) , $i \in I$, of quasi-uniform spaces. Then f is quasi-uniformly continuous if and only if $p_i \circ f$ is quasi-uniformly continuous for each $i \in I$, where p_i denotes the projection Y to Y_i .

Proof: If f is quasi-uniformly continuous, then $p_i \circ f$ is quasi-uniformly continuous as the composition of quasi-uniformly continuous mappings for each $i \in I$. If $p_i \circ f$ is quasi-uniformly continuous for each $i \in I$, then $(p_i \circ f)_2^{-1}(U_i) \in \mathcal{U}$ for each $U_i \in \mathcal{V}_i$. But $(p_i \circ f)_2^{-1}(U_i) = f_2^{-1}(S(i, U_i))$. Hence the inverse under f_2 of each member of a subbase for the product quasi-uniformity belongs to \mathcal{U} and so by Definition 3.1.2.1 (b), f is quasi-uniformly continuous.

Let (X, \mathcal{U}) be a quasi uniform space and let τ be the topology induced by \mathcal{U} . On the product set $X \times X$ a topology, related to τ , can be given in many possible ways. The most natural, perhaps, is the topology $\tau \times \tau$. We shall call this the **product topology** and will not always mention it explicitly. The other natural choices are $\tau \times \tau^{-1}$ and $\tau^{-1} \times \tau$, where τ^{-1} is the topology induced by \mathcal{U}^{-1} . We call these topology **hybrid topologies**.

We will presently see that the lack of symmetry in quasi-uniform structures is partially overcome by considering the hybrid topologies instead of the product topology as it is usually done. As just one example, we will mention the following.

Unlike in uniform spaces, an entourage in a quasi-uniformity need not be a neighborhood (in the product topology) of the diagonal. In Example 3.1.1.1(a), the set $\{(x, y) : x \leq y\}$ is an entourage but not a neighborhood of Δ in the product topology. This situation is remedied by replacing the product topology by the appropriate hybrid topology (see Proposition 3.1.2.9 below).

Lemma 3.1.2.3 If L, M, N are subsets of $X \times X$, then

$$L \circ M \circ N = \bigcup_{(x,y) \in M} N^{-1}[x] \times L[y].$$

Proof: $(a, b) \in L \circ M \circ N$ if and only if there exist x, y such that $(a, x) \in N$, $(x, y) \in M$ and $(y, b) \in L$ if and only if there exists an $(x, y) \in M$ such that $a \in N^{-1}[x]$ and $b \in L[y]$ if and only if $(a, b) \in N^{-1}[x] \times L[y]$ for some $(x, y) \in M$.

Corollary 3.1.2.2 If \mathcal{U}, \mathcal{V} are quasi-uniformities on X, and $L \in \mathcal{U}, N \in \mathcal{V}$, then $L \circ M \circ N$ is a neighborhood of M in the topology of $\mathcal{V}^{-1} \times \mathcal{U}$.

Proposition 3.1.2.6 Let τ_1 and τ_2 be topologies on a set X and let (Y, \mathcal{V}) be a quasi-uniform space. If $f : (X, \tau_1) \to (Y, \tau_{\mathcal{V}})$ and $f : (X, \tau_2) \to (Y, \tau_{\mathcal{V}^{-1}})$ are continuous mappings, then, for each $V \in \mathcal{V}$, $f_2^{-1}(V)$ is a $\tau_2 \times \tau_1$ -neighborhood of Δ .

Proof: Let $V \in \mathcal{V}$ and let $W \in \mathcal{V}$ such that $W \circ W \subset V$. Let $x \in X$. There exists $G_1 \in \tau_1$ and $G_2 \in \tau_2$ such that $x \in G_1 \cap G_2$, $f(G_1) \subset W[f(x)]$ and $f(G_2) \subset W^{-1}[f(x)]$. It is easily verified that $(x, x) \in G_2 \times G_1 \subset f_2^{-1}(V)$.

Proposition 3.1.2.7 Let \mathcal{B} and \mathcal{C} be bases for the quasi-uniformities \mathcal{U} and \mathcal{V} respectively on X. If $M \subset X \times X$ is any set, then the closure of M in the topology of $\mathcal{U} \times \mathcal{V}$ is given by

$$\overline{M} = \bigcap C^{-1} \circ M \circ B \,,$$

where $B \in \mathcal{B}$ and $C \in \mathcal{C}$.

Proof: A point $(x, y) \in \overline{M}$ if and only if for each $B \in \mathcal{B}$, $C \in \mathcal{C}$, $(B[x] \times C[y]) \cap M \neq \emptyset$ if and only if there exists an $(a, b) \in M$ such that $a \in B[x]$, $b \in C[y]$ for each B, C if and only if $(x, y) \in C^{-1} \circ M \circ B$ for each $B \in \mathcal{B}$ and $C \in \mathcal{C}$.

Corollary 3.1.2.3 If $\mathcal{U} = \mathcal{V}$, then $\overline{M} = \bigcap B^{-1} \circ M \circ B$, where $B \in \mathcal{B}$.

Proof: By Proposition 3.1.2.7,

$$\overline{M} = \bigcap_{B,C \in \mathcal{B}} C^{-1} \circ M \circ B \subset \bigcap_{B \in \mathcal{B}} B^{-1} \circ M \circ B \,.$$

But for each $B, C \in \mathcal{B}, D^{-1} \circ M \circ D \subset C^{-1} \circ M \circ B$ where $D = B \cap C$ and hence the reverse inclusion follows.

Proposition 3.1.2.8 Let \mathcal{U} and \mathcal{V} be quasi-uniformities an X and let G be a subset of $X \times X$. Then G is open in $\mathcal{U} \times \mathcal{V}$ if and only if G^{-1} is open in $\mathcal{V} \times \mathcal{U}$.

Proof: A point $(x, y) \in G^{-1}$ if and only if $(y, x) \in G$ if and only if there exist $U \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $U[y] \times V[x] \subset G$ if and only if $V[x] \times U[y] \subset G^{-1}$.

Corollary 3.1.2.4 The reflexive mapping $(x, y) \to (y, x)$ is a homeomorphism of $(X \times X, \mathcal{U} \times \mathcal{V})$ to $(X \times X, \mathcal{V} \times \mathcal{U})$.

Corollary 3.1.2.5 If $M \subset X \times X$ and \mathcal{P} is a topological property, then M has \mathcal{P} in $\mathcal{U} \times \mathcal{V}$ topology if and only if M^{-1} has \mathcal{P} in $\mathcal{V} \times \mathcal{U}$ topology. In particular, M is closed in $\mathcal{U} \times \mathcal{V}$ if and only if M^{-1} is closed in $\mathcal{V} \times \mathcal{U}$.

Corollary 3.1.2.6

$$Int_{\mathcal{U}\times\mathcal{V}}M^{-1} = \left(Int_{\mathcal{V}\times\mathcal{U}}M\right)^{-1},$$
$$Cl_{\mathcal{U}\times\mathcal{V}}M^{-1} = \left(Cl_{\mathcal{V}\times\mathcal{U}}M\right)^{-1} \clubsuit.$$

Proposition 3.1.2.9 If (X, U) is a quasi-uniform space and U is an entourage, then

 $Int_{\mathcal{U}^{-1}\times\mathcal{U}}U\in\mathcal{U}$.

Proof: There exists an entourage V such that $V \circ V \circ V \subset U$. By Corollary 3.1.2.2, $V \circ V \circ V$ is a neighborhood of V in $\mathcal{U}^{-1} \times \mathcal{U}$. Hence U is a neighborhood of V and $V \subset \operatorname{Int}_{\mathcal{U}^{-1} \times \mathcal{U}} U$, which is then an entourage by Definition 3.1.1.1 (d).

Corollary 3.1.2.7 Entourages which are open in $\mathcal{U}^{-1} \times \mathcal{U}$ form a base for quasi-uniformity \mathcal{U} .

Proposition 3.1.2.10 Entourages which are closed in $\mathcal{U} \times \mathcal{U}^{-1}$ form a base for quasi-uniformity \mathcal{U} .

Proof: For an entourage U, let V be an entourage such that $V \circ V \circ V \subset U$. By Proposition 3.1.2.7, $V \subset \operatorname{Cl}_{\mathcal{U} \times \mathcal{U}^{-1}} V \subset V \circ V \circ V \subset U$ and so $\operatorname{Cl} V \in \mathcal{U}$.

Corollary 3.1.2.8 If \mathcal{U} and \mathcal{U}^{-1} are compatible, then the induced topology is regular.

Proof: Under the hypothesis, the two hybrid topologies coincide with the usual product topology. Since closed entourages form a base for \mathcal{U} and closed neighborhoods of x form a local base, then the topology is regular.

Proposition 3.1.2.11 If $\tau_{\mathcal{U}} \subset \tau_{\mathcal{U}^{-1}}$, then $\tau_{\mathcal{U}^{-1}}$ is uniformizable and hence completely regular.

Proof: Let $\mathcal{W} = \sup\{\mathcal{U}, \mathcal{U}^{-1}\}$. Then \mathcal{W} is a uniformity. By hypothesis, for each $x \in X$, and $U \in \mathcal{U}$, there exists a $V \in \mathcal{U}$, $V \subset U$, such that $V^{-1}[x] \subset U[x]$. This shows that $V^{-1}[x] \subset U \cap U^{-1}[x]$. Hence, $\tau_{\mathcal{W}} \subset \tau_{\mathcal{U}^{-1}}$. But obviously $\tau_{\mathcal{U}^{-1}} \subset \tau_{\mathcal{W}}$. Hence $\tau_{\mathcal{U}^{-1}} = \tau_{\mathcal{W}}$ is uniformizable and hence completely regular.

Corollary 3.1.2.9 If a quasi-uniformity is compatible with its conjugate, then the induced topology is completely regular. \clubsuit

Theorem 3.1.2.1 Let (X, τ) be a compact Hausdorff space and let G be a closed partial order on X. There exists exactly one quasi-uniformity \mathcal{U} on X such that $\cap \mathcal{U} = G$ and $\tau_{\mathcal{U}^*} = \tau$.

Proof: Let us suppose that \mathcal{U} is a quasi-uniformity on X such that $\cap \mathcal{U} = G$ and $\tau_{\mathcal{U}^*} = \tau$. We will show that \mathcal{U} consists of all $\tau \times \tau$ -neighborhoods of G. By Corollary 3.1.2.7, all members of \mathcal{U} are $\tau \times \tau$ -neighborhoods of G. Let us suppose that there exists a $\tau \times \tau$ -neighborhood V of G that is not a member of \mathcal{U} . Then $\{U - V : U \in \mathcal{U}\}$ is a base for a filter \mathcal{V} on $X \times X$. Since (X, τ) is compact, \mathcal{V} has a $\tau \times \tau$ -cluster point (x, y) that does not belong to G. Since \mathcal{U} is coarser than \mathcal{V} , (x, y) is a cluster point of \mathcal{U} . It follows from Corollary 3.1.2.10 that the intersection of the $\tau \times \tau$ closures of members of \mathcal{U} is G, which is a contradiction.

The proof may be completed by establishing that the family \mathcal{U} of all $\tau \times \tau$ -neighborhoods of G is a quasi-uniformity on X such that $\cap \mathcal{U} = G$ and $\tau_{\mathcal{U}^*} = \tau$. Clearly \mathcal{U} is a filter on $X \times X$, and $\cap \mathcal{U} = G$. Let us suppose that (QU_2) is not satisfied; then there exists a $\tau \times \tau$ -open set $U \in \mathcal{U}$ such that for all $V \in \mathcal{U}, V \circ V - U \neq \emptyset$. For each $V \in \mathcal{U}$ let $V' = \{((x, y), z) \in X^2 \times X : (x, y) \notin U, (x, z) \in V, (z, y) \in V\}$. It follows that $\mathcal{B} = \{V' : V \in \mathcal{U}\}$ is a filter base on $(X \times X - U) \times X$. Since $(X \times X - U) \times X$ is compact, \mathcal{B} has a cluster point ((a, b), c). We assert that $(a, c) \in G$. Let us suppose that $(a, c) \notin G$; as G is compact, there exist disjoint open sets V and H such that $G \subset V$ and $(a, c) \in H$. Let us set $W = \{((x, y), z) : (x, z) \in H\}$. Then $((a, b), c) \in W$ and $W \cap V' = \emptyset$, which is a contradiction. Thus $(a, c) \in G$ and it follows as above that $(c, b) \in G$. Since G is transitive, $(a, b) \in G \subset U$, which is also a contradiction. Finally it is evident that $\tau_{\mathcal{U}^*} \subset \tau$; as $\cap \mathcal{U}$ is a partial order, $\tau_{\mathcal{U}^*}$ is a Hausdorff topology. Hence $\tau_{\mathcal{U}^*} = \tau$.

Corollary 3.1.2.10 Let (X, τ) be a compact Hausdorff space. The unique uniformity compatible with τ is the family of all neighborhoods of Δ .

Theorem 3.1.2.2 Let (X, \mathcal{U}) and (Y, \mathcal{V}) be quasi-uniform spaces and let us suppose that (X, \mathcal{U}^*) is a compact Hausdorff space. If $f : (X, \tau_{\mathcal{U}}) \to (Y, \tau_{\mathcal{V}})$ and $f : (X, \tau_{\mathcal{U}^{-1}}) \to (Y, \tau_{\mathcal{V}^{-1}})$ are continuous mappings, then $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ (Y, \mathcal{V}) is quasi-uniformly continuous.

Proof: Let $V \in \mathcal{V}$. By Proposition 3.1.2.6 $f_2^{-1}(V)$ is a $\tau_{\mathcal{U}^{-1} \times \mathcal{U}}$ -neighborhood of Δ and hence of $\cap \mathcal{U}$. By the proof of Theorem 3.1.2.1, $f_2^{-1}(V) \in \mathcal{U}$.

3.1.3 Quasi-uniformities and pseudo-quasi-metrics

Definition 3.1.3.1 A real-valued function d on $X \times X$ is called a **pseudo** quasi-metric on X if

(a) $d(x,y) \ge 0$ for every $x, y \in X$,

(b) d(x, x) = 0 for every $x \in X$,

(c) $d(x,z) \leq d(x,y) + d(y,z)$ for every $x, y, z \in X$.

If, in addition,

(d) d(x, y) = 0 implies x = y,

then d is called a quasi-metric. A pseudo quasi-metric d satisfying

(e) d(x,y) = d(y,x) for every $x, y \in X$

is called a *pseudo-metric*.

Let d be a pseudo-quasi-metric on X. For $a \in X$ and $\varepsilon > 0$ the set $S(a, \varepsilon) = \{x \in X : d(x, a) < \varepsilon\}$ is called an ε -sphere with center at a. The family $\{S(a, \varepsilon) : a \in X, \varepsilon > 0\}$ is a base for a topology on X, called the **pseudo-quasi-metric topology of** d and is denoted by τ_d .

If $U_{\varepsilon} = \{(x, y) : d(x, y) < \varepsilon\}$, then the family $\{U_{\varepsilon} : \varepsilon \in \mathbb{R}^+\}$ satisfies conditions of Proposition 3.1.1.1 and hence is a quasi-uniform base. The quasi-uniformity generated in such a way is the **quasi-uniformity of** d, and denoted by \mathcal{U}_d . Let us note that $U_{\varepsilon}[x] = S(x, \varepsilon)$ and therefore the topology induced by the quasi-uniformity of d is precisely the τ_d .

If d is a pseudo-quasi-metric on X, so is d^* , where

$$d^*(x,y) = d(y,x) \,.$$

We call d and d^* conjugate pseudo-quasi-metrics. It is easy to see that the quasi-uniformities and topologies derived from d and d^* are conjugate.

Example 3.1.3.1 In examples (a) - (c), X is any set linearly ordered by <, and in (d), (e) the set of real numbers.

(a)
$$d(x,y) = \begin{cases} 0, & \text{if } x \leq y, \\ 1, & \text{if } x > y, \end{cases}$$

(b) $d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x = y, \end{cases}$

(b)
$$d(x,y) = \begin{cases} 1, & \text{if } x < y \\ 2, & \text{if } x > y \end{cases}$$

(c)
$$d(x,y) = \begin{cases} \rho(x,y), & \text{if } x \leq y, \\ 1, & \text{if } x > y. \end{cases}$$
 (ρ is any metric bounded by 1)

if $r \leq u$

(d)
$$d(x,y) = \max\{0, y-x\}$$

(e)
$$d(x,y) = \begin{cases} 0, & \text{if } x < y, \\ \min\{1, |x-y|\}, & \text{if } x > y. \end{cases}$$

A quasi-uniform space (X, \mathcal{U}) is said to be **pseudo-quasi-metrizable** if there exists a pseudo-quasi-metric d on X whose quasi-uniformity coincides with \mathcal{U} . Such a quasi-uniformity has a countable base; for, the family $\{U_{\varepsilon}\}$ as ε runs through all positive rationales is a base.

We omit the proof of the following lemma on the assumption that the elegant presentation of this result given in [164] is one argument concerning quasi-uniform spaces that are easily accessible and well known.

Lemma 3.1.3.4 Let (U_n) be a sequence of reflexive relations on a set X such that $U_n \circ U_n \circ U_n \subset U_n$ for each $n \in \mathbb{N}$. Then there exists a quasipseudo-metric d for X such that $U_{n+1} \subset \{(x,y) : d(x,y) < 1/2^n\} \subset U_n$ for each $n \in \mathbb{N}$. If each U_n is symmetric, d can be taken to be a pseudo-metric.

Theorem 3.1.3.1 A quasi-uniform space (X, U) is pseudo-quasi-metrizable if and only if U has a countable base.

The proof of this theorem is omitted, since it is too long, but easily accessible (see [259]).

Definition 3.1.3.2 If d, e are pseudo-quasi-metrics on X and Y respectively, then the **product pseudo-quasi-metric** $d \times e$ on $X \times Y$ is defined by

 $(d \times e)(u, v) = (d^2(x_1, x_2) + e^2(y_1, y_2))^{1/2},$

where $u = (x_1, y_1)$ and $v = (x_2, y_2)$ are points of $X \times Y$.

That the function $d \times e$ defined in such a way is a pseudo-quasi-metric is easily established. The quasi-uniformity and the topology derived form $d \times e$ coincide with the product of the quasi-uniformities and topologies respectively, derived from d and e.

A particular interest for us in the following subsection are the products $d \times d^*$ and $d^* \times d$.

There are several other ways of generating new pseudo-quasi-metrics from the given ones. One of them is the following. If $\{d_i\}$ is a family of pseudo-quasi-metrics on X, let us define d by

$$d(x,y) = \sup_{i} d_i(x,y) \,.$$

Then d is a pseudo-quasi-metric on X, called the **supremum** of $\{d_i\}$. It is generally not true that the quasi-uniformity $\sup_i d_i$ coincides with the supremum of the quasi-uniformities of d_i , unless the family is finite. However, by virtue of Theorem 3.1.3.1 we have the following

Proposition 3.1.3.1 The supremum of countable many pseudo-quasi-metrizable spaces is pseudo-quasi-metrizable. \clubsuit

Definition 3.1.3.3 A real-valued function on quasi-uniform space (X, U)is **quasi-uniformly upper semi-continuous** if for each $\varepsilon > 0$, there is an entourage $U \in U$ such that $(x, y) \in U$ implies $f(y) < f(x) + \varepsilon$.

A quasi-uniformly lower semi-continuous function is defined similarly by reversing the last inequality and by replacing ε with $-\varepsilon$. Obviously a function is quasi-uniformly upper semi-continuous on (X, \mathcal{U}) if and only if it is quasi-uniformly lower semi-continuous on (X, \mathcal{U}^{-1}) .

Proposition 3.1.3.2 Let d be a pseudo-quasi-metric on X. Then d is quasi-uniformly upper semi-continuous in $\mathcal{U}_{d^* \times d}$ and consequently quasi-uniformly lower semi-continuous in $\mathcal{U}_{d \times d^*}$.

Proof: Let $V_{\varepsilon} = \{(u, v) \in X^2 \times X^2 : (d^* \times d)(u, v) < \varepsilon\}$. Let us recall that the sets V_{ε} form a base for the quasi-uniformity of $d^* \times d$. Then $((x_1, x_2), (x_3, x_4)) \in V_{\varepsilon}$ implies $d^*(x_1, x_3) < \varepsilon$ and $d(x_2, x_4) < \varepsilon$. This in turn implies $d(x_3, x_4) \leq d(x_3, x_1) + d(x_2, x_4) < d(x_1, x_2) + 2\varepsilon$. Hence d is quasi-uniformly upper semi-continuous relative to the quasi-uniformity of $d^* \times d$.

Corollary 3.1.3.1 Let d be a pseudo-quasi-metric on X. Then for every x, d(x, y) is a quasi-uniformly upper semi-continuous function of y and for every y, d(x, y) is a quasi-uniformly lower semi-continuous function of x (relative to the quasi-uniformity of d).

We will now prove one generalization of this corollary.

Proposition 3.1.3.3 For any fixed set $A \subset X$, d(x, A) and d(A, x) are respectively quasi-uniformly lower semi-continuous and quasi-uniformly upper semi-continuous functions of x.

Proof: Let $U_{\varepsilon} = \{(x, y) \in X^2 : d(x, y) < \varepsilon\}$. Then $(x, y) \in U_{\varepsilon}$ implies $d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a) \leq d(y, a) + \varepsilon$, for all $a \in A$. Hence $d(y, A) \geq d(x, A) - \varepsilon$. This proves one part; the second part is a consequence of the first.

Theorem 3.1.3.2 If, for each $x \in X$, the pseudo-quasi-metric d is d^* -continuous in y, then τ_{d^*} is pseudo-metrizable.

Proof: For any $\varepsilon > 0$, the inverse image of $(0, \varepsilon)$ is d^* -open. That is, $\{y : d(x, y) < \varepsilon\}$ is d^* -open. This means $\tau_d \subset \tau_{d^*}$. By Proposition 3.1.2.11 τ_{d^*} is pseudo-metrizable.

3.1.4 Separation axioms

A pseudo-metric space is always normal and hence given the T_0 -axiom, other separation axioms T_1 , T_2 , T_3 and complete regularity are at once satisfied. A similar situation prevails in a uniform space since it is necessarily completely regular. In a pseudo-quasi-metric or a quasi-uniform spaces, however, no separation axiom need be satisfied. In this subsection we will attempt to find pseudo-quasi-metric and quasi-uniformly characterizations of the various separation and "regularity" axioms.

To avoid repetitions and cumbersome notation, we will make the following conventions for this subsection. In all propositions, it is understood that (X, \mathcal{U}) is a quasi-uniform space, \mathcal{B} is any base for \mathcal{U} , d is a pseudo-quasimetric on X and \cap stands for the intersection as U runs through \mathcal{B} .

Proposition 3.1.4.1 (a) (X, U) is T_0 if and only if $\cap U$ is an anti-symmetric set. (b) (X, d) is T_0 if and only if d(x, y) + d(y, x) = 0 implies x = y.

Proof: (a) Let $S = \cap U$. Then $(x, y) \in S \cap S^{-1}$ if and only if $x \in U[y]$ and $y \in U[x]$ for all entourages $U \in \mathcal{U}$. Hence X is T_0 if and only if S is an anti-symmetric set.

(b) Follows from the fact that τ_d is T_0 if and only if $x \neq y$ implies that at least one of d(x, y) and d(y, x) is non-zero.

Proposition 3.1.4.2 (a) (X, U) is T_1 if and only if $\Delta = \cap U$. (b) (X, d) is T_1 if and only if d(x, y) = 0 implies x = y. That is, a pseudo-quasi-metric is T_1 if and only if it is a quasi-metric.

Proof: Equality $\Delta = \cap U$ holds if and only if $x = \cap U[x]$ for each $x \in X$, which in turn is equivalent to T_1 . The proof of (b) is omitted.

The condition $\Delta = \cap U$ is the well-known characterization of T_2 in uniform spaces, but it characterizes T_1 in quasi-uniform spaces. This, however, is not so surprising since a uniform space is T_2 whenever it is T_1 . The following characterization of T_1 quasi-uniform spaces, though less elegant than the characterization given in Proposition 3.1.4.2, proves useful.

Proposition 3.1.4.3 Let (X, U) be a quasi-uniform space. Then (X, U) is a T_1 space if and only if the following condition holds.

If a filter \mathcal{F} converges in $(X, \tau_{\mathcal{U}^{-1}})$ to x, then \mathcal{F} has no $\tau_{\mathcal{U}}$ cluster point different from x.

Proposition 3.1.4.4 (X, \mathcal{U}) is T_2 if and only if $\Delta = \cap (U^{-1} \circ U)$.

Proof: X is T_2 if and only if $\Delta = \overline{\Delta}$. Since by Corollary 3.1.2.3 $\overline{\Delta} = \cap (U^{-1} \circ \Delta \circ U)$, we have the proof of the proposition.

We now pass on to the "regularity" axioms, of which R_0 and R_1 were first introduced by Davis [64]. It will be found that the R_0 axiom is particularly well-behaved in the sense that several characterizations are obtainable.

Proposition 3.1.4.5 Let (X, τ) be a topological space. Then the following statements are equivalent:

(a) (X, τ) is R_0 ;

(b) there exists a compatible quasi-uniformity \mathcal{U} such that for each $x \in X$ and an entourage $U \in \mathcal{U}$, there exists a symmetric entourage $V \in \mathcal{U}$ such that $V[x] \subset U[x]$;

(c) there exists a compatible quasi-uniformity \mathcal{U} such that for each $x \in X$, the sets V[x] as V runs through all symmetric entourages of \mathcal{U} , form a local base at x;

(d) there exists a compatible quasi-uniformity \mathcal{U} such that for each $x \in X$ and an entourage $U \in \mathcal{U}$, there exists an entourage $V \in \mathcal{U}$ with $V^{-1}[x] \subset U[x]$;

(e) there exists a compatible quasi-uniformity \mathcal{U} such that $\tau = \tau_{\mathcal{U}} \subset \tau_{\mathcal{U}^{-1}}$.

Proof: $(a) \Rightarrow (b)$: Let \mathcal{U} be the Pervin quasi-uniformity of τ . For any given point x and an entourage $U \in \mathcal{U}$, U[x] is a neighborhood of x and hence contains an open neighborhood G of x. Let us define $V = (G \times G) \cup ((X - \overline{x}) \times (X - \overline{x}))$. Then, since $\overline{x} \subset G$, $S_G \cap S_{X-\overline{x}} \subset V$. Hence V is an entourage and obviously symmetric. Also $V[x] = G \subset U[x]$.

 $(b) \Leftrightarrow (c) \Rightarrow (d) \Leftrightarrow (e)$ is obvious.

 $(d) \Rightarrow (a)$: By Proposition 3.1.1.9 $\overline{x} = \cap U^{-1}[x] \subset \cap U[x]$, which means that \overline{x} is contained in every neighborhood of x.

By combining Proposition 3.1.2.11 and Proposition 3.1.4.5, we come to the following:

Corollary 3.1.4.1 (X, τ) is R_0 if and only if τ has a completely regular conjugate topology.

Proposition 3.1.4.6 (a) (X, U) is R_0 if and only if $\cap U$ is symmetric. (b) (X, d) is R_0 if and only if for all $x, y \in X$, $d(x, y) = 0 \Leftrightarrow d(y, x) = 0$.

Proof: The space is R_0 if and only if for each $x \in X$, $\cap U^{-1}[x] = \overline{x} \subset \cap U[x]$ (by Proposition 3.1.1.9 and definition of R_0), if and only if $\cap U^{-1} \subset \cap U$. Reflecting both sides, $\cap U \subset \cap U^{-1}$. The proof of (b) is omitted.

By combining Proposition 3.1.4.1 (a) and 3.1.4.6 (a) and by observing that Δ is the only set in $X \times X$ which is both symmetric and anti-symmetric, the following holds:

Corollary 3.1.4.2 A quasi-uniform space is T_1 if and only if it is T_0 and R_0 .

Proposition 3.1.4.7 (X, U) is R_0 if and only if for each $x \in X$, $\overline{x} = \cap U[x]$.

Proof: Since $\cap U = \cap U^{-1}$, $\overline{x} = \cap U^{-1}[x] = \cap U[x]$.

Proposition 3.1.4.8 A quasi-uniform space (X, U) is R_1 if and only if $\widetilde{\Delta} = \{(x, y) : \overline{x} = \overline{y}\} = \cap U^{-1} \circ U.$

Proof: Follows from Proposition 2.2.4.1 and Corollary 3.1.2.3.

Corollary 3.1.4.3 $T_2 = T_1 + R_1 = T_0 + R_1$.

Theorem 3.1.4.1 Let (X, τ) be a topological space. Then the following statements are equivalent:

(a) (X, τ) is a regular space;

(b) there exists a compatible quasi-uniformity \mathcal{U} such that for each point x and each entourage $U \in \mathcal{U}$, there exists a symmetric entourage $V \in \mathcal{U}$ with $V \circ V[x] \subset U[x]$;

(c) there exists a compatible quasi-uniformity \mathcal{U} such that for each point x and each entourage $U \in \mathcal{U}$, there exists an entourage $V \in \mathcal{U}$ such that $V^{-1} \circ V[x] \subset U[x]$.

Proof: $(a) \Rightarrow (b)$: We will show that the Pervin quasi-uniformity \mathcal{U} of τ has the required property. Let $x \in X$ and $U \in \mathcal{U}$ be arbitrary, and let $G = \operatorname{Int} U[x]$. Since X is a regular space, there exist open neighborhoods I, H of x such that $\overline{I} \subset H \subset \overline{H} \subset G$. Let $W_1 = (G \times G) \cup ((X - \overline{H}) \times (X - \overline{H}))$, and $W_2 = (H \times H) \cup ((X - \overline{I}) \times (X - \overline{I}))$. Clearly $V = W_1 \cap W_2$ is symmetric, and $V \circ V[x] = G \subset U[x]$. Also it can be shown that $S_G \cap S_H \cap S_{X - \overline{H}} \subset W_1$ and $S_H \cap S_I \cap S_{X - \overline{I}} \subset W_2$, which proves that V is an entourage.

 $(b) \Rightarrow (c)$: Obvious.

 $(c) \Rightarrow (a)$: For a given point x and an entourage $U \in \mathcal{U}$, let V be such that $V^{-1} \circ V[x] \subset U[x]$. Hence by Proposition 3.1.1.9 we have that

$$\overline{V[x]} = \bigcap_{W \in \mathcal{U}} W^{-1} \circ V[x] \subset V^{-1} \circ V[x] \subset U[x] \,,$$

so that each neighborhood of x contains a closed neighborhood and the space is regular. \clubsuit

Proposition 3.1.4.9 If (X, τ) is regular, then there exists a compatible quasi-uniformity \mathcal{U} such that

$$\cap U^{-1} \circ U = \cap U = \cap U \circ U^{-1}.$$

Proof: Let \mathcal{U} be the quasi-uniformity in Theorem 3.1.4.1 (b). It suffices to prove that $\cap U^{-1} \circ U \subset \cap U$ and $\cap U \circ U^{-1} \subset \cap U$. Let $(x, y) \in \cap U^{-1} \circ U$ and let W be an arbitrary entourage. Then there exists a symmetric entourage V with $V \circ V[x] \subset W[x]$. Hence $y \in V^{-1} \circ V[x] = V \circ V[x] \subset W[x]$. That is, $(x, y) \in W$. Since W was arbitrary, $(x, y) \in W$. Since W was arbitrary, $(x, y) \in W$. Since W was arbitrary, $(x, y) \in W$. Since W was arbitrary, $(x, y) \in W$. Since W was arbitrary, $(x, y) \in W$.

Corollary 3.1.4.4 If a topology is T_3 , then it has a conjugate T_2 topology.

Proposition 3.1.4.10 A topological space (X, τ) is completely regular if and only if τ equals one of its conjugates.

Proof: A completely regular space is uniformizable and hence self-conjugate. Conversely, if τ has a compatible quasi-uniformity \mathcal{U} such that $\tau_{\mathcal{U}} = \tau_{\mathcal{U}^{-1}}$, then by Proposition 3.1.2.11, $\tau_{\mathcal{U}}$ is completely regular.

The relation (*), given in the commentary after Proposition 3.1.1.9, which holds in every uniform space, does not necessarily hold in a quasiuniform space. For example, we have seen in Proposition 3.1.4.7 that this condition for singletons holds only in R_0 -spaces. We will now show that (*)characterizes uniformizable spaces.

Theorem 3.1.4.2 A topological space (X, τ) is completely regular if and only if there exists a compatible quasi-uniformity \mathcal{U} such that for every $A \subset X$

$$\overline{A} = \bigcap U[A] \,.$$

Proof: The necessity of the condition is obvious. To prove sufficiency, we observe that, by virtue of Proposition 3.1.1.9, the condition implies that the closures of A in the topologies of \mathcal{U} and \mathcal{U}^{-1} are identical. Hence the two topologies are also identical. Each of them is then completely regular by Corollary 3.1.2.9.

Historical and bibliographic notes

The study of quasi-uniformities begins with L. Nachbin in 1948 [232], who calls these structures semi-uniformities. The term quasi-uniformity is suggested by A. Császár, [61]. Alternate characterizations of quasi-uniform spaces in terms of families of quasi-pseudo-metrics and in terms of families of covers are given by T. E. Ganter and R. C. Steinlage [122] and Császár [61]. Asymmetric distance functions are considered by F. Hausdorff [133] and V. Niemytzki [240]. The term "neighbornet" and the concept of an unsymmetric neighbornet are due to H. J. K. Junnila [158]. Theorem 3.1.1.1 was first proved by Császár (see [61]) in terms of syntopogenic structures. The proof is far from simple. Subsequently W. J. Pervin [255] gave a more direct and simpler proof, which is presented here. Theorem 3.1.3.1 was proved by Nachbin in 1948 [231] (see also [233], [259]). The concept of an R_0 space is due to N. A. Shanin [289], who calls these spaces weakly regular spaces. The terminology " R_0 -space" is introduced by A. S. Davis [64]. Many other properties of quasi-uniform spaces can be found in [110], [236], [174].

3.2 Quasi-proximity spaces

3.2.1 Elementary properties of quasi-proximities

Definition 3.2.1.1 A relation δ in P(X) is a quasi-proximity for a set X if it satisfies the following conditions:

 $(QP_1) X \delta \emptyset and \ \emptyset \delta X;$

 $(QP_2) C\delta(A \cup B)$ if and only if $C\delta A$ or $C\delta B$,

 $(A \cup B)\delta C$ if and only if $A\delta C$ or $B\delta C$;

 (QP_3) {x} δ {x} for each $x \in X$;

 (QP_4) if $A\overline{\delta}B$, then there exists $C \subset X$ such that $A\overline{\delta}C$ and $X - C\overline{\delta}B$.

The pair (X, δ) is called a quasi-proximity space.

If δ is a quasi-proximity on X, then so is δ^{-1} . A quasi-proximity δ is a proximity if $\delta = \delta^{-1}$. Let A and B be subsets of a quasi-proximity space (X, δ) . If $A\delta B$, then A is said to be **near** B and if $A\overline{\delta}B$, then A is said to be **far** from B.

Proposition 3.2.1.1 Let (X, δ) be a quasi-proximity space. Then

- (a) if $A \cap B \neq \emptyset$, then $A\overline{\delta}B$;
- (b) if $A \subseteq B \subseteq X$ and $A\delta C$, then $B\delta C$;
- (c) if $A \subseteq B \subseteq X$ and $B\overline{\delta}C$, then $A\overline{\delta}C$;
- (d) for each set $A \subseteq X$, $A\overline{\delta}\emptyset$ and $\emptyset\overline{\delta}A$ holds;

(e) if $\{A_j\}$ and $\{B_k\}$ are the finite family of subsets of the space X for which $(\bigcup_i A_j)\delta(\bigcup_k B_k)$, then $A_j\delta B_k$ for some j and k.

Definition 3.2.1.2 A subset B of a quasi-proximity space (X, δ) is a δ -neighborhood of a set A if $A\overline{\delta}X - B$.

Proposition 3.2.1.2 Let (X, δ) be a quasi-proximity space and let \ll be the relation on P(X) defined by $A \ll B$ if and only if B is a δ -neighborhood of A. Then \ll satisfies the following conditions:

- (a) $X \ll X$ and $\emptyset \ll \emptyset$;
- (b) if $A \ll B$, then $A \subset B$;
- (c) if $A \subset B \ll C \subset D$, then $A \ll D$;
- (d) if $A \ll B_1$ and $A \ll B_2$, then $A \ll B_1 \cap B_2$;
- (e) if $A_1 \ll B$ and $A_2 \ll B$, then $A_1 \cup A_2 \ll B$;

(f) if $A \ll B$, then there exists $C \subset X$ such that $A \ll C \ll B$.

Conversely, if a relation \ll defined on P(X) satisfies conditions (a)-(f), then the relation δ , defined by $A\overline{\delta}B$ provided $A \ll X-B$, is a quasi-proximity on X. Further, B is a δ -neighborhood of A if and only if $A \ll B$. **Proposition 3.2.1.3** If (X, δ) is a quasi-proximity space, then the function $c_{\delta} : P(X) \to P(X)$ defined by $c_{\delta}(A) = \{x : x \delta A\}$ is a Kuratowski closure operator on X.

If (X, δ) is a quasi-proximity space, then the topology induced by δ (or simply the topology τ_{δ} of δ) is the topology generated by the closure operator defined in the previous proposition. A topological space (X, τ) is said to **admit** a quasi-proximity (and δ is said to be **compatible** with τ) provided δ induces τ . If $x \in X$, the τ_{δ} -neighborhoods of x are precisely the δ -neighborhoods of x. Every neighborhood of a compact set is a δ neighborhood. If A is not a compact set, a neighborhood of A need not be a δ -neighborhood of A. We note that if $A\overline{\delta}B$, then $clB \subset X - A$. Moreover since int(X - B) = X - clB, it also follows that if $A\overline{\delta}B$ then $A \subset int(X - B)$.

Proposition 3.2.1.4 Let (X, δ) be a quasi-proximity space. If $\{x\}\overline{\delta}A$, then there exists a τ_{δ} -neighborhood G of x such that $G\overline{\delta}A$.

Proof: Since $\{x\}\overline{\delta}A$, there exists a set C such that $\{x\}\overline{\delta}C$ and $X - C\overline{\delta}A$. Let us set G = int(X - C). As noted above, $x \in G$ and $G\overline{\delta}A$.

3.2.2 Quasi-proximities induced by a quasi-uniformity

Proposition 3.2.2.1 Let \mathcal{U} be a quasi-uniformity on X and let $\delta_{\mathcal{U}}$ denote the relation on P(X) defined by $A\delta_{\mathcal{U}}B$ provided that for each $U \in \mathcal{U}$, $(A \times B) \cap U \neq \emptyset$ holds. Then $\delta_{\mathcal{U}}$ is a quasi-proximity on X and $\tau_{\mathcal{U}} = \tau_{\delta_{\mathcal{U}}}$. Moreover the relation $\ll_{\mathcal{U}}$ corresponding to $\delta_{\mathcal{U}}$ is given by $A \ll_{\mathcal{U}} B$ if and only if there exists a $U \in \mathcal{U}$ such that $U[A] \subset B$.

If (X, \mathcal{U}) is a quasi-uniform space, the quasi-proximity $\delta_{\mathcal{U}}$ induced by \mathcal{U} is the quasi-proximity defined in Proposition 3.2.2.1 A quasi-uniformity \mathcal{U} is said to be **compatible** with a quasi-proximity δ if $\delta_{\mathcal{U}} = \delta$. If δ is a quasi-proximity on X, then $\pi(\delta)$ denotes the class of all quasi-uniformities compatible with δ . Two quasi-uniformities that belong to the same quasi-proximity class are said to be **qp-equivalent**.

Proposition 3.2.2.2 Let \mathcal{U} and \mathcal{V} be quasi-uniformities on a set X. If $\mathcal{U} \subset \mathcal{V}$, then $\tau_{\mathcal{U}} \subset \tau_{\mathcal{V}}$ and $\delta_{\mathcal{V}} \subset \delta_{\mathcal{U}}$.

Proposition 3.2.2.3 Let δ be a quasi-proximity on a set X and let Y be a subset of X. Then $\delta_Y = \delta \cap (P(Y) \times P(Y))$ is a quasi-proximity on Y and $\tau_{\delta_Y} = \tau_{\delta} | Y$. Further, if \mathcal{U} induces δ , then $\mathcal{U} | Y \times Y$ induces δ_Y .

Let (X, δ) be a quasi-proximity space. We proceed to show that there exists a unique totally bounded quasi-uniformity compatible with δ and that this quasi-uniformity is the coarsest quasi-uniformity compatible with δ . The result is fundamental. It establishes a one-to-one correspondence between the quasi-proximities and the totally bounded quasi-uniformities that are compatible with a given topology. One reason for an interest in quasiproximities is that some important properties of quasi-uniform spaces are qp-invariant in the sense that if \mathcal{U} and \mathcal{V} are qp-equivalent quasi-uniformities, then \mathcal{U} has the given property if and only if \mathcal{V} does.

Lemma 3.2.2.1 Let δ be a quasi-proximity on X and let us suppose that $A\delta B$ and $E\overline{\delta}F$. Then $(A - E)\delta B$ or $A\delta(B - F)$.

Proof: Let us suppose $(A - E)\overline{\delta}B$. Since $A = (A \cap E) \cup (A - E)$ and $A\delta B$, then $(A \cap E)\delta B$. Since $B = (B \cap F) \cup (B - F)$ and $E\overline{\delta}F$, then $A \cap E\delta(B - F)$ so that $A\delta(B - F)$.

Let X be a set and let $(A, B) \in P(X) \times P(X)$. We let T(A, B) denote $X \times X - A \times B$.

Theorem 3.2.2.1 Let (X, δ) be a quasi-proximity space. The collection S of all sets of the form T(A, B), where $A\overline{\delta}B$, is a subbase for a totally bounded quasi-uniformity \mathcal{U}_{δ} , which is compatible with δ . Moreover, \mathcal{U}_{δ} is the coarsest quasi-uniformity in $\pi(\delta)$ and is the only totally bounded member of $\pi(\delta)$.

Proof: First we shall prove that S is a subbase for a quasi-uniformity \mathcal{U}_{δ} . Let $T(A, B) \in S$. Since $A\overline{\delta}B$, there exists a subset C of X such that $A\overline{\delta}C$ and $X - C\overline{\delta}B$. It follows that $[T(A, C) \cap T(X - C, B)]^2 \subset T(A, B)$. Each T(A, B) in S is reflexive, and the result follows from Proposition 3.2.1.1.

Let $T(A, B) \in \mathcal{S}$. Since $(X - A) \cup (X - B) = X$ and $[(X - A) \times (X - A)] \cup [(X - B) \times (X - B)] \subset T(A, B), \mathcal{U}_{\delta}$ is totally bounded.

 \mathcal{U}_{δ} is compatible with δ . Indeed, let α denote the quasi-proximity induced by \mathcal{U}_{δ} . If $A\overline{\delta}B$, then $(A \times B) \cap T(A, B) = \emptyset$ and $A\overline{\alpha}B$. We show that $A\alpha B$ whenever $A\alpha B$ by establishing the following:

For each (A, B) with $A\delta B$ and each subfamily $\{T(E_i, F_i) : 1 \leq i \leq n\}$ of $S, n \in \mathbb{N}, (A \times B) \cap [\cap \{T(E_i, F_i) : 1 \leq i \leq n\}] \neq \emptyset$ holds.

The proof is by induction. The case n = 1 follows easily from the preceding lemma. Let $n \in \mathbb{N}$ and let us suppose that the statement holds for all k < n. Let $A\delta B$ and let $\{T(E_i, F_i) : 1 \leq i \leq n\}$ be a subfamily of S. For each $k \leq n$ set $G_k = \cap\{T(E_i, F_i) : 1 \leq i \leq k\}$. Then $(A \times B) \cap G_n = [(A \times B) \cap (X - E_n) \times X \cap G_{n-1}] \cup [(A \times B) \cap X \times (X - F_n) \cap G_{n-1}] =$

 $[((A - E_n) \times B) \cap G_{n-1}] \cup [((A \times (B - F_n)) \cap G_{n-1}]]$. By the preceding lemma and the inductive hypothesis, one of the terms in the union given above is nonempty.

To prove that \mathcal{U}_{δ} is the coarsest quasi-uniformity compatible with δ , let us suppose that $\mathcal{U} \in \pi(\delta)$. If $A\overline{\delta}B$, there exists $U \in \mathcal{U}$ such that $(A \times B) \cap U = \emptyset$. Thus $U \subset T(A, B)$, so that $\mathcal{U}_{\delta} \subset \mathcal{U}$.

Finally, \mathcal{U}_{δ} is the unique totally bounded quasi-uniformity compatible with δ . Let \mathcal{U} be a totally bounded quasi-uniformity compatible with δ . Let $U \in \mathcal{U}$ and let us choose $W \in \mathcal{U}$ such that $W \circ W \subset U$. Since \mathcal{U} is totally bounded, there exists a finite cover $\{A_i : 1 \leq i \leq n\}$ of X such that $A_i \times A_i \subset W$, whenever $1 \leq i \leq n$. Since $A_i \times (X - W[A_i]) \cap W = \emptyset$, it follows that $A_i \overline{\delta}(X - W[A_i])$. Let $V = \bigcap \{T(A_i, X - W[A_i]) : 1 \leq i \leq n\}$. Then $V \in \mathcal{U}_{\delta}$ and it follows that $V \subset \cup \{A_i \times W[A_i] : 1 \leq i \leq n\} \subset W \circ W \subset U$. Therefore $U \in \mathcal{U}_{\delta}$ and $\mathcal{U} \subset \mathcal{U}_{\delta}$.

Corollary 3.2.2.1 Let (X, δ) be a proximity space. The collection of all sets of the form T(A, B), where $A\overline{\delta}B$, is a subbase for a totally bounded uniformity \mathcal{U}_{δ} , which is compatible with δ . Moreover \mathcal{U}_{δ} is the coarsest quasiuniformity in $\pi(\delta)$ and is the only totally bounded member of $\pi(\delta)$.

Corollary 3.2.2.2 If δ_1 and δ_2 are quasi-proximities on X such that $\delta_1 \subset \delta_2$, then $\mathcal{U}_{\delta_2} \subset \mathcal{U}_{\delta_1}$.

Corollary 3.2.2.3 Let (X, δ) be a quasi-proximity space. A base for \mathcal{U}_{δ} is the collection \mathcal{V} of all sets of the form $\cup \{A_i \times B_i : 1 \leq i \leq n\}$, where $\{A_i : 1 \leq i \leq n\}$ is a finite cover of X and $A_i \ll B_i$ holds for every i, $1 \leq i \leq n$.

Proof: Let $V = \bigcup \{A_i \times B_i : 1 \leq i \leq n\}$ be an element of \mathcal{V} . Let $U = \bigcap \{T(A_i, X - B_i) : 1 \leq i \leq n\}$; then $U \in \mathcal{U}_{\delta}$ and $U[A_i] \subset B_i$. Therefore $U \subset V$ and $\mathcal{V} \subset \mathcal{U}_{\delta}$.

Let $A\overline{\delta}B$. Then $X - A \ll X$ and $A \ll X - B$. As $T(A, B) = ((X - A) \times X) \cup (A \times (X - B))$, then $T(A, B) \in \mathcal{V}$. Hence \mathcal{V} is finer than \mathcal{U}_{δ} .

If \mathcal{U} is a quasi-uniformity on X, then \mathcal{U}_{ω} denotes the totally bounded member of $\pi(\delta_{\mathcal{U}})$. The notation \mathcal{U}_{ω} is standard and, as we will not use ω to denote a quasi-proximity, no ambiguity will arise. It follows from Theorem 3.2.2.1 that \mathcal{U}_{ω} is the finest totally bounded quasi-proximity on X that is coarser than \mathcal{U} and that two quasi-proximities \mathcal{U} and \mathcal{V} are qp-equivalent if and only if $\mathcal{U}_{\omega} = \mathcal{V}_{\omega}$.

Let us consider the family of all quasi-proximities on a set X to be partial ordered by reverse set inclusion. We will say that δ_1 is **coarser** than δ_2 (and δ_2 is **finer** than δ_1) provided $\delta_2 \subset \delta_1$. In light of Proposition 3.2.2.2, this partial order squares with the usual partial order on the family of all quasiproximities on the set X. Every set X has the finest quasi-proximity : the discrete quasi-proximity given by $A\delta B$ if and only if $A \cap B \neq \emptyset$. It also has the coarsest quasi-proximity given by $A\delta B$ if and only if $A \neq \emptyset$ and $B \neq \emptyset$.

Let X be a set. Since the union of a family of quasi-uniformities on X is a subbase for the supremum of that family, it appears at first glance that the supremum of a family of quasi-proximities on X might be the intersection of all the quasi-proximities of the family. Unfortunately, it is unlikely even for the family $\{\delta, \delta^{-1}\}$ that $\delta \cap \delta^{-1}$ is a quasi-proximity; condition (QP_2) of the definition of quasi-proximity is nearly always violated. Nonetheless every family of quasi-proximities on X has a supremum and it follow from Proposition 3.2.2.4 that the supremum of a nonempty family of quasi-proximities compatible with a given topology, is a quasi-proximity that is compatible with that topology. The infimum of a family $\{\delta_i : i \in I\}$ of quasi-proximity on X always exists; it is the supremum of the family of all quasi-proximities that are coarser than each δ_i .

Proposition 3.2.2.4 Let $\{\delta_i : i \in I\}$ be a nonempty family of quasiproximities on a set X and let δ_0 be defined by $A\delta_0B$ if and only if for each finite cover \mathcal{A} of A and each finite cover \mathcal{B} of B there exist $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$ such that for each $i \in I$, $A'\delta_iB'$. Then δ_0 is a quasi-proximity on X and δ_0 is the coarsest quasi-proximity that is finer than δ_i for each $i \in I$.

As a special case, the preceding proposition establishes that for each quasi-proximity δ , there exists the coarsest proximity that is finer than both δ and δ^{-1} . This proximity is denoted by δ^* . Evidently $\delta^* = \sup\{\delta, \delta^{-1}\}$. Thus $A\delta^*B$ provided that, if \mathcal{A} is a finite cover of A and if \mathcal{B} is a finite cover of B, then there exist $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$ such that $A'\delta B'$ and $B'\delta A'$.

Let $\{\mathcal{U}_i : i \in I\}$ be a collection of quasi-uniformities on a set X and let $\mathcal{U} = \sup\{\mathcal{U}_i : i \in I\}$. It might appear reasonable to hope that $\sup\{\delta_{\mathcal{U}_i} : i \in I\} = \delta_{\mathcal{U}}$. While this hope is unfounded, we do have the following proposition which relies upon Theorem 3.2.2.1.

Proposition 3.2.2.5 Let X be a set, let $\{\mathcal{U}_i : i \in I\}$ be a collection of totally bounded quasi-uniformities on X, let $\delta_0 = \sup\{\delta_{\mathcal{U}_i} : i \in I\}$ and let $\mathcal{U} = \sup\{\mathcal{U}_i : i \in I\}$. Then $\delta_0 = \delta_{\mathcal{U}}$.

Proof: For each $i \in I$, $\mathcal{U}_i \subset \mathcal{U}$ and $\delta_{\mathcal{U}} \subset \delta_{\mathcal{U}_i}$. Thus $\delta_{\mathcal{U}} \subset \delta_0$. Let us note that if $A\overline{\delta}_{\mathcal{U}_i}B$, then $A\overline{\delta}_0B$, and that, if $T(A, B) \in \mathcal{U}_i$, then $T(A, B) \in \mathcal{U}_{\delta_0}$. By

Theorem 3.2.2.1, for each $i \in I$, $\mathcal{U}_i \subset \mathcal{U}_{\delta_0}$. Thus $\mathcal{U} = \sup\{\mathcal{U}_i : i \in I\} \subset \mathcal{U}_{\delta_0}$ so that $\delta_0 \subset \delta_{\mathcal{U}}$.

Corollary 3.2.2.4 If $\{\delta_i : i \in I\}$ is a family of quasi-proximities on a set X, then $\sup\{\delta_i : i \in I\}$ induces $\sup\{\tau_{\delta_i} : i \in I\}$.

Theorem 3.2.2.2 establishes that $\inf \{ \delta_i : i \in I \}$ need not induce $\inf \{ \tau_{\delta_i} : i \in I \}$.

Lemma 3.2.2.2 Let (X, δ) be a quasi-proximity space and let V be a subset of $X \times X$ such that for each subset C of $X, C \ll V[C]$. Then $C \ll (U \cap V)[C]$ whenever $C \subset X$ and $U \in \mathcal{U}_{\delta}$.

Proof: Let $C \subset X$ and let $U \in \mathcal{U}_{\delta}$. In view of Theorem 3.2.2.1 we may assume that $U = \bigcap_{i=1}^{n} T(A_i, B_i)$ where, for each $i, A_i \overline{\delta}B_i$. We will first verify that $C \ll (U \cap V)[C]$ for the case n = 1, that is for the case U = T(A, B) = $(X - A \times X) \cup (A \times X - B)$. In this case, $(U \cap V)[C] = (U \cap V)[C \cap A] \cup (U \cap V)[C - A] = [(X - B) \cap V[C \cap A]] \cup V[C - A]$. By hypothesis $C \cap A \ll V[C \cap A]$ and $C - A \ll V[C - A]$, and since $A\overline{\delta}B, C \cap A \ll X - B$ holds. Thus $C \ll (U \cap V)[C]$. The result now follows by induction on n.

Proposition 3.2.2.6 Let \mathcal{U} and \mathcal{V} be quasi-uniformities on a set X. If \mathcal{U} is totally bounded, then $\delta_{\mathcal{U}\vee\mathcal{V}} = \delta_{\mathcal{U}} \vee \delta_{\mathcal{V}}$.

Proof: It is clear that $\delta_{\mathcal{U}\vee\mathcal{V}} \subset \delta_{\mathcal{U}} \vee \delta_{\mathcal{V}}$. For notational convenience we will let \ll denote the strong inclusion determined by $\delta_{\mathcal{U}} \vee \delta_{\mathcal{V}}$ and where \ll' denotes the strong inclusion determined by $\delta_{\mathcal{U}\vee\mathcal{V}}$. We will show that $\delta_{\mathcal{U}} \vee \delta_{\mathcal{V}} \subset \delta_{\mathcal{U}\vee\mathcal{V}}$ by establishing that $A \ll B$ whenever $A \ll' B$. Let us suppose that $A \ll' B$. Then there exist $U \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $(U \cap V)[A] \subset B$. For each subset C of $X, C \ll V[C]$; and \mathcal{U} is coarser than the totally bounded member of $\pi(\delta_{\mathcal{U}} \vee \delta_{\mathcal{V}})$. By the preceding lemma $A \ll (U \cap V)[A] \subset B$.

Corollary 3.2.2.5 Let X be a set and let $\{\mathcal{U}_i : i \in I\}$ be a collection of a quasi-uniformities on X and let suppose that for one $i \in I$ at the most, \mathcal{U}_i is not totally bounded. Then the quasi-proximity induced by $\sup\{\mathcal{U}_i : i \in I\}$ is the supremum of $\{\delta_{\mathcal{U}_i} : i \in I\}$.

An interesting consequence of the previous corollary is that if \mathcal{U} and \mathcal{V} are quasi-uniformities on a set X, then $\mathcal{U}_{\omega} \vee \mathcal{V}$ and $\mathcal{U} \vee \mathcal{V}_{\omega}$ are qp-equivalent. It is also a consequence of the previous proposition that if \mathcal{U} is totally bounded,

then $\delta_{\mathcal{U}^*} = (\delta_{\mathcal{U}})^*$. We will show later that $(\mathcal{U}^*)_{\omega}$ is not necessarily equal to $(\mathcal{U}_{\omega})^*$ so that, unless \mathcal{U} is totally bounded, $\delta_{\mathcal{U}^*}$ is not necessarily equal to $(\delta_{\mathcal{U}})^*$. The following equalities, however, are easily verified.

- (a) For each quasi-uniformity \mathcal{U} , $(\delta_{\mathcal{U}})^{-1} = \delta_{\mathcal{U}^{-1}}$.
- (b) For each quasi-uniformity $\mathcal{U}, \ (\mathcal{U}_{\omega})^{-1} = (\mathcal{U}^{-1})_{\omega}.$
- (c) For each quasi-proximity δ , $(\mathcal{U}_{\delta})^{-1} = \mathcal{U}_{\delta^{-1}}$.
- (d) For each quasi-proximity δ , $(\mathcal{U}_{\delta})^* = \mathcal{U}_{\delta^*}$.

Proposition 3.2.2.7 Let (X, δ) be a quasi-proximity space. The following statements are equivalent:

- (a) $A\delta B$;
- (b) $cl_{\delta^*}A \,\delta \, cl_{\delta^*}B;$
- (c) $cl_{\delta^{-1}}A\delta cl_{\delta}B$.

Proof: It follows from Proposition 3.2.1.1 (b) that $(a) \Rightarrow (b) \Rightarrow (c)$. Let us suppose that $cl_{\delta^{-1}}A \,\delta \, cl_{\delta}B$ and that $A\overline{\delta}B$. Then there exists a subset C of X such that $A\overline{\delta}C$ and $X - C\overline{\delta}B$. It follows that $cl_{\delta}B \subset C$ so that $A\overline{\delta}cl_{\delta}B$. Consequently, $cl_{\delta}B\overline{\delta}^{-1}A$ and there exists a subset D of X such that $cl_{\delta}B\overline{\delta}^{-1}D$ and $X - D\overline{\delta}^{-1}A$. As $cl_{\delta^{-1}}A \subset D$, $cl_{\delta}B\overline{\delta}^{-1}cl_{\delta^{-1}}A$ holds so that $cl_{\delta^{-1}}A\overline{\delta}cl_{\delta}B$, which is a contradiction.

Proposition 3.2.2.8 Let (X, τ) be a normal Hausdorff space. The relation δ defined by $A\delta B$ if and only if $\overline{A} \cap \overline{B} \neq \emptyset$ is the finest proximity compatible with τ .

The following proposition is easily obtained from the result of Proposition 3.2.1.4.

Proposition 3.2.2.9 Let (X, δ) be a quasi-proximity space. If A is a τ_{δ} compact subset of X and if $B \subset X$, then the following statements are equivalent:

- (a) there exists $x \in A$ such that $\{x\}\delta B$;
- (b) $A \cap cl_{\delta}B \neq \emptyset$;
- (c) $A\delta B$.

Proposition 3.2.2.10 Let (X, τ) be a compact topological space, let δ be a quasi-proximity on X such that $\tau(\delta^*) \subset \tau$ and let A and B be subsets of X. Then the following statements are equivalent:

- (a) $A\delta B$;
- (b) there exist $x \in cl_{\tau}A$ and $y \in cl_{\tau}B$ such that $\{x\}\delta\{y\}$;
- (c) $cl_{\delta^{-1}}A \cap cl_{\delta}B \neq \emptyset$.

Proof: $(a) \Rightarrow (b)$: Let \overline{A} denote $cl_{\tau}A$ and let \overline{B} denote $cl_{\tau}B$. Since $\tau_{\delta} \subset \tau$, \overline{A} is τ_{δ} -compact so that by Proposition 3.2.2.9 there exists an $x \in \overline{A}$ such that $\{x\}\delta\overline{B}$. Since \overline{B} is $\tau_{\delta^{-1}}$ -compact, by Proposition 3.2.2.9, there exists a $y \in \overline{B}$ such that $\{y\}\delta^{-1}\{x\}$. Thus there exist $x \in \overline{A}$ and $y \in \overline{B}$ such that $\{x\}\delta\{y\}$.

 $(b) \Rightarrow (c)$: Let $x \in cl_{\tau}A$ and $y \in cl_{\tau}B$ such that $\{x\}\delta\{y\}$. Then $\{x\}\delta cl_{\delta}B$ and since $\{x\}$ is compact, it follows from Proposition 3.2.2.9 that $x \in cl_{\delta^{-1}}A \cap cl_{\delta}B$.

 $(c) \Rightarrow (a)\,$: This implication is the consequence of Proposition 3.2.2.7.

Corollary 3.2.2.6 The only proximity compatible with the topology of compact Hausdorff space is the one defined in Proposition 3.2.2.8.

Proposition 3.2.2.11 Let (X, τ) be a compact Hausdorff space. Then the proximity δ defined in Proposition 3.2.2.8 is the coarsest quasi-proximity compatible with the topology τ , and \mathcal{U}_{δ} is the coarsest quasi-uniformity compatible with the topology τ .

Proof: Let δ_1 be any quasi-proximity compatible with τ . If $A\delta B$, then by Proposition 3.2.2.7, $\overline{A}\,\overline{\delta}\,\overline{B}$. By the preceding proposition $\overline{A}\,\overline{\delta}_1\,\overline{B}$ and so $A\overline{\delta}_1B$. By Theorem 3.2.2.1 and Corollary 3.2.2.2, \mathcal{U}_{δ} is the coarsest quasiuniformity compatible with τ .

Corollary 3.2.2.7 Let (X, \mathcal{U}) be a compact Hausdorff space, let (Y, \mathcal{V}) be a uniform space, and let $f : (X, \tau_{\mathcal{U}}) \to (Y, \tau_{\mathcal{V}})$ be a continuous mapping. Then $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is a quasi-uniformly continuous mapping.

Proof: As in the previous proposition, let δ denote the coarsest quasiproximity compatible with $\tau_{\mathcal{U}}$. By Theorem 3.1.2.2, $f : (X, \mathcal{U}_{\delta}) \to (Y, \mathcal{V})$ is uniformly continuous. By the previous proposition, $\mathcal{U}_{\delta} \subset \mathcal{U}$ so the result follows.

The results of Corollary 3.2.2.6 and Proposition 3.2.2.11 naturally suggest two questions: which topological spaces admit only one quasi-proximity and which spaces admit a coarser quasi-proximity? We do not know the complete answer to these questions and we will postpone considering the first until the next section. The following theorem provides the answer to the second question for Tychonoff spaces.

Lemma 3.2.2.3 Let (X, τ) be a locally compact Hausdorff space and let \ll denote the relation defined by $A \ll B$ provided that B = X or provided that

there exists an open set G and a compact set K such that $A \subset K \subset G \subset B$. Then \ll satisfies conditions (a) - (f) of Proposition 3.2.1.2 and the associated quasi-proximity is compatible with τ .

Proof: The verification that \ll satisfies the conditions (a) - (e) is straightforward. To prove that \ll satisfies condition (f) let us suppose that $A \ll B$ and $B \neq X$. Then there exists a compact set K and an open set G such that $A \subset K \subset G \subset B$. For each $x \in K$ let C_x be an open set containing x such that \overline{C}_x is compact and $\overline{C}_x \subset G$. There is a finite subset I of K such that $K \subset \cup \{C_x : x \in I\}$. Let us set $C = \cup \{C_x : x \in I\}$. Then $A \subset K \subset C \subset \overline{C} \subset G \subset B$ so that $A \ll C$ and $C \ll B$. If B = X, let us choose C = X.

Theorem 3.2.2.2 Let (X, τ) be a Tychonoff space. The following statements are equivalent:

- (a) (X, τ) admits the coarsest quasi-proximity;
- (a') (X, τ) admits the coarsest quasi-uniformity;
- (b) (X, τ) admits the coarsest proximity;
- (b') (X, τ) admits the coarsest uniformity;

(c) (X, τ) is locally compact.

Proof: By Theorem 3.2.2.1 and Corollary 3.2.2.2, conditions (a) and (a') are equivalent and conditions (b) and (b') are equivalent as well. It therefore suffices to prove that conditions (a), (b) and (c) are equivalent.

 $(a) \Rightarrow (b)$: Let δ be the coarsest quasi-proximity compatible with τ and let δ_1 denote any proximity compatible with τ . Then $\delta_1 \subset \delta$ so that $\delta_1 \subset \delta^*$. Thus δ^* is the coarsest proximity compatible with τ .

 $(b) \Rightarrow (c)$: This implication is a well-known result from the theory of proximities.

 $(c) \Rightarrow (a)$: Let δ be the quasi-proximity determined by the strong inclusion \ll of the preceding lemma and let δ_1 be any quasi-proximity compatible with τ . Let us suppose that $A\overline{\delta}B$. Then $B = \emptyset$ or there exists a compact set K and a closed set F such that $A \subset K$, $B \subset F$, and $K \cap F = \emptyset$. By Proposition 3.2.2.9, $A\overline{\delta}_1B$. Thus $\delta_1 \subset \delta$.

Proposition 3.2.2.12 Let (X, τ) be a Hausdorff topological space. Then (X, τ) is compact if and only if (X, τ) is locally compact and the coarsest quasi-proximity compatible with τ is a proximity.

Proof: Let δ be the coarsest quasi-proximity compatible with τ . In light of Proposition 3.2.2.11 it suffices to show that if δ is a proximity, then (X, τ)

is compact. Let C be an open cover of X and let C be a nonempty member of $C, C \neq X$. Let $x \in C$ and let us set F = X - C. Since $\{x\} \cap F = \emptyset$, $\{x\}\overline{\delta}F$ and so $F\overline{\delta}\{x\}$. It follows from the previous lemma that there is a compact set containing F. In particular, F is compact and so C has a finite subcover.

3.2.3 qp-continuous mappings

Let (X, δ) and (Y, ρ) be quasi-proximity spaces and let $f : X \to Y$. The relation δ' given by $A\delta'B$ if and only if $f(A)\rho f(B)$ is a quasi-proximity on X. A mapping $f : (X, \delta) \to (Y, \rho)$ is said to be **qp-continuous** provided that $\delta \subset \delta'$. Evidently δ' is the coarsest quasi-proximity for which $f : X \to (Y, \rho)$ is *qp*-continuous. If $\mathcal{V} \in \pi(\rho)$ and \mathcal{U} denotes the quasi-uniformity for which $\{f_2^{-1}(V) : V \in \mathcal{V}\}$ is a base, then $\mathcal{U} \in \pi(\delta')$. Thus the following proposition holds.

Proposition 3.2.3.1 Let $f: X \to Y$ and let \mathcal{V} be a quasi-uniformity on Y. Let \mathcal{U} be the quasi-uniformity for which $\{f_2^{-1}(V): V \in \mathcal{U}\}$ is a base. Then $\{f_2^{-1}(V): V \in \mathcal{V}_{\omega}\}$ is a base for \mathcal{U}_{ω} .

The composition of two qp-continuous mappings is a qp-continuous mapping. A bijection f such that f and f^{-1} are qp-continuous, is a **qp-isomorphism**. A quasi-proximity space (X, δ) is **embedded** in a quasi-proximity space (Y, ρ) by a **qp-embedding** f provided that f is a qp-isomorphism from X onto some subspace of Y.

Proposition 3.2.3.2 If $f : (X, \delta) \to (Y, \rho)$ is qp-continuous, then so are $f : (X, \delta^{-1}) \to (Y, \rho^{-1})$ and $f : (X, \delta^*) \to (Y, \rho^*)$.

Proposition 3.2.3.3 Let $f : (X, \delta) \to (Y, \rho)$ be a qp-continuous mapping. Then $f : (X, \tau_{\delta}) \to (Y, \tau_{\rho}), f : (X, \tau_{\delta^{-1}}) \to (Y, \tau_{\rho^{-1}})$ and $f : (X, \tau_{\delta^*}) \to (Y, \tau_{\rho^*})$ are continuous mappings.

Proposition 3.2.3.4 If $f : (X, U) \to (Y, V)$ is quasi-uniformly continuous, then $f : (X, \delta_U) \to (Y, \delta_V)$ is qp-continuous.

Proposition 3.2.3.5 Let (X, δ) and (Y, ρ) be quasi-proximity spaces and let us suppose that (X, δ^*) is a compact Hausdorff space. If $f : (X, \tau_{\delta}) \to (Y, \tau_{\rho})$ and $f : (X, \tau_{\delta^{-1}}) \to (Y, \tau_{\rho^{-1}})$ are continuous, then $f : (X, \delta) \to (Y, \rho)$ is qpcontinuous. **Proof:** By Theorem 3.1.2.2, $f : (X, \mathcal{U}_{\delta}) \to (Y, \mathcal{V}_{\rho})$ is quasi-uniformly continuous. Thus by Proposition 3.2.3.4, $f : (X, \delta) \to (Y, \rho)$ is *qp*-continuous.

Definition 3.2.3.1 Let $\{(X_i, \delta_i) : i \in I\}$ be a nonempty collection of quasiproximity spaces and let $X = \prod \{X_i : i \in I\}$. The **product quasiproximity** is the coarsest quasi-proximity on X such that the projection $\pi_i : X \to X_i$ is qp-continuous for each $i \in I$.

For each $i \in I$, let δ'_i denote the coarsest quasi-proximity for which π_i is *qp*-continuous. Then $\sup\{\delta'_i : i \in I\}$ is the product quasi-proximity and in light of Proposition 3.2.2.4, the product quasi-proximity δ may be characterized explicitly as follows. If A and B are subsets of X, then $A\delta B$ provided that, if \mathcal{A} and \mathcal{B} are finite covers of A and B, then there exist $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$ so that $\pi_i(A')\delta_i\pi_i(B')$ for each $i \in I$. It is false in general that the product quasi-uniformity induces the product quasi-proximity even though both do induce the product topology. We have, however, the following proposition as a consequence of Corollary 3.2.2.5.

Proposition 3.2.3.6 For each $i \in I$ let \mathcal{U}_i be a quasi-uniformity that induces a quasi-proximity δ_i and let us suppose that at most one \mathcal{U}_i fails to be totally bounded. Then $\prod \{\mathcal{U}_i : i \in I\}$ induces $\prod \{\delta_i : i \in I\}$.

Definition 3.2.3.2 Let (X, U) be a quasi-uniform space. A subset A of X is **uniformly discrete** provided that $U|A \times A$ is the discrete uniformity on A.

An obvious consequence of the following proposition is that a uniform space is totally bounded if and only if every uniformly discrete subset is finite.

Proposition 3.2.3.7 Let (X, U) and (Y, V) be quasi-uniform spaces. Each of the statements (a) and (b) implies its successor, and if (X, U) and (Y, V) are uniform spaces, then the following statements are equivalent:

(a) one of \mathcal{U} and \mathcal{V} is totally bounded;

- (b) $\delta_{\mathcal{U}} \times \delta_{\mathcal{V}} = \delta_{\mathcal{U} \times \mathcal{V}};$
- (c) one of (X, \mathcal{U}) and (Y, \mathcal{V}) has no infinite uniformly discrete subset.

Proof: We have noted in Proposition 3.2.3.6 that $(a) \Rightarrow (b)$.

 $(b) \Rightarrow (c)$: Let us suppose that $A = \{a_n : n \in \mathbb{N}\}$ and $B = \{b_n : n \in \mathbb{N}\}$ are uniformly discrete subsets of X and Y such that $a_m \neq a_n$ and $b_m \neq b_n$ unless m = n. If $C = \{(a_m, b_n) : m \neq n\}$ and $D = \{(a_n, b_n) : n \in \mathbb{N}\}$, then $C\delta_{\mathcal{U}} \times \delta_{\mathcal{V}}D$ whereas $C\delta_{\mathcal{U}\times\mathcal{V}}D$ is false.

 $(c) \Rightarrow (a)$: Let us suppose that \mathcal{U} is not totally bounded. Then there exists a symmetric entourage $U \in \mathcal{U}$ such that for each finite set $F, U[F] \neq X$ holds. Let us choose a sequence (x_n) in X such that for each i < n, $x_n \notin U[x_i]$. Then $\{x_n : n \in \mathbb{N}\}$ is an infinite uniformly discrete subset of X.

An example to show that $(b) \neq (a)$ is given in Example 3.3.2.1.

Proposition 3.2.3.8 Let $f : (X, \delta) \to (Y, \rho)$ be a qp-continuous mapping and let $\mathcal{V} \in \pi(\rho)$. Then there exists $\mathcal{U} \in \pi(\delta)$ such that $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is quasi-uniformly continuous.

Proof: Let \mathcal{W} denote the quasi-uniformity for which $\{f_2^{-1}(V) : V \in \mathcal{V}\}$ is a base. Then $\delta_{\mathcal{W}}$ is the coarsest quasi-proximity on X for which $f : X \to (Y, \rho)$ is *qp*-continuous. Thus $\delta \subset \delta_{\mathcal{W}}$. Let $\mathcal{U} = \mathcal{U}_{\delta} \vee \mathcal{W}$. By Proposition 3.2.2.6, $\mathcal{U} \in \pi(\delta)$. Clearly $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is quasi-uniformly continuous.

If in the preceding proposition \mathcal{V} is assumed to be totally bounded, then \mathcal{W} is totally bounded. Thus by Theorem 3.2.2.1 the quasi-uniformity \mathcal{U} obtained in the preceding proposition is \mathcal{U}_{δ} and the following corollary holds.

Corollary 3.2.3.1 Let (X, U) be a quasi-uniform space and let (Y, V) be a totally bounded quasi-uniform space. Then $f : (X, U) \to (Y, V)$ is quasi-uniformly continuous if and only if $f : (X, \delta_U) \to (Y, \delta_V)$ is qp-continuous.

For each $\varepsilon > 0$ and $f : X \to \mathbb{R}$, we will let $U_{(\varepsilon,f)} = f_2^{-1}(Q_{\varepsilon}) = \{(x,y) : f(x) - f(y) < \varepsilon\}$. Let \mathcal{U} be a quasi-uniformity on X. Then $f : (X,\mathcal{U}) \to (\mathbb{R},\mathcal{W})$ is quasi-uniformly continuous if and only if $U_{(\varepsilon,f)} \in \mathcal{U}$ for each $\varepsilon > 0$.

Let (X, \mathcal{U}) be a quasi-uniform space. Then $Q(\mathcal{U})$ $(QB(\mathcal{U}))$ denotes the set of all (bounded) quasi-uniformly continuous functions from (X, \mathcal{U}) to $(\mathbb{R}, \mathcal{W})$. Similarly, if (X, δ) is a quasi-proximity space, $Q(\delta)$ $(QB(\delta))$ denotes the set of all (bounded) *qp*-continuous functions from (X, δ) to $(\mathbb{R}, \delta_{\mathcal{W}})$. We could just as well have taken $Q(\mathcal{U})$ to be the set of all quasi-uniformly continuous functions from (X, \mathcal{U}) to $(\mathbb{R}, \mathcal{W}^{-1})$; our choice of definition is motivated by certain applications of quasi-uniformities to topological ordered spaces.

Let \mathcal{U} and \mathcal{V} be quasi-uniformities on a set X and let δ be a quasiproximity on X. We will summarize some immediate consequences of previous results in the terminology established above. (a) If $\mathcal{U} \subset \mathcal{V}$, then $Q(\mathcal{U}) \subset Q(\mathcal{V})$ and $QB(\mathcal{U}) \subset QB(\mathcal{V})$.

(b) $Q(\mathcal{U}) \subset Q(\delta_{\mathcal{U}})$ (Proposition 3.2.3.4).

(c) If $f \in Q(\mathcal{U})$, then $f : (X, \mathcal{U}^*) \to (\mathbb{R}, \mathcal{E})$ is continuous (Proposition 3.1.2.3).

(d) If $f \in Q(\delta)$, then $f : (X, \tau_{\delta^*}) \to (\mathbb{R}, \tau_{\mathcal{E}})$ is continuous (Proposition 3.2.3.3).

(e) If $f \in Q(\mathcal{U})$ (or $Q(\delta)$), then f is lower semi-continuous.

(f) $QB(\delta) \subset QB(\mathcal{U}_{\delta})$ (Proposition 3.2.3.8).

We will omit the proof of the following lemma for two reasons. First, a proof of the lemma can be given in a way that differs only slightly from the well-known proof of Urysohn's Lemma. Second, for (X, δ) a T_0 -space, we will show in subsection 5.5. that this lemma is a corollary to a generalization of Urysohn's Lemma.

Lemma 3.2.3.4 Let (X, δ) be a quasi-proximity space, and let A and B be subsets of X such that $A\overline{\delta}B$. Then there exists $f : X \to [0,1]$ such that $f \in QB(\delta), f(A) = 1$ and f(B) = 0.

Theorem 3.2.3.1 Let (X, δ) be a quasi-proximity space. Then $\mathcal{U} \in \pi(\delta)$ if and only if $QB(\mathcal{U}) = QB(\delta)$.

Proof: Let $\mathcal{U} \in \pi(\delta)$. We show that $Q(\mathcal{U}_{\delta}) = QB(\mathcal{U}) = QB(\delta)$. Let $f \in Q(\mathcal{U}_{\delta})$. As \mathcal{U}_{δ} is totally bounded, there exists a finite cover $\{A_i : 1 \leq i \leq m\}$ of X such that $\cup \{A_i \times A_i : 1 \leq i \leq m\} \subset U_{(1,f)}$. For each *i* with $1 \leq i \leq m$ let us choose $a_i \in A_i$ and set $F = \{a_i : 1 \leq i \leq m\}$. Then $U_{(1,f)}[F] = X$ so that $1 + \max\{f(x) : x \in F\}$ is an upper bound for *f* and $-1 + \min\{f(x) : x \in F\}$ is a lower bound for *f*. Thus $Q(\mathcal{U}_{\delta}) = QB(\mathcal{U}_{\delta})$. Since $QB(\mathcal{U}_{\delta}) \subset QB(\mathcal{U}), QB(\mathcal{U}) \subset QB(\delta)$ and $QB(\delta) \subset QB(\mathcal{U}_{\delta}), Q(\mathcal{U}_{\delta}) = QB(\mathcal{U}) = QB(\delta)$ holds.

Now let us suppose that $QB(\mathcal{U}) = QB(\delta)$. It follows from the preceding lemma that $\delta_{\mathcal{U}} = \delta$.

Corollary 3.2.3.2 Let (X, δ) be a quasi-proximity space. Then $\{U_{(\varepsilon, f)} : \varepsilon > 0 \text{ and } f \in QB(\delta)\}$ is a subbase for \mathcal{U}_{δ} .

Proof: Let \mathcal{U} denote the quasi-uniformity on X for which $\{U_{(\varepsilon, f)} : \varepsilon > 0, f \in QB(\delta)\}$ is a subbase. As $QB(\delta) = QB(\mathcal{U}_{\delta})$, it is clear that $\mathcal{U} \subset \mathcal{U}_{\delta}$ and that $QB(\mathcal{U}) \subset QB(\delta) \subset QB(\mathcal{U})$. By the previous theorem, $\mathcal{U} \in \pi(\delta)$. It remains to show that \mathcal{U} is totally bounded. Let $f \in QB(\delta)$, let $\varepsilon > 0$, and let us assume without loss of generality that $f(x) \ge 0$ for all $x \in X$. For each nonnegative integer i, let $A_i = \{t : |f(t) - i\varepsilon/2| < \varepsilon/2\}$. There is a positive

integer m such that $\bigcup_{i=0}^{m} A_i = X$. Let j be an integer with $0 \leq j \leq m$ and let $(x, y) \in A_j \times A_j$. Then $|f(x) - j\varepsilon/2| < \varepsilon/2$ and $|f(y) - j\varepsilon/2| < \varepsilon/2$. Consequently, $f(x) - f(y) \leq |f(x) - f(y)| < \varepsilon$ so that $(x, y) \in U_{(\varepsilon, f)}$. Thus \mathcal{U} is totally bounded and since there exists only one totally bounded quasiuniformity in $\pi(\delta)$, $\mathcal{U} = \mathcal{U}_{\delta}$.

We have seen that every quasi-proximity class has the coarsest member. The basic information concerning the product quasi-proximity given in Proposition 3.2.3.7 is all that is required to provide an example of a quasi-proximity class that has no the finest member.

Example 3.2.3.1 Let \mathcal{U} be the discrete uniformity on an infinite set X. By Proposition 3.2.3.7, $\mathcal{U} \times \mathcal{U}_{\omega}$ and $\mathcal{U}_{\omega} \times \mathcal{U}$ both induce $\delta_{\mathcal{U}} \times \delta_{\mathcal{U}}$. This proximity is strictly coarser than the discrete proximity on $X \times X$, which is induced by $\mathcal{U} \times \mathcal{U}$. Since $\mathcal{U} \times \mathcal{U}$ is the only quasi-uniformity on $X \times X$ that is finer than both $\mathcal{U} \times \mathcal{U}_{\omega}$ and $\mathcal{U}_{\omega} \times \mathcal{U}$, the quasi-proximity class $\pi(\delta_{\mathcal{U}} \times \delta_{\mathcal{U}})$ has no finest member. Indeed $\pi(\delta_{\mathcal{U}} \times \delta_{\mathcal{U}})$ has no the finest uniformity.

Historical and bibliographic notes

The term quasi-proximity appears first in W. J. Pervin [257] and E. F. Steiner [306]. Quasi-proximities are considered in terms of a strong inclusion by C. H. Dowker [90] and as a part of a more general study of A. Császár [61]. Theorem 3.2.2.1 is established by W. N. Hunsaker and W. F. Lindgren [143] in 1970. A proof of the corresponding result for totally bounded syntopogenous structures is given by Császár [61]. Proposition 3.2.2.6 was proved by E. M. Alfsen and O. Njastad [15]. The proof of Lemma 3.2.2.3 is based upon an argument of Gál given in [119]. The equivalence of (b') and (c) of Theorem 3.2.2.2 is due to P. Samuel [285]. Proposition 3.2.3.6 was proved by Dowker [90] and Proposition 3.2.3.7 was proved by J. R. Isbell [150]. The Lemma 3.2.3.4 is the asymmetric analogue of Ju. M. Smirnoff [294], Lemma 3, p. 23. The analogue for syntopogenous structures is given by Császár [61]. Theorem 3.2.3.1 is due to Dowker [90].

3.3 Approximations of symmetry

3.3.1 Transitive quasi-uniformities

Definition 3.3.1.1 A (sub)base \mathcal{B} for a quasi-uniformity is **transitive** provided that each $B \in \mathcal{B}$ is a transitive relation. A quasi-uniformity with a

transitive (sub)base is called a transitive quasi-uniformity.

The problem of determining which topological spaces admit a quasiuniformity, has as its antitype the classical result that a topological space admits a uniformity if and only if it is completely regular. This problem for quasi-uniform spaces is simply disposed of; all topological spaces admit a quasi-uniformity.

Proposition 3.3.1.1 The Pervin quasi-uniformity for a topological space (X, τ) is totally bounded transitive quasi-uniformity compatible with τ .

Proof: That the family $S = \{S_A : A \in \tau\}$ is a subbase for the Pervin quasiuniformity \mathcal{U} compatible with the topology τ , we have proved in Theorem 3.1.1.1. Since $S_A \circ S_A = S_A$ for each $A \in \tau$, \mathcal{U} is a transitive quasi-uniformity on X. To see that \mathcal{U} is totally bounded, it suffices to note that for each $S_A \in S$, $(A \times A) \cup ((X - A) \times (X - A)) \subset S_A$ and $A \cup (X - A) = X$ hold.

Let (X, τ) be a topological space. The supremum of all quasi-uniformities compatible with τ is called the **fine quasi-uniformity** for (X, τ) and is denoted by **FINE**. Similarly the supremum of all the uniformities compatible with a completely regular space is called the **fine uniformity**. Since the topology induced by the supremum of a family $\{\mathcal{U}_i : i \in I\}$ of quasiuniformities is the supremum of the topologies induced by the \mathcal{U}_i , it follows from Proposition 3.3.1.1 that **FINE** is compatible with τ . The fine quasiuniformity can also be characterized as that quasi-uniformity \mathcal{U} on X with the property that, whenever (Y, \mathcal{V}) is a quasi-uniform space and $f : (X, \tau) \rightarrow$ $(Y, \tau_{\mathcal{V}})$ is continuous mapping, then f is a quasi-uniformly continuous mapping. Finally, we state yet another characterization of **FINE**, which depends upon Lemma 3.1.3.4. Let D denote the family of all quasi-pseudo-metrics dfor X such that for each $x \in X$, d(x, y) is an upper semi-continuous function of y. The family $\{\{(x, y) : d(x, y) < \varepsilon\} : d \in D$ and $\varepsilon > 0\}$ is a base for **FINE**.

We will omit the proof of the following proposition whose proof is similar to the proof of Proposition 3.3.1.1.

Proposition 3.3.1.2 Let (X, τ) be a noncompact topological space and let \mathcal{F} be a filter on X that has no cluster point. Then $\{S_G : G \in \tau \text{ and } X - G \in \mathcal{F}\}$ is a subbase for a totally bounded transitive quasi-uniformity that is compatible with τ .

A collection C of subsets of a topological space is **interior preserving** provided that if $C' \subset C$, then $\operatorname{Int} \cap \{C : C \in C'\} = \cap \{\operatorname{Int} C : C \in C'\};$ also, a collection C is **closure preserving** provided that if $C' \subset C$, then $\overline{\cup\{C : C \in C'\}} = \cup\{\overline{C} : C \in C'\}$. A collection C of open subsets is interior preserving if and only if for each subcollection C' of C, $\cap C'$ is open; a collection C of closed subsets is closure preserving if and only if for each subcollection C' of C, $\cup C'$ is closed. The following proposition is a straightforward consequence of de Morgan's laws.

Proposition 3.3.1.3 A collection C of a topological space (X, τ) is interior preserving if and only if $\{X - C : C \in C\}$ is a closure-preserving collection.

If \mathcal{C} is a collection of subsets of a set X and $x \in X$, then \mathcal{C}_x denotes $\{C \in \mathcal{C} : x \in C\}$ so that $\cap \mathcal{C}_x = \cap \{C \in \mathcal{C} : x \in \mathcal{C}\}$. In terms of this notation, a collection \mathcal{C} of open subsets of a topological space (X, τ) is interior preserving if and only if for each $x \in X$, $\cap \mathcal{C}_x \in \tau$ holds. If \mathcal{C} is a collection of subsets of a set X, then $U_{\mathcal{C}}$ denotes the reflexive transitive relation $\{(x, y) : x \in X \text{ and } y \in \cap \mathcal{C}_x\}$; also if \mathcal{A} is a nonempty family of collections of subsets of X, then $\mathcal{U}_{\mathcal{A}}$ denotes the quasi-uniformity on X for which $\{U_{\mathcal{C}} : \mathcal{C} \in \mathcal{A}\}$ is a subbase. For the nonce we make use of the following notation: if V is a neighbornet of a space (X, τ) , then \mathcal{C}_V denotes $\{V[x] : x \in X\}$.

Theorem 3.3.1.1 Let (X, τ) be a topological space and let \mathcal{A} be a family of interior-preserving open collections such that $\cup \mathcal{A}$ is a subbase for τ . Then $\mathcal{U}_{\mathcal{A}}$ is a transitive quasi-uniformity that is compatible with τ . Moreover, if \mathcal{U} is any transitive quasi-uniformity compatible with τ , there is a collection \mathcal{A} of interior-preserving open covers of X such that $\cup \mathcal{A}$ is a subbase for τ and $\mathcal{U} = \mathcal{U}_{\mathcal{A}}$.

Proof: For each $\mathcal{C} \in \mathcal{A}$, $U_{\mathcal{C}}$ is transitive so that \mathcal{U} is a transitive quasiuniformity on X. Let $G \in \tau_{\mathcal{U}_{\mathcal{A}}}$ and let $x \in G$. Then there exists a finite subcollection $\{\mathcal{C}_i : 1 \leq i \leq n\}$ of \mathcal{A} such that $x \in \bigcap_{i=1}^n U_{\mathcal{C}_i}[x] \subset G$. For each $i, 1 \leq i \leq n, \mathcal{C}_i$ is an interior-preserving open collection so that $U_{\mathcal{C}_i}[x] \in \tau$. Thus $\bigcap_{i=1}^n U_{\mathcal{C}_i}[x] \in \tau$, so $\tau_{\mathcal{U}_{\mathcal{A}}} \subset \tau$.

Let $G \in \tau$ and let $x \in G$. Since $\cup \mathcal{A}$ is a subbase for τ , there exist $A_1, A_2, \ldots, A_n \in \cup \mathcal{A}$ such that $x \in \bigcap_{i=1}^n A_i \subset G$. For each $i, 1 \leq i \leq n$, let $\mathcal{C}_i \in \mathcal{A}$ such that $A_i \in \mathcal{C}_i$. Then $\bigcap_{i=1}^n U_{\mathcal{C}_i} \in \mathcal{U}_{\mathcal{A}}$ and $\bigcap_{i=1}^n U_{\mathcal{C}_i}[x] \subset G$. Thus $\tau \subset \tau_{\mathcal{U}_{\mathcal{A}}}$ so that $\mathcal{U}_{\mathcal{A}}$ is compatible with τ .

Now let us suppose that \mathcal{U} is a quasi-uniformity that is compatible with τ and let \mathcal{B} be a transitive base for \mathcal{U} . Let $V \in \mathcal{B}$, let $y \in X$ and let $G = \cap\{V[x] : y \in V[x], x \in X\}$. Let $p \in G$. For each $x \in X$ such that $y \in V[x], V[p] \subset V[x]$. Hence $V[p] \subset G$ and $G \in \tau$. Thus for each $V \in \mathcal{B}$, \mathcal{C}_V is an interior-preserving open cover of X. Let $\mathcal{A} = \{\mathcal{C}_V : V \in \mathcal{B}\}$. In order to see that $\mathcal{U} = \mathcal{U}_{\mathcal{A}}$, it suffices to observe that for each $x \in X$ and each $V \in \mathcal{B}, V[x] = U_{\mathcal{C}_V}[x]$ so that for each $V \in \mathcal{B}, V = U_{\mathcal{C}_V}$. It is obvious that $\cup \mathcal{A}$ is a subbase for τ .

Corollary 3.3.1.1 Let (X, τ) be a topological space and let \mathcal{A} be the collection of all interior-preserving open covers of X. Then $\mathcal{U}_{\mathcal{A}}$ is the finest transitive quasi-uniformity compatible with τ .

The finest transitive quasi-uniformity for a space (X, τ) , whose existence is guaranteed by Corollary 3.3.1.1, is denoted by **FT** and is called the **fine transitive quasi-uniformity** for (X, τ) . We note that the collection of all transitive neighbornets of a space is a base for **FT**. If (X, τ) is a topological space and \mathcal{A} is the collection of all point-finite (locally finite) open covers of X, then $\mathcal{U}_{\mathcal{A}}$ is called the **point-finite (locally finite) covering quasi-uniformity** for (X, τ) and is denoted by **PF(LF)**. The next proposition establishes that **P** can also be obtained from the construction given in Theorem 3.3.1.1.

Proposition 3.3.1.4 Let (X, τ) be a topological space and let \mathcal{A} be the collection of all finite open covers of X. Then $\mathcal{U}_{\mathcal{A}} = \mathbf{P}$.

Proposition 3.3.1.5 Let (X, τ) be a topological space and let \mathcal{A} be a family of interior-preserving open collections such that $\cup \mathcal{A}$ is a subbase for τ . A necessary and sufficient condition that $\mathcal{U}_{\mathcal{A}}$ is totally bounded is that each member of \mathcal{A} is a finite collection.

Proof: If each member of \mathcal{A} is a finite collection, then, by the previous proposition, $\mathcal{U}_{\mathcal{A}} \subset \mathbf{P}$. Since \mathbf{P} is totally bounded, $\mathcal{U}_{\mathcal{A}}$ is totally bounded as well.

Now let us suppose that $\mathcal{U}_{\mathcal{A}}$ is totally bounded and let $\mathcal{C} \in \mathcal{A}$. For each $x \in X$ let us set $G_x = \cap \mathcal{C}_x$, $U = U_{\mathcal{C}}$, $\mathcal{G} = \{G_x : x \in X\}$, $H_x = \{y : G_x = G_y\}$ and $\mathcal{H} = \{H_x : x \in X\}$. Then \mathcal{H} is a partition of X and card $(\mathcal{H}) = \text{card}(\mathcal{G})$. Since $\mathcal{U}_{\mathcal{A}}$ is totally bounded, there exists a collection $\{A_i : i = 1, 2, \ldots, n\}$ such that $\cup \{A_i : i = 1, 2, \ldots, n\} = X$ and $\cup \{A_i \times A_i : i = 1, 2, \ldots, n\} \subset U$. If \mathcal{H} (equivalently \mathcal{G}) is infinite, there exist A_i , H_x , H_y such that $H_x \neq H_y$, and there exist $w \in A_i \cap H_x$ and $z \in A_i \cap H_y$. Since $(w, z) \in A_i \times A_i \subset U$, $z \in U[w] = G_w$; hence $G_z \subset G_w$. Since $(z, w) \in A_i \times A_i \subset U$, $w \in U[z] = G_z$ holds; hence $G_w \subset G_z$ and $G_w = G_z$. Now $w \in H_x$ so that $G_w = G_x$, and $z \in H_y$ so that $G_z = G_y$. Then $G_x = G_w = G_z = G_y$ and $H_x = H_y$ which is a contradiction. We have proved that \mathcal{H} is finite; thus \mathcal{G} is finite. Since, for each $G \in \mathcal{C}$, $G = \bigcup \{G_x : x \in G\}$, \mathcal{C} must be finite as well.

Corollary 3.3.1.2 Let (X, τ) be a topological space. Then $\mathbf{P} = \mathbf{FT}$ if and only if every interior-preserving open cover of X is finite.

Let (X, τ) be a topological space. We will use the usual notation that C(X) denotes the set of all continuous real-valued function on X and $C^*(X)$ denotes the set of all bounded members of C(X). If (X, τ) is a completely regular space, then $\{U_{(\varepsilon,f)} : \varepsilon > 0, f \in C(X)\}$ is a subbase for a uniformity on X compatible with τ . This uniformity is denoted by C(X) or just C when the topological space (X, τ) is understood.

Proposition 3.3.1.6 Let (X, τ) be a topological space and let E be a collection of lower semi-continuous functions on X such that, if $G \in \tau$ and $x \in G$, then there exists $f \in E$ such that f(x) = 1 and f(X - G) = 0. Let $S = \{U_{(\varepsilon, f)} : f \in E, \varepsilon > 0\}$. Then S is a subbase for a quasi-uniformity that is compatible with τ .

Proof: Since for each $f \in E$ and $\varepsilon > 0$, $\Delta \subset U_{(\varepsilon,f)}$ and $U_{(\varepsilon/2,f)} \circ U_{(\varepsilon/2,f)} \subset U_{(\varepsilon,f)}$, S is a subbase for a quasi-uniformity \mathcal{U} on X. Let $f \in E$, $\varepsilon > 0$ and $x \in X$. Then $U_{(\varepsilon,f)}[x] = \{y : f(y) > f(x) - \varepsilon\} = f^{-1}(f(x) - \varepsilon, +\infty)$ and $f^{-1}(f(x) - \varepsilon, +\infty) \in \tau$, since f is lower semi-continuous. Hence $\tau_{\mathcal{U}} \subset \tau$. Let $G \in \tau$, let $x \in G$ and let $f \in E$ such that f(x) = 1 and f(X - G) = 0. Then $x \in U_{(1,f)}[x] \subset G$ so that $\tau \subset \tau_{\mathcal{U}}$ and \mathcal{U} is compatible with τ .

The quasi-uniformity generated in Proposition 3.3.1.6 by taking E to be the set of all lower semi-continuous functions is called the **semi-continuous quasi-uniformity** for (X, τ) and is denoted by **SC**. This quasi-uniformity is the coarsest quasi-uniformity on X for which each continuous function $f: X \to (\mathbb{R}, \mathcal{W})$ is quasi-uniformly continuous.

Proposition 3.3.1.7 Let (X, τ) be a topological space and let E be the collection of all bounded lower semi-continuous functions on X. Then $\{U_{(\varepsilon,f)} : \varepsilon > 0, f \in E\}$ is a subbase for \mathbf{P} .

Proof: Let \mathcal{U} be the quasi-uniformity for which $\{U_{(\varepsilon, f)} : \varepsilon > 0, f \in E\}$ is a subbase. Since E contains all characteristic functions of open sets, it

follows from Proposition 3.3.1.6 that $\tau_{\mathcal{U}} = \tau$. By Corollary 3.2.3.2, $\mathbf{P} \subset \mathcal{U}$. As \mathcal{U} is totally bounded, it follows from remarks after Proposition 3.3.1.1 that $\mathcal{U} = \mathbf{P}$.

Since $\delta_{\mathbf{P}}$ is the finest quasi-proximity compatible with a given topology τ , any quasi-uniformity compatible with τ that contains \mathbf{P} belongs to the quasi-proximity class $\pi(\delta_{\mathbf{P}})$. By the previous proposition, \mathbf{SC} as well as \mathbf{PF} , \mathbf{LF} , \mathbf{FT} and \mathbf{FINE} are members of $\pi(\delta_{\mathbf{P}})$.

Although Corollary 3.2.3.2 shows that not every quasi-uniformity defined as in Proposition 3.3.1.6 is transitive, we now establish that **SC** is transitive.

Definition 3.3.1.2 An open spectrum a in a set X is a family $\{A_n : n \in \mathbb{Z}\}$ of open subsets of X such that for each $n \in \mathbb{Z}$, $A_n \subset A_{n+1}$, $\bigcap_{n \in \mathbb{Z}} A_n = \emptyset$ and $\bigcup_{n \in \mathbb{Z}} A_n = X$.

Each open spectrum is an interior-preserving open cover. If A is an open spectrum, there is an integer n such that $A_n = X$ if and only if A is a point-finite open cover. In this case we call A a **point-finite open spectrum**. If A is an open spectrum, it is easily verified that $U_a = \bigcup_{n \in \mathbb{Z}} (A_n - A_{n-1}) \times A_n$.

Theorem 3.3.1.2 Let \mathcal{A} be the collection of all open spectra in a space (X, τ) . Then $\mathcal{U}_{\mathcal{A}} = \mathbf{SC}$.

Proof: Let *E* be the collection of all lower semi-continuous functions. It suffices to show that $\{U_a : a \in \mathcal{A}\}$ and $\{U_{(\varepsilon,f)} : \varepsilon > 0, f \in E\}$ are equivalent subbases. Let $f \in E$ and $\varepsilon > 0$ be given; let us take $x_0 \in X$. For each $n \in \mathbb{Z}$, let us set $A_n = \{x : f(x) > f(x_0) - (n+1)\varepsilon\}$ and let us set $a = \{A_n : n \in \mathbb{Z}\}$. Let $(x, y) \in U_a$. There exists an integer *n* such that $f(x_0) - n\varepsilon \ge f(x) > f(x_0) - (n+1)\varepsilon$. Then $y \in U_a[x] = A_n$ so that $f(y) > f(x_0) - (n+1)\varepsilon$. Hence $f(x) - f(y) < f(x_0) + n\varepsilon - (f(x_0) - (n+1)\varepsilon) = \varepsilon$ and $(x, y) \in U_{(\varepsilon, f)}$.

Let $a \in \mathcal{A}$. For each $x \in X$ let us define f(x) = -n where n is the integer for which $x \in A_n - A_{n-1}$. Then f is lower semi-continuous. Let $(x, y) \in U_{(1, f)}$ and let us suppose that $x \in A_n - A_{n-1}$. Then f(y) > f(x) - 1 = -(n+1). Hence $y \in A_n$ and $(x, y) \in (A_n - A_{n-1}) \times A_n \subset U_a$.

Corollary 3.3.1.3 Let (X, τ) be a topological space. Then **SC** is a transitive quasi-uniformity.

Corollary 3.3.1.4 Let (X, τ) be a countable compact topological space. Then each open spectrum of X is a point-finite open spectrum and $\mathbf{SC} \subset \mathbf{PF}$. **Corollary 3.3.1.5** Let (X, τ) be a topological space. For each $U \in \mathbf{SC}$ there exists a countable subset D of X such that U[D] = X.

We will omit the proof of the following theorem, which is similar to the proof of Theorem 3.3.1.2.

Theorem 3.3.1.3 Let (X, τ) be a topological space and let \mathcal{A} be the collection of all point-finite open spectra. For each lower semi-continuous function f that is bounded below, $f : (X, \mathcal{U}_{\mathcal{A}}) \to (\mathbb{R}, \mathcal{W})$ is quasi-uniformly continuous, and $\mathcal{U}_{\mathcal{A}}$ is the coarsest quasi-uniformity that has this property.

Proposition 3.3.1.8 Let $f: (X, \tau) \to (Y, \tau')$ be a continuous mapping, let \mathcal{U} denote the Pervin (resp. point finite, locally finite, semi-continuous, fine transitive, fine) quasi-uniformity on X and let \mathcal{V} denote the corresponding quasi-uniformity on Y. Then $f: (X, \mathcal{U}) \to (Y, \mathcal{V})$ is quasi-uniformly continuous.

Proof: The proof of the case that \mathcal{U} and \mathcal{V} are the fine transitive quasiuniformities on X and Y respectively is representative. Let $V \in \mathcal{V}$ be a transitive entourage. Then $\mathcal{C} = \{f^{-1}(V[y]) : y \in Y\}$ is an interiorpreserving open cover of X and so $U_{\mathcal{C}} \in \mathcal{U}$. If $(a, b) \in U_{\mathcal{C}}$, then $(f(a), f(b)) \in$ V; hence f is quasi-uniformly continuous.

Since each of **P**, **PF**, **LF**, **SC**, **FT** and **FINE** is defined for every topological space, there is a rule, defined for each of these quasi-uniformities, that assigns to each topological space the corresponding quasi-uniformity. Let f denote the rule corresponding to any of **P**, **PF**, **LF**, **SC**, **FT** or **FINE** and let X and Y be topological spaces. Then $f(X) \times f(Y) \subset f(X \times Y)$ and if A is a subspace of X, $f(X)|A \subset f(A)$. While, in general, these two comparisons are not equalities, it is easily seen that if f is the rule corresponding to the Pervin quasi-uniformity and if A is a subspace of X, then f(X)|A = f(A). Even for this rule, however, it is not in general true that $f(X) \times f(Y) = f(X \times Y)$. Indeed the following proposition shows that if (X, τ) is a Hausdorff space, f is any one of the rules discussed in this subsection and if $f(X) \times f(Y) = f(X \times Y)$, then f(X) is the discrete uniformity. In particular **FINE** is the fine quasi-uniformity only when **FINE** is the discrete uniformity.

Proposition 3.3.1.9 Let (X, τ) be a Hausdorff space and let \mathcal{U} be a quasiuniformity compatible with τ such that $\delta_{\mathcal{U}\times\mathcal{U}}$ is the Pervin quasi-proximity on $X \times X$. Then \mathcal{U} is the discrete uniformity. **Proof:** As $\delta_{\mathcal{U}\times\mathcal{U}}$ is the Pervin quasi-proximity on $X \times X$, $(X \times X - \Delta)\overline{\delta}_{\mathcal{U}\times\mathcal{U}}\Delta$ so that there exists an entourage $U \in \mathcal{U}$ such that $\{((a, b), (c, d)) : (a, c), (b, d) \in U\} \cap (X \times X - \Delta) \times \Delta = \emptyset$. It follows that for each $x \in X$, $U^{-1}[x] = \{x\}$ and so $U = \Delta$.

Corollary 3.3.1.6 Let (X, τ) be a Hausdorff space such that $\mathbf{P}_X \times \mathbf{P}_X = \mathbf{P}_{X \times X}$. Then X is a finite set.

Proposition 3.3.1.10 Let X be a topological space and let S be an open or closed subspace of X. Every normal neighbornet of S is the restriction to S of a normal neighbornet of X. Moreover, every transitive neighbornet of S is the restriction to S of transitive neighbornet of X.

Proof: Let V be a normal neigbornet of S. If S is open, let us set that $V^* = V \cup ((X - S) \times X)$ and if S is closed, let us set that $V^* = V \cup (X \times (X - S))$. Then V^* is the required neighbornet of X and it is easily seen that V^* is transitive whenever V is transitive.

Corollary 3.3.1.7 Let X be a topological space and let a subset S of X be the intersection of an open set and a closed set. Then $\mathbf{FINE}_X | S \times S =$ \mathbf{FINE}_S and $\mathbf{FT}_X | S \times S = \mathbf{FT}_S$.

We will close this subsection with a technical theorem which shows, in particular, that the restriction of the fine quasi-uniformity on \mathbb{R} to the subset Q is not the fine quasi-uniformity on Q.

Theorem 3.3.1.4 Let X be a T_1 space and let $D = \{x_n : n \in \mathbb{N}\}$ be a dense subspace of X such that X - D is of the second category in X. Then there exists a normal neighbornet U of D so that for each neighbornet V of X, $V^2 \cap D \times D - U \neq \emptyset$.

Proof: For each $m, n \in \mathbb{N}$ let us set $g(m, x_n) = D - \{x_i : i \leq m, x_i \neq x_n\}$ and let $U = \bigcup_{n=1}^{\infty} \{x_n\} \times g(n, x_n)$. Since U is a member of the point-finite quasi-uniformity on D, U is a normal neighbornet of D. Let us suppose that there exists a neighbornet V of X such that $V^2 \cap D \times D \subset U$. For each $n \in \mathbb{N}$ let us set that $A_n = \{x \in X - D : \text{ for each } y \in D, V[x] \cap D - g(n, y) \neq \emptyset\}$. Then $\bigcup_{n=1}^{\infty} A_n = X - D$ so that there exists an $n \in \mathbb{N}$ such that $\operatorname{int} (\overline{A}_n) \neq \emptyset$. There exists an m > n so that $x_m \in D \cap \operatorname{int}(\overline{A}_n)$. Let $x \in V[x_m] \cap A_n$. Then $V[x] \cap D \subset V^2[x_m] \cap D \subset U[x_m] = g(m, x_m) \subset g(n, x_m)$, which is a contradiction.
3.3.2 Point-symmetry and local symmetry

Although the difference between proximity spaces and arbitrary quasi-proximity spaces is only a matter of assuming an axiom of symmetry, the difference between two classes of spaces is considerable. We now investigate some approximations of symmetry, which serve to narrow the gap between quasiproximities and proximities.

Definition 3.3.2.1 A quasi-proximity δ on a set X is point-symmetric provided that $\{x\}\delta A$ whenever $A\delta\{x\}$.

Proposition 3.3.2.1 Let (X, U) be a quasi-uniform space. Then the following statements are equivalent:

(a) $(X, \delta_{\mathcal{U}})$ is point-symmetric;

(b) for each $U \in \mathcal{U}$ and $x \in X$, there exists a symmetric $V \in \mathcal{U}$ such that $V[x] \subset U[x]$;

(c) for each $U \in \mathcal{U}$ and $x \in X$, there exists a $V \in \mathcal{U}$ such that $V^{-1}[x] \subset U[x]$;

$$(d) \ \tau(\mathcal{U}) \subset \tau(\mathcal{U}^{-1}).$$

Proof: We will prove only $(a) \Rightarrow (b)$, since the remaining implications are apparent. Let $U \in \mathcal{U}$ and let $x \in X$. Then $\{x\}\overline{\delta}_{\mathcal{U}}X - U[x]$ so that $X - U[x]\overline{\delta}_{\mathcal{U}}\{x\}$. Let us set that $V = T(\{x\}, X - U[x]) \cap T(X - U[x], \{x\})$. By Theorem 3.2.2.1, $V \in \mathcal{U}_{\omega} \subset \mathcal{U}$. Further V is symmetric and $V[x] \subset U[x]$.

We say that a quasi-uniformity is **point-symmetric** provided it satisfies any of the conditions of the previous proposition. Let us note that if δ is a point-symmetric quasi-proximity, then $\tau_{\delta^{-1}} = \tau_{\delta^*}$. Thus (X, τ_{δ}) is completely regular if δ^{-1} is point-symmetric. In addition, $\tau_{\delta} = \tau_{\delta^{-1}}$ if and only if both δ and δ^{-1} are point-symmetric. It follows from the preceding proposition that a topological space that admits a point-symmetric quasiproximity is an R_0 space. Moreover, if (X, τ) is an R_0 space, then $\delta_{\mathbf{P}}$ is a point-symmetric quasi-proximity.

Definition 3.3.2.2 A quasi-proximity δ on a set X is **locally symmetric** provided that $\{x\}\delta A$ whenever $A\delta G$ for each τ_{δ} -neighborhood G of x.

By Proposition 3.2.1.4 every proximity is locally symmetric. Another consequence of this proposition is that if δ is locally symmetric and $\{x\}\overline{\delta}A$, there exists an open set G containing x such that $A\overline{\delta}G$ and $G\overline{\delta}A$. If ρ is locally symmetric (point symmetric), $\delta \subset \rho$ and $\tau_{\delta} = \tau_{\rho}$, then δ is locally symmetric (point symmetric). Both point symmetry and locally symmetry are hereditary properties.

Proposition 3.3.2.2 Let (X, U) be a quasi-uniform space. The following statements are equivalent:

(a) $(X, \delta_{\mathcal{U}})$ is locally symmetric;

(b) for each $U \in \mathcal{U}$ and $x \in X$, there exists a symmetric $V \in \mathcal{U}$ such that $V^2[x] \subset U[x]$;

(c) for each $U \in \mathcal{U}$ and $x \in X$, there exists a $V \in \mathcal{U}$ such that $V^{-1}[V[x]] \subset U[x]$;

(d) for each $x \in X$, $\{U^{-1}[U[x]] : U \in \mathcal{U}\}$ is a base for the $\tau_{\mathcal{U}}$ -neighborhood filter of x.

Proof: We will prove only that $(a) \Rightarrow (b)$ since the remaining implications are apparent. Let $U \in \mathcal{U}$ and let $x \in X$. Then $\{x\}\overline{\delta}_{\mathcal{U}}X - U[x]$ so that there exists a $G \in \mathcal{N}_x$ (\mathcal{N}_x is a neighborhood filter of the point x) such that $X - U[x]\overline{\delta}_{\mathcal{U}}G$ and $G\overline{\delta}_{\mathcal{U}}X - U[x]$. Since $\{x\}\overline{\delta}_{\mathcal{U}}X - G$, there exists an $H \in \mathcal{N}_x$ such that $X - G\overline{\delta}_{\mathcal{U}}H$ and $H\overline{\delta}_{\mathcal{U}}X - G$. Let us set $V = T(X - U[x], G) \cap$ $T(G, X - U[x]) \cap T(H, X - G) \cap T(X - G, H)$. Then V is a symmetric member of \mathcal{U} and $V[V[x]] \subset V[G] \subset U[x]$.

We say that a quasi-uniformity is **locally symmetric** provided it satisfies any of the conditions of the previous proposition. It follows from this proposition that a topological space that admits a locally symmetric quasiuniformity is a regular space. Furthermore, if (X, τ) is a regular space, then $\delta_{\mathbf{P}}$ is a locally symmetric quasi-proximity. Manifestly, point-symmetry and local symmetry are qp-invariant properties.

We have considered two qp-invariant properties possessed by the Pervin quasi-proximity of any regular topological space and not possessed by any quasi-proximity of a topological space that fails to be an R_0 space. These two properties are related to the third qp-invariant property that is possessed by the Pervin quasi-proximity of any topological space. A quasi-proximity δ on a set X is **equinormal** provided that, if A and B are disjoint closed subsets of X, then $A\bar{\delta}B$. A quasi-uniformity \mathcal{U} is **equinormal** provided $\delta_{\mathcal{U}}$ is equinormal. A quasi-uniform space (X, \mathcal{U}) is equinormal if and only if for each closed set F of X, and each open set G containing F, there exists a $U \in \mathcal{U}$ such that $U[F] \subset G$.

Proposition 3.3.2.3 A quasi-uniform space (X, U) is equinormal if (X, U) is compact or (X, U) is countably compact and U has a countable base.

Proof: Let A and B be disjoint closed sets of a quasi-uniform space (X, \mathcal{U}) satisfying hypothesis stated above. Let us suppose that $A\delta_{\mathcal{U}}B$. For each $U \in \mathcal{U}, (A \times B) \cap U \neq \emptyset$ and, by hypothesis, filter for which $(\{A \cap U^{-1}[B] : U \in \mathcal{U}\})$ is a subbase, has a cluster point p. Then $p \in A \cap (\cap \{U^{-1}[B] : U \in \mathcal{U}\}) = A \cap \overline{B} = A \cap B$, which is a contradiction.

We have seen that point symmetry is related to the R_0 separation axiom and that local symmetry is related to regularity: similarly, equinormality and normality are related concepts. Indeed, the following proposition shows that for a normal Hausdorff space (X, τ) , the only equinormal proximity compatible with τ , is the fine proximity. Proposition 3.3.2.5 provides a more significant comparison of equinormality, point symmetry and local symmetry.

Proposition 3.3.2.4 Let (X, τ) be a normal Hausdorff space and let β be the finest proximity compatible with τ . A quasi-proximity δ compatible with τ is equinormal if and only if $\delta \subset \beta$.

Proof: Let us suppose first that $\delta \subset \beta$. It follows from Proposition 3.2.2.8 that β , and hence δ , is an equinormal quasi-proximity.

Now let us suppose that δ is equinormal and that A and B are subsets of X such that $A\overline{\beta}B$. Then $\overline{A}\overline{\beta}\overline{B} = \emptyset$ so that $\overline{A} \cap \overline{B} = \emptyset$. Since δ is equinormal, $\overline{A}\overline{\delta}\overline{B}$, so $A\overline{\delta}B$.

Proposition 3.3.2.5 Let (X, δ) be an equinormal quasi-proximity space.

(a) If X is an R_0 space, δ is point symmetric.

(b) If X is a regular space, δ is locally symmetric.

Proof: Let x be an element of X and let A be a subset of X such that $\{x\}\overline{\delta}A$. By Proposition 3.2.1.4 there exists an open set G such that $x \in G$ and $G\overline{\delta}A$.

Let us suppose first that X is an R_0 space. Then $\overline{\{x\}} \subset G$ so that $\overline{\{x\}} \overline{\delta} \overline{A}$ and $A\overline{\delta}\{x\}$.

Now let us suppose that X is a regular space. There exists an open set H such that $x \in H \subset \overline{H} \subset G$. Since $G\overline{\delta}A, \overline{A} \cap \overline{H} = \emptyset$, so $A\overline{\delta}H$.

Proposition 3.3.2.6 Let (X, τ) be a topological space for which every quasi-proximity compatible with τ is point-symmetric. Then (X, τ) is compact. Moreover, if (X, δ) is a subspace of a compact R_0 (regular) quasi-proximity space, then δ is point-(locally) symmetric. **Proof:** Let us suppose that (X, τ) is not compact and let \mathcal{F} be a filter on X that has no cluster point. By Proposition 3.3.1.2, $\{T(G, X - G) : G \in \tau \text{ and } X - G \in \mathcal{F}\}$ is a subbase for a quasi-uniformity \mathcal{U} compatible with τ . Let F be a closed set such that $F \in \mathcal{F} - \{X\}$ and let $x \in X - F$. Then $F\delta_{\mathcal{U}}\{x\}$, and since $\delta_{\mathcal{U}}$ is point-symmetric, then $\{x\}\delta_{\mathcal{U}}F$, which is a contradiction. The remaining implications follow from Propositions 3.3.2.3 and 3.3.2.5.

Proposition 3.3.2.7 A topological space that admits a point-symmetric quasi-metric is a developable space.

Proof: Let X be a topological space that admits a point-symmetric quasimetric d. Let (U_n) be a nested base (A family $\mathcal{J} \subseteq P(X)$ is **nested**, if for all $G, H \in \mathcal{J} \subseteq G \subseteq H$ or $H \subseteq G$) of open neighbornets for the quasi-uniformity generated by d and for each $n \in \mathbb{N}$ let us set that $\mathcal{G}_n = \{U_n[x] : x \in X\}$. Let $x \in X$ and let G be an open set containing x. There is an $n \in \mathbb{N}$ such that $U_n[x] \subset G$, and an $m \in \mathbb{N}$ such that $U_m^2 \subset U_n$, and a k > m such that $U_k^{-1}[x] \subset U_m[x]$. It is easily verified that $st(x, \mathcal{G}_k) \subset G$.

The following proposition is an immediate consequence of Propositions 3.3.2.3 and 3.3.2.5.

Proposition 3.3.2.8 Let (X, τ) be a countably compact space. Every quasimetric, compatible with τ , is a point-symmetric quasi-metric.

Corollary 3.3.2.1 Every countably compact quasi-metrizable space is compact.

Proof: Let X be a countably compact quasi-metrizable space. It follows from the previous two propositions that X is developable, and every countably compact developable space is compact. \clubsuit

Let X be a countable infinite set and let τ be the cofinite topology on X. Then (X, τ) is a compact quasi-metrizable space that is not a Hausdorff space. It is well known and easy to prove, however, that every compact Moore (regular developable) space is metrizable. Thus, since every locally metrizable paracompact space is metrizable, the previous two propositions yield the following simple metrization theorem.

Corollary 3.3.2.2 Every locally compact paracompact Hausdorff quasi-metrizable space is metrizable space. In order to characterize those topological spaces that admit a locally symmetric quasi-metric, it is convenient to state the following consequence of Theorem 3.1.3.1 in terms of development.

Proposition 3.3.2.9 Let (X, τ) be a T_1 space and let (\mathcal{G}_n) be a development for X. Let us suppose that for each $n \in \mathbb{N}$, if two members of \mathcal{G}_{n+1} have nonempty intersection, then some member of \mathcal{G}_n contains their union. Then (X, τ) is metrizable.

Proof: For each $n \in \mathbb{N}$ let us set that $U_n = \{(x, y) : y \in st(x, \mathcal{G}_n)\}$. It is easily verified that (U_n) is a base for a uniformity compatible with τ .

Theorem 3.3.2.1 A topological space that admits a locally symmetric quasi-metric is metrizable.

Proof: Let (X, τ) be a topological space that admits a locally symmetric quasi-metric and let (U_n) be a decreasing sequence of open neighbornets of Xsuch that (U_n) is a base for a locally symmetric quasi-uniformity compatible with τ . For each $x \in X$ and $n \in \mathbb{N}$ let g[n, x] be the least natural number j > n such that $U_j \circ U_j^{-1} \circ U_j[x] \subset U_n[x]$. For each $x \in X$ let us choose a sequence of natural numbers as follows. Let us set that f(1, x) = 1, $f(2, x) = g[f(1, x), x], \ldots, f(r + 1, x) = g[f(r, x), x]$. It follows easily by induction that for each $x \in X$ and $n \in \mathbb{N}$, $f(n, x) \ge n$. For each $n \in \mathbb{N}$, let us set that $V_n = \{(x, y) : y \in U_{f(n, x)}[x]\}$ and $\mathcal{G}_n = \{V_n[x] : x \in X\}$.

We will show first that the sequence (\mathcal{G}_n) of open covers of X is a development for X. Let $G \in \tau$, let $a \in G$ and let $r \in \mathbb{N}$ such that $U_r[a] \subset G$. Let us set that m = g[r, a] and let $b \in st(a, \mathcal{G}_m)$. There exists an $x \in X$ such that $\{a, b\} \subset V_m[x] = U_{f(m,x)}[x] \subset U_m[x]$. Thus $b \in U_m[x] \subset U_m^{-1} \circ U_m[a] \subset U_r[a]$ so that $st(a, \mathcal{G}_m) \subset G$.

To see that (\mathcal{G}_n) satisfies the conditions of Proposition 3.3.2.9, let $m \in \mathbb{N}$ and let $p, q \in X$ such that $V_{m+1}[p] \cap V_{m+1}[q] \neq \emptyset$. For notational convenience let us set f(m+1,p) = j and f(m+1,q) = k and we assume without loss of generality that $j \ge k$. Then $V_{m+1}[p] \cup V_{m+1}[q] = U_j[p] \cup U_k[q] \subset$ $U_k \circ U_k^{-1} \circ U_k[q]$ and as $k = f(m+1,q) = g[f(m,q),q], U_k \circ U_k^{-1} \circ U_k[q] \subset$ $U_{f(m,q)}[q] = V_m[q] \in \mathcal{G}_m$.

If (X, τ) is a regular space that admits an equinormal quasi-metric, then by Proposition 3.3.2.5, (X, τ) admits a locally symmetric quasi-metric and so is metrizable.

Theorem 3.3.2.2 For a Tychonoff space X, the following statements are equivalent:

(a) X admits an equinormal quasi-metric;

(b) X is metrizable and the set of all nonisolated points of X is a compact subset of X;

(c) the fine uniformity of X has a countable base;

(d) X admits an equinormal metric.

Proof: $(a) \Rightarrow (b)$: Let d be an equinormal quasi-metric and let (U_n) be a nested base for the quasi-metric generated by d. Let I denote the set of all isolated points of X. In light of the previous theorem it suffices to show that X - I is countably compact. Let us suppose that X - I is not countably compact and let D be a closed infinite discrete subset of X - I. We may assume that $D = \{x_n : n \in \mathbb{N}\}$ where $x_n = x_m$ only if n = m. There exists a family $\{G_n : n \in \mathbb{N}\}$ of pairwise disjoint open subsets of Xsuch that for each $n \in \mathbb{N}$, $x_n \in G_n$. Since no point of D is isolated, for each $n \in \mathbb{N}$ there exists a $y_n \neq x_n$ such that $y_n \in U_n[D] \cap G_n$. Let us set that $F = \overline{\{y_n : n \in \mathbb{N}\}}$. Since D and F are disjoint closed sets, there exists an $n \in \mathbb{N}$ such that $U_n[D] \cap F = \emptyset$, which is a contradiction.

 $(b) \Rightarrow (c)$: Let $K = \{(x, x) : x \text{ is not an isolated point of } X\}$, let d be a metric compatible with the topology on $X \times X$ and for each $n \in \mathbb{N}$ let us set $G_n = \{y \in X \times X : d(y, K) < 1/n\}$. For each $n \in \mathbb{N}$ let us set $U_n = G_n \cup \{(x, x) : x \in I\}$. Evidently $\{U_n : n \in \mathbb{N}\}$ is a base for the fine uniformity, which consists of all neighborhoods of the diagonal.

 $(c) \Rightarrow (d)$: Since X is metrizable, it is an immediate consequence of Proposition 3.3.2.4 that the fine uniformity is equinormal.

 $(d) \Rightarrow (a)$: This implication is evident.

We will now characterize those regular T_1 spaces for which the fine quasiuniformity has a countable base. Our characterization relies upon Theorem 3.3.2.2 (b), which it should be compared to.

Proposition 3.3.2.10 Let (X, τ) be a regular T_1 space. The fine quasiuniformity for X has a countable base if and only if X is a metric space with only finitely many nonisolated points.

Proof: Suppose that (X, τ) is a metric space with only finitely many nonisolated points x_1, x_2, \ldots, x_n and for $1 \leq i \leq n$ let $\{U_j[x_i] : j \in \mathbb{N}\}$ be a base for \mathcal{N}_{x_i} with the property that $x_i \in U_j[x_k]$ only if i = k. Then $\{\Delta \cup [\bigcup_{i=1}^n \{x_i\} \times U_j[x_i]] : j \in \mathbb{N}\}$ is a subbase for **FINE**. Now let us suppose that **FINE** has a countable base. Since this quasi-uniformity belongs to the Pervin quasi-proximity class, it is equinormal and so, by Theorem 3.3.2.2, the set X' of all nonisolated points of X is a compact metric space. Let G' be an open subset of X' and let $G \in \tau$ such that $G' = X' \cap G$. Let I denote the set of isolated points of G and let K = G - I. By Corollary 3.3.1.7, **FINE** $|G \times G =$ **FINE** $_G$ so that by Theorem 3.3.2.2, K is compact. Consequently, $G' = X' \cap G = X' \cap (K \cup I) = X' \cap K$, which is compact. Thus, every open subset of X' is compact and so X' is finite.

It is natural to pose the problem of determining those topological spaces for which \mathbf{P} is the only compatible quasi-uniformity. We have postponed considering this problem until now, since it is inextricably tied to the study of those topological spaces for which every compatible quasi-uniformity is (locally) symmetric.

Proposition 3.3.2.11 Let (X, τ) be a topological space. If **P** is a uniformity, then for each $x \in X$, $\overline{\{x\}}$ is the smallest open set containing x.

Proof: Let us suppose that **P** is a uniformity, let $x \in X$ and let $U = T(X - \overline{\{x\}}, \overline{\{x\}})$. Then $U^{-1} \in \mathbf{P}$; hence $\overline{\{x\}} = U^{-1}[x]$ is a neighborhood of x. Since (X, τ) is completely regular, $\overline{\{x\}}$ is a subset of every neighborhood of x. It follows that $\overline{\{x\}}$ is the smallest open set containing x.

Corollary 3.3.2.3 Let (X, τ) be a T_0 space. Then **P** is a uniformity if and only if τ is the discrete topology.

Theorem 3.3.2.3 Let (X, τ) be a topological space. Each of the statements (a) through (d) implies its successor, and if (X, τ) is an R_1 space, then the following statements are equivalent:

- (a) τ is finite;
- (b) **P** is the only quasi-uniformity compatible with τ ;
- (c) every interior-preserving open collection is finite;
- (d) (X, τ) is hereditarily compact;
- (e) $\delta_{\mathbf{P}}$ is the only quasi-proximity compatible with τ .

Proof: We will first prove that $(d) \Rightarrow (c)$, since this implication expedites the proof that $(a) \Rightarrow (b)$. Let us suppose that (X, τ) is hereditarily compact and let δ be a quasi-proximity that is compatible with τ . Then $\delta_{\mathbf{P}} \subset \delta$. Let us suppose that $A\overline{\delta}_{\mathbf{P}}B$. Then $A \cap \overline{B} \neq \emptyset$ and since A is compact, it follows from Proposition 3.2.2.9 that $A\overline{\delta}B$.

 $(a) \Rightarrow (b)$: Let \mathcal{U} be a quasi-uniformity compatible with τ . As (X, τ) is hereditarily compact, $\delta_{\mathcal{U}} = \delta_{\mathbf{P}}$ so that $\mathbf{P} \subset \mathcal{U}$. Since τ is finite, $U_{\tau} \in \mathbf{P}$. Furthermore, every neighbornet of X contains U_{τ} ; hence $\mathcal{U} \subset \mathbf{P}$.

 $(b) \Rightarrow (c)$: Since **FT** = **P**, from Corollary 3.3.1.2, every interiorpreserving open collection is finite. $(c) \Rightarrow (d)$: It is well known that a topological space is hereditarily compact if and only if every strictly increasing sequence of open sets is finite. As every increasing sequence of open sets is interior preserving, (X, τ) is hereditarily compact.

 $(e) \Rightarrow (a)$: Now let us suppose that (X, τ) is an R_1 space for which $\delta_{\mathbf{P}}$ is the only quasi-proximity compatible with τ . By Proposition 3.3.2.6 we can see that (X, τ) is compact. As every compact R_1 space is completely regular, \mathbf{P} is a uniformity. Thus by Proposition 3.3.2.11, $\mathcal{C} = \{\overline{\{x\}} : x \in X\}$ is an open cover; (X, τ) is compact and R_1 , so \mathcal{C} is finite. For each $G \in \tau$, $G = \cup\{\overline{\{x\}} : x \in G\}$. Hence τ is finite.

Corollary 3.3.2.4 Let (X, τ) be a Hausdorff topological space. Then the following statements are equivalent:

- (a) X is finite;
- (b) **P** is the only quasi-uniformity compatible with τ ;
- (c) $\delta_{\mathbf{P}}$ is the only quasi-proximity compatible with τ .

Example 3.3.2.1 A topological space that admits more than one quasiuniformity, for which $\delta_{\mathbf{P}}$ is the only compatible quasi-proximity.

Let (\mathbb{N}, τ) be the set of positive integers with the cofinite topology. Since (\mathbb{N}, τ) is hereditarily compact, by Theorem 3.3.2.3, $\delta_{\mathbf{P}}$ is the only compatible quasi-proximity. Let us set that $G_1 = \mathbb{N}$ and for each n > 1, let $G_n = \mathbb{N} - \{1, 2, \ldots, n-1\}$. Then $\mathcal{G} = \{G_n : n \in \mathbb{N}\}$ is an infinite interior-preserving open cover of \mathbb{N} . By Proposition 3.3.1.5, **FT** is not totally bounded and so **P** is not the only quasi-uniformity compatible with τ . Since the product of two hereditarily compact spaces is hereditarily compact, $\delta_{\mathbf{FT}} \times \delta_{\mathbf{FT}} = \delta_{\mathbf{FT}}$ and so the present example shows that conditions (a) and (b) of Proposition 3.2.3.7 are not generally equivalent.

Historical and bibliographic notes

In 1955 V. S. Krishnan shows that every topological space admits a quasi-uniformity (see [170]); subsequent proofs of this result were obtained by A. Császár in [61] and W. J. Pervin in [255]. The proof given here is Pervin's. A comparison of the constructions of Császár and Pervin is given by C. Votaw in [327]. In [257] Pervin defines $\delta_{\mathbf{P}}$. This quasi-proximity had been defined previously in terms of strong inclusion by Császár [61]. The characterization of **FINE** in terms of quasi-pseudo-metrics follows from Császár in [61]. Theorem 3.3.1.2 was proved by W. N. Hunsaker and W. F. Lindgren in 1970 (see [143]). The term interior preserving is due to H. J. K.

Junnila [158]. Theorem 3.3.1.1 was proved by P. Fletcher in 1971 in [106]. Proposition 3.3.1.5 was proved by Fletcher and Lindgren in [108], and Proposition 3.3.1.6 was proved by Krishnan in [170]. The semi-continuous quasiuniformity is defined in [40] and [108] in terms of upper semi-continuous functions. Proposition 3.3.1.7 was proved by Hunsaker and Lindgren in [143]. The concept of a spectrum was introduced by Leader in [193]. Theorem 3.3.1.2 is Theorem 2.1 by Fletcher and Lindgren in [108]. Proposition 3.3.2.1 is given by M. G. Murdeshwar and S. A. Naimpally in their book (see [236]). Proposition 3.3.2.2 and the terminology loccaly symmetric quasiproximity are suggested by Theorem 3.17. by Murdeshwar and Naimpally in [236]. Proposition 3.3.2.7 is due to R. Stoltenberg [311]. Theorem 3.3.2.1 is from Fletcher [105]. A list of characterizations of those metrizable spaces for which the fine uniformity has a countable base, is given by J. Rainwater [265]. Proposition 3.3.2.10 was proved by Lindgren and Fletcher in [198] and Theorem 3.3.2.3 was proved by Lindgren in [195].

3.4 Completness

3.4.1 Cauchy filters

We will now consider the theory of completeness and completions for quasiuniform spaces. In Subsection 2 of this section we establish a satisfactory analogue of the completion theory of uniform spaces. The results of Subsection 2 extend the usual theory. In this analogue, as in its uniform space counterpart, we have an ideal economy; every space has a completion and no space has two essentially different completions.

Definition 3.4.1.1 Let (X, U) be a quasi-uniform space and let \mathcal{F} be a filter on X. \mathcal{F} is said to be \mathcal{U} -Cauchy if for every entourage $U \in \mathcal{U}$ there exists an x = x(U) such that $U[x] \in \mathcal{F}$.

Definition 3.4.1.2 A filter base (or subbase) on a quasi-uniform space (X, \mathcal{U}) is a \mathcal{U} -Cauchy filter base (subbase) if the generated filter is a Cauchy filter.

A necessary and sufficient condition for a filter base \mathcal{B} to be a Cauchy filter base is that, for every entourage $U \in \mathcal{U}$, there exists an x = x(U) and a member $B \in \mathcal{B}$ such that $B \subset U[x]$. When there is no danger of ambiguity, we also say "Cauchy filter" instead of " \mathcal{U} -Cauchy filter". This definition is due to Sieber-Pervin in [284] and is equivalent to the usual definition of a Cauchy filter in a uniform structure.

Proposition 3.4.1.1 A filter \mathcal{F} on a uniform space (X, \mathcal{U}) is Cauchy if and only if for each entourage U, there exists an element $F \in \mathcal{F}$ such that $F \times F \subset U$.

Proof: For any entourage $U \in \mathcal{U}$, there exists a symmetric entourage $V \in \mathcal{U}$ with $V \circ V \subset U$ and a point x such that $V[x] \in \mathcal{F}$. Taking F = V[x], $F \times F \subset V \circ V \subset U$.

Conversely, if $F \times F \subset U$ and $x \in F$, then $F \subset U[x]$, so $U[x] \in \mathcal{F}$.

The reason for adopting Definition 3.4.1.1 is the fact that the usual definition of Cauchy filters has the disadvantage that a convergent filter need not be a Cauchy filter. For instance, in Example 3.1.3.1 (a), taking X to be the set of real numbers with the usual order, the sequence (1/n), $n \in \mathbb{N}$, is convergent but not Cauchy in the usual sense. On the other hand, it is easily verified that a convergent filter is Cauchy according to Definition 3.4.1.1.

It is easy to prove that

(a) every $\tau_{\mathcal{U}}$ -convergent filter base is a \mathcal{U} -Cauchy filter base;

(b) every filter base that is finer than a Cauchy filter base is a Cauchy filter base;

(c) if \mathcal{U} and \mathcal{V} are quasi-uniformities on X and $\mathcal{U} \subset \mathcal{V}$, then every \mathcal{V} -Cauchy filter base is a \mathcal{U} -Cauchy filter base. In particular, every \mathcal{U}^* -Cauchy filter base is a \mathcal{U} -Cauchy and \mathcal{U}^{-1} -Cauchy filter base.

Extension of a Cauchy filter is a Cauchy filter. More precisely, if \mathcal{F} is a \mathcal{U} -Cauchy filter on X and $X \subset Y$, where X is a quasi-uniform subspace of a quasi-uniform space (Y, \mathcal{V}) , then $ext_Y \mathcal{F}$ is a \mathcal{V} -Cauchy filter.

Proposition 3.4.1.2 Let (X, U) be a quasi-uniform space and let \mathcal{B} be a filter base on X. Then \mathcal{B} is a \mathcal{U}^* -Cauchy filter base if and only if for each $U \in \mathcal{U}$ there exists $B \in \mathcal{B}$ such that $B \times B \subset U$.

Proof: Let us suppose that for each $U \in \mathcal{U}$ there exists $B \in \mathcal{B}$ such that $B \times B \subset U$. Let $U \in \mathcal{U}$ and let $B \in \mathcal{B}$ such that $B \times B \subset U$. For each $x \in B$, $B \subset (U \cap U^{-1})[x]$ so that \mathcal{B} is a \mathcal{U}^* -Cauchy filter base.

Conversely, let \mathcal{B} be a \mathcal{U}^* -Cauchy filter base and let $U \in \mathcal{U}$. Let us choose $V \in \mathcal{U}$ such that $V \circ V \subset U$. There exists $B \in \mathcal{B}$ and $x \in X$ such that $B \subset (V \cap V^{-1})[x]$. Let $(a, b) \in B \times B$. Then $(a, x) \in V$ and $(x, b) \in V$; hence $(a, b) \in V \circ V$ so that $B \times B \subset U$.

Example 3.4.1.1 Let $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$; for each $n \in \mathbb{N}$ let $U_n = \Delta \cup \{(0, 1/i) : i > n\}$ and let \mathcal{U} be the quasi-uniformity on X generated by (U_n) . Then $\tau_{\mathcal{U}}$ -neighborhood filter of 0 is a $\tau_{\mathcal{U}}$ -convergent filter which is not a \mathcal{U}^* -Cauchy filter.

Proposition 3.4.1.3 Let $f : (X, U) \to (Y, V)$ be a quasi-uniformly continuous mapping and let \mathcal{B} be a \mathcal{U} -Cauchy filter base. Then $f(\mathcal{B})$ is a \mathcal{V} -Cauchy filter base.

Proof: Since $f(A \cap B) \subset f(A) \cap f(B)$, $f(\mathcal{B})$ is a filter base. Let $V \in \mathcal{V}$. There exists an entourage $U \in \mathcal{U}$ such that $(f(x), f(y)) \in V$ whenever $(x, y) \in U$, and since \mathcal{B} is a \mathcal{U} -Cauchy filter base, there exists $B \in \mathcal{B}$ and $x \in X$ such that $B \subset U[x]$. Then $f(B) \subset V[f(x)]$ and $f(\mathcal{B})$ is a \mathcal{V} -Cauchy filter base.

Proposition 3.4.1.4 Let X be a set, let $\{(Y_i, \mathcal{V}_i) : i \in I\}$ be a family of uniform spaces, and for each $i \in I$ let $f_i : X \to Y_i$. Let \mathcal{U} denote the coarsest uniformity on X for which each f_i is uniformly continuous. Then a filter base \mathcal{B} on X is a \mathcal{U} -Cauchy filter base if and only if, for each $i \in I$, $f_i(\mathcal{B})$ is a \mathcal{V}_i -Cauchy filter base.

Corollary 3.4.1.1 Let (X, U) be a subspace of a quasi-uniform space (Y, V). If a V^* -Cauchy filter on Y induces a filter on X, this induced filter is a U^* -Cauchy filter.

Example 3.4.1.1 shows that the induced filter of a Cauchy filter is not necessarily a Cauchy filter. Thus Proposition 3.4.1.4 does not hold for quasiuniform spaces even when there is only one mapping involved. The following two propositions demonstrate that two important particular cases of Proposition 3.4.1.4 do generalize to quasi-uniform spaces.

Proposition 3.4.1.5 Let $\{(X_i, \mathcal{V}_i) : i \in I\}$ be a family of quasi-uniform spaces and let (X, \mathcal{U}) denote the product quasi-uniform space. A filter base \mathcal{B} on X is a \mathcal{U} -Cauchy filter base if and only if $\pi_i(\mathcal{B})$ is a \mathcal{V}_i -Cauchy filter base for each $i \in I$.

Proof: Let \mathcal{B} be a filter base on X. It follows from Proposition 3.4.1.3 that if \mathcal{B} is a \mathcal{U} -Cauchy filter base, then, for each $i \in I$, $\pi_i(\mathcal{B})$ is a \mathcal{V}_i -Cauchy filter base. Conversely, let us suppose that for each $i \in I$, $\pi_i(\mathcal{B})$ is a \mathcal{V}_i -Cauchy filter base. Let $U \in \mathcal{U}$. Then U contains an entourage of the form $\{(x,y): \text{ for each } j \in J, (\pi_j(x), \pi_j(y)) \in V_j\}$ where J is a finite subset of I and for each $j \in J$, $V_j \in \mathcal{V}_j$. For each $j \in J$ there exists $x_j \in X_j$ such that $V_j[x_j] \in \pi_j(\mathcal{B})$. Let us choose $x \in X$ such that for each $j \in J$, $\pi_j(x) = x_j$. Then $U[x] \in \mathcal{B}$.

Proposition 3.4.1.6 Let X be a set and let (Y, V) be a quasi-uniform space. Let f be a mapping from X onto Y and let \mathcal{U} denote the coarsest quasiuniformity on X that makes f quasi-uniformly continuous. A filter base \mathcal{B} on X is a \mathcal{U} -Cauchy filter base whenever $f(\mathcal{B})$ is \mathcal{V} -Cauchy filter base.

Proof: Let \mathcal{B} be a filter base on X and let us suppose that $f(\mathcal{B})$ is a \mathcal{V} -Cauchy filter base. Let $V \in \mathcal{V}$ and let $U = f_2^{-1}(V)$. There exists a $y \in Y$ such that $V[y] \in f(\mathcal{B})$. Let $x \in f^{-1}(y)$. Then $f^{-1}(V[y]) = (f_2^{-1}(V))[x] =$ U[x]. Thus $U[x] \in \mathcal{B}$ and \mathcal{B} is a \mathcal{U} -Cauchy filter base.

Example 3.4.1.2 Let $X = Y_1 = Y_2 = \mathbb{R}$ and let $f_1 = f_2$ be the identity function on \mathbb{R} . Let $\mathcal{V}_1 = \{V : \text{ for some } x \in \mathbb{R}, \ \Delta \cup (x, +\infty) \times \mathbb{R} \subset V\}$ and let $V_2 = \{V : \text{ for some } x \in \mathbb{R}, \ \Delta \cup (-\infty, x) \times \mathbb{R} \subset V\}$. Let \mathcal{U} denote the coarsest quasi-uniformity making both f_1 and f_2 quasi-uniformly continuous. Then \mathcal{U} is the discrete uniformity on \mathbb{R} so $\mathcal{B} = \{X\}$ is not a \mathcal{U} -Cauchy filter even though for $i = 1, 2, \ f_i(\mathcal{B}) = \mathcal{B}$ is a \mathcal{V}_i -Cauchy filter.

3.4.2 Completeness of quasi-uniform spaces

Definition 3.4.2.1 A quasi-uniform space (X, U) is said to be complete provided that every U-Cauchy filter has a cluster point. If every U-Cauchy filter converges, the space is said to be convergence complete.

The usual definition of a complete uniform structure requires each Cauchy filter to be convergent but we have preferred a less stringent condition for quasi-uniform structure. Since, in a uniform space, a Cauchy filter converges to each of its cluster points (if they exist), the foregoing definition is equivalent to the usual definition of a complete uniform space.

A quasi-uniform space is complete provided every Cauchy ultrafilter converges, and a regular quasi-uniform space is complete provided every open Cauchy filter has a cluster point.

Example 3.4.2.1 Let X = [0,1]. For each $\varepsilon > 0$ let us set $V_{\varepsilon} = \Delta \cup \{0\} \times [0,\varepsilon) \cup \{1\} \times (1-\varepsilon,1] \cup (1/2-\varepsilon,1/2) \times ((0,\varepsilon) \cup (1-\varepsilon,1))$. Note that for $\varepsilon < 1/4$ if $(x,y), (y,z) \in V_{\varepsilon}$, then x = y or y = z. Let \mathcal{U} denote the quasi-uniformity for which $\{V_{\varepsilon} : \varepsilon > 0\}$ is a (transitive) base. Then $\tau_{\mathcal{U}^{-1}}$ is

the discrete topology on X so that (X, \mathcal{U}) is point-symmetric. To see that (X, \mathcal{U}) is complete let \mathcal{F} be a Cauchy ultrafilter. Let us suppose that \mathcal{F} does not converge to 0 or 1. There exists $\varepsilon > 0$ so that $[\varepsilon, 1 - \varepsilon] \in \mathcal{F}$. Let $x \in X$ such that $V_{\varepsilon}[x] \in \mathcal{F}$. Then $V_{\varepsilon}[x] = \{x\}$ and \mathcal{F} converges to x.

Clearly \mathcal{U} is not convergence complete, for $\{(0,\varepsilon) \cup (1-\varepsilon,1) : \varepsilon > 0\}$ is a nonconvergent Cauchy filter base.

Proposition 3.4.2.1 Let (X, U) be a locally symmetric quasi-uniform space. Every cluster point of a U-Cauchy filter \mathcal{F} is a limit point of \mathcal{F} .

Proof: Let p be a cluster point of a \mathcal{U} -Cauchy filter \mathcal{F} , let $U \in \mathcal{U}$ and let V be a symmetric entourage such that $V^3[p] \subset U[p]$. Since \mathcal{F} is a Cauchy filter, there exists $x \in X$ such that $V[x] \in \mathcal{F}$; and since p is a cluster point of \mathcal{F} , then $V[x] \cap V[p] \neq \emptyset$. Thus $V[x] \subset U[p]$ so that \mathcal{F} converges to p.

Corollary 3.4.2.1 Every locally symmetric complete quasi-uniform space is convergence complete.

Proposition 3.4.2.2 Let (X, U) be a (convergence) complete quasi-uniform space and let F be a closed subset of X. Then $(F, U|F \times F)$ is a (convergence) complete space.

Example 3.4.2.2 A complete subspace of a (locally symmetric) Hausdorff quasi-uniform space need not be closed. Let (X, \mathcal{U}) be the quasi-uniform space of Example 3.4.1.1. Then \mathcal{U} induces the discrete quasi-uniformity on $Y = \{1/n : n \in \mathbb{R}\}$ and therefore Y is a complete subspace that is not closed.

Proposition 3.4.2.3 Let $\{(X_i, U_i) : i \in I\}$ be a family of quasi-uniform spaces and let (X, U) denote the product quasi-uniform space. Then (X, U) is (convergence) complete if and only if (X_i, U_i) is (convergence) complete for each $i \in I$.

Proof: In order to prove the parenthetical result, let \mathcal{B} be a \mathcal{U} -Cauchy filter base. By Proposition 3.4.1.3, for each $i \in I$, $\pi_i(\mathcal{B})$ is a \mathcal{V}_i -Cauchy filter base and thus converges. It follows that \mathcal{B} is convergent. Conversely, let us suppose that (X, \mathcal{U}) is convergence complete, let $j \in I$ and let \mathcal{B}_j be a \mathcal{V}_j -Cauchy filter base. For each $i \neq j$, let \mathcal{B}_i be a \mathcal{V}_i -Cauchy filter on X_i . By Proposition 3.4.1.5, the product filter base $\prod \mathcal{B}_i$ is a \mathcal{U} -Cauchy filter base and hence converges. Therefore \mathcal{B}_j also converges.

A similar proof for completeness may be given using the characterization of completeness in terms of ultrafilters. \clubsuit

Proposition 3.4.2.4 Let $f : (X, \tau) \to (Y, \tau')$ be a perfect mapping, let \mathcal{U} denote the Pervin (resp. point-finite, locally finite, semi-continuous, fine transitive, fine) quasi-uniformity on (X, τ) and let \mathcal{V} denote the corresponding quasi-uniformity on (Y, τ') . Then (X, \mathcal{U}) is complete whenever (Y, \mathcal{V}) is complete.

Proof: Let us suppose that (Y, \mathcal{V}) is complete and let \mathcal{F} be a \mathcal{U} -Cauchy filter that has no cluster point. By Propositions 3.3.1.8 and 3.4.1.3, $f(\mathcal{F})$ is a \mathcal{V} -Cauchy filter base and has a cluster point, say p. Since \mathcal{F} has no cluster point, there exists a closed set $F \in \mathcal{F}$ such that $F \cap f^{-1}(p) = \emptyset$. Then f(F) is closed and $p \notin f(F)$, which is a contradiction.

3.4.3 Precompactness and total boundedness in quasi-uniform spaces

Definition 3.4.3.1 A quasi-uniform space (X, \mathcal{U}) is

(a) **precompact** provided that for each $U \in \mathcal{U}$ there exists a finite subset F of X such that X = U[F];

(b) **totally-bounded** if for each entourage U there exist a covering $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ of X such that $A_i \times A_i \subset U$ for each $i = 1, 2, \ldots, n$.

It is clear that in this definition we may replace the term "entourage" with the term "(sub) base element".

In a uniform space, the concepts of precompactness and total boundedness are equivalent. This, however, need not be the case in a quasi-uniform spaces.

Proposition 3.4.3.1 Every totally bounded quasi-uniform space (X, U) is precompact, but not conversely.

Proof: Let U be any entourage of totally bounded quasi-uniform structure \mathcal{U} . Then there exist sets A_1, A_2, \ldots, A_n such that $A_i \times A_i \subset U$ and $\bigcup_{i=1}^n A_i = X$. Let us choose $x_i \in A_i, i = 1, 2, \ldots, n$. Then $A_i = A_i \times A_i[x_i] \subset U[x_i]$ and $\bigcup_{i=1}^n U[x_i] = X$. But then U[A] = X, where $A = \{x_1, x_2, \ldots, x_n\}$.

Let us take X to be the ordered set of real numbers, with the quasiuniformity defined by $V_{0,1}$ as in Example 3.1.1.1 (c). This quasi-uniformity is precompact, but not totally bounded.

Some basic properties of precompactness and total boundedness are listed below without proofs.

Proposition 3.4.3.2 Total boundedness is

(a) hereditary;

(b) contractive, i.e. if (X, U) is a totally bounded quasi-uniform space, then (X, V) is totally bounded for each $V \subset U$;

(c) invariant under any quasi-uniformly continuous mapping;

(d) productive and projective, i.e. a product space is totally bounded if and only if each factor space is totally bounded. Also,

(e) the inverse image of a totally bounded structure is totally bounded;

(f) supremum of totally bounded quasi-uniformities is a totally bounded quasi-uniformity;

(g) (X, \mathcal{U}) is totally bounded if and only if (X, \mathcal{U}^{-1}) is totally bounded; and

(h) (X, \mathcal{U}) is totally bounded if and only if $(X, \mathcal{U} \vee \mathcal{U}^{-1})$ is totally bounded.

All the statements in this proposition except (a), (f), (g) and (h) are true if "totally bounded" is replaced by "precompact". Although precompactness of quasi-uniform spaces was not defined in Section 2. of this chapter, the proof of Theorem 3.2.3.1 establishes the following proposition.

Proposition 3.4.3.3 If (X, U) is a precompact quasi-uniform space, then every member of Q(U) has a lower bound.

The converse of the preceding proposition fails since by Theorem 3.4.3.1, if X is countably compact but not compact, **FT** is not precompact.

Proposition 3.4.3.4 Let (X, U) be a quasi-uniform space.

(a) (X, \mathcal{U}) is precompact if and only if every ultrafilter on X is a \mathcal{U} -Cauchy filter.

(b) (X, \mathcal{U}) is totally bounded if and only if every ultrafilter on X is a \mathcal{U}^* -Cauchy filter.

Proof: (a) Let (X, \mathcal{U}) be a precompact quasi-uniform space, let \mathcal{H} be an ultrafilter on X, and let $U \in \mathcal{U}$. There exists a finite set F such that U[F] = X. Since \mathcal{H} is an ultrafilter, there exists an $x \in F$ such that $U[x] \in \mathcal{H}$ so that \mathcal{H} is a \mathcal{U} -Cauchy filter. Conversely, let us suppose that every ultrafilter on X is a \mathcal{U} -Cauchy filter. If (X,\mathcal{U}) is not precompact, there exists $U \in \mathcal{U}$ such that for each finite subset F of X, $U[F] \neq X$. It follows that $\mathcal{B} = \{X - U[F] : F$ is a finite subset of $X\}$ is a filter base on X. Let \mathcal{H} be an ultrafilter on X that contains \mathcal{B} . There exists an $x \in X$ such that $U[x] \in \mathcal{H}$ and $X - U[x] \in \mathcal{H}$, which is a contradiction.

A similar proof may be given for (b) using Proposition 3.4.1.2.

Corollary 3.4.3.1 Let (X, U) be a uniform space. Then (X, U) is precompact if and only if it is totally bounded.

Proposition 3.4.3.5 Let (X, \mathcal{U}) be a complete quasi-uniform space that contains a dense precompact subspace (Y, \mathcal{V}) . Then every open filter on X has a cluster point.

Proof: Let \mathcal{F} be an open filter on X. Then $\mathcal{F}|Y$ is contained in an ultrafilter \mathcal{H} on Y, which is, by Proposition 3.4.3.4, a \mathcal{V} -Cauchy filter. Therefore, \mathcal{H} is a \mathcal{U} -Cauchy filter base and so \mathcal{H} has a cluster point in X. Since $\mathcal{F} \subset \mathcal{H}$, \mathcal{F} also has a cluster point.

Corollary 3.4.3.2 If X is a regular complete quasi-uniform space that has a dense precompact subspace, then X is compact. \clubsuit

It is dishearteningly to give examples of precompact quasi-uniform spaces that are not hereditarily precompact. On the other hand, we have already seen that the Pervin quasi-uniformity is totally bounded; hence every topological space admits a compatible hereditarily precompact quasi-uniformity. We will now introduce a transitive quasi-uniformity, which is like the Pervin quasi-uniformity hereditarily precompact but which is not totally bounded for infinite Hausdorff spaces.

In the following lemma, for each $n \in \mathbb{N}$, let \mathcal{A}_n denote $\bigcup_{i=1}^n \mathbf{a}_i$.

Lemma 3.4.3.1 Let (X, τ) be a topological space. For each $n \in \mathbb{N}$, for each collection $\{a_1, a_2, \ldots, a_n\}$ of point-finite open spectra in X, and for each $Y \subset X$, the collection $\{\cap(\mathcal{A}_n)_y : y \in Y\}$ contains a finite subcover of Y.

Proof: For each $i \in \mathbb{N}$, let us set $a_i = \{A(i, j) : j \in \mathbb{Z}\}$ and let us suppose without loss of generality that for all $i \in \mathbb{N}$ and $j \ge 0$, A(i, j) = X. The proof comes by induction on n. For n = 1 let $Y \subset X$ and let a_1 be a point-finite open spectrum in X. Let us set that $m = \sup\{j : Y \cap (A(1, j) - A(1, j - 1)) \neq \emptyset\}$ and let $y \in Y \cap (A(1, m) - A(1, m - 1))$; then $Y \subset \cap (A_1)_y$.

Now let $Y \subset X$ and let $\{a_1, a_2, \ldots, a_n, a_{n+1}\}$ be a collection of point-finite open spectra in X. By the inductive hypothesis there exist $m \in \mathbb{N}$ and $y_i \in Y$ $(1 \leq i \leq m)$ such that $Y \subset \bigcup_{i=1}^m (\cap(\mathcal{A}_n)_{y_i})$. For each $i \leq m$ there exists j(i) such that $y_i \in A(n+1, j(i)) - A(n+1, j(i)-1)$. Let us set that $M = \min\{j(i) : 1 \leq i \leq m\}$. Then $Y \cap A(n+1, M) \subset \bigcup_{i=1}^m (\cap(\mathcal{A}_{n+1})_{y_i})$. Using the inductive hypothesis again, we see that for each k $(M < k \leq 0)$ such that $Y \cap (A(n+1,k) - A(n+1,k-1)) \neq \emptyset$, there exist m_k and y(k,i) $(i = 1, 2, \ldots, m_k)$ such that $y(k,i) \in Y \cap (A(n+1,k) - A(n+1,k-1))$ and $Y \cap (A(n+1,k) - A(n+1,k-1)) \subset \bigcup_{i=1}^{m_k} \cap (\mathcal{A}_n)_{y(k,i)}$. Hence for each $k, Y \cap (A(n+1,k) - A(n+1,k-1)) \subset \bigcup_{i=1}^{m_k} \cap (\mathcal{A}_{n+1})_{y(k,i)}$. Since $Y = Y \cap A(n+1,M) \cup \bigcup_{k=M+1}^{0} Y \cap (A(n+1,k) - A(n+1,k-1)), \{\cap (\mathcal{A}_{n+1})_{y_i} : 1 \leq i \leq m\} \cup \{\cap (\mathcal{A}_{n+1})_{y(k,i)} : M < k \leq 0; 1 \leq i \leq m_k\}$ is the required finite subcover of $\{\cap (\mathcal{A}_{n+1})_y : y \in Y\}$.

Proposition 3.4.3.6 Let (X, τ) be a topological space and let \mathcal{A} denote the collection of all point-finite open spectra in (X, τ) . Then every subspace of $(X, \mathcal{U}_{\mathcal{A}})$ is precompact.

Proof: Let $U \in \mathcal{U}_{\mathcal{A}}$ and let $Y \subset X$. There exist $a_1, a_2, \ldots, a_n \in \mathcal{A}$ such that $\bigcap_{i=1}^n U_{\mathbf{a}_i} \subset U$. By the preceding lemma there exists a finite set $F \subset Y$ such that $Y \subset (\bigcap_{i=1}^n U_{\mathbf{a}_i})[F] \subset U[F]$.

Proposition 3.4.3.7 Let (X, τ) be a topological space. The following statements are equivalent:

(a) (X, τ) is countably compact;

(b) every countably interior-preserving open cover of X has a finite subcover;

- (c) **SC** is hereditarily precompact;
- (d) **SC** is precompact;
- (e) every lower semi-continuous function has a lower bound.

Proof: It is evident that $(a) \Rightarrow (b)$, and the implication $(b) \Rightarrow (c)$ follows from Theorem 3.3.1.2 and the preceding proposition. Clearly $(c) \Rightarrow (d)$ and the implication $(d) \Rightarrow (e)$ follows from Proposition 3.4.3.3. It is well known that $(e) \Rightarrow (a)$.

In order to establish the following proposition, it is useful to note that, if (X, τ) is a topological space, then **PF** is precompact if and only if every point-finite open cover of X has a finite subcover.

Proposition 3.4.3.8 Let (X, τ) be a regular T_1 topological space. Then **PF** is precompact if and only if (X, τ) is countably compact.

Proof: Let us suppose first that (X, τ) is not countably compact. Then there exists a closed infinite discrete set $D = \{x_i\}_{i=1}^{\infty}$. Since (X, τ) is regular and T_1 , there exists a sequence (V_i) of pairwise disjoint closed sets such that for each $i \ge 1$, V_i is a neighborhood of x_i . Then $\{X - D\} \cup \{\text{int } V_i\}_{i=1}^{\infty}$ is a point-finite open cover of X that has no finite subcover. Hence **PF** is not precompact. Now let us suppose that (X, τ) is countably compact and let \mathcal{C} be a point-finite open cover of X. It suffices to show that \mathcal{C} has a finite subcover. Let \mathcal{G} be the collection of all subcollections \mathcal{A} of \mathcal{C} such that $\mathcal{C} - \mathcal{A}$ is a cover of X. Then \mathcal{G} is partially ordered by inclusion, and by Zorn's lemma there is a maximal subcollection $\mathcal{M} \in \mathcal{G}$. As in the well-known proof of the Arens-Dugundji theorem, $\mathcal{C} - \mathcal{M}$ is a finite subcover of \mathcal{C} .

Proposition 3.4.3.9 Let (X, τ) be a T_1 space. The following statements are equivalent:

- (a) **LF** is totally bounded;
- (b) **LF** is precompact;
- (c) every locally finite open cover of X has a finite subcover.

Prof: We will prove only $(c) \Rightarrow (a)$, since the remaining implications are evident. Let us suppose that C is an infinite locally finite open cover of X. Without loss of generality we will assume that $C = \{C_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$ let us choose $x_n \in C_n$ and set $G_n = C_n - \{x_i : x_i \neq x_n\}$. Then $\{G_n : n \in \mathbb{N}\}$ is a locally finite open cover of X that has no finite subcover, which is a contradiction. Thus every locally finite open cover of X is finite and so $\mathbf{LF}=\mathbf{P}$, which is totally bounded. \clubsuit

In a Tychonoff space, condition (c) of the previous proposition is equivalent to pseudocompactness, and in a normal Hausdorff space, pseudocompactness is equivalent to countable compactness. Thus in a normal Hausdorff space all the conditions of Propositions 3.4.3.7, 3.4.3.8 and 3.4.3.9 are equivalent. Even in the class of Tychonoff spaces, the preceding propositions provide striking evidence that the quasi-uniformities discussed in subsection 3.1. of this chapter are ill behaved with respect to subspaces. We will pause, therefore, to justify the assertions of the chart given below.

| The quasi-uniformity | Р | PF | \mathbf{LF} | \mathbf{SC} | \mathbf{FT} | FINE |
|-------------------------|-----|-----|---------------|---------------|---------------|------|
| indicated to the right, | | | | | | |
| restricts to | | | | | | |
| arbitrary subspaces | Yes | No | No | No | No | No |
| open subspaces | Yes | Yes | No | No | Yes | Yes |
| closed subspaces | Yes | No | No | No | Yes | Yes |

It is evident that, if G is an open subspace of a space X, then $\mathbf{PF}_X|G \times G = \mathbf{PF}_G$, and the remaining affirmative statements of the chart have already been observed in subsection 3.1. of this chapter. It follows from Theorem 3.3.1.4 that neither **FT** nor **FINE** restricts to arbitrary subspaces,

and as the closed unit interval has an open subspace that is not countably compact, we have from Propositions 3.4.3.7 and 3.4.3.9 that neither **SC** nor **LF** restricts to open subspaces. Further, as pseudocompactness is not a closed-hereditary property, it follows from Proposition 3.4.3.9 that **LF** does not restrict to closed subspaces. Thus the example given below concludes the verification of the assertions of the chart.

Example 3.4.3.1 Let Q be the set of all rational numbers of the Michael line $(\mathbb{R}, \tau_{\mathcal{M}})$. Then Q is a closed subspace of the Michael line such that $\mathbf{PF}_{\mathbb{R}}|Q \times Q \neq \mathbf{PF}_Q$ and $\mathbf{SC}_{\mathbb{R}}|Q \times Q \neq \mathbf{SC}_Q$.

Let us suppose that $\mathbf{PF}_{\mathbb{R}}|Q \times Q = \mathbf{PF}_Q$; set $Q = \{x_i : i \in I\}$ and for each $n \in \mathbb{N}$ let us set that $G_n = Q - \{x_i : i < n\}$. Since $\{G_n : n \in \mathbb{N}\}$ is a point-finite open cover of Q, there is a point-finite open cover C of \mathbb{R} such that for each $x_i \in Q$, $U_C[x_i] \cap Q \subset G_i$. For each $i \in \mathbb{N}$ let H_i denote the Euclidean interior of $U_C[x_i]$, let $C' = \{H_i : i \in \mathbb{N}\} \cup \{\mathbb{R}\}$ and let $V = U_{C'}$. Then $V \in \mathbf{PF}_{\mathbb{R}}$ and for each $q \in Q$, $V[q] \subset U_C[q]$. Since C' is a countable point-finite open cover of \mathbb{R} , there is a countable subset $\{t_n : n \in \mathbb{N}\}$ of \mathbb{R} such that $\{V[t_n] : n \in \mathbb{N}\} = \{V[x] : x \in \mathbb{R}\}$. For each $n \in \mathbb{N}$, let us set $A_n = \{x : V[x] = V[t_n]\}$. By the Baire Category Theorem there exists an $m \in \mathbb{N}$ and an Euclidean open interval that is a subset of the Euclidean closure of A_m . It follows that there exists an infinite subset S of \mathbb{N} so that for each $j \in S$, $V[x_j] \cap A_m \neq \emptyset$. Thus $V[t_m] \cap Q \subset \cap_{i \in S} V[x_i] \cap Q \subset \cap \{G_n : n \in \mathbb{N}\}$, which is a contradiction.

Now let us suppose that $\mathbf{SC}_{\mathbb{R}}|Q \times Q = \mathbf{SC}_Q$. Since the family $\{G_n : n \in \mathbb{N}\}$ given above defines an open spectrum in Q, there exists a collection $\{a_i : 1 \leq i \leq m\}$ of open spectra in \mathbb{R} such that, if $V = \bigcap_{i=1}^m V_{a_i}$, then for each $x_n \in Q$, $V[x_n] \cap Q \subset G_n$. For each $n \in \mathbb{N}$ let H_n denote the Euclidean interior of $V[x_n]$. Let $\mathcal{C}' = \{H_n : n \in \mathbb{N}\} \cup \{\mathbb{R}\}$ and let $W = U_{\mathcal{C}'}$. Since, for each rational $x_j, x_j \in H_n$ only if $j > n, W[x_j]$ is an Euclidean neighborhood of x_j . There exists a countable subset $\{t_n : n \in \mathbb{N}\}$ of \mathbb{R} such that $\{W[x] : x \in \mathbb{R}\} = \{W[t_n] : n \in \mathbb{N}\}$. Using the Baire Category Theorem we can establish, as in the previous argument, that for some $m \in \mathbb{N}$ there is an infinite subset S of \mathbb{N} so that for each $j \in S, W[x_j] \cap A_m \neq \emptyset$; then $W[t_m] \cap Q \subset \cap \{G_n : n \in \mathbb{N}\}$, which is a contradiction.

Since we have just observed that **SC** is particularly ill behaved with respect to subspaces, the following proposition is some consolation.

Proposition 3.4.3.10 Let F be a closed G_{δ} -subspace of a topological space (X, τ) . Then $\mathbf{SC}_X | F \times F = \mathbf{SC}_F$.

Proof: It suffices to show that $\mathbf{SC}_F \subset \mathbf{SC}_X | F \times F$. Let $\{G_n : n \in \mathbb{Z}\}$ be an open spectrum in F and let (A_n) be a nested sequence of open sets such that $F = \bigcap_{n=1}^{\infty} A_n$. Let (H_n) be a nested sequence of open subsets of Xsuch that $H_n \cap F = G_{-n}$. Let us define an open spectrum $\{K_n : n \in \mathbb{Z}\}$ on X as follows. For $n \ge 0$ let us set that $K_n = G_n \cup (X - F)$ and for n > 0let us set that $K_{-n} = H_n \cap A_n \cap K_0$. For each $n \in \mathbb{Z}$, $K_n \cap F = G_n$. Thus $\mathbf{SC}_F \subset \mathbf{SC}_X | F \times F$.

The proposition given below is in marked contrast to Proposition 3.4.3.9.

Proposition 3.4.3.11 3.4.3.12 Let (X, τ) be a Hausdorff space. The following statements are equivalent:

- (a) **SC** is totally bounded;
- (b) **PF** is totally bounded;
- (c) X is a finite set.

Proof: $(a) \Rightarrow (b)$: If **SC** is totally bounded, then by Theorem 3.2.3.1 every lower semi-continuous function on X is bounded. It follows that every point-finite open cover of X is finite; by Proposition 3.3.1.5, **PF=P**.

 $(b) \Rightarrow (c)$: If **PF** is totally bounded, **PF=P** and every point-finite open cover of X is finite. Let us suppose that X is infinite. Then X has a countable infinite collection C of pairwise disjoint open sets, and $C \cup \{X\}$ is an infinite point-finite open cover of X.

 $(c) \Rightarrow (a)$: Obvious.

A remarkable consequence of Propositions 3.4.3.7 and 3.4.3.12 is that if X is an infinite countably compact Hausdorff space, then **SC** is hereditarily precompact but not totally bounded.

Proposition 3.4.3.7 suggests the possibility that a topological space is compact (Lindelöf) if and only if every interior-preserving open cover of X has a finite (countable) subcover.

Proposition 3.4.3.12 Let (X, τ) be a topological space. If every open cover of X that is well ordered by set inclusion has a finite subcover, then (X, τ) is compact.

Proof: Let \mathcal{C} be an open cover of X and let $m = \min\{|\mathcal{C}'| : \mathcal{C}' \text{ is a subcover of } \mathcal{C}\}$. Let us suppose that m is infinite and let $\mathcal{C}' = \{G_{\alpha} : \alpha < m\}$ be a subcover of \mathcal{C} having cardinality m. For each $\alpha < m$ let $H_{\alpha} = \bigcup_{\beta < \alpha} G_{\beta}$. Then $\{H_{\alpha} : \alpha < m\}$ is a well-ordered open cover of X that has no finite subcover, which is a contradiction.

Theorem 3.4.3.1 Let (X, τ) be a topological space. The following statements are equivalent:

(a) every admissible quasi-uniformity is complete;

(b) there exists an admissible quasi-uniformity that is complete and precompact;

(c) every admissible quasi-uniformity is complete and precompact;

(d) every admissible quasi-uniformity is precompact;

- (e) **FT** is precompact;
- (f) (X, τ) is compact.

Proof: It is evident that $(f) \Rightarrow (a) \Rightarrow (b)$ and $(f) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e)$. By Proposition 3.4.3.12, $(e) \Rightarrow (f)$. We will complete the proof by showing that $(b) \Rightarrow (f)$. Let \mathcal{U} be a quasi-uniformity compatible with τ that is complete and precompact and let \mathcal{F} be an ultrafilter on X. By Proposition 3.4.3.4, \mathcal{F} is a \mathcal{U} -Cauchy filter; since (X, \mathcal{U}) is complete, \mathcal{F} has a cluster point.

This theorem remains true if complete is replaced by convergence complete in (a), (b) or (c). It follows from Proposition 3.4.3.7 and Theorem 3.4.3.1 that if X is a countably compact topological space that is not compact, then $\mathbf{SC} \subset \mathbf{FT}$, but $\mathbf{SC} \neq \mathbf{FT}$. If X is also a regular T_1 space, then by Proposition 3.4.3.8, $\mathbf{PF} \subsetneq \mathbf{FT}$.

The following table summarizes characterizations of precompactness and total boundedness of certain familiar quasi-uniformities.

| | Precompactness | Total Boundedness |
|----|------------------------|--------------------|
| P | Always | Always |
| LF | Pseudocompactness | Pseudocompactness |
| | (Tychonoff spaces) | (Tychonoff spaces) |
| PF | Countable compactness | Finite ground set |
| | (regular T_1 spaces) | (Hausdorff spaces) |
| SC | Countable compactness | Finite ground set |
| | | (Hausdorff spaces) |
| FT | Compactness | Finite ground set |
| | | (Hausdorff spaces) |

Except for the Pervin quasi-uniformity and the semi-continuous quasiuniformity, little is known about topological characterizations of those topological spaces for which one of the quasi-uniformities listed on this page is (convergence) complete. Indeed it would be reasonable to pose a problem that the table given on this page be extended to completeness and convergence completeness. Instead of pursuing this broad and vaguely posed problem, we will pause to consider one fragment of it, which can be stated explicitly.

Does there exist a regular space for which no compatible quasi-uniformity is complete? (Note that the fine quasi-uniformity of a regular space is convergence complete whenever it is complete.) The corresponding problem for uniform spaces is solved by exhibiting a noncompact topological space, like ω_1 with the order topology, for which every compatible uniformity is precompact. This approach for quasi-uniform spaces is blocked by Theorem 3.4.3.1 (d). The following observations may prove useful in answering the question. Every Hausdorff space can be embedded as a closed subspace of a minimal Hausdorff space and every minimal Hausdorff space is a closed subspace of any Hausdorff space in which it is embedded. Consequently, if every minimal Hausdorff space admits a (convergence) complete quasi-uniformity, then every Hausdorff space admits a (convergence) complete quasi-uniformity.

It is difficult to construct a topological space for which $\mathbf{FT} \neq \mathbf{FINE}$ and so, as might be expected, it is also an unsolved problem whether there exists a regular space for which \mathbf{FT} is not complete. We observe that an ultrafilter \mathcal{F} on a space (X, τ) is an \mathbf{FT} -Cauchy ultrafilter if and only if every closurepreserving subcollection of \mathcal{F} has a cluster point. A modification of the proof of Proposition 3.4.3.12 also provides some insight into the problem of determining whether the fine transitive quasi-uniformity of a regular space must be complete.

Proposition 3.4.3.13 Let (X, τ) be a topological space. Then every closed **FT**-Cauchy filter has a cluster point.

Proof: Let us suppose that \mathcal{F} is a closed **FT**-Cauchy filter that has no cluster point. Let $m = \min\{|\mathcal{E}| : \mathcal{E} \text{ is a closed subcollection of } \mathcal{F} \text{ and } \cap \mathcal{E} = \emptyset\}$. Let $\mathcal{E} = \{E_{\alpha} : \alpha < m\}$ be a closed subcollection of \mathcal{F} such that $\cap \mathcal{E} = \emptyset$. For each $\alpha < m$ let $F_{\alpha} = \cap\{E_{\beta} : \beta < \alpha\}$. Since $\{X - F_{\alpha} : \alpha < m\}$ is an interior-preserving open cover, there exists $\alpha < m$ and a closed set $F \in \mathcal{F}$ such that $F \subset X - F_{\alpha}$. Then $\mathcal{E}' = \{F\} \cup \{E_{\beta} : \beta \leq \alpha\}$ is a closed subcollection of \mathcal{F} onsisting of fewer than m sets and $\cap \mathcal{E}' = \emptyset$, which is a contradiction.

Corollary 3.4.3.3 Let (X, τ) be a T_1 space such that for each interiorpreserving open cover C of X there exists an interior-preserving open cover \mathcal{R} such that $\{\overline{R} : R \in \mathcal{R}\}$ refines C. Then **FT** is convergence complete.

Proof: Let \mathcal{F} be an **FT**-Cauchy filter. It follows from the hypothesis that $\mathcal{B} = \{\overline{F} : F \in \mathcal{F}\}$ is an **FT**-Cauchy filter base. By the previous proposition,

 \mathcal{B} (hence \mathcal{F}) has a cluster point. Since the hypothesis implies that (X, τ) is regular, **FT** is convergence complete by Corollary 3.4.2.1.

Example 3.4.3.2 A T_1 space that admits no convergence complete quasiuniformity.

For each nonnegative integer n let $X_n = \mathbb{R} \times \{n\}$ and let $X = \bigcup_{n=0}^{\infty} X_n$. Let us define $g : \mathbb{N} \times X \to P(X)$ as follows. For each $x \in X$, $m \in \mathbb{N}$ and p = (x, 0), let $g(m, p) = \{p\} \cup \bigcup_{i=m}^{\infty} X_i - \{(x, i)\}$ and for each $p = (x, n) \in \mathbb{R} \times \mathbb{N}$ and $m \in \mathbb{N}$ let $g(m, p) = (x - 1/m, x + 1/m) \times \{n\}$. Let τ be the topology for which $\{g(n, p) : n \in \mathbb{N}, p \in X\}$ is a base.

Let $\mathcal{F} = \{F : \text{ there exists an } n \in \mathbb{N} \text{ so that } \bigcup_{i=n}^{\infty} X_i \subset F\}$ and let $\mathcal{U} = \mathbf{FINE}$. We will assert that \mathcal{F} is a \mathcal{U} -Cauchy filter. Let us not suppose; then there exists $U \in \mathcal{U}$ such that for all $p \in X$, $U[p] \notin \mathcal{F}$. Let $V \in \mathcal{U}$ such that $V^2 \subset U$. Then for each $x \in \mathbb{R}$, $\{n : U[(x,0)] \cap X_n = V[(x,0)] \cap X_n = X_n - \{(x,n)\}\}$ is infinite. For each $n \in \mathbb{N}$ let $A_n = \{x \in \mathbb{R} : U[(x,0)] \cap X_n = V[(x,0)] \cap X_n = X_n - \{(x,0)\} \cap X_n = X_n - \{(x,n)\}\}$. There exists an $n \in \mathbb{N}$ so that A_n is uncountable and so there exists $z \in \mathbb{R}$ such that for each $\varepsilon > 0$, $(z - \varepsilon, z + \varepsilon) \cap A_n - \{z\} \neq \emptyset$. Let $A = A_n - \{z\}$. Then $(z, n) \in \bigcap_{x \in A} V[(x, 0)]$ and $X_n \cap V[(z, n)] \subset X_n \cap [\bigcap_{x \in A} V^2[(x, 0)] \subset (\mathbb{R} - A) \times \{n\}$, which is a contradiction. Thus \mathcal{F} is a nonconvergent \mathcal{U} -Cauchy filter.

3.4.4 Bicompleteness

In this subsection we will present a theory of bicompleteness for quasiuniform spaces, which extends the usual theory of completeness for uniform spaces, and which is in every sense analogous to this theory. As it would be expected, the keystone of the theory is the construction of the bicompletion of a quasi-uniform space. Our presentation of this construction closely parallels a standard construction of the completion of a uniform space by means of minimal Cauchy filters, and the reader who is familiar with this standard construction will understand that in our view the uniform completion is seen as if through binoculars. Such a reader may wish to omit many proofs. On the other hand, the reader who is not familiar with the standard construction, is assured that the construction presented herein is no more difficult than the standard construction and contains the usual completion of a uniform space as a special case.

Definition 3.4.4.1 A quasi-uniform space (X, U) is said to be **bicomplete** provided that every U^* -Cauchy filter has a τ_{U^*} -limit point.

By Proposition 3.4.2.1, a quasi-uniform space (X, \mathcal{U}) is bicomplete if and only if (X, \mathcal{U}^*) is complete. Thus (X, \mathcal{U}) is bicomplete if and only if (X, \mathcal{U}^{-1}) is bicomplete. The results on completeness established in previous subsections may therefore be regarded as the results on bicompleteness.

If X is a set and \mathcal{U} is the only quasi-uniformity on X under consideration, then for convenience we will adopt the following notation. For each $x \in X$, \mathcal{N}_x^* denotes the $\tau_{\mathcal{U}^*}$ -neighborhood filter of x.

Proposition 3.4.4.1 Let (X, U) be a T_0 quasi-uniform space and let (Y, V) be a bicomplete subspace of (X, U). Then Y is a closed subspace of (X, τ_{U^*}) .

Proof: Let us suppose that Y is not closed in $(X, \tau_{\mathcal{U}^*})$ and let $p \in \overline{Y} - Y$. Then $\mathcal{N}_p^*|Y$ is a \mathcal{V}^* -Cauchy filter and therefore $\mathcal{N}_p^*|Y$ converges to a point $q \in Y$. Since $(X, \tau_{\mathcal{U}^*})$ is a Hausdorff space, p = q, which is a contradiction.

Theorem 3.4.4.1 Let (X, \mathcal{U}) be a quasi-uniform space, let (Y, \mathcal{V}) be a bicomplete T_0 quasi-uniform space, let D be a dense subset of $(X, \tau_{\mathcal{U}^*})$ and let $f: (D, \mathcal{U}|D \times D) \to (Y, \mathcal{V})$ be a quasi-uniformly continuous mapping. Then there exists a unique continuous extension $g: (X, \tau_{\mathcal{U}^*}) \to (Y, \tau_{\mathcal{V}^*})$, and g is quasi-uniformly continuous.

Proof: We will show first that f has a $\tau_{\mathcal{U}^*} - \tau_{\mathcal{V}^*}$ continuous extension. Let $x \in X$. As is remarked in subsection 3.4.1. of this section, \mathcal{N}_x^* is a \mathcal{U}^* -Cauchy filter so that by Corollary 3.4.1.1, $\mathcal{F}_X = \{D \cap R : R \in \mathcal{N}_x^*\}$ is a Cauchy filter on D. Since f is quasi-uniformly continuous, it follows from Proposition 3.4.1.3 that $\{f(F) : F \in \mathcal{F}_X\}$ is a base for a \mathcal{V}^* -Cauchy filter \mathcal{G}_X . The space (Y, \mathcal{V}) is bicomplete and $(Y, \tau_{\mathcal{V}^*})$ is a Hausdorff space so that \mathcal{G}_X has a unique limit, denoted by g(x). It remains to be shown that $g : X \to Y$, defined in the manner described above, is a quasi-uniformly continuous extension of f.

To show that g extends f, let $x \in D$ and let G be a $\tau_{\mathcal{V}^*}$ -neighborhood of f(x). As $f^{-1}(G)$ is a $\tau_{\mathcal{U}^*}|D$ -neighborhood of x, there exists a $\tau_{\mathcal{U}^*}$ neighborhood R of x such that $f(D \cap R) \subset G$. Thus $G \in \mathcal{G}_x$ and \mathcal{G}_x converges to f(x); that is g(x) = f(x).

To show that g is quasi-uniformly continuous, let $V \in \mathcal{V}$. By Corollary 3.1.2.3 there exists $V_1 \in \mathcal{V}$ such that V_1 is closed in $\tau_{\mathcal{V} \times \mathcal{V}^{-1}}$ and $V_1^3 \subset V$. Since f is quasi-uniformly continuous, there exists $U \in \mathcal{U}$ such that $(f(x), f(y)) \in V_1$ whenever $(x, y) \in U \cap D \times D$. Let $U_1 \in \mathcal{U}$ such that $U_1^3 \subset U$. The proof will be completed by showing that $(g(u), g(v)) \in V$ whenever $(u, v) \in U_1$. Let $(u, v) \in U_1$, let $U_2 = U_1 \cap U_1^{-1}$ and let $V_2 = V_1 \cap V_1^{-1}$. We assert that if $u' \in D \cap U_2[u]$, then $g(u) \in V_2[f(u')]$. Let H be a $\tau_{\mathcal{V}^*}$ -neighborhood of g(u); then there exists a $\tau_{\mathcal{U}^*}$ -neighborhood R of u such that $f(D \cap R) \subset H$. Let $w \in D \cap R \cap U_2[u]$, then $f(w) \in H \cap V_2[f(u')]$. Thus $H \cap V_2[f(u')] \neq \emptyset$, so that $g(u) \in cl_{\tau_{\mathcal{V}^*}}(V_2[f(u')]) = V_2[f(u')]$ as was asserted.

Now, let $u' \in D \cap U_2[u]$ and let $v' \in D \cap U_2[v]$; then $(u', v') \in U$ and $(f(u'), f(v')) \in V_1$. By the preceding assertion, $(g(u), f(u')) \in V_2 \subset V_1$ and $(f(v'), g(v)) \in V_2 \subset V_1$. Thus $(g(u), g(v)) \in V_1^3 \subset V$ as was to be shown.

Finally, since $\tau_{\mathcal{V}^*}$ is a Hausdorff topology, if h is a $\tau_{\mathcal{U}^*} - \tau_{\mathcal{V}^*}$ continuous mapping that agrees with g on the dense subspace D of X, then g = h. Thus g is the only $\tau_{\mathcal{U}^*} - \tau_{\mathcal{V}^*}$ continuous extension of f.

Corollary 3.4.4.1 Let (X, U) and (Y, V) be bicomplete T_0 quasi-uniform spaces and let D and E be dense subspaces of (X, τ_{U^*}) and (Y, τ_{V^*}) respectively. Let $f : (D, U|D \times D) \to (E, V|E \times E)$ be a quasi-unimorphism. Then f can be extended to a quasi-unimorphism between (X, U) and (Y, V).

Proof: By Theorem 3.4.4.1, f has a quasi-uniformly continuous extension $g_1: X \to Y$, and f^{-1} has a quasi-uniformly continuous extension $g_2: Y \to X$. Then $g_2 \circ g_1$ is a continuous extension of the identity mapping on D. Likewise, $g_1 \circ g_2$ is a continuous extension of the identity mapping on E. As the identity mappings on X and Y are the unique continuous extensions of the identity mappings on D and E respectively, $g_2 \circ g_1$ is the identity mapping on X and $g_1 \circ g_2$ is the identity mapping on Y. It follows that g_1 is a quasi-unimorphism from (X, \mathcal{U}) onto (Y, \mathcal{V}) .

Let (X, \mathcal{U}) be a T_0 quasi-uniform space and let us suppose that (Y, \mathcal{V}) is a T_0 quasi-uniform space such that (Y, \mathcal{V}^*) is a complete Hausdorff uniform space that contains (X, \mathcal{U}^*) as a dense subspace. If (X, \mathcal{U}) and (Y, \mathcal{V}) are both uniform spaces, then (Y, \mathcal{V}) must be the uniform completion of (X, \mathcal{U}) . Thus any completion that assigns to each T_0 quasi-uniform space (X, \mathcal{U}) a T_0 quasi-uniform space (Y, \mathcal{V}) , and that preserves symmetry (in the sense that \mathcal{V} is a uniformity whenever \mathcal{U} is a uniformity), must include the construction of the completion of a Hausdorff uniform space as a special case. In this subsection we present just such a construction. While it is comforting that our heretical construction recovers the canonical Hausdorff completion as a special case, it will become apparent in the next section that the importance of the construction is founded as much upon its preservation of transitivity as upon its preservation of symmetry. Indeed if $\cap \mathcal{U}$ is a linear order on X, then $\cap \mathcal{V}$ is a linear order on Y; and it is this remarkable attribute of the construction that establishes the importance of quasi-uniform spaces in the study of linearly ordered topological spaces and their subspaces.

Definition 3.4.4.2 A bicompletion of a quasi-uniform space (X, \mathcal{U}) is a bicomplete quasi-uniform space (Y, \mathcal{V}) that has a $\tau_{\mathcal{V}^*}$ dense subspace quasi-unimorphic to (X, \mathcal{U}) .

The key to our construction of a bicompletion of a quasi-uniform space is the use of minimal \mathcal{U}^* -Cauchy filters.

Definition 3.4.4.3 A \mathcal{U}^* -Cauchy filter on a quasi-uniform space (X, \mathcal{U}) is said to be **minimal** provided that it contains no \mathcal{U}^* -Cauchy filter other than itself.

Proposition 3.4.4.2 Let \mathcal{F} be a \mathcal{U}^* -Cauchy filter on a quasi-uniform space (X,\mathcal{U}) . There exists exactly one minimal \mathcal{U}^* -Cauchy filter that is coarser than \mathcal{F} . Furthermore, if \mathcal{B} is any base for \mathcal{F} , then $\mathcal{B}_0 = \{U[B] : B \in \mathcal{B} \text{ and } U \text{ is a symmetric member of } \mathcal{U}^*\}$ is a base for the minimal \mathcal{U}^* -Cauchy filter coarser than \mathcal{F} .

Proof: Let \mathcal{B} be a base for \mathcal{F} and let $U_1[B_1]$ and $U_2[B_2]$ be members of \mathcal{B}_0 . Let $U = U_1 \cap U_2$ and let $B \in \mathcal{B}$ such that $B \subset B_1 \cap B_2$; then $U[B] \subset U_1[B_1] \cap U_2[B_2]$. Thus \mathcal{B}_0 is a base for a filter \mathcal{F}_0 . Given $V \in \mathcal{U}^*$, let U be a symmetric member of \mathcal{U}^* such that $U^3 \subset V$. By Proposition 3.4.1.2 there exists $B \in \mathcal{B}$ such that $B \times B \subset U$. Then $U[B] \times U[B] \subset V$; hence \mathcal{F}_0 is a \mathcal{U}^* -Cauchy filter that is coarser than \mathcal{F} . Let \mathcal{G} be a \mathcal{U}^* -Cauchy filter coarser than \mathcal{F} and let $U[B] \in \mathcal{B}_0$. By Proposition 3.4.1.2 there exists $G \in \mathcal{G}$ such that $G \times G \subset U$. Since $G \in \mathcal{F}$, $G \cap B \neq \emptyset$. It follows that $G \subset U[B]$ and so $U[B] \in \mathcal{G}$. As any \mathcal{U}^* -Cauchy filter coarser than \mathcal{F} contains $\mathcal{F}_0, \mathcal{F}_0$ is a minimal \mathcal{U}^* -Cauchy filter and is the only minimal Cauchy filter coarser than \mathcal{F} .

Corollary 3.4.4.2 Let (X, U) be a quasi-uniform space. For each $x \in X$, \mathcal{N}_x^* is a minimal \mathcal{U}^* -Cauchy filter.

Theorem 3.4.4.2 Let (X, U) be a quasi-uniform space. Every minimal U^* -Cauchy filter has a base consisting of τ_{U^*} -open sets.

Proof: Let $V \in \mathcal{U}^*$. By Corollary 3.1.2.7 there exists a symmetric entourage $U \in \mathcal{U}^*$ such that $U \subset V$ and such that for each $x \in X$, $U[x] \in \tau_{\mathcal{U}^*}$. For each subset A of X, U[A] is a $\tau_{\mathcal{U}^*}$ -open subset of V[A]; hence the result follows from Proposition 3.4.4.2.

Proposition 3.4.4.3 Let (X, U) be a quasi-uniform space and let D be a dense subset of (X, τ_{U^*}) . If every Cauchy filter on $(D, U^* | D \times D)$ converges in (X, τ_{U^*}) , then (X, U) is bicomplete.

Proof: It suffices to show that every minimal \mathcal{U}^* -Cauchy filter \mathcal{F} on X converges. Since D is dense in $(X, \tau_{\mathcal{U}^*})$ and since, by Theorem 3.4.4.2, every member of \mathcal{F} has a nonempty interior, $\mathcal{F}|D$ is a Cauchy filter on $(D, \mathcal{U}^*|D \times D)$ and so converges in $(X, \tau_{\mathcal{U}^*})$. As \mathcal{F} is coarser than the filter on X determined by $\mathcal{F}|D$, it follows from Proposition 3.4.2.1 that \mathcal{F} converges.

Theorem 3.4.4.3 Every T_0 quasi-uniform space has a T_0 bicompletion.

Proof: Let (X, \mathcal{U}) be a quasi-uniform space and let \widetilde{X} be the set of all minimal \mathcal{U}^* -Cauchy filters on X. For each $U \in \mathcal{U}$, let $\widetilde{U} = \{(\mathcal{F}, \mathcal{G}) :$ there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subset U\}$. We will show that $\{\widetilde{U} : U \in \mathcal{U}\}$ is a base for a quasi-uniformity $\widetilde{\mathcal{U}}$ on \widetilde{X} . As every member of \widetilde{X} is a \mathcal{U}^* -Cauchy filter, for each $U \in \mathcal{U}$ and for each $\mathcal{F} \in \widetilde{X}, (\mathcal{F}, \mathcal{F}) \in \widetilde{U}$. Let $V_1, V_2 \in \mathcal{U}$; then $V = V_1 \cap V_2 \in \mathcal{U}$ and $\widetilde{V} = \widetilde{V}_1 \cap \widetilde{V}_2$. Given $U \in \mathcal{U}$ let $V \in \mathcal{U}$ such that $V^2 \subset U$. Let $(\mathcal{F}, \mathcal{G}), (\mathcal{G}, \mathcal{H}) \in \widetilde{V}$; then there exist $F \in \mathcal{F}, G_1 \in \mathcal{G}$ such that $F \times G_1 \subset V$ and $G_2 \in \mathcal{G}, H \in \mathcal{H}$ such that $G_2 \times H \subset V$. Since $G_1 \cap G_2 \neq \emptyset, F \times H \subset V^2$; thus $\widetilde{V}^2 \subset \widetilde{U}$.

To show that (X, \mathcal{U}) is a T_0 quasi-uniform space, let \mathcal{F} and \mathcal{G} be minimal \mathcal{U}^* -Cauchy filters such that for each $U \in \mathcal{U}, (\mathcal{F}, \mathcal{G}) \in \widetilde{\mathcal{U}}$ and $(\mathcal{G}, \mathcal{F}) \in \widetilde{\mathcal{U}}$. Then $\mathcal{F} \cap \mathcal{G}$ is a \mathcal{U}^* -Cauchy filter. As \mathcal{F} and \mathcal{G} are minimal Cauchy filters, and $\mathcal{F} \cap \mathcal{G}$ is coarser than both \mathcal{F} and \mathcal{G} , then $\mathcal{F} = \mathcal{F} \cap \mathcal{G} = \mathcal{G}$.

Let $i: X \to \tilde{X}$ be the mapping defined by $i(x) = \mathcal{N}_x^*$. It follows from Corollary 3.4.4.2 that $\mathcal{N}_x^* \in \tilde{X}$ for each $x \in X$. Thus *i* is well defined. Given $U \in \mathcal{U}$ let $V \in \mathcal{U}$ such that $V^3 \subset U$. Then $(x, y) \in U$ whenever $(\mathcal{N}_x^*, \mathcal{N}_y^*) \in \tilde{U}$ and $(\mathcal{N}_x^*, \mathcal{N}_y^*) \in \tilde{U}$ whenever $(x, y) \in V$. Thus *i* is a quasiuniform embedding.

Since $(\widetilde{\mathcal{U}})^* = (\widetilde{\mathcal{U}}^*)$, we will use the symbol $\widetilde{\mathcal{U}}^*$ unambiguously hereafter. We assert that if $\mathcal{F} \in \widetilde{X}$, then $i(\mathcal{F})$ converges to \mathcal{F} in $(\widetilde{X}, \tau_{\widetilde{\mathcal{U}}^*})$. It follows at once from this assertion that i(X) is dense in $(\widetilde{X}, \tau_{\widetilde{\mathcal{U}}^*})$. Given $\mathcal{F} \in \widetilde{X}$ and $U_1 \in \mathcal{U}$, let $V_1 \in \mathcal{U}$ and $F \in \mathcal{F}$ such that $V_1^2 \subset U_1$ and $F \times F \subset V_1$. Let $U = U_1 \cap U_1^{-1}$ and $V = V_1 \cap V_1^{-1}$. For each $x \in F$, $F \times V[x] \subset U$; thus for each $x \in F$, $i(x) = \mathcal{N}_x^* \in \widetilde{U}[\mathcal{F}]$ and so $i(F) \subset \widetilde{U}[\mathcal{F}]$.

By Proposition 3.4.4.3 in order to show that (X, \mathcal{U}) is bicomplete, it is sufficient to show that for every \mathcal{U}^* -Cauchy filter \mathcal{F} on X, $i(\mathcal{F})$ converges in $(\widetilde{X}, \tau_{\widetilde{\mathcal{U}}^*})$. Let \mathcal{F} be a \mathcal{U}^* -Cauchy filter. By Proposition 3.4.4.2 there exists a minimal \mathcal{U}^* -Cauchy filter \mathcal{G} coarser than \mathcal{F} . By the assertion of the previous paragraph, $i(\mathcal{G})$ converges to \mathcal{G} . As $i(\mathcal{F})$ is finer than $i(\mathcal{G})$, $i(\mathcal{F})$ converges as well.

We call the mapping $i: X \to \widetilde{X}$ defined in Theorem 3.4.4.3 the **canoni**cal embedding of (X, \mathcal{U}) into $(\widetilde{X}, \widetilde{\mathcal{U}})$. As a direct consequence of Corollary 3.4.4.1 we have the following uniqueness property for T_0 bicompletions.

Theorem 3.4.4.4 If (X, U) is a T_0 quasi-uniform space, any T_0 bicompletion of (X, U) is quasi-unimorphic to $(\widetilde{X}, \widetilde{U})$.

When (X, \mathcal{U}) is a T_0 quasi-uniform space, $(\widetilde{X}, \widetilde{\mathcal{U}})$ is said to be the bicompletion of (X, \mathcal{U}) , and we may identify X with i(X). When this identification is made, the minimal \mathcal{U}^* -Cauchy filters on X are the traces of the $\tau_{\widetilde{\mathcal{U}}^*}$ -neighborhood filters of the points of \widetilde{X} .

Theorem 3.4.4.5 Let (X, \mathcal{U}) be a Hausdorff uniform space. Then (X, \mathcal{U}) is a complete Hausdorff uniform space, and any complete Hausdorff uniform space that has a dense subspace unimorphic to (X, \mathcal{U}) is unimorphic to $(\widetilde{X}, \widetilde{\mathcal{U}})$.

Proposition 3.4.4.4 Let (X, U) be a T_0 quasi-uniform space. Then (X, U) is totally bounded if and only if $(\widetilde{X}, \widetilde{U}^*)$ is compact.

Proof: Let us suppose that $(\widetilde{X}, \widetilde{\mathcal{U}}^*)$ is compact. By Theorem 3.4.3.1 $(\widetilde{X}, \widetilde{\mathcal{U}}^*)$ is totally bounded. As total boundedness is hereditary property, (X, \mathcal{U}) is totally bounded.

The converse is an immediate consequence of Corollary 3.4.3.2. \clubsuit

The following proposition, which relies upon Corollary 3.1.2.10 and Theorem 3.4.4.5, establishes a one-to-one correspondence between the Hausdorff compactifications of a Tychonoff space (X, τ) and the totally bounded uniformities on X that are compatible with τ .

Proposition 3.4.4.5 Let (X, τ) be a Tychonoff space and let cX be a Hausdorff compactification of X. There exists exactly one uniformity \mathcal{U} compatible with τ such that $\tau_{\widetilde{\mathcal{U}}}$ is the topology of cX.

Let (X, δ) be a T_0 quasi-proximity space. By Theorem 3.4.4.3, $(X, \mathcal{U}_{\delta})$ has a bicompletion $(\widetilde{X}, \widetilde{\mathcal{U}_{\delta}})$. Let $\widetilde{\delta}$ denote the quasi-proximity induced by $\widetilde{\mathcal{U}_{\delta}}$, the bicompletion of \mathcal{U}_{δ} .

It is easy to prove that $(\widetilde{\mathcal{U}^{-1}}) = (\widetilde{\mathcal{U}})^{-1}$ and that $(\widetilde{\mathcal{U}})^* = (\widetilde{\mathcal{U}^*})$; indeed the second equality is observed in the proof of Theorem 3.4.4.3. These results together with equalities (a) through (d) of subsection 3.2.2. after Corollary 3.2.2.5 are sufficient to establish that for each quasi-proximity δ , $(\tilde{\delta})^{-1} = (\tilde{\delta}^{-1})$ and $(\tilde{\delta})^* = \tilde{\delta^*}$. The reader is warned that $(\tilde{\mathcal{U}})_{\delta} \neq \tilde{\mathcal{U}}_{\delta}$, $\tilde{\mathcal{U}}_{\omega} \neq (\tilde{\mathcal{U}})_{\omega}$ and $\delta_{\tilde{\mathcal{U}}} \neq (\tilde{\delta}_{\mathcal{U}})$.

Theorem 3.4.4.6 Let (X, δ) be a T_0 quasi-proximity space. Then

(a) the canonical embedding $i : (X, \mathcal{U}_{\delta}) \to (\widetilde{X}, \widetilde{\mathcal{U}}_{\delta})$ is a qp-embedding of (X, δ) into $(\widetilde{X}, \widetilde{\delta})$;

(b) (X, δ^*) is compact;

(c) $A\delta B$ if and only if $cl_{\tilde{\delta}^{-1}}(i(A)) \cap cl_{\tilde{\delta}}(i(B)) \neq \emptyset$.

Proof: (a) Proposition 3.2.3.4.

(b) Proposition 3.4.4.4.

(c) If $cl_{\tilde{\delta}^{-1}}(i(A)) \cap cl_{\tilde{\delta}}(i(B)) \neq \emptyset$, by Proposition 3.2.2.7, $i(A)\tilde{\delta}i(B)$ so that $A\delta B$. If $A\delta B$, then $i(A)\tilde{\delta}i(B)$ so that by Proposition 3.2.2.10 $cl_{\tilde{\delta}^{-1}}(i(A)) \cap cl_{\tilde{\delta}}(i(B)) \neq \emptyset$ holds. \clubsuit

3.4.5 Completions and compactifications

Let (X, \mathcal{U}) be a T_1 quasi-uniform space. A completion of (X, \mathcal{U}) is a complete T_1 quasi-uniform space (Y, \mathcal{V}) that has a dense subspace quasiunimorphic (relative to \mathcal{U} and \mathcal{V}) to (X,\mathcal{U}) . In the theory of uniform spaces it is common to insist that a completion of uniform spaces be a T_1 space, since this restriction simplifies some of the proofs and makes it possible to strengthen the conclusion of several important theorems. In the study of quasi-uniform spaces it is necessary to insist that completions be T_1 spaces, for, if we do not do so, every space (X, \mathcal{U}) admits the following trivial "compactification". Let $Y = X \cup \{\infty\}$ and for each $U \in \mathcal{U}$ let $U' = U \cup \{\infty\} \times Y$. Then $\{U': U \in \mathcal{U}\}$ is a base for a complete precompact quasi-uniformity \mathcal{V} on Y, and (X,\mathcal{U}) is a dense subspace of (Y,\mathcal{V}) . Since we do insist that a completion must be a T_1 space, it is a consequence of Corollary 3.4.3.1 that a precompact quasi-uniformity compatible with a topological space that is not completely regular, has no regular completion. This consequence of Corollary 3.4.3.1 serves as an early warning that the theory of completions for quasi-uniform spaces is not nearly so tidy as either the theory of completions for uniform spaces or the theory of bicompletions for quasi-uniform spaces.

In this subsection we present a completion, which is based on the construction of the Katetov extension of a Hausdorff space. In this construction we take certain nonconvergent filters as adjoined points and build open sets about an adjoined filter in terms of its members. To this end let (X, \mathcal{U}) be a quasi-uniform space, let $\mathbb{F} = \{\mathcal{F} : \mathcal{F} \text{ is a } \mathcal{U}-\text{Cauchy filter on } X$ that has no cluster point}, and let $\hat{X} = X \cup \mathbb{F}$. We need the collection Φ of all choice functions that pick a member of each filter in \mathbb{F} . Formally $\Phi = \{\phi : \phi :$ $\mathbb{F} \to P(X) \text{ and } \phi(\mathcal{F}) \in \mathcal{F}\}$. For each $U \in \mathcal{U}$ and each $\phi \in \Phi$, let us set $S(U, \phi) = U \cup \Delta \cup \{(\mathcal{F}, x) \in \mathbb{F} \times X : x \in U[\phi(\mathcal{F})]\}$ and let $\hat{\mathcal{U}}$ be the quasiuniformity on \hat{X} for which $\{S(U, \phi) : U \in \mathcal{U}, \phi \in \Phi\}$ is a base. Then $(\hat{X}, \hat{\mathcal{U}})$ contains (X, \mathcal{U}) as a dense open subset.

Lemma 3.4.5.2 Let (X, \mathcal{U}) be a T_1 quasi-uniform space. Then $(\widehat{X}, \widehat{\mathcal{U}})$ is a T_1 space if and only if for each $\mathcal{F} \in \mathbb{F}$, $adh_{\tau_{\mathcal{U}-1}}\mathcal{F} = \emptyset$.

Proof: Let us suppose that $(\widehat{X}, \widehat{\mathcal{U}})$ is a T_1 space and let $F \in \mathbb{F}$ and $p \in X$. Then there exist $U \in \mathcal{U}$ and $\phi \in \Phi$ such that $p \notin S(U, \phi)[\mathcal{F}]$. Let $F = \phi(\mathcal{F})$; then $p \notin U[F]$ and $U^{-1}[p] \cap F = \emptyset$. Now let us suppose that for each $\mathcal{F} \in \mathbb{F}$, $adh_{\tau_{\mathcal{U}^{-1}}}\mathcal{F} = \emptyset$ and let $p \in \widehat{X}$. If $p \in X$, $\cap \widehat{\mathcal{U}}[p] = \cap \mathcal{U}[p] = \{p\}$. If $p = \mathcal{F} \in \mathbb{F}$, then $\cap \widehat{\mathcal{U}}[p] = \{p\} \cup X \cap (\cap \widehat{\mathcal{U}}[p])$. Let us suppose that there exists $x \in X \cap (\cap \widehat{\mathcal{U}}[p])$. Since $adh_{\tau_{\mathcal{U}^{-1}}}\mathcal{F} = \emptyset$, there exist $F \in \mathcal{F}$ and $U \in \mathcal{U}$ such that $U^{-1}[x] \cap F = \emptyset$. Let $\phi \in \Phi$ be such that $\phi(\mathcal{F}) = F$. Then $x \notin S(U, \phi)[p]$, which is a contradiction. \clubsuit

Lemma 3.4.5.3 Let (X, \mathcal{U}) be a quasi-uniform space and let \mathcal{M} be a nonconvergent $\widehat{\mathcal{U}}$ -Cauchy ultrafilter on \widehat{X} . Then $X \in \mathcal{M}$ and $\mathcal{M}|X$ is a nonconvergent \mathcal{U} -Cauchy ultrafilter on X.

Proof: Let $U_1 \in \mathcal{U}$ and let $\phi_1 \in \Phi$. Since \mathcal{M} is a $\hat{\mathcal{U}}$ -Cauchy filter, there exists $p_1 \in \hat{X}$ such that $S(U_1, \phi_1)[p_1] \in \mathcal{M}$. Since \mathcal{M} does not converge to p_1 , there exist $U_2 \in \mathcal{U}, \phi_2 \in \Phi$ and $p_2 \neq p_1$ such that $S(U_2, \phi_2)[p_2] \in \mathcal{M}$. Then $S(U_1, \phi_1)[p_1] \cap S(U_2, \phi_2)[p_2] \subset X$ and $X \in \mathcal{M}$. It follows that $\mathcal{M}|X \subset \mathcal{M}$ so that $\mathcal{M}|X$ does not converge. To show that $\mathcal{M}|X$ is a \mathcal{U} -Cauchy filter, let $U \in \mathcal{U}$ and let $V \in \mathcal{U}$ such that $V^2 \subset U$. Let us choose $\phi \in \Phi$ such that for each $\mathcal{F} \in \mathbb{F}, \phi(\mathcal{F}) = V[x]$, where $V[x] \in \mathcal{F}$. Since \mathcal{M} is a $\hat{\mathcal{U}}$ -Cauchy filter, there exists $p \in \hat{X}$ such that $S(V, \phi)[p] \in \mathcal{M}$. If $p \in X$, $S(V, \phi)[p] = V[p] \in \mathcal{M}|X$; if $p = \mathcal{F} \in \mathbb{F}$, then $S(V, \phi)[p] = \{\mathcal{F}\} \cup V[\phi(\mathcal{F})]$. There exists $x \in X$ so that $V[x] \in \mathcal{F}$ and $\phi(\mathcal{F}) = V[x]$; thus $U[x] \in \mathcal{M}|X$.

Theorem 3.4.5.1 Let (X, U) be a T_1 quasi-uniform space. The following statements are equivalent:

- (a) (X, \mathcal{U}) has a completion;
- (b) for each \mathcal{U} -Cauchy filter \mathcal{F} , $adh_{\tau_{\mathcal{U}}^{-1}}\mathcal{F} \subset adh_{\tau_{\mathcal{U}}}\mathcal{F}$; (c) for each \mathcal{U} -Cauchy filter \mathcal{F} , if $adh_{\tau_{\mathcal{U}}^{-1}}\mathcal{F} \neq \emptyset$, then $adh_{\tau_{\mathcal{U}}} \mathcal{F} \neq \emptyset$.

Proof: $(a) \Rightarrow (b)$: Let us suppose that $i: (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$ is a dense embedding of (X, \mathcal{U}) in a complete T_1 space (Y, \mathcal{V}) . Let \mathcal{F} be a \mathcal{U} -Cauchy filter and let $x \in adh_{\tau_{\mathcal{U}^{-1}}}\mathcal{F}$. Then \mathcal{F} is contained in an ultrafilter \mathcal{G} that converges to x in $\tau_{\mathcal{U}^{-1}}$. By Propositions 3.4.1.3 and 3.1.2.2, $i(\mathcal{G})$ is a \mathcal{V} -Cauchy ultrafilter base that converges to i(x) in $\tau_{\mathcal{V}^{-1}}$. Since (Y, \mathcal{V}) is complete, $i(\mathcal{G})$ converges in $\tau_{\mathcal{V}}$ to a point y. By Proposition 3.1.4.3, y = i(x). Moreover $i^{-1}(i(\mathcal{G}))$ is a filter base on X, coarser than \mathcal{G} , that converges to x in $\tau_{\mathcal{U}}$; hence $x \in adh_{\tau_{1}}\mathcal{F}$.

 $(c) \Rightarrow (a)$: We will now suppose that for each \mathcal{U} -Cauchy filter \mathcal{F} , $adh_{\tau_{\mathcal{U}}}\mathcal{F} \neq \emptyset$ whenever $adh_{\tau_{\mathcal{U}}-1}\mathcal{F} \neq \emptyset$, and let us show that (X,\mathcal{U}) is a completion of (X, \mathcal{U}) . By Lemma 3.4.5.2, $(\widehat{X}, \widehat{\mathcal{U}})$ is a T_1 space. As noted, (X,\mathcal{U}) is a dense open subspace of $(\widehat{X},\widehat{\mathcal{U}})$. To show that $(\widehat{X},\widehat{\mathcal{U}})$ is complete, let us suppose that there exists a nonconvergent $\hat{\mathcal{U}}$ -Cauchy ultrafilter \mathcal{M} . By Lemma 3.4.5.3, $\mathcal{M}|X$ is a nonconvergent \mathcal{U} -Cauchy ultrafilter. Then $\mathcal{M}|X$, as a member of \mathbb{F} , is a $\tau_{\hat{\mathcal{U}}}$ -cluster point of \mathcal{M} , which is a contradiction.

Corollary 3.4.5.1 Let (X, τ) be a T_1 space and let \mathcal{U} and \mathcal{V} be quasiuniformities that are compatible with τ . If $\mathcal{U} \subset \mathcal{V}$ and (X, \mathcal{U}) has a completion, (X, \mathcal{V}) also has a completion.

Corollary 3.4.5.2 Every point-symmetric T_1 quasi-uniform space has a completion. 🐥

The following proposition, which extends Theorem 3.4.3.1, shows that any topological space that admits a noncomplete quasi-uniformity, also admits a quasi-uniformity that has no completion.

Proposition 3.4.5.1 Let (X, τ) be a topological space for which every quasi-uniformity compatible with τ has a completion. Then (X, τ) is compact.

Proof: Let us suppose that \mathcal{F} is a filter on X that has no cluster point. By Proposition 3.3.1.2 the quasi-uniformity \mathcal{U} generated by $\{T(X-F,F):$ $F \in \mathcal{F}$ and $X - F \in \tau$ is compatible with τ . For each $U \in \mathcal{U}$ there is an $x \in X$ such that U[x] = X, and so \mathcal{F} is a \mathcal{U} -Cauchy filter. To show that (X, \mathcal{U}) has no completion, by virtue of Theorem 3.4.5.1, it suffices to show that $adh_{\tau_{1,-1}}\mathcal{F}\neq \emptyset$. We will show indeed that $adh_{\tau_{1,-1}}\mathcal{F}=X$. Let $x \in X, U \in \mathcal{U}$ and $F \in \mathcal{F}$. There exist closed sets F_1, F_2, \ldots, F_n such that $\bigcap_{i=1}^{n} T(X - F_i, F_i) \subset U$. For each $i, T(F_i, X - F_i)[x] = X$ or F_i . Hence $\emptyset \neq F \cap (\bigcap_{i=1}^{n} F_i) \subset F \cap [\bigcap_{i=1}^{n} T(F_i, X - F_i)[x]] \subset F \cap U^{-1}[x]$.

Definition 3.4.5.1 Let (X, U) be a Tychonoff quasi-uniform space. A compactification of (X, U) is a compact Hausdorff quasi-uniform space (Y, V) that has a dense subspace quasi-unimorphic to (X, U).

Lemma 3.4.5.4 Let (X, \mathcal{V}) be a totally bounded Tychonoff quasi-uniform space, let cX be a Hausdorff compactification of X, and let \mathcal{U} be the unique uniformity that is compatible with the topology of cX. Then \mathcal{V} has an extension to a quasi-uniformity on cX that is compatible with $\tau_{\mathcal{U}}$ if and only if $\mathcal{U}|X \times X \subset \mathcal{V}$.

Proof: Let us suppose that $\mathcal{U}|X \times X \subset \mathcal{V}$. Since \mathcal{V} is totally bounded, by Corollary 3.2.3.2, $\{U_{(\varepsilon, f)} : \varepsilon > 0, f \in QB(\mathcal{V})\}$ is a subbase for \mathcal{V} . For each $f \in QB(\mathcal{V})$ we will define a bounded lower semi-continuous extension f' of f to cX as follows: $f'(x) = \min\{\sup_{z \in X} f(z), \sup\{\inf_{z \in G} f(z) : G \in \mathcal{N}_x\}\}$. Let \mathcal{V}' be the quasi-uniformity on cX for which $\{U_{(\varepsilon, f')} : \varepsilon > 0, f \in QB(\mathcal{V})\}$ is a subbase. Then $\mathcal{V}'|X \times X = \mathcal{V}$ and $\tau_{\mathcal{V}'} \subset \tau_{\mathcal{U}}$ so that $\mathcal{U} \vee \mathcal{V}'$ is a quasi-uniformity on cX that is compatible with $\tau_{\mathcal{U}}$. Moreover, since $\mathcal{U}|X \times X \subset \mathcal{V}, (\mathcal{U} \vee \mathcal{V}')|X \times X = \mathcal{V}$ holds.

Now let us suppose that \mathcal{W} is an extension of \mathcal{V} to cX such that $\tau_{\mathcal{U}} = \tau_{\mathcal{W}}$. By Proposition 3.2.2.11, $\mathcal{U} \subset \mathcal{W}$. Thus $\mathcal{U}|X \times X \subset \mathcal{W}|X \times X = \mathcal{V}$.

Theorem 3.4.5.2 A totally bounded Tychonoff quasi-uniform space (X, \mathcal{V}) has a compactification if and only if \mathcal{V} contains a uniformity compatible with $\tau_{\mathcal{V}}$.

Proposition 3.4.5.2 Let (X, τ) be a locally compact Hausdorff space and let \mathcal{U} be a quasi-uniformity compatible with τ . Then (X, \mathcal{U}) has a compactification if and only if (X, \mathcal{U}) is locally symmetric.

Proof: If (X, \mathcal{U}) has a compactification, (X, \mathcal{U}) is locally symmetric by Proposition 3.3.2.6.

Now let us suppose that (X, \mathcal{U}) is locally symmetric. Let $X \cup \{\infty\}$ be the one-point compactification of X. For each $U \in \mathcal{U}$ and each compact subset K of X, let us define $W(U, K) = U \cup (\{\infty\} \times U[X - K]) \cup \{(\infty, \infty)\}$. Let \mathcal{W} be the quasi-uniformity on $X \cup \{\infty\}$ for which $\{W(U, K) : U \in \mathcal{U}, K \text{ is a compact subspace of } X\}$ is a base. Evidently $\mathcal{W}|X \times X = \mathcal{U}$ and $\tau_{\mathcal{W}}$ is coarser than the compact Hausdorff topology of $X \cup \{\infty\}$. Therefore, we need only observe, using the local symmetry of \mathcal{U} , that \mathcal{W} is a Hausdorff quasi-uniformity. \clubsuit **Corollary 3.4.5.3** Let (X, τ) be a locally compact Hausdorff space. A quasi-uniformity W compatible with τ is locally symmetric if and only if W contains a uniformity compatible with τ .

Lemma 3.4.5.4 and Proposition 3.4.5.2 are evidence of the over-abundance of compactifications. Furthermore, it follows from a remark in subsection 3.3.1. of this chapter that if (X, τ) is a Tychonoff space and cX is any Hausdorff compactification of X, then (cX, \mathbf{P}) is a compactification of (X, \mathbf{P}) . Moreover Proposition 3.4.5.2 establishes that a complete quasi-uniformity on noncompact space can have a compactification; hence in the theory of quasi-uniformities, the terminology "precompactness" is misleading. Obviously, many questions concerning completions and compactifications remain unsolved. The following two questions are representative.

Theorem 3.4.5.1 characterizes those quasi-uniform spaces that have a completion. How can we characterize those quasi-uniform spaces that have a Hausdorff (completely regular) completion? Theorem 3.4.5.2 characterizes those totally bounded quasi-uniform spaces that have a compactification. Under what conditions does an arbitrary quasi-uniform space have a compactification?

Historical and bibliographic notes

It is natural to define a Cauchy filter as a filter \mathcal{F} with the property that for each $U \in \mathcal{U}$ there is an $F \in \mathcal{F}$ such that $F \times F \subset U$. Example 3.4.1.1 shows that with this definition a $\tau(\mathcal{U})$ -convergent filter would not necessarily be such a "Cauchy" filter. For this reason J. L. Siber and W. J. Pervin [284] choose the definition of Cauchy filter as given in the text.

Proposition 3.4.3.5 was proved by M. G. Murdeshwar and S. A. Naimpally in [236]. Lemma 3.4.3.1 and Proposition 3.4.3.6 were proved by P. Fletcher and W. F. Lindgren in [108]. C. Barnhill, P. Flecher and W. F. Lindgren proved Proposition 3.4.3.7 in [40] and [108]. Proposition 3.4.3.8 was proved by K. Iseki and S. Kasahara in [154] (see also [20]). Proposition 3.4.3.9 was proved by Flecher and Lindgren in [108]. Proposition 3.4.3.12 was proved by P. S. Alexandroff and P. Urysohn in [12].

The term bicomplete is the same as A. Császár's doubly complete in book [61]. Our presentation of the bicompletion of a quasi-uniform space is modelled upon the presentations of the completion of a uniform space given by N. Bourbaki in [38]. All results in subsection 4.4. was proved by J. W. Carlson and T. L. Hicks in [55]. Theorem 3.4.5.1 was proved by Ward in [331].

3.5 Ordered structures

3.5.1 Topological ordered spaces

In this section we will examine the symbiotic relationship existing between topologies and partial orders. We establish theories of completeness, compactness, and normality that include the usual uniform and topological theories in the special case that the partial order under consideration is equality. Although quasi-uniformities do not make their appearance until the third subsection of this section, their role is central to study of the interdependence between topologies and orders. Indeed it is the theory of quasi-uniformities that enables us to consider order completions and compactifications of ordered spaces and to develop for generalized ordered spaces a theory of uniformities that reflects both the topological and order structures of these spaces.

Definition 3.5.1.1 A topological ordered space or a T_2 -ordered topological spaces is a triple (X, τ, \leq) where X is a set, τ is a topology on X, \leq is a partial order on X, and $G(\leq) = \{(x, y) : x \leq y\}$ is a closed set of $X \times X$.

Definition 3.5.1.2 Let X be an ordered set. A subset $A \subset X$ is said to be decreasing if $a \leq b$ and $b \in A$ imply $a \in A$.

Every subset $A \subset X$ determines, in a unique fashion, a decreasing set d(A) which is the smallest one among the decreasing sets containing A; a point a belongs to d(A) if and only if it is possible to determine a point $b \in A$ such that $a \leq b$. Dually, one defines the concept of an **increasing** set and of the smallest increasing set i(A) containing a given subset $A \subset X$. A subset A of X is increasing if and only if X - A is decreasing.

Proposition 3.5.1.1 Let (X, τ) be a topological space and let \leq be an order on X. Then the following conditions are equivalent:

(a) (X, τ, \leq) is a topological ordered space;

(b) for each pair of elements $a \not\leq b$ in X, there exist τ -open sets U containing a and V containing b such that $x \in U$ and $y \in V$ together imply $x \nleq y$;

(c) for each pair of elements $a \leq b$ in X there exist disjoint τ -neighborhoods U, V of a, b respectively such that U is increasing, and V is decreasing in X;

(d) when nets $\{x_{\alpha} : \alpha \in \Gamma\}$ and $\{y_{\alpha} : \alpha \in \Gamma\}$ in X, where $x_{\alpha} \leq y_{\alpha}$ for each $\alpha \in \Gamma$, converge to a and b respectively, then $a \leq b$.

Proof: The equivalence of conditions (b) and (c) is obvious, while the equivalence of conditions (a) and (b) is given in [233]. The equivalence of conditions (b) and (d) may be obtained using an argument analogous to that in the proof of [164], Theorem 3, p. 67.

Corollary 3.5.1.1 If (X, τ, \leq) is a topological ordered space, then the sets i(a) and d(a) are closed for every point $a \in X$.

Proof: If there is given a point $a \in X$, then, if $b \in X - i(a)$, $a \leq b$ is false. We will apply the first of the proposition and determine an increasing neighborhood V of a and a decreasing neighborhood W of b which are disjoint. Now $i(a) \subset V$ implies $W \cap i(a) = \emptyset$ which proves that i(a) is closed. We reason analogously for d(a) and thereby complete the proof. \clubsuit

Corollary 3.5.1.2 Every topological ordered space (X, τ, \leq) is a Hausdorff space.

Proof: Let us consider any two distinct points $a, b \in X$. Then one of the two relations $a \leq b$ or $b \leq a$ is false. Let us suppose that the first one is false. The application of Proposition 3.5.1.1 shows that a and b have disjoint neighborhoods as we have desired.

Proposition 3.5.1.2 If a topological space is equipped with a quasi-order such that the set consisting of the open decreasing and open increasing sets is an open subbase, then the set of convex neighborhoods of every point is a base for the neighborhood system of this point.

Proof: Taking into account that the intersection of a finite number of open decreasing sets is an open decreasing set and similarly for an open increasing sets, we see that the hypothesis of the proposition signifies that the set of subsets of the form $V \cap W$, where V is an open decreasing and W is an open increasing set, is an open base. The proposition then results from the observation that each subset $V \cap W$ is open and convex.

Definition 3.5.1.3 If Y is a subset of a topological ordered space (X, τ, \leq) and \leq_Y denotes a partial order induced on Y, then $(Y, \tau|Y, \leq_Y)$ is a topological ordered space, called the **topological ordered subspace**. **Definition 3.5.1.4** Let (X, τ, \leq) and (X', τ', \leq') be topological ordered spaces. A mapping $f : X \to X'$ is said to be **increasing (decreasing)** provided that $f(x) \leq' f(y)$ whenever $x \leq y$ ($y \leq x$). A one-to-one continuous increasing function f mapping X onto X' is an **ordered homeomorphism** provided that f^{-1} is also continuous and increasing. If f is an order homeomorphism of X onto a subspace X', then f is said to be an **ordered embedding** of X into X'.

We let $C(X,\uparrow) = \{f : f : X \to \mathbb{R}, f \text{ is continuous and increasing}\}$ and $C^*(X,\uparrow) = \{f : f \in C(X,\uparrow) \text{ and } f \text{ is bounded}\}$. It is easy to see that $C(X,\uparrow)$ is closed under the operations meet and join.

Definition 3.5.1.5 A topological ordered space X is said to be a normally ordered space if, whenever A and B are disjoint closed subsets of X such that A is decreasing and B is increasing, there exist disjoint open sets G and H such that G is decreasing and contains A, and H is increasing and contains B.

Let (X, τ, \leq) be a topological ordered space. The family of all increasing (decreasing) open sets is a subtopology of τ . For each subset A of X, there exists the smallest closed increasing (decreasing) subset $cl_i(A)(cl_d(A))$ that contains A. Moreover, cl_i and cl_d are Kuratowski closure operators on X.

Lemma 3.5.1.1 A topological ordered space X is normally ordered if and only if for each closed decreasing subset E of X and each open decreasing subset F of X that contains E, there exists a decreasing open set G of X such that $E \subset G$ and $cl_d(G) \subset F$.

Prof: Let us suppose that the condition is satisfied. Let A be a closed decreasing subset of X that is disjoint from some closed increasing set B. Then X-B is an open decreasing set containing A so that, by the condition, there exists a decreasing open set G of X such that $A \subset G$ and $cl_d(G) \subset X-B$. Since $X-cl_d(G)$ is increasing and contains B, we have shown that X is normally ordered.

Now let us suppose that X is normally ordered and let E be a closed and decreasing subset of an open and decreasing set F. Then X - F is closed and increasing; thus there exists an open decreasing set G containing E and an open increasing set H containing X - F such that $G \cap H = \emptyset$. Since X - H is closed and decreasing, $E \subset G \subset cl_d(G) \subset X - H \subset F$ as was to be shown.
Lemma 3.5.1.2 Let (X, τ, \leq) be a topological ordered space and let D be a dense subset of the positive real numbers. For each $t \in D$, let F(t) be an open subset of X such that $\cup \{F(t) : t \in D\} = X$ and $cl_d(F(s)) \subset F(t)$ whenever s < t. Then the function f, defined by $f(x) = \inf\{t : x \in F(t)\}$, is continuous and increasing.

Proof: The proof that f is a continuous mapping is well known. To show that f is increasing, let $x \leq y$ and let us suppose that f(y) < f(x). Then there exists $s, t \in D$ such that f(y) < s < t < f(x) and $y \in F(s)$. Thus $x \in cl_d(F(s)) \subset F(t)$, which is a contradiction.

Theorem 3.5.1.1 (Urysohn's Lemma) Let (X, τ, \leq) be a topological ordered space. Then X is normally ordered if and only if for any two disjoint closed sets A and B, where A is decreasing and B is increasing, there exists $f \in C^*(X, \uparrow)$ such that f(A) = 0 and f(B) = 1.

Proof: Let us suppose that X is normally ordered and let A and B be disjoint closed sets, where A is decreasing and B is increasing. Let $D = \{k \cdot 2^{-n} : k, n \in \mathbb{N}\}$. For $t \in D$ and t > 1 let F(t) = X, let F(1) = X - B and let F(0) be a decreasing open set containing A such that $cl_d(F(0)) \cap B = \emptyset$. For $t \in D$ and 0 < t < 1, let us write t in the form $t = (2m+1) \cdot 2^{-n}$ and let us use Lemma 3.5.1.1 to choose, inductively on n, F(t) to be a decreasing open set containing $cl_d(F(2m \cdot 2^{-n}))$ such that $cl_d(F(t)) \subset F((2m+2) \cdot 2^{-n})$. Let us define $f(x) = \inf\{t : x \in F(t)\}$. By the previous lemma, $f \in C^*(X, \uparrow)$. For each $t \in D$, $A \subset F(t)$ so that f(A) = 0 and f(B) = 1 since $F(t) \subset X - B$ for $t \leq 1$ and F(t) = X for t > 1.

Now let us suppose that X satisfies the condition and let A and B be disjoint closed sets where A is decreasing and B is increasing. By hypothesis there exists $f \in C^*(X,\uparrow)$ such that f(A) = 0 and f(B) = 1. Let $G = \{x \in X : f(x) < 1/2\}$ and $H = \{x \in X : f(x) > 1/2\}$. Then $G \cap H = \emptyset$, G is decreasing and contains A, and H is increasing and contains B. Therefore, X is normally ordered.

Theorem 3.5.1.2 Let (X, τ, \leq) be a topological ordered space. Then the following conditions are equivalent:

(a) if g and h are increasing real-valued functions on X, g is upper semicontinuous and h is lower semicontinuous, and if $g(x) \leq h(x)$ for each $x \in X$, then there exists a continuous increasing function f on X such that $g(x) \leq f(x) \leq h(x)$ for each $x \in X$;

(b) X is a normally ordered space.

Proof: Let us first assume that the condition (a) holds. Let F, H be closed disjoint subsets of X, where F is decreasing and H is increasing. Then the set G = X - F is open and increasing; moreover, $H \subset G$. Therefore, χ_H is increasing and upper semicontinuous, χ_G is increasing and lower semicontinuous, and $\chi_H \leq \chi_G$. Thus, by (a), there exists a continuous increasing function f such that $\chi_H \leq f \leq \chi_G$. It is clear that f(x) = 1 for each $x \in H$ and f(x) = 0 for each $x \in F$. Consequently, the sets $U = \{x : f(x) < 1/3\}$ and $V = \{x : f(x) > 2/3\}$ have desired properties.

Let us now suppose that the condition (b) holds. Since there exists an order-preserving homeomorphism from the extended real line into [0, 1], we may assume that $0 \leq g(x) \leq h(x) \leq 1$ for each $x \in X$. The proof is similar to the one of Urysohn's lemma. Let \mathbb{Q} denote the set of rational numbers of [0, 1]. If $q \in \mathbb{Q}$, let $F(q) = \{x \in X : h(x) \leq q\}$ and $V(q) = \{x \in X : g(x) < q\}$. It is obvious that F(q) is closed, V(q) is open, and both are decreasing; moreover, if q < r, then

(1)
$$F(q) \subset V(r), \quad F(q) \subset F(r), \quad V(q) \subset V(r).$$

We shall show that for each $q \in \mathbb{Q}$ we can find an open decreasing set W(q) such that

(2)
$$F(q) \subset W(r) \quad \text{if} \quad q < r ,$$

$$\begin{array}{ccc} (3) & cl_d(W(q)) \subset W(r) & \text{if } q < r \\ (4) & cl_d(W(q)) \subset V(r) & \text{if } q < r \end{array}$$

(4)
$$cl_d(W(q)) \subset V(r)$$
 if $q <$

Let q_1, q_2, \ldots be a sequence of all elements of \mathbb{Q} such that $q_1 = 0, q_2 = 1, q_n \neq q_m$ if $n \neq m$. We proceed by induction. First we will define $W(0) = \emptyset, W(1) = V(1)$, and then we will suppose that the open decreasing sets $W(q_1), \ldots, W(q_k)$ have been defined and satisfy (2), (3), (4) for q, r in $T_k = \{q_1, \ldots, q_k\}$. If we denote $p = \sup\{q : q \in T_k \text{ and } q < t_{k+1}\}$ and $p' = \inf\{q : q \in T_k \text{ and } q > t_{k+1}\}$, then by (1), $F(p) \subset V(p')$. Moreover, $F(p) \subset W(p'), cl_d(W(p)) \subset W(p'), and cl_d(W(p)) \subset V(p')$. Thus, $F(p) \cup cl_d(W(p))$ is a closed decreasing set contained in the open decreasing set $W(p') \cap V(p')$. Consequently, by Lemma 3.5.1.1, there exists an open decreasing set U such that $F(p) \cup cl_d(W(p)) \subset U$ and $cl_d(U) \subset W(p') \cap V(p')$. This set U will be denoted by $W(q_{k+1})$. It is obvious that conditions (2), (3) and (4) are satisfied for $q \in T_{k+1}$.

Let us now denote $f(x) = \sup\{q : q \in \mathbb{Q} \text{ and } x \notin W(q)\}$ for each $x \in X$. It is clear that $0 \leq f(x) \leq 1$. We shall prove the continuity of f at an arbitrary point $x \in X$. Let us first consider the case where 0 < f(x) < 1. Let r, s be any rational numbers such that 0 < r < f(x) < s < 1. Then $x \in W(s)$. If r < q < f(x), then $x \notin W(q)$; consequently, $x \notin cl_d(W(r))$. Therefore $U = W(s) - cl_d(W(r))$ is a neighborhood of x. Let $y \in U$. Then $y \in W(s)$; hence s < f(y). Similarly, $y \notin cl_d(W(r))$ implies $y \notin W(r)$ and $f(y) \ge r$. We have shown that $y \in U$ implies $r \le f(y) < s$. Thus, f is continuous at x. If f(x) = 0 or f(x) = 1, the proof is similar.

The function f is increasing. Indeed, let $x \leq y$ and $q \in \mathbb{Q}$. Since X - W(q) is increasing, the set $\{q : x \notin W(q)\}$ is contained in $\{q : y \notin W(q)\}$. Consequently, $f(x) \leq f(y)$.

All that remains to be shown is that $g(x) \leq f(x) \leq h(x)$ for each $x \in X$. Suppose, if possible, that f(x) > h(y). Then there exist $q, r \in \mathbb{Q}$ such that f(x) > r > q > h(x). Consequently, $x \notin W(r)$ and $x \in F(q)$. This contradicts (2). The proof of the inequality $g(x) \leq f(x)$ is similar. This concludes the proof of the theorem.

Theorem 3.5.1.3 Let X be a normally ordered space. Let F be a closed subset of X, and let f be a bounded real-valued function which is continuous and increasing on F. We shall denote by $A(\xi)$ the set of all points $x \in F$ such that $f(x) \leq \xi$, and by $B(\xi)$ the set of all points such that $f(x) \geq \xi$, where ξ is a real number.

The function f may be extended to X in such a way as to become a continuous bounded real-valued function of X if and only if $\xi < \xi'$ implies $cl_d A(\xi) \cap cl_i B(\xi') = \emptyset$.

The proof of this theorem is omitted, since it is too long (see [233], p. 36).

Definition 3.5.1.6 A topological ordered space X is perfectly normally ordered, if for any two closed disjoint subsets A and B of X, where A is decreasing and B is increasing, there exists a continuous and increasing function $f: X \to [0,1]$, which satisfies the following conditions:

 $(f(x) = 0 \Leftrightarrow x \in A)$ and $(f(x) = 1 \Leftrightarrow x \in B)$.

According to Theorem 3.5.1.1 it is obvious that every perfectly normally ordered space is normally ordered. Also, by Definition 3.5.1.6, the following statement is obvious.

Proposition 3.5.1.3 Let A be a closed subset of a perfectly normally ordered space X. If A is a decreasing (increasing) set, then there exists a continuous increasing function $f: X \to [0,1]$ such that f(x) = 0 (f(x) = 1) if and only if $x \in A$. **Definition 3.5.1.7** Let X be topological space equipped with order. A subset $A \subset X$ is a decreasing (increasing) G_{δ} -set, if it is a countable intersection of decreasing (increasing) open sets.

Lemma 3.5.1.3 Let A be a closed, decreasing G_{δ} -set of a normally ordered space X. If B is a closed, increasing set disjoint with A, then there exists a continuous, increasing function $f: X \to [0, 1]$ such that

$$(f(x) = 0 \Leftrightarrow x \in A)$$
 and $(x \in B \Rightarrow f(x) = 1)$.

Proof: Since A is a decreasing G_{δ} -set, it is a countable intersection of open decreasing sets: $A = G_1 \cap G_2 \cap \ldots$ Without loss of generality, we can assume that $G_n \cap B = \emptyset$, because on the contrary, we can exchange the set G_n with the set $G_n \cap (X - B)$, which is an open, decreasing set. Since A is a decreasing set, by Theorem 3.5.1.1, for each $n \in \mathbb{N}$ there exists an increasing, continuous function $f_n : X \to [0, 1]$ such that

$$f_n(x) = \begin{cases} 0, & \text{for } x \in A, \\ 1, & \text{for } x \in X - G_n. \end{cases}$$

Let us define a function f with

$$f(x) = \sum_{n=1}^{+\infty} \frac{1}{2^n} f_n(x).$$

Since $\sum_n f_n(x)/2^n$ is a uniformly convergent series of continuous functions, the function f(x) is continuous. But $0 \leq f_n \leq 1$ which implies that $f: X \to [0, 1]$. For each $n \in \mathbb{N}$, $f_n(x) \leq f_n(y)$ whenever $x \leq y$, and so f is an increasing function.

Let us suppose that $x \in B$. Since $B \subset X - G_n$ for each $n \in \mathbb{N}$, there follows that f(x) = 1. Thus the implication $x \in B \Rightarrow f(x) = 1$ holds.

Let us prove the equivalence $x \in A \Leftrightarrow f(x) = 0$. Since $A \subset G_n$, $n \in \mathbb{N}$, then $f_n(x) = 0$ for each $x \in A$. Therefore, f(x) = 0. If $x \in X - A$, then there exists a number $n \in \mathbb{N}$ for which $x \in X - G_n$ holds. But then $f_n(x) = 1$. Thus f(x) > 0.

Theorem 3.5.1.4 A topological ordered space X is perfectly normally ordered if and only if it is normally ordered, and if every decreasing closed set is a decreasing G_{δ} -set, and every closed increasing set is an increasing G_{δ} -set. **Proof:** Let X be a perfectly normally ordered space. Then it is normally ordered. If $A \subset X$ is a closed decreasing set, then, by Proposition 3.5.1.3, there exists an increasing continuous function $f : X \to [0,1]$ such that f(x) = 0 if and only if $x \in A$. Let us define the sets $G_n = \{x : f(x) < 1/n\}, n \in \mathbb{N}$, which are open and decreasing. Since

$$\cap G_n = \cap \{x : f(x) < 1/n\} = \{x : f(x) < 1/n \text{ for each } n\} = \{x : f(x) = 0\} = A,$$

every closed decreasing set is a decreasing G_{δ} -set.

Conversely, let us suppose that every closed decreasing (increasing) set of a normally ordered space X is a decreasing (increasing) G_{δ} -set. Let A and B be any two closed, disjoint sets, where A is decreasing, and B is increasing. Then, by Lemma 3.5.1.3, there exists an increasing continuous function $g: X \to [0, 1]$ such that

$$(x \in A \Leftrightarrow g(x) = 0)$$
 and $(x \in B \Rightarrow g(x) = 1)$.

Let us define sets

$$G = \{x: g(x) < 1/2\}, \ F = \{x: g(x) = 1/2\}, \ H = \{x: g(x) > 1/2\}.$$

Then $G \cup F$ and $H \cup F$ are closed sets, where the first set is decreasing, and the second one is increasing. Since $(G \cup F) \cap B = \emptyset$, by the lemma which is dual to Lemma 3.5.1.3, there exists an increasing continuous function $h: X \to [1/2, 1]$ such that

$$(x \in B \Leftrightarrow h(x) = 1)$$
 and $(x \in G \cup F \Rightarrow h(x) = 1/2)$.

Let us define a function

$$f(x) = \begin{cases} g(x), & \text{for } x \in G \cup F, \\ h(x), & \text{for } x \in H \cup F. \end{cases}$$

Since $(G \cup F) \cap (H \cup F) = F$ and g(x) = h(x) = 1/2 for $x \in F$, the function f(x) is continuous. That $f: X \to [0,1]$ is true, follows from the fact that $(G \cup F) \cup (H \cup F) = X$. The functions g and h are increasing, g(x) = h(x) for each $x \in F$ and therefore the function f is increasing. It is easy to see that $f(x) = 0 \Leftrightarrow x \in A$ and $f(x) = 1 \Leftrightarrow x \in B$.

Definition 3.5.1.8 Let (X, d) be a metric space and let \leq be an order on X. The distance function d is **compatible** with the order \leq if the following conditions are satisfied:

(a) if A is a closed increasing subset of X and $x \leq y$, then $d(x, A) \geq d(y, A)$;

(b) if B is a closed decreasing subset of X and $x \leq y$, then $d(x, B) \leq d(y, B)$.

Theorem 3.5.1.5 Let (X, τ_d, \leq) be a topological ordered space. If the distance function d is compatible with the order, then the space X is perfectly normally ordered.

Proof: First will we prove that X is normally ordered. Indeed, if F and H are closed disjoint subsets of X, where F is decreasing, and H is increasing, then the function

$$f(x) = \frac{d(x, F)}{d(x, F) + d(x, H)}$$

is continuous and increasing on X. Moreover, $0 \leq f(x) \leq 1$ for $x \in X$, f(x) = 0 for $x \in F$ and f(x) = 1 for $x \in H$. It is clear that the sets $U = \{x : f(x) < 1/3\}$ and $V = \{x : f(x) > 2/3\}$ are open, U is decreasing, V is increasing, $F \subset U$, $H \subset V$ and $U \cap V = \emptyset$.

Let F be any closed, decreasing subset of X. Then g(x) = d(x, F) is an increasing, continuous function because the metric d is compatible with the order \leq . For every $n \in \mathbb{N}$, let us define the sets $G_n = \{x : g(x) < 1/n\}$. Since g is an increasing, continuous function, G_n is open and decreasing set. Moreover,

$$\bigcap_{n=1}^{+\infty} G_n = \bigcap_{n=1}^{+\infty} \{x : g(x) < 1/n\} = \{x : g(x) < 1/n \text{ for each } n \in \mathbb{N}\} = \{x : g(x) = 0\} = \{x : d(x, F) = 0\} = F.$$

This proves that each closed, decreasing set is a decreasing G_{δ} -set. In analogous manner it can be proved that each closed, increasing set is an increasing G_{δ} -set. Thus X is perfectly normally ordered by Theorem 3.5.1.4.

Proposition 3.5.1.4 Let (X, τ, \leq) be a topological ordered space and let K be a compact subset of X. Then d(K) and i(K) are closed so that $d(K) = cl_d(K)$ and $i(K) = cl_i(K)$.

Proof: To show that d(K) is closed, let $p \in X - d(K)$. Then $p \leq x$ is false for each $x \in K$. Since the graph $G(\leq)$ is closed, for each $x \in K$ there exists an increasing neighborhood U_x of p and a decreasing neighborhood V_x of xsuch that $U_x \cap V_x = \emptyset$. By the compactness of the set K there exist points $x_1, x_2, \ldots, x_n \in K$ such that $K \subset \bigcup_{i=1}^n V_{x_i}$. Let $U = \bigcap_{i=1}^n U_{x_i}$. Clearly, U is an increasing neighborhood of the point p. Furthermore,

$$U \cap K \quad \subset U \cap (V_{x_1} \cup \ldots \cup V_{x_n}) = (U \cap V_{x_1}) \cup \ldots \cup (U \cap V_{x_n}) \subset (U_{x_1} \cap V_{x_1}) \cup \ldots \cup (U_{x_n} \cap V_{x_n}) = \emptyset,$$

that is, $K \subset X - U$. Noting that the set X - U is decreasing, we obtain $d(K) \subset X - U$, that is, $U \cap d(K) = \emptyset$. Thus every point of the complement of d(K) possesses a neighborhood which is disjoint from d(K), that is, d(K) is a closed set. A similar proof establishes that i(K) is a closed set.

Proposition 3.5.1.5 Let (X, τ, \leq) be a compact topological ordered space. If F is a decreasing (increasing) subset of X, and G is an open set containing F, then there exists a decreasing (increasing) open set H such that $F \subset H \subset G$.

Proof: We have established this result in the case when F is decreasing; the proof in the case when F is increasing is similar. Let us set H = X - i(X - G). By the preceding proposition, H is open, and since i(X - G) is increasing, H is decreasing. Clearly $H \subset G$. Let us suppose that F is not a subset of H. Then there exists $x \in F \cap i(X - G)$, so that for some $y \in X - G$, $y \leq x$. Since F is decreasing, $y \in F$, which is a contradiction.

Theorem 3.5.1.6 Every compact topological ordered space is a normally ordered space.

Proof: Let (X, τ, \leq) be a compact topological ordered space. Let E be closed and decreasing, let F be open and decreasing and let $E \subset F$. Since (X, τ) is a normal space, there exists an open set H such that $E \subset H \subset \overline{H} \subset F$. By the previous proposition there exists a decreasing open set G such that $E \subset G \subset H$. By Proposition 3.5.1.4 $d(\overline{G}) = cl_d(\overline{G})$. Thus $E \subset G \subset cl_d(G) \subset cl_d(\overline{G}) = d(\overline{G}) \subset d(\overline{H}) \subset F$. But then X is a normally ordered space by Lemma 3.5.1.1.

Corollary 3.5.1.3 Let (X, τ, \leq) be a compact topological space and $S = \{G : G \in \tau \text{ and } G \text{ is increasing or decreasing}\}$. Then S is a subbase for τ .

Proof: Let us prove that S is a separating family in the following sense: if x and y are two distinct points, then there exist sets $U, V \in S$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Indeed, let x and y be any two distinct points of X. Without loss of generality we can assume that $y \leq x$ is false. Then $d(\{x\})$

and $i(\{y\})$ are disjoint sets. By Proposition 3.5.1.4, $d(\{x\})$ and $i(\{y\})$ are closed. Since (X, τ, \leq) is normally ordered, there exists a decreasing open set G containing $d(\{x\})$ and an increasing open set H containing $i(\{y\})$ such that $G \cap H = \emptyset$. Since, in a compact space, any separating family consisting of open subsets is an open subbase, the corollary is proved.

Corollary 3.5.1.4 Let F be a compact subset of a normally ordered space X. Then every continuous and bounded real-valued function increasing on F has a continuous, bounded and increasing extension on X.

Definition 3.5.1.9 A completely regularly quasi-ordered space is a topological space X equipped with a quasi-order which satisfies the following two conditions:

(a) if $a \in X$ and G is a neighborhood of a, then there exist two continuous real-valued functions f and g on X, where f is increasing and g is decreasing, such that

$$\begin{array}{l} 0 \leqslant f \leqslant 1 \,, \ 0 \leqslant g \leqslant 1 \,, \\ f(a) = 1 \,, \ g(a) = 1 \,, \\ \inf\{f(x) \,, \ g(x)\} = 0 \ \ if \ x \in X - G \,; \end{array}$$

(b) if $a, b \in X$ and $a \leq b$, there exists a continuous increasing, real-valued function f on X such that f(a) > f(b).

Proposition 3.5.1.6 Every completely regularly quasi-ordered space is a uniformizable space. The space is topological quasi-ordered, and the set formed of the open decreasing and the open increasing subsets is an open subbase.

Proof: Let X be the space under consideration. We will denote by $a \in X$ a point of this space and by G a neighborhood of this point. Let us determine the functions f and g in accordance with condition (a) of the preceding definition. Then we can introduce a function h defined by the equation

$$h(x) = \sup\{0, f(x) + g(x) - 1\}, x \in X.$$

Clearly, the function h is continuous and $h \ge 0$. Furthermore, $f(x) + g(x) - 1 \le 1 + 1 - 1 = 1$ whence $h \le 1$, and f(a) + g(a) - 1 = 1 + 1 - 1 = 1, that is, h(a) = 1. Finally, if $x \in X - G$, the last part of condition (a) shows that either f(x) = 0 or g(x) = 0. Let us consider the first case (the second is analogous). Then $f(x) + g(x) - 1 = g(x) - 1 \le 0$ whence h(x) = 0. We can, therefore, conclude that X is a uniformizable space.

We now consider two points $a, b \in X$ such that $a \leq b$. We refer to condition (b) and determine the function f indicated there. Let r be a real number such that f(a) > r > f(b). Let us define two sets V and W by the following equations: $V = \{x \in X : f(x) > r\}, W = \{x \in X : f(x) < r\}$. Making use of the fact that f is continuous and increasing, we verify that V is an increasing neighborhood of a and W a decreasing neighborhood of b. Moreover, V and W are obviously disjoint. Applying Proposition 3.5.1.1 we see that X is a topological quasi-ordered space.

Finally, let a point $a \in X$ and a neighborhood V of a be given. We once more apply the preceding condition (a) and let f and g be the corresponding functions. We will define the sets $W_1 = \{x \in X : f(x) > 0\}$ and $W_2 = \{x \in$ $X : g(x) > 0\}$. Clearly, W_1 is an open increasing set and W_2 is an open decreasing set. Furthermore, condition (a) shows that $a \in W_1 \cap W_2 \subset V$; this justifies the last assertion made in the proposition.

Definition 3.5.1.10 A completely regularly ordered space is a completely regularly quasi-ordered space which has one of the following equivalent properties:

- (a) the Hausdorff axiom is satisfied;
- (b) the quasi-order of the space is an order.

The condition (a) and (b) are equivalent. Indeed, let us suppose that (a) is satisfied. We will then consider two distinct points $a, b \in X$. By (a) we can determine a neighborhood V of a which does not contain b. Let us next apply condition (a) of the definition of a completely regular quasi-ordered space and let f and g be the corresponding functions. The last part of (a) shows that either f(b) = 0 or g(b) = 0; in the first case, we see that $a \leq b$ is false since $a \leq b$ would imply $1 = f(a) \leq f(b) = 0$ which is impossible; and in the second one that $a \geq b$ is false for a similar reason. Consequently, (b) is satisfied. If we, consequently, assume that the space has property (b), it suffices to apply Proposition 3.5.1.6 proved above and to refer to Corollary 3.5.1.2 to obtain (a).

In view of the definitions of a topological subspace and quasi-ordered subspace, we easily obtain the following results.

Proposition 3.5.1.7 Every topological and ordered subspace of a completely regularly ordered space is a space with the same properties.

Proposition 3.5.1.8 Every topological and ordered subspace of a compact topological ordered space is a completely regularly ordered space.

Prof: Let X be a given space. By virtue of Proposition 3.5.1.7, it suffices to establish that this compact topological ordered space itself is a completely regular ordered space. For this purpose, let us consider a point $a \in X$ and a neighborhood V of a. On the basis of Corollary 3.5.1.3 and of an observation made in the course of the proof of Proposition 3.5.1.2, there exist two open subsets W_1 and W_2 of X, of which the first is decreasing and the second increasing, such that

$$(1) a \in W_1 \cap W_2 \subset V.$$

We note that d(a) is decreasing and closed by Corollary 3.5.1.1, which, because of $a \in W_1$, is disjoint from the increasing and closed subset $X - W_1$. We determine a continuous increasing real-valued function q' on X such that

$$0 \le g' \le 1$$
, $g'(x) = 0$ if $x \in d(a)$ and $g'(x) = 1$ if $x \in x - W_1$.

From the second condition it follows, in particular, that g'(a) = 0. We recall that such a function g' exists by virtue of Theorems 3.5.1.6 and 3.5.1.1. We now set g = 1 - g'. Clearly, then

(2)
$$0 \leqslant g \leqslant 1, \ g(a) = 1 \text{ and } g(x) = 0 \text{ if } x \in X - W_1,$$

and g is a continuous decreasing function. In a corresponding fashion, observing that i(a) is a closed increasing subset that is disjoint from the closed decreasing subset $X - W_2$, we can determine a continuous real-valued function f on X such that

(3)
$$1 \leq f \leq 1, \ f(a) = 1 \text{ and } f(x) = 0 \text{ if } x \in X - W_2.$$

These two functions f and g show that condition (a) of the definition of a completely regular quasi-ordered space is satisfied. Indeed, it suffices to take into account (2) and (3) and to note that (1) implies $X - V \subset (X - W_1) \cup (X - W_2)$, and from this, the third part of (a) obviously, follows.

We will now consider points $a, b \in X$ such that $a \notin b$. Then $i(a) \cap d(b) = \emptyset$. Applying, once more, the propositions and theorems mentioned above, we will obtain a continuous increasing real-valued function f such that f(x) = 0 if $x \in d(b)$ and f(x) = 1 if $x \in i(a)$. In particular, f(a) = 1 and f(b) = 0, that is, f(a) > f(b) as we desired. Finally, let us note that X is an ordered space. The theorem is proved.

Let (X, τ) be a topological space equipped with an order \leq . A subset Z of X is called a decreasing (increasing) zero set in (X, τ) when there is an increasing (decreasing) continuous function f from X into \mathbb{R} such that $Z = \{x \in X : f(x) \leq 0\}$. The set of all decreasing (increasing) zero sets in (X, τ) is denoted by $\mathcal{A}_0(\mathcal{B}_0)$.

Lemma 3.5.1.4 If (X, τ) be a topological space endowed with a partial order \leq . Then there exist topologies τ_1 and τ_2 on X such that

(a) $\tau_1, \tau_2 \leq \tau$; (b) $x \leq y$ implies $x \in \overline{y}^{\tau_1}$ and $y \in \overline{x}^{\tau_2}$;

(c) if τ'_1, τ'_2 is another pair of topologies fulfilling (a) and (b), then $\tau'_1 \leq \tau_1$ and $\tau'_2 \leq \tau_2$.

Proof: Let us define topologies τ_1 , τ_2 on X by G is $\tau_1(\tau_2)$ -open if G is τ open and increasing (decreasing) in X. Then $\tau_1 \leq \tau$ and $\tau_2 \leq \tau$. If $x \notin \overline{y}^{\tau_1}$,
then there exists a τ_1 -open neighborhood U of x such that $y \notin U$. Thus $x \leq y$ since otherwise $y \in U$. In an analogous manner $x \leq y$ implies $y \in \overline{x}^{\tau_2}$.

Let τ'_1, τ'_2 be another pair of topologies fulfilling the conditions (a) and (b) and let G be a τ'_1 -open set. Then if $y \in G$ and $x \ge y$ we have $y \in \overline{x}^{\tau'_1}$ and thus $x \in G$. It follows that G is an increasing set and hence $\tau'_1 \le \tau_1$. In a similar manner we have that $\tau'_2 \le \tau_2$.

Definition 3.5.1.11 If (X, τ) is a topological space equipped with an order \leq . If τ_1, τ_2 are topologies on X, then (τ_1, τ_2) is called an **ordered defining pair** when $\tau_1, \tau_2 \leq \tau$ and the following three statement are equivalent:

(a)
$$x \in \overline{y}^{\tau_1}$$
, (b) $x \leq y$, (c) $y \in \overline{x}^{\tau_2}$.

If (X, τ) is a topological space with an order on X, then $\mathcal{A}_0(\mathcal{B}_0)$ is a base for the closed sets of a topology $\tau_{\mathcal{A}_0}(\tau_{\mathcal{B}_0})$ on X. Theorem 3.5.1.7 shows that a completely regularly ordered space can be characterized by these topologies. First we have the following result.

Proposition 3.5.1.9 Let (X, τ) be a topological space endowed with an order. If a set A(B) belongs to $A_0(\mathcal{B}_0)$, then there exists a continuous increasing (decreasing) function g(h) from X into \mathbb{R} such that $g \ge 0$ $(h \ge 0)$ and $A = \{x \in X : g(x) = 0\} (B = \{x \in X : h(x) = 0\}).$

Theorem 3.5.1.7 A topological space (X, τ) endowed with an order is completely regularly ordered if and only if (τ_{A_0}, τ_{B_0}) is an ordered defining pair and $\tau_{A_0} \vee \tau_{B_0} = \tau$.

Proof: Let (X, τ, \leq) be a completely regularly ordered space. Then if $x \leq y$ clearly there follows that $x \in \overline{y}^{\tau_{\mathcal{A}_0}}$ and $y \in \overline{x}^{\tau_{\mathcal{B}_0}}$ since each member of $\mathcal{A}_0(\mathcal{B}_0)$ is decreasing (increasing). If $x \leq y$, there exists an increasing continuous function f from X into \mathbb{R} such that f(y) < f(x). We can assume without loss of generality that f(y) = 0 and f(x) = 1. Then, if $A = \{u \in X : f(u) \leq 0\}$,

 $A \in \mathcal{A}_0, y \in A, x \notin A$ and thus $x \notin \overline{y}^{\tau_{\mathcal{A}_0}}$. By a dual argument $x \leq y$ if and only if $y \in \overline{x}^{\tau_{\mathcal{B}_0}}$.

If G is τ -open and $x \in G$, then there exist two continuous functions f, g from X into \mathbb{R} , where f is increasing, g is decreasing and $0 \leq f, g \leq 1$, $f(x) = 1 = g(x), f(y) \wedge g(y) = 0$ if $y \in X - G$. Let us put that $A = \{u \in X :$ $f(u) \leq 0\}$ and $B = \{u \in X : g(u) \leq 0\}$; then $x \in (X - A) \cap (X - B) \subseteq G$ and thus $\tau \leq \tau_{\mathcal{A}_0} \vee \tau_{\mathcal{B}_0}$. Since clearly $\tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0} \leq \tau, \tau = \tau_{\mathcal{A}_0} \vee \tau_{\mathcal{B}_0}$ holds.

On the other hand let $(\tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0})$ be an order defining pair and $\tau_{\mathcal{A}_0} \vee \tau_{\mathcal{B}_0} = \tau$. Then, if $x \leq y, x \notin \overline{y}^{\tau_{\mathcal{A}_0}}$ and thus there exists an $A \in \mathcal{A}_0$ such that $y \in A$ and $x \notin A$. Then, since $A \in \mathcal{A}_0$, there exists an increasing continuous function from X into \mathbb{R} such that f(x) > f(y).

If $x \in X$ and Y is a τ -neighborhood of x, then, since $\tau_{\mathcal{A}_0} \vee \tau_{\mathcal{B}_0} = \tau$, there exist sets A, B in $\mathcal{A}_0, \mathcal{B}_0$ respectively such that $x \in (X - A) \cap (X - B) \subseteq Y$. Now by Proposition 3.5.1.9 there is an increasing continuous function f from X into \mathbb{R} and a decreasing continuous function g from X into \mathbb{R} such that $f, g \ge 0, A = \{u \in X : f(u) = 0\}$ and $B = \{u \in X : g(u) = 0\}$. Now let us put that $h = f/f(x) \wedge 1, k = g/g(x) \wedge 1$; then h(k) is an increasing (decreasing) continuous function from X into \mathbb{R} such that $0 \le h, k \le 1$, h(x) = 1 = k(x) and $u \in X - Y$ implies $h(u) \wedge k(u) = 0$.

In theorem 3.5.1.8 we will show that a completely regularly ordered space has a normally ordered subbase but first we state the following simple result which is similar to (1.15) of [123].

Lemma 3.5.1.5 If (X, τ) is a topological space equipped with an order and if A, B belongs to \mathcal{A}_0 and \mathcal{B}_0 respectively where $A \cap B = \emptyset$, then there exists an increasing continuous function f from X into \mathbb{R} such that f(A) = 0 and f(B) = 1.

Corollary 3.5.1.5 If (X, τ) is an topological space equipped with order and if A, B are as in above lemma, then there exist sets A', B' in $\mathcal{A}_0, \mathcal{B}_0$ respectively such that $A \subseteq A', B \subseteq B', A' \cup B' = X, A \cap B' = \emptyset = A' \cap B$.

Definition 3.5.1.12 Let (X, τ) be a topological space equipped with order and let $\mathcal{A}(\mathcal{B})$ be a family of decreasing (increasing) τ -closed sets of X. Thus $\mathcal{A} \cap \mathcal{B}$ is called a **normally ordered subbase** for (X, τ) where

(a) $\mathcal{A}(\mathcal{B})$ is a base for the closed sets of topologies $\tau_{\mathcal{A}}(\tau_{\mathcal{B}})$ on X such that $\tau_{\mathcal{A}} \vee \tau_{\mathcal{B}} = \tau$ and $(\tau_{\mathcal{A}}, \tau_{\mathcal{B}})$ is an order defining pair;

(b) given any $\tau_{\mathcal{A}}(\tau_{\mathcal{B}})$ -closed set $F \subseteq X$ and $x \in X - F$, there is a set B(A) in $\mathcal{B}(\mathcal{A})$ such that $x \in B(A)$ and $B \cap F = \emptyset$ $(A \cap F = \emptyset)$;

(c) given A, B in A, \mathcal{B} respectively with $A \cap B = \emptyset$, there are sets A', B'in A, \mathcal{B} respectively such that $A \subseteq A', B \subseteq B', A \cap B' = \emptyset = A' \cap B$ and $A' \cup B' = X$.

Theorem 3.5.1.8 If (X, τ, \leq) is a completely regularly ordered space, then (X, τ, \leq) has a normally ordered subbase.

Proof: Let us show that $\mathcal{A}_0 \cup \mathcal{B}_0$ is a normally ordered subbase for (X, τ, \leq) . By Theorem 3.5.1.7 and corollary of Lemma 3.5.1.5 we need only show that it satisfies condition (b) of the definition of a normally ordered subbase.

If F is $\tau_{\mathcal{A}_0}$ -closed and $x \in X - F$, then there exists an $A \in \mathcal{A}_0$ such that $x \notin A$ and $F \subseteq A$. Since $A \in \mathcal{A}_0$, there exists an increasing continuous function f from X into \mathbb{R} such that $A = \{y \in X : f(y) \leq 0\}$. Now let us put $B = \{y \in X : f(y) \geq f(x)\}$; then $x \in B$, $B \in \mathcal{B}_0$ and $B \cap F = \emptyset$, since $y \in B \cap F$ implies $f(x) \leq f(y) \leq 0$ contrary to $x \notin A$.

A dual argument suffices if F is $\tau_{\mathcal{B}_0}$ -closed.

3.5.2 Ordered compactification of ordered topological spaces

Throughout this subsection we shall assume that (X, τ, \leq) is a completely regularly ordered space.

If $\mathcal{D}(\mathcal{I})$ is a non-empty subset of $\mathcal{A}_0(\mathcal{B}_0)$ then the ordered pair $(\mathcal{D}, \mathcal{I})$ is called an $[\mathcal{A}_0, \mathcal{B}_0]$ family where $D \in \mathcal{D}, I \in \mathcal{I}$ implies $D \cap I \neq \emptyset$, for all $D, I \in \mathcal{D}, \mathcal{I}$ respectively. If we write $(\mathcal{D}, \mathcal{I}) \subseteq (\mathcal{D}', \mathcal{I}')$ if and only if $\mathcal{D} \subseteq \mathcal{D}'$ and $\mathcal{I} \subseteq \mathcal{I}'$, where $(\mathcal{D}, \mathcal{I})$ and $(\mathcal{D}', \mathcal{I}')$ are $[\mathcal{A}_0, \mathcal{B}_0]$ families, then, clearly, this is a partial ordering on the collection of $[\mathcal{A}_0, \mathcal{B}_0]$ families.

Proposition 3.5.2.1 If $(\mathcal{D}, \mathcal{I})$ is an $[\mathcal{A}_0, \mathcal{B}_0]$ family, then there exists a maximal $[\mathcal{A}_0, \mathcal{B}_0]$ family $(\mathcal{D}', \mathcal{I}')$ such that $(\mathcal{D}, \mathcal{I}) \subseteq (\mathcal{D}', \mathcal{I}')$.

Proof: This follows from a direct application of Zorn's Lemma.

Lemma 3.5.2.6 If $(\mathcal{D}, \mathcal{I})$ is a maximal $[\mathcal{A}_0, \mathcal{B}_0]$ family and A(B) is a member of $\mathcal{A}_0(\mathcal{B}_0)$ such that $A \cap I \neq \emptyset$ for all $I \in \mathcal{I}(B \cap D \neq \emptyset$ for all $D \in \mathcal{D})$, then $A(B) \in \mathcal{D}(\mathcal{I})$.

Proof: Let $A \in \mathcal{A}_0$ be such that $A \cap I \neq \emptyset$ for all $I \in \mathcal{I}$, and let us put $\mathcal{D}' = \mathcal{D} \cup \{A\}$. Then $(\mathcal{D}', \mathcal{I})$ is an $[\mathcal{A}_0, \mathcal{B}_0]$ family and thus, by the maximality of $(\mathcal{D}, \mathcal{I}), A \in \mathcal{D}$. A similar argument suffices for \mathcal{B}_0 .

If $x \in X$, then we will denote the family of members of $\mathcal{A}_0(\mathcal{B}_0)$ containing x by $\mathcal{A}_0^x(\mathcal{B}_0^x)$.

Lemma 3.5.2.7 $(\mathcal{A}_0^x, \mathcal{B}_0^x)$ is a maximal $[\mathcal{A}_0, \mathcal{B}_0]$ family for each $x \in X$.

Proof: It is easily seen that $(\mathcal{A}_0^x, \mathcal{B}_0^x)$ is an $[\mathcal{A}_0, \mathcal{B}_0]$ family and thus we need only show the maximality.

If $(\mathcal{A}_0^x, \mathcal{B}_0^x)$ is not a maximal $[\mathcal{A}_0, \mathcal{B}_0]$ family, then there exists an $[\mathcal{A}_0, \mathcal{B}_0]$ family $(\mathcal{D}, \mathcal{I})$ such that $(\mathcal{A}_0^x, \mathcal{B}_0^x) \subseteq (\mathcal{D}, \mathcal{I})$.

If $\mathcal{A}_0^x \neq \mathcal{D}$, then there exists a $D \in \mathcal{D}$ such that $D \notin \mathcal{A}_0^x$, thus $x \notin D$. By postulate (b) of the definition of a normally ordered subbase there is a $B \in \mathcal{B}_0$ such that $x \in B$ and $B \cap D = \emptyset$. Then $B \in \mathcal{B}_0^x$ and thus $B \in \mathcal{I}$, $D \in \mathcal{D}, B \cap D = \emptyset$ contrary to $(\mathcal{D}, \mathcal{I})$ being an $[\mathcal{A}_0, \mathcal{B}_0]$ family.

A similar argument is applied if $\mathcal{B}_0^x \neq \mathcal{I}$.

Proposition 3.5.2.2 If $x \in X$, then $\cap \mathcal{A}_0^x = [\leftarrow, x]$ and $\cap \mathcal{B}_0^x = [x, \rightarrow]$.

Proof: This follows immediately from postulate (a) of the definition of a normally ordered subbase.

Corollary 3.5.2.1 If $x \in X$, then $\cap (\mathcal{A}_0^x \cup \mathcal{B}_0^x) = \{x\}$.

Let $\{(\mathcal{A}_0^p, \mathcal{B}_0^p)\}, p \in P$, denote the collection of maximal $[\mathcal{A}_0, \mathcal{B}_0]$ families. It is clear from Lemma 3.5.2.7 and the Corollary of Proposition 3.5.2.2 that there is a one-to-one correspondence between the maximal $[\mathcal{A}_0, \mathcal{B}]$ families of the form $(\mathcal{A}_0^x, \mathcal{B}_0^x), x \in X$ and the points of X. We can thus assume without loss of generality that $X \subseteq P$, and we will put $C_{\mathcal{A}_0}^{\mathcal{B}_0}(X) = P$. We shall now define a partial order and a topology u on $C_{\mathcal{A}_0}^{\mathcal{B}_0}(X)$ such that $(C_{\mathcal{A}_0}^{\mathcal{B}_0}(X), u)$ is a compact ordered topological space and (X, τ) is a topological ordered subspace of $(C_{\mathcal{A}_0}^{\mathcal{B}_0}(X), u)$.

Lemma 3.5.2.8 If $p, q \in C_{\mathcal{A}_0}^{\mathcal{B}_0}(X)$, then $\mathcal{A}_0^q \subseteq \mathcal{A}_0^p$ if and only if $\mathcal{B}_0^p \subseteq \mathcal{B}_0^q$.

Proof: Let $\mathcal{A}_0^q \subseteq \mathcal{A}_0^p$ and let $B \in \mathcal{B}_0^p$. Since $(\mathcal{A}_0^q, \mathcal{B}_0^q \cup \{B\})$ is an $[\mathcal{A}_0, \mathcal{B}_0]$ family, then by the maximality of $(\mathcal{A}_0^q, \mathcal{B}_0^q)$, there follows that $B \in \mathcal{B}_0^q$. In a similar manner $\mathcal{B}_0^p \subseteq \mathcal{B}_0^q$ implies $\mathcal{A}_0^q \subseteq \mathcal{A}_0^p$.

Let us now define a relation R on $C_{\mathcal{A}_0}^{\mathcal{B}_0}(X)$ by

p R q if and only if $\mathcal{A}_0^q \subseteq \mathcal{A}_0^p$.

Theorem 3.5.2.1 The relation R is a partial order on $C_{\mathcal{A}_0}^{\mathcal{B}_0}(X)$ and further, for $x, y \in X$, $x \leq y$ if and only if x R y.

Proof: It is clear that p R p for all $p \in C^{\mathcal{B}_0}_{\mathcal{A}_0}(X)$. Now if p R q and q R p, then, by the definition of R and Lemma 3.5.2.8, $(\mathcal{A}^p_0, \mathcal{B}^p_0) = (\mathcal{A}^q_0, \mathcal{B}^q_0)$ and hence p = q. It is easily seen that R is transitive and thus it is a partial order on $C^{\mathcal{B}_0}_{\mathcal{A}_0}(X)$.

Now let $x \leq y, x, y \in X$ and let $A \in \mathcal{A}_0^y$; then $y \in A$ and since A is decreasing (in X), $x \in A$ and thus $A \in \mathcal{A}_0^x$. It follows that x R y.

On the other hand, if x R y, $x, y \in X$, then $\mathcal{A}_0^y \subseteq \mathcal{A}_0^x$ and thus, by Proposition 3.5.2.2, $(\leftarrow, x] = \cap \mathcal{A}_0^x \subseteq \cap \mathcal{A}_0^y = (\leftarrow, y]$; hence $x \leq y$.

In view of Theorem 3.5.2.1 we shall denote the partial order on $C_{\mathcal{A}_0}^{\mathcal{B}_0}(X)$ by \leq .

Theorem 3.5.2.2 $C_{\mathcal{A}_0}^{\mathcal{B}_0}(X)$ is a distributive lattice.

Proof: Let $p, q \in C^{\mathcal{B}_0}_{\mathcal{A}_0}(X)$ and let $(\mathcal{A}^p_0, \mathcal{B}^p_0)$, $(\mathcal{A}^q_0, \mathcal{B}^q_0)$ be the corresponding maximal $[\mathcal{A}_0, \mathcal{B}_0]$ families. Now let us put $\mathcal{D} = \mathcal{A}^p_0 \cap \mathcal{A}^q_0$, $\mathcal{I} = \mathcal{B}^p_0 \cup \mathcal{B}^q_0$; then \mathcal{I} is not empty. Further, \mathcal{D} is not empty for, if $A^p \in \mathcal{A}^p_0$, $A^q \in \mathcal{A}^q_0$, then we will show that $A^p \cup A^q \in \mathcal{D}$. If $A^p \cup A^q$ is not in \mathcal{A}^p_0 , then by Lemma 3.5.2.8 there exists a set $B^p \in \mathcal{B}^p_0$ such that $B^p \cap (A^p \cup A^q) = \emptyset$; thus $B^p \cap A^p = \emptyset$ contrary to $(\mathcal{A}^p_0, \mathcal{B}^p_0)$ being an $[\mathcal{A}_0, \mathcal{B}_0]$ family. A similar argument will suffice if $A^p \cup A^q$ is not in \mathcal{A}^q_0 .

Now, if $(\mathcal{D}, \mathcal{I})$ is not a maximal $[\mathcal{A}_0, \mathcal{B}_0]$ family, then there exists an $[\mathcal{A}_0, \mathcal{B}_0]$ family $(\mathcal{D}', \mathcal{I}')$ such that $(\mathcal{D}, \mathcal{I}) \subset (\mathcal{D}', \mathcal{I}')$.

If $\mathcal{D} \subset \mathcal{D}'$, then there exists an $A' \in \mathcal{D}'$ such that A' is not in \mathcal{D} and hence A' is not \mathcal{A}_0^p or A' is not in \mathcal{A}_0^q . If A' is not in \mathcal{A}_0^p , by Lemma 3.5.2.8, there exists a set $B^p \in \mathcal{B}_0^p$ such that $A' \cup B^p = \emptyset$ contrary to $(\mathcal{D}', \mathcal{I}')$ being an $[\mathcal{A}_0, \mathcal{B}_0]$ family. A similar argument suffices if A' is not in \mathcal{A}_0^q .

If $\mathcal{I} \subset \mathcal{I}'$ there is a B' in \mathcal{I}' such that B' is not in \mathcal{I} and thus B' is not in \mathcal{B}_0^p and B' is not in \mathcal{B}_0^q . Hence, by Lemma 3.5.2.8, there exist sets A^p, A^q in $\mathcal{A}_0^p, \mathcal{A}_0^q$ respectively such that $B' \cap A^p = \emptyset = B' \cap A^q$ and thus $B' \cap (A^p \cup A^q) = \emptyset$ contrary to $(\mathcal{D}', \mathcal{I}')$ being an $[\mathcal{A}_0, \mathcal{B}_0]$ family.

Since $(\mathcal{D}, \mathcal{I})$ is a maximal $[\mathcal{A}_0, \mathcal{B}_0]$ family there is an $r \in C^{\mathcal{B}_0}_{\mathcal{A}_0}(X)$ such that $(\mathcal{D}, \mathcal{I}) = (\mathcal{A}^r_0, \mathcal{B}^r_0)$. We will show now that $p \lor q = r$.

Since $\mathcal{A}_0^r \subseteq \mathcal{A}_0^p$, $\mathcal{A}_0^r \subseteq \mathcal{A}_0^q$, $p, q \leq r$ holds. Now let $p, q \leq s$ where s belongs to $C_{\mathcal{A}_0}^{\mathcal{B}_0}(X)$, then $\mathcal{A}_0^s \subseteq \mathcal{A}_0^p$ and $\mathcal{A}_0^s \subseteq \mathcal{A}_0^q$; thus $\mathcal{A}_0^s \subseteq \mathcal{A}_0^r$, hence $r \leq s$. It follows that $p \lor q = r$.

In a similar manner we can show that $(\mathcal{A}_0^p \cup \mathcal{A}_0^q, \mathcal{B}_0^p \cap \mathcal{B}_0^q)$ is a maximal $[\mathcal{A}_0, \mathcal{B}_0]$ family. $(\mathcal{A}_0^t, \mathcal{B}_0^t)$, say, there is t in $C_{\mathcal{A}_0}^{\mathcal{B}_0}(X)$ and that $p \wedge q = t$.

Finally the lattice is distributive since for $p, q, r \in C^{\mathcal{B}_0}_{\mathcal{A}_0}(X)$ the maximal $[\mathcal{A}_0, \mathcal{B}_0]$ family corresponding to $p \wedge (q \vee r)$ is $(\mathcal{A}_0^p \cup (\mathcal{A}_0^q \cap \mathcal{A}_0^r), \mathcal{B}_0^p \cap (\mathcal{B}_0^q \cup \mathcal{B}_0^r))$

and this is clearly equal to $((\mathcal{A}_0^p \cup \mathcal{A}_0^q) \cap (\mathcal{A}_0^p \cup \mathcal{A}_0^r), (\mathcal{B}_0^p \cap \mathcal{B}_0^q) \cup (\mathcal{B}_0^p \cap \mathcal{B}_0^r))$ which is the maximal $[\mathcal{A}_0, \mathcal{B}_0]$ family corresponding to $(p \land q) \lor (p \land r)$.

Now for each A(B) in $\mathcal{A}_0(\mathcal{B}_0)$ respectively we define

$$\phi(A) = \{ p \in C^{\mathcal{B}_0}_{\mathcal{A}_0}(X) : A \in \mathcal{A}^p_0 \} (\psi(B) = \{ p \in C^{\mathcal{B}_0}_{\mathcal{A}_0}(X) : B \in \mathcal{B}^p_0 \}).$$

Proposition 3.5.2.3 If $A(B) \in \mathcal{A}_0(\mathcal{B}_0)$, then $\phi(A)(\psi(B))$ is an ideal (dual ideal) in $C^{\mathcal{B}_0}_{\mathcal{A}_0}(X)$.

Proof: This is clear from the definition of \lor and \land given in the proof of the above theorem.

Lemma 3.5.2.9 If A, B belong to \mathcal{A}_0 , \mathcal{B}_0 respectively and $A \cup B = X$, then $\phi(A) \cup \psi(B) = C^{\mathcal{B}_0}_{\mathcal{A}_0}(X)$.

Proof: Let $p \in C^{\mathcal{B}_0}_{\mathcal{A}_0}(X)$ and let us suppose that p is not in $\phi(A) \cup \psi(B)$. Then $A \notin \mathcal{A}^p_0$ and $B \notin \mathcal{B}^p_0$. Thus, by Lemma 3.5.5.19, there are sets A^p , B^p in \mathcal{A}^p_0 , \mathcal{B}^p_0 respectively such that $A \cap B^p = \emptyset = B \cap A^p$. Then $A^p \cap B^p = (A^p \cap B^p) \cap X = (A^p \cap B^p) \cap (A \cup B) = \emptyset$ contrary to $(\mathcal{A}^p_0, \mathcal{B}^p_0)$ being a maximal $[\mathcal{A}_0, \mathcal{B}_0]$ family.

Lemma 3.5.2.10 The family $\{\phi(A), \psi(B)\}$, all $A \in \mathcal{A}_0$, $B \in \mathcal{B}_0$ is a subbase for the closed sets of a topology u on $C^{\mathcal{B}_0}_{\mathcal{A}_0}(X)$.

Proof: We can assume that X contains more than one element; then there exists an $A \in \mathcal{A}_0$ or a $B \in \mathcal{B}_0$ such that $A \neq X$ or $B \neq X$. If $A \neq X$, then there exists an $x \in X - A$ and hence, by postulate (b) of the definition of a normally ordered subbase, there is a B' in \mathcal{B}_0 such that $x \in B'$, $B' \cap A = \emptyset$. Thus $\phi(A) \cap \psi(B) = \emptyset$. If $B \neq X$, a similar argument will suffice.

Theorem 3.5.2.3 $(C_{\mathcal{A}_0}^{\mathcal{B}_0}(X), u, \leqslant)$ is a compact ordered topological space.

Proof: Let $\{\phi(A_k), \psi(A_l)\}, k \in K, l \in L$, have the finite intersection property where $A_k(B_k) \in \mathcal{A}_0(\mathcal{B}_0)$, for all $k \in K, l \in L$. Now let us put $\mathcal{D} =$ $\{A_k : k \in K\}, \mathcal{I} = \{B_l : l \in L\}$; then $(\mathcal{D}, \mathcal{I})$ is an $[\mathcal{A}_0, \mathcal{B}_0]$ family and by Proposition 3.5.2.1 there is a maximal $[\mathcal{A}_0, \mathcal{B}_0]$ family $(\mathcal{A}_0^p, \mathcal{B}_0^p), p \in C_{\mathcal{A}_0}^{\mathcal{B}_0}(X)$, such that $(\mathcal{D}, \mathcal{I}) \subseteq (\mathcal{A}_0^p, \mathcal{B}_0^p)$. Thus $p \in \phi(A_k) \cap \psi(B_l)$ for each $k \in K, l \in L$ and hence the space is compact.

Let $p \notin q$, $p, q \in C^{\mathcal{B}_0}_{\mathcal{A}_0}(X)$; then $\mathcal{A}^q_0 \notin \mathcal{A}^p_0$ and hence there is an $A^q \in \mathcal{A}^q_0$ and such that $A^q \notin \mathcal{A}^p_0$. Since $A^q \notin \mathcal{A}^p_0$ by Lemma 3.5.5.19 there exists a $B^p \in \mathcal{B}^p_0$ such that $A^q \cap B^p = \emptyset$. Then by postulate (c) of the definition of normally ordered subbase there exist sets A, B in $\mathcal{A}_0, \mathcal{B}_0$ respectively such that $A^q \subseteq A$, $B^p \subseteq B$, $A \cup B = X$ and $A^q \cap B = \emptyset = A \cap B^p$. Now let us put $U = C^{\mathcal{B}_0}_{\mathcal{A}_0}(X) - \phi(A)$, $V = C^{\mathcal{B}_0}_{\mathcal{A}_0}(X) - \psi(B)$. Then U, V are *u*-open neighborhoods of p, q respectively. Further by Lemma 3.5.2.9, $U \cap V = \emptyset$ and by Proposition 3.5.2.1 U(V) is increasing (decreasing) in $C_{A_0}^{\mathcal{B}_0}(X)$.

We show now that if f is a bounded increasing continuous function from X into \mathbb{R} , then f can be extended to a function F from $C_{\mathcal{A}_0}^{\mathcal{B}_0}(X)$ into \mathbb{R} such that F is increasing and continuous.

Lemma 3.5.2.11 Let f be an increasing continuous function from X into \mathbb{R} such that $0 \leq f \leq 1$; if we define a function F from $C_{\mathcal{A}_0}^{\mathcal{B}_0}(X)$ into \mathbb{R} by

$$F(p) = \inf_{A^p \in \mathcal{A}_0^p} \{ \sup f(A^p) \},\$$

then F is increasing and F(x) = f(x) for all $x \in X$.

Proof: Let $p, q \in C^{\mathcal{B}_0}_{\mathcal{A}_0}(X)$ with $p \leq q$, then $\mathcal{A}^q_0 \subseteq \mathcal{A}^p_0$. Thus, if $S = \{\sup f(A^q) : A^q \in \mathcal{A}^p_0\}, T = \{\sup f(A^p) : A^p \in \mathcal{A}^p_0\}, \text{then } S \subseteq T \text{ and hence} \}$ $F(p) = \inf T \leq \inf S = F(q).$

If $x \in X$, let us put a = f(x); then $A = \{y \in X : f(y) \leq a\}$ is a member of \mathcal{A}_0^x and thus $F(x) \leq \sup f(A) \leq a$.

Conversely, $F(x) = \inf_{A^x \in \mathcal{A}_0^x} \{\sup f(A^x)\} \ge f(x).$

Lemma 3.5.2.12 Let f be an increasing continuous function from X into \mathbb{R} such that $0 \leq f \leq 1$; if we define a function G from $C_{\mathcal{A}_0}^{\mathcal{B}_0}(X)$ into \mathbb{R} by

$$G(p) = \sup_{B^p \in \mathcal{B}_0^p} \{ \inf f(B^p) \},\$$

then F = G.

Proof: If $p \in C^{\mathcal{B}_0}_{\mathcal{A}_0}(X)$, then for all $A^p \in \mathcal{A}^p_0$, $B^p \in \mathcal{B}^p_0$ it follows that

 $\sup f(A^p) \ge \sup f(A^p \cap B^p) \ge \inf f(A^p \cap B^p) \ge \inf f(B^p).$

Thus $\inf_{A^p \in \mathcal{A}_0^p} \{ \sup f(A^p) \} \ge \inf f(B^p) \text{ and hence } F \ge G.$ Conversely, for a given $\varepsilon > 0$ let us put $a_{\varepsilon} = F(p) - \varepsilon$, $A_{\varepsilon} = \{ x \in X : z \in X \}$ $f(x) \leq a_{\varepsilon}$ and $B_{\varepsilon} = \{x \in X : f(x) \geq a_{\varepsilon}\}$. Then, since $A_{\varepsilon} \cup B_{\varepsilon} = X$, by Lemma 3.5.2.9, $\phi(A_{\varepsilon}) \cup \psi(B_{\varepsilon}) = C_{\mathcal{A}_0}^{\mathcal{B}_0}(X)$ and hence either $p \in \phi(A_{\varepsilon})$ or $p \in \psi(B_{\varepsilon}).$

If $p \in \phi(A_{\varepsilon})$, then A_{ε} belongs to \mathcal{A}_0^p and thus $F(p) \leq \sup f(A_{\varepsilon}) \leq a_{\varepsilon} < \varepsilon$ F(p). Hence, for each $\varepsilon > 0$ B_{ε} belongs to \mathcal{B}_0^p and thus $G(p) \ge \inf f(B_{\varepsilon}) \ge$ $a_{\varepsilon} = F(p) - \varepsilon$. It follows that $G \ge F$.

Theorem 3.5.2.4 Let f be an increasing continuous function from X into \mathbb{R} such that $0 \leq f \leq 1$; then there is an increasing continuous function F from $C_{\mathcal{A}_0}^{\mathcal{B}_0}(X)$ into \mathbb{R} such that $0 \leq F \leq 1$ and f(x) = F(x) for each $x \in X$.

Proof: Let F be the same as in Lemma 3.5.2.11; let us show that F has the desired properties. By Lemma 3.5.2.12 F is increasing and F(x) = f(x) for all $x \in X$. Further, $0 \leq F \leq 1$; thus we need only show the continuity of F.

Let $p \in C_{\mathcal{A}_0}^{\mathcal{B}_0}(X)$ and let K be a neighborhood of F(p) where K is of the form $(\leftarrow, a], a \in \mathbb{R}$. If $K' = [b, \rightarrow)$, where $b \in \mathbb{R}$ and F(p) < b < a, then $K \cup K' = \mathbb{R}$; thus $f^{-1}(K) \cup f^{-1}(K') = X$. Now, since $f^{-1}(K) \in \mathcal{A}_0$, $f^{-1}(K') \in \mathcal{B}_0$ by Lemma 3.5.2.9 so it follows that

(1)
$$\phi(f^{-1}(K)) \cup \psi(f^{-1}(K')) = C_{\mathcal{A}_0}^{\mathcal{B}_0}(X) \,.$$

Now $p \notin \psi(f^{-1}(K'))$ since otherwise $f^{-1}(K') \in \mathcal{B}_0^p$ and thus

$$G(p) \ge \inf f(f^{-1}(K')) \ge \inf K' > F(p)$$

which is contrary to Lemma 3.5.2.12. Then if $U = C_{\mathcal{A}_0}^{\mathcal{B}_0}(X) - \psi(f^{-1}(K'))$ clearly U is a u-neighborhood of p.

If $q \in U$, then, by (1), $q \in \phi(f^{-1}(K))$ and hence $f^{-1}(K) \in \mathcal{A}_0^q$; thus $F(q) \leq \sup f(f^{-1}(K)) \leq \sup K$ and hence $F(q) \in K$. It follows that $p \in U \subseteq F^{-1}(K)$.

If K is a neighborhood of F(p) of the form $[a, \rightarrow)$, $a \in \mathbb{R}$, then by an argument similar to the above, there exists a *u*-neighborhood V of p such that $p \in V \subseteq F^{-1}(K)$ and thus F is continuous.

Corollary 3.5.2.2 If (X,τ) is a completely regularly ordered space and $\alpha(X) = \overline{X}^u$ (where u is the topology on $C_{\mathcal{A}_0}^{\mathcal{B}_0}(X)$), then $(\alpha(X), u_{\alpha(X)})$ is the ordered Czech-Stone compactification of (X,τ) .

Proof: This follows immediately from page 104 of [233] and Theorem 3.5.2.4 above.

Let (X, τ, \leq) be a topological space equipped with an order and let (X^*, τ^*) denote a one point compactification of (X, τ) , where $X^* = X \cup \{w\}$, $w \notin X$.

If G denotes the graph of the partial order on X and if there is defined a partial order on X^* whose graph is G^* such that $G^* \cap (X \times X) = G$, then (X^*, τ^*) is called a **one point ordered compactification** of (X, τ) . If with this partial order on X^* , (X^*, τ^*) is a T_2 -ordered, then we say that (X^*, τ^*) is a **point** T_2 -ordered compactification. **Definition 3.5.2.1** A topological space (X, τ) , equipped with an order \leq is called strongly locally compact, when given $x, y \in X$, $x \neq y$, then either:

(a) there exists an increasing τ -open neighborhood G of x such that \overline{G}^{τ} is compact, or

(b) there exists a decreasing τ -open neighborhood H of y such that \overline{H}^{τ} is compact.

Theorem 3.5.2.5 If (X, τ) is a topological space equipped with a partial order, then a partial order can be defined on X^* such that (X^*, τ^*) is a one point T_2 -ordered compactification of (X, τ) if and only if (X, τ) is strongly locally compact and T_2 -ordered.

Proof: Let us assume first that a partial order can be defined on X^* such that (X^*, τ^*) is a one point T_2 -ordered compactification of (X, τ) . Let $x, y \in X, x \not\leq y$, then either $x \notin w$ or $w \notin y$. Now by Theorem 3.5.1.6, (X^*, τ^*) is normally ordered and hence $x \notin w$ implies that there is a decreasing τ^* -open neighborhood H of w and an increasing τ^* -open neighborhood G of x, such that $G \cap H = \emptyset$. Then $x \in G \subseteq X^* - H$ and it follows that \overline{G}^{τ} is compact. Since the property of being T_2 -ordered is hereditary, (X, τ) is T_2 -ordered. If $w \notin y$, a dual argument suffices.

On the other hand, let (X, τ) be strongly locally compact and T_2 -ordered. We will now define a partial order \leq^* on X^* in the following manner.

For $x, y \in X$, $x \leq^* y$ if and only if $x \leq y$, where \leq is the original partial order on X; $x <^* w$ if and only if there exists a decreasing τ -open neighborhood G of x with \overline{G}^{τ} compact, but there does not exist an increasing τ -open neighborhood of x whose closure is compact. $w <^* x$ if and only if there exists an increasing τ -open neighborhood H of x such that \overline{H}^{τ} is compact, but there does not exist a decreasing τ -open neighborhood of xwhose closure is compact.

We will show now that \leq^* is in fact a partial order on X. Clearly, we need only show that \leq^* is transitive.

If $x < y, y <^* w, x, y \in X$, then there exists a decreasing τ -open neighborhood H of y whose closure is compact. Since H is decreasing, $x \in H$ and thus $x <^* w$ for, if there is an increasing τ -open neighborhood K of x whose closure is compact, then $y \in K$, contrary $y <^* w$. Clearly the case $w <^* x, x < y$ is analogous.

If $x <^{*} w <^{*} y, x, y \in X$, then, if $x \not< y$, since (X, τ) is strongly locally compact, there exists either (1) an increasing τ -open neighborhood G of xsuch that \overline{G}^{τ} is compact; or (2) a decreasing τ -open neighborhood H of ysuch that \overline{H}^{τ} is compact. Now (1) clearly contradicts $x <^* w$ and (2) is contrary to $w <^* y$; thus x < y.

Finally we will show that (X^*, τ^*) is T_2 -ordered. If, for $x \in X$, $x \notin w$, then either $w <^* x$ or w || x.

If $w <^* x$, then there exists an increasing τ -open neighborhood G of x such that \overline{G}^{τ} is compact. Now G is increasing in X^* for, let $g \in G$, $g <^* y$ then $y \neq w$ since G is an increasing τ -open neighborhood of g whose closure is compact. Thus $y \in X$ and hence $y \in G$.

Put $H = X^* - G$, then $H \supseteq X^* - \overline{G}^{\tau}$ and thus G is an increasing τ^* -neighborhood of x, H is a decreasing τ^* -neighborhood of w with $G \cap H = \emptyset$. An argument similar to the above will suffice if w || x.

Theorem 3.5.2.6 If the graph of the partial order \leq^* on X^* (as defined in Theorem 3.5.2.5) is denoted by G^* and the graph of the partial order on X is denoted by G, then G^* is the smallest graph G' of a partial order on X^* such that

(a) G' is closed in $(X^* \times X^*, \tau^* \times \tau^*)$,

(b)
$$G' \cap (X \times X) = G$$
.

Proof: Let G' be the graph of a partial order \leq' on X^* with properties (a) and (b) and let $G^* \not\subseteq G'$. Then there exists an element $(x, y) \in X \times X$ such that $(x, y) \in G^* - G'$ and hence either x = w or y = w. The space (X^*, τ^*) is compact and T_2 -ordered when partially ordered by \leq' ; thus, if y = w, then $x \not\leq' w$ and by an argument similar to the one in the first paragraph of Theorem 3.5.2.5, there exists an increasing τ -open neighborhood of x whose closure is compact contrary to $(x, w) \in G^*$. If x = w by a similar argument, we again have a contradiction and hence $G^* \subseteq G'$.

Proposition 3.5.2.4 If (X, τ) is a Hausdorff locally compact ordered topological space with the property that the sets i(K) and d(K) are closed for each compact set K, then (X, τ) is T_2 -ordered.

Proof: Let us note first that $d(x) = (\leftarrow, x]$ and $i(x) = [x, \rightarrow)$ are closed for all $x \in X$.

Let $x \notin y$ and let us put $G = X - (\leftarrow, y]$. Since (X, τ) is locally compact, Hausdorff space, there is a compact τ -neighborhood K of x such that $K \subseteq G$. Now let us put H = X - i(K), then i(K) and H are increasing and decreasing τ -neighborhoods of x, y respectively such that $i(K) \cap H = \emptyset$. **Proposition 3.5.2.5** A locally compact Hausdorff space (X, τ) endowed with a partial order is strongly locally compact if the sets i(K) and d(K) are compact for each compact set $K \subseteq X$.

Proof: Let (x^*, τ^*) denote a one point compactification of (X, τ) . Then, if we define w || x for all $x \in X$, it follows from Theorem 3.5.2.4 above and from Theorem 5. of [213], that (X^*, τ^*) is a one point ordered compactification of (X, τ) . The result then follows immediately from Theorem 3.5.2.5.

3.5.3 Uniform ordered spaces

In the present subsection, it is our objective to analyze the concept of a uniform ordered space. This generalization of a uniform space is obtained by omitting one of the axioms governing uniform structures and by appropriately interpreting the mathematical structure resulting in such a way.

Let \mathcal{U} be a quasi-uniform structure on X. We will denote by \mathcal{U}^* the set of all subsets of X of the form $V \cap W^{-1}$, where $V, W \in \mathcal{U}$. It is not difficult to prove that \mathcal{U}^* is a uniform structure. We shall call \mathcal{U}^* a uniform structure **generated** by, or **associated** with, the given quasi-uniform structure \mathcal{U} .

If \mathcal{U} is a quasi-uniformity on X, then the set G defined by $G(\leqslant) = \cap \{U : u \in \mathcal{U}\}$ is the graph of a quasi-order on X. Indeed, if $x \in X$, then $(x,x) \in \Delta \subset U$ for every $U \in \mathcal{U}$ from where follows $(x,x) \in G(\leqslant)$, that is, $x \leqslant x$. Now let us assume that $x \leqslant y$ and $y \leqslant z$, that is, $(x,y) \in G(\leqslant)$ and $(y,z) \in G(\leqslant)$. For any member $U \in \mathcal{U}$, let us determine $V \in \mathcal{U}$ in such a way that $V \circ V \subset U$. Now $(x,y) \in G(\leqslant) \subset V$, $(y,z) \in G(\leqslant) \subset V$, from where we conclude that $(x,z) \in V \circ V \subset U$ for every $U \in \mathcal{U}$. Thus $(x,z) \in G(\leqslant)$ which signifies that $x \leqslant z$ and completes the proof of the assertion that $G(\leqslant)$ is the graph of a quasi-order on the set X. We shall call this quasi-order **generated** by, or **associated** with, the quasi-uniformity \mathcal{U} .

Definition 3.5.3.1 A uniform quasi-ordered space is a uniform space (X, \mathcal{U}) which is, at the same time, a quasi-ordered set in such a way that there exists at least one quasi-uniformity which generates the uniform structure \mathcal{U} and the quasi-order given on the set X.

Definition 3.5.3.2 A uniform ordered space is a uniform quasi-ordered space which satisfies one of the two following equivalent conditions:

- (a) the uniform structure of the space is a Hausdorff uniform structure;
- (b) the quasi-order is an order.

These conditions are equivalent since, by definition, they are respectively expressed by the equalities

$$\Delta = \cap \{ U \cap V^{-1} : U, V \in \mathcal{U} \}, \qquad \Delta = G \cap G^{-1},$$

which are, in turn, equivalent since G is the intersection of all the sets $U \in \mathcal{U}$.

Proposition 3.5.3.1 If (X, \mathcal{U}^*, \leq) is a uniform quasi-ordered space, then $(X, \tau_{\mathcal{U}^*}, \leq)$ is a topological quasi-ordered space.

Proof: Let us suppose that \mathcal{U} is a quasi-uniformity which generates quasiorder \leq and uniformity \mathcal{U}^* . Let a, b be two points of X such that $a \not\leq b$, that is, $(a, b) \in X^2 - G(\leq)$. Then there exists a set $V \in \mathcal{U}$ such that $(a, b) \in X^2 - V$. Let us choose $V_1 \in \mathcal{U}$ such that $V_1 \circ V_1 \subset V$, and then, $W \in \mathcal{U}$ such that $W \circ W \subset V_1$. Let us define the sets A and B by A = i(W[a])and $B = d(W^{-1}[b])$. From $W[a] \subset A$ it follows that A is an increasing neighborhood of the point a; similarly, $W^{-1}[b] \subset B$ shows that B is a decreasing neighborhood of the point b. By virtue of Proposition 3.5.1.1, it suffices to show that the sets A and B are disjoint. Let us suppose that there exists a point $z \in A \cap B$. Since $z \in A$, there exists a point $x \in W[a]$ such that $x \leq z$. Then $(a, x) \in W$, $(x, z) \in G \subset W$ from where it follows that

(1)
$$(a,z) \in W \circ W \subset V_1$$
.

Similarly, since $z \in B$, there exists a point $y \in W^{-1}[b]$ such that $z \leq y$. Then $(z, y) \in G \subset W$, $(b, y) \in W^{-1}$ or $(y, b) \in W$ which furnishes

$$(2) (z,b) \in W \circ W \subset V_1$$

Combining (1) and (2), we obtain $(a,b) \in V_1 \circ V_1 \subset V$ which contradicts $(a,b) \in X^2 - V$. The proposition is, thus, proved.

Proposition 3.5.3.2 The topology of every uniform quasi-ordered space (X, \mathcal{U}^*, \leq) is locally convex, that is, the set of convex neighborhoods of every point of X is a base for the neighborhood system of this point.

Proof: Let us suppose that the uniformity \mathcal{U}^* and the quasi-order \leq are generated by the quasi-uniformity \mathcal{U} . Let $a \in X$ be a point of X and let A be a neighborhood of a. By virtue of the definition of the topology associated with a uniform structure, we can determine $W \in \mathcal{U}^*$ such that A = W[a].

Recalling the definition of \mathcal{U}^* , we easily see that there exists a set $V \in \mathcal{U}$ such that

(1)
$$V \cap V^{-1} \subset W.$$

Let us then determine $V_1 \in \mathcal{U}$ in such a manner that $V_1 \circ V_1 \subset V$. Let us set $W_1 = V_1 \cap V_1^{-1}$, $B = k(W_1[a])$, where k indicates the convex hull of the corresponding set. Since $W_1[a] \subset B$, $W_1 \in \mathcal{U}^*$, we can see that B is a convex neighborhood of a. The proposition will be proved if we show that $B \subset A$. For this purpose, we will consider a point $x \in B$. There exist, then, two points $x', x'' \in W_1[a]$ such that $x' \leq x \leq x''$. Now $(a, x') \in W_1 \subset V_1$, $(x', x) \in G(\leq) \subset V_1$ show that

$$(2) (a,x) \in V_1 \circ V_1 \subset V.$$

Similarly, $(a, x'') \in W_1 \subset V_1^{-1}$ or $(x'', a) \in V_1$, $(x, x'') \in G(\leq) \subset V_1$ show that

(3)
$$(x,a) \in V_1 \circ V_1 \subset V$$
 or $(a,x) \in V^{-1}$.

Thus, by virtue of (1), (2) and (3) it follows that $(a, x) \in V \cap V^{-1} \subset W$, that is, $x \in W[a] = A$ as we have wished to show.

Proposition 3.5.3.3 Let (X, \mathcal{U}, \leq) be a uniform ordered space and Y be a uniform subspace of the space X with the induced order. Then Y is a uniform ordered space.

Proof: Let $G(\leq)$ be a graph of the order of the space X. Let us denote by $\mathcal{U}|Y$ and $G_Y(\leq)$ the uniform structure and graph of the order on Y which are generated by the uniform structure and the order of the space X. Since \mathcal{U} is a uniform ordered structure, then there exists a quasi-uniform structure \mathcal{V} on X, such that $\mathcal{V}^* = \mathcal{U}$ and $\cap \mathcal{V} = G(\leq)$. Let $\mathcal{V}|Y$ denote the trace of \mathcal{V} on $Y \times Y$, i.e. $\mathcal{V}|Y = \{V \cap (Y \times Y) : V \in \mathcal{V}\}$. Then $\mathcal{V}|Y$ is a quasi-uniform structure on Y. It is easy to prove that $(\mathcal{V}|Y)^* = \mathcal{U}|Y$. Since

$$\cap(\mathcal{V}|Y) = \cap_{V \in \mathcal{V}}(V \cap (Y \times Y)) = (\cap \mathcal{V}) \cap (Y \times Y) = G(\leqslant) \cap (Y \times Y) = G_Y(\leqslant),$$

the order generated by the quasi-uniform structure $\mathcal{V}|Y$ is identical with the order which is generated by the order of the space X.

We will consider now a quasi-pseudo-metric space X. Every quasipseudometric d on X determines a quasi-uniformity \mathcal{U} on X as follows. Let us denote by d_{ε} the set of all points $(x, y) \in X^2$ such that $d(x, y) \leq \varepsilon$, where $\varepsilon > 0$. Let \mathcal{U} be the family of the subsets of X^2 which contain at least one subset of the form d_{ε} , where $\varepsilon > 0$. Clearly, \mathcal{U} is a quasi-uniform structure on X^2 .

The topology and the quasi-order generated by the quasi-uniform structure \mathcal{U} described above, shall be called the topology and the quasi-order associated with the quasi-pseudo-metric d.

Proposition 3.5.3.4 Let d be a quasi-pseudo-metric on X. For every point $b \in X$, the function d(x,b) of x is continuous and increasing according to the topology and the quasi-order associated with d. Similarly, for every $a \in X$, the function d(a, x) of x is continuous and decreasing.

Proof: First we shall prove the continuity of the function d(x,b). Given a point $x_0 \in X$ in which we want to prove the continuity of this function, let us note that $d(x,b) \leq d(x,x_0) + d(x_0,b)$, $d(x_0,b) \leq d(x_0,x) + d(x,b)$. It follows that

(1)
$$-d(x_0, x) \leq d(x, b) - d(x_0, b) \leq d(x, x_0).$$

Let us consider an arbitrary number $\varepsilon > 0$ and let us define the set A by

(2)
$$A = \{x \in X : d(x, x_0) \leq \varepsilon, \ d(x_0, x) \leq \varepsilon\}.$$

In view of definition of the topology associated with a quasi-pseudo-metric, we see that A is a neighborhood of x_0 according to the topology associated with d. Now, (1) and (2) show that $|d(x,b) - d(x_0,b)| \leq \varepsilon$ provided that $x \in A$, and this establishes the continuity of d(x,b) relative to the variable x.

We will now prove that d(x, b) is an increasing function of x. For this purpose, we will consider two points $x', x'' \in X$ such that $x' \leq x''$, that is, such that the point (x', x'') belongs to the intersection of all the sets $U \in \mathcal{U}$. By the definition of \mathcal{U} , we can see that $(x', x'') \in d_{\varepsilon}$ or $d(x'', x') \leq \varepsilon$, for every $\varepsilon > 0$. Consequently, d(x', x'') = 0. It follows that

$$d(x',b) \leqslant d(x',x'') + d(x'',b) = d(x'',b),$$

as we have wished to show. The proof of that part of the statement relating to the function d(a, x) is analogous.

Theorem 3.5.3.1 If (X, \mathcal{U}^*, \leq) is a uniform quasi-ordered space, then $(X, \tau_{\mathcal{U}^*}, \leq)$ is a completely regular quasi-ordered space. Conversely, every completely regular quasi-ordered space (X, τ, \leq) can be equipped with a uniform structure \mathcal{U}^* in such a way that $\tau = \tau_{\mathcal{U}^*}$ and (X, \mathcal{U}^*, \leq) is a uniform quasi-ordered space.

Proof: Let (X, \mathcal{U}^*, \leq) be a uniform quasi-ordered space. Let us denote by \mathcal{U} a quasi-uniformity which generates the uniformity \mathcal{U}^* and the quasi-order \leq on X. We then consider a point $a \in X$ and a neighborhood V of a in the topology $\tau_{\mathcal{U}^*}$ of X. By the definition of $\tau_{\mathcal{U}^*}$, there exists a set $W \in \mathcal{U}$ such that $W[a] \cap W^{-1}[a] \subset V$.

Let us set $W_1 = W$ and let us assume that $W_n \in \mathcal{U}$ have already been defined. We will then determine $W_{n+1} \in \mathcal{U}$ in such a way that $W_{n+1} \circ$ $W_{n+1} \subset W_n$, and let us indicate by \mathcal{U}' the filter of subsets of X^2 which admits W_1, \ldots, W_n, \ldots as a base. It is clear that the sets W_n can be taken as the base of a filter and that the filter \mathcal{U}' obtained in such a way is a quasi-uniformity on X which admits a countable base.

We will apply Theorem 3.1.3.1 and construct a quasi-pseudo-metric d on X which defines the quasi-uniformity \mathcal{U}' . Let us introduce the real-valued functions f' and g' on X defined as follows:

$$f'(x) = d(x, a), \quad g'(x) = d(a, x)$$

Noting that every member of the filter \mathcal{U}' is also a member of the filter \mathcal{U} and applying Proposition 3.5.3.4, we can see that f' is a continuous increasing function and g' is a continuous decreasing function on X.

Now $W \in \mathcal{U}'$ and d defines \mathcal{U}' ; thus there exists an $\varepsilon > 0$ such that $d_{\varepsilon} \subset W$. We will next define two real-valued functions, f'' and g'' on X, in the following manner:

$$f''(x) = \sup\{0, 1 - d(a, x)/\varepsilon\}, \ g''(x) = \sup\{0, 1 - d(x, a)/\varepsilon\}.$$

It is clear that f'' is continuous and increasing and g'' continuous and decreasing. Furthermore, it follows that

(1)
$$f'' \ge 0, g'' \ge 0, f''(a) = 1, g''(a) = 1$$

If $x \in X - V$, then

(2)
$$\inf\{f''(x), g''(x)\} = 0$$

Indeed, if we had f''(x) > 0 and g''(x) > 0, we should, by the definition of f'' and g'', have $1 - d(a, x)/\varepsilon > 0$, $1 - d(x, a)/\varepsilon > 0$, that is, $d(a, x) < \varepsilon$, $d(x, a) < \varepsilon$. Consequently, $(a, x) \in d_{\varepsilon} \subset W$, $(x, a) \in d_{\varepsilon} \subset W$ or $(a, x) \in W^{-1}$, so that $x \in W[a] \cap W^{-1}[a] \subset V$, which contradicts the hypothesis that $x \in X - V$, so (2) is proved.

If we now define the functions f and g by

$$f(x) = \inf\{1, f''(x)\}, \ g(x) = \inf\{1, g''(x)\},\$$

and take into account relations (1) and (2) given above, we can see that these functions f and g have all the properties indicated in the first condition of the definition of a completely regular quasi-ordered space.

We now go on to consider two points $a, b \in X$ such that $a \notin b$. By the definition of the quasi-order associated with a quasi-uniformity \mathcal{U} , there exists a set $W \in \mathcal{U}$ such that $(a, b) \in X^2 - W$.

We repeat the construction used in the previous case, setting $W_1 = W$ and determining W_n , $n \in \mathbb{N}$, \mathcal{U}' , d and ε in the manner would be indicated. If there followed d(a, b) = 0, we should have $d(a, b) \leq \varepsilon$, whence the result $(a, b) \in d_{\varepsilon} \subset W$, and this would contradict the choice of W. Therefore, d(a, b) > 0. Let us introduce the continuous increasing real-valued function f defined by f(x) = d(x, b). Since f(a) = d(a, b) > 0 = d(b, b) = f(b), we can see that the second condition of the definition of a completely regular quasi-ordered space is satisfied. The proof of the first part of the theorem is, thus, complete.

Conversely, let us consider a completely regular quasi-ordered space X. Let us denote by f an arbitrary continuous increasing real-valued function on X and let us introduce the set $W_f \subset X^2$ defined by $W_f = \{(x, y) :$ $f(x) - f(y) < 1\}$. Clearly, $\Delta \subset W_f$. The collection of all sets of the form W_f can, therefore, be taken as a subbase of a filter \mathcal{U} on X^2 . It is a quasi-uniformity, and this detail will be omitted. It remains to show that \mathcal{U} generates precisely the topology and the quasi-order given on the set X.

Since for every point $a \in X$ and for every continuous increasing realvalued function f defined on X, the sets $\{y : (a, y) \in W_f\}$, $\{x : (x, a) \in W_f\}$, are (according to the given topology) neighborhoods of a, we see that every subset which is open in the topology generated by \mathcal{U} , is also open in the given topology. Conversely, let us consider a point $a \in X$ and one of its neighborhoods V in the given topology. Making use of the fact that X is a completely regular quasi-ordered space, we can determine two continuous real-valued functions f and g, where f is increasing and g is decreasing, such that

$$\begin{array}{l} 0 \leqslant f \leqslant 1 \,, \ 0 \leqslant g \leqslant 1 \,, \\ f(a) = 1 \,, \ g(a) = 1 \,, \\ \inf\{f(x), g(x)\} = 0 \ \text{if} \ x \in X - V \,. \end{array}$$

We assert that

(3) $W_f[a] \cap W_{1-a}^{-1}[a] \subset V;$

this is true since, if $x \in W_f[a] \cap W_{1-g}^{-1}[a]$, then $x \in W_f[a]$ furnishes $(a, x) \in W_f$ or f(a) - f(x) < 1 whence it follows f(x) > 0 as f(a) = 1. Furthermore,

 $x \in W_{1-g}^{-1}[a]$ signifies that $(a, x) \in W_{1-g}^{-1}$, that is, $(x, a) \in W_{1-g}$ or [1-g(x)] - [1-g(a)] < 1 whence it follows that g(x) > 0 since g(a) = 1. We can, thus, assert that $\inf\{f(x), g(x)\} > 0$ and this, by the third property of f and g, requires that $x \in V$ as we have wished to show.

Now the inclusion relation (3) implies that V is a neighborhood of a according to the topology generated by the quasi-uniformity \mathcal{U} . Consequently, every subset that is open according to the given topology is also open according to the topology generated by \mathcal{U} . Combining this fact with the converse observation made earlier, we conclude that the two topologies are identical.

As the last step, we will prove that the quasi-order generated by \mathcal{U} is identical with the given quasi-order. If $a \leq b$ where $a, b \in X$, then $f(a) \leq f(b)$ whence $f(a) - f(b) \leq 0 < 1$. This shows us that $(a, b) \in W_f$ for every continuous real-valued function on X which is increasing according to the given quasi-order; but, then, $(a, b) \in W$ for every $W \in \mathcal{U}$; that is, $a \leq b$ according to the quasi-order determined by \mathcal{U} . Furthermore, if $a \leq b$ is false according to the given quasi-order, there exists a continuous real-valued function which is increasing according to that quasi-order such that f(a) > f(b). Without loss of generality, we can assume that f(a) - f(b) = 1 since, on the contrary, it suffices to substitute for f the function defined by the expression

$$\frac{f(x) - f(b)}{f(a) - f(b)}$$

Clearly, then, $(a, b) \in X^2 - W_f$ and thus, $a \leq b$ is false according to the quasi-order determined by \mathcal{U} . Again combining this fact with the converse observation made earlier, we can see that the two quasi-orders are identical. The theorem is, therefore, proved.

As soon as the definition of a uniform ordered space is formulated, the following problem arises. Let X be, at the same time, a uniform space and a quasi-ordered space. Under what condition of interdependence between the uniform structure and the quasi-order is X a uniform quasi-ordered space? An interesting result in this direction, of which we shall make application later, is the following:

Proposition 3.5.3.5 Let X be a uniform space which is, at the same time, a quasi-ordered space. Let \mathcal{U}^* stand for the filter of subsets of X^2 which define the uniform structure of X and G for the graph of the quasi-order of X in X^2 . In order that X be a uniform quasi-ordered space it is sufficient that

(a) given $V \in \mathcal{U}^*$, there exists a set $W \in \mathcal{U}^*$ such that $W \circ G \subset G \circ V$;

(b) given $V \in \mathcal{U}^*$, there exists a set $W \in \mathcal{U}^*$ such that $(G \circ W) \cap (W \circ G^{-1}) \subset V$;

(c) for every
$$a \in X$$
, the set $i(a) = \{x \in X : x \ge a\}$ is closed.

Proof: Let us suppose that (a), (b) and (c) are satisfied. Since $\Delta \subset G \circ V$ and $G \circ (V_1 \cap V_2) \subset (G \circ V_1) \cap (G \circ V_2)$, the sets of the form $G \circ V$, where $V \in \mathcal{U}^*$, can be taken as a base of a filter \mathcal{U} on X^2 . It is our objective to establish that \mathcal{U} is a quasi-uniform structure which determines the given uniform structure and quasi-order.

From $\Delta \subset G \circ V$, where $V \in \mathcal{U}^*$, we can see that every member of \mathcal{U} contains Δ . In order to complete the proof that \mathcal{U} is a quasi-uniformity, it therefore suffices to show that, if $V \in \mathcal{U}^*$, there exists a set $W \in \mathcal{U}^*$ such that

(1)
$$(G \circ W) \circ (G \circ W) \subset G \circ V$$
.

Now in terms of its graph G, the transitive property of a quasi-order signifies that $G \circ G \subset G$. We determine a set $V' \in \mathcal{U}^*$ in such a manner that $V' \circ V' \subset V$. Then, making use of the condition (a) as stated in the theorem, let us determine a set $V'' \in \mathcal{U}^*$ such that $V'' \circ G \subset G \circ V'$. Setting $W = V' \cap V''$, it is clear that $W \in \mathcal{U}^*$ and that

$$G \circ W \circ G \circ W \subset G \circ V'' \circ G \circ V' \subset G \circ G \circ V' \circ V' \subset G \circ V.$$

This proves (1). So \mathcal{U} is a quasi-uniformity on X.

We are going now to show that the uniformity associated with \mathcal{U} is identical with the given uniformity on X. For this purpose, we will establish two facts: in the first place, that $G \circ V \in \mathcal{U}^*$ for every $V \in \mathcal{U}^*$, and, in the second place, that corresponding to every set $V \in \mathcal{U}^*$, there exists a set $W \in \mathcal{U}^*$ such that

(2)
$$(G \circ W) \cap (G \circ W)^{-1} \subset V.$$

The first fact follows in a simple manner from $V = \Delta \circ V \subset G \circ V$ and from one of the properties of filters. In order to establish the relation (2), we will make use of condition (b) in the statement of the theorem and, once $V \in \mathcal{U}^*$ is given, let us determine $W' \in \mathcal{U}^*$ in such a way that $(G \circ W') \cap (W' \circ G^{-1}) \subset V$. Setting $W = W' \cap W'^{-1}$, it follows that $W \in \mathcal{U}^*$ and the inclusion (2) is obviously verified.

Finally, let us prove that the quasi-order determined by \mathcal{U} is identical with the given quasi-order, or, in equivalent terms, that

(3)
$$G = \bigcap_{V \in \mathcal{U}^*} G \circ V.$$

Indeed, let us note that $G = G \circ \Delta \subset G \circ V$ and thus, relation (3) is valid provided that we replace the = sign by \subset . Let us assume that $(a, b) \in X^2 - G$, that is, that $a \leq b$ is false. By hypothesis, the set i(a) is closed and $b \in X - i(a)$ so that there exists a neighborhood B of b such that $B \cap i(a) = \emptyset$. Let us now determine a set $V \in \mathcal{U}^*$ in such a way that $V^{-1}[b] = B$. We assert that $(a, b) \in X^2 - G \circ V$, since, on the contrary, there would exist a point $x \in X$ such that $(a, x) \in G$, $(x, b) \in V$, whence it would follow $a \leq x$ or $x \in i(a)$, $(b, x) \in V^{-1}$ or $x \in V^{-1}[b]$, in contradiction to the hypothesis that B and i(a) are disjoint. Thus equality (3) is proved.

A first application of the preceding theorem is given in the following result.

Proposition 3.5.3.6 Every Hausdorff uniform space X which is, at the same time, a sup-lattice such that $x \lor y$ is a uniformly continuous function of (x, y), is a uniform ordered space.

Proof: Our procedure will simply be to verify that conditions (a), (b) and (c) of Proposition 3.5.3.5 are satisfied in the case in question.

We will indicate by \mathcal{U}^* the filter on X^2 which defines the uniformity on X, and by G the graph of the order of X.

In order to establish (a), let us consider an arbitrary set $V \in \mathcal{U}^*$ and let us determine $W \in \mathcal{U}^*$ in such a way that, if

(1)
$$(x', x'') \in W, \ (y', y'') \in W,$$

then

$$(x' \lor y', x'' \lor y'') \in V,$$

this being possible by the uniform continuity of the supremum with respect to its two arguments. We assert that $W \circ G \subset G \circ V$. Indeed, if (x, y) is a point belonging to the first member of this inclusion relation, then there exists a point $t \in X$ such that $(x,t) \in W$, $(t,y) \in G$ or $t \leq y$. Noting that $(x,t) \in W$, $(y,y) \in W$, and taking into account (1) and (2), we obtain $(x \lor y, t \lor y) \in V$, or

$$(3) \qquad \qquad (x \lor y, y) \in V$$

since $t \lor y = y$. On the other hand, we note that $x \leq x \lor y$, that is,

$$(4) (x, x \lor y) \in G$$

Combining (3) and (4), we conclude that the point (x, y) also belongs to $G \circ V$, and this completes the proof of (a).

In order to establish (b), we consider a set $V \in \mathcal{U}^*$ and, then determine $W_1 \in \mathcal{U}^*$ in such a way that $W_1 \circ W_1^{-1} \circ W_1 \subset V$. Making use once more of the uniform continuity of the supremum, let us select a set $W_2 \in \mathcal{U}^*$ such that

$$(5) \qquad (x' \lor y', x'' \lor y'') \in W_1$$

whenever

(6)
$$(x', x'') \in W_2, \ (y', y'') \in W_2.$$

Setting $W = W_1 \cap W_2$, we assert that $(G \circ W) \cap (W \circ G^{-1}) \subset V$.

Indeed, if (x, y) is a point belonging to the first member of this relation, then there follows directly from $(x, y) \in G \circ W$ the existence of a point $u \in X$ such that $(x, u) \in G$ or $x \leq u$, $(u, y) \in W$, and similarly, there follows from $(x, y) \in W \circ G^{-1}$ that there exists a point $v \in X$ such that $(x, v) \in W$, $(v, y) \in G^{-1}$ or $y \leq v$.

Now the relations $(x, v) \in W \subset W_2$, $(u, y) \in W \subset W_2$, together with (5) and (6), imply that $(x \lor u, v \lor y) \in W_1$, that is, $(u, v) \in W_1$ since $x \lor u = u$ and $v \lor y = v$. Combining the relations $(x, v) \in W \subset W_1$, $(v, u) \in W_1^{-1}$, $(u, y) \in W \subset W_1$, it follows that $(x, y) \in W_1 \circ W_1^{-1} \circ W_1 \subset V$ whereby the proof of (b) is completed.

Finally, we note that $x \vee y$ is a uniformly continuous function of the two variables x and y simultaneously and, thus, a continuous function of each variable separately. It follows from this that, for every point $a \in X$, the set $i(a) = \{x \in X : x \vee a = x\}$ is closed since X is a Hausdorff space (and therefore, the diagonal of X^2 is closed). If follows that condition (c) is satisfied and the proposition is proved.

Proposition 3.5.3.7 Every compact topological ordered space is a uniform ordered space.

Proof: Let us consider a compact topological ordered space X and let G be the graph of its order. The general theory of uniform spaces teaches us that the filter \mathcal{U}^* of the neighborhoods of the diagonal Δ of X^2 is a uniformity on X which is compatible with the topology of X and which, more precisely, is the only uniformity with this property. We will consider the filter \mathcal{U} of the neighborhoods of G in X^2 . We shall show that \mathcal{U} is a quasi-uniformity which generates the uniformity \mathcal{U}^* and the order G.

We will first show that \mathcal{U} is a quasi-uniformity on X. If $V \in \mathcal{U}$, then $\Delta \subset V$ since $G \subset V$. Moreover, given $V \in \mathcal{U}$, there exists a set $W \in \mathcal{U}$ such

that $W \circ W \subset V$, and it suffices to establish this fact under the assumption that V is open. Let us suppose, for a moment, that it is impossible to determine $W \in \mathcal{U}$ in such a way that $W \circ W \subset V$. In other words, given any set $W \in \mathcal{U}$, there exist points $x, y \in X$ such that $(x, y) \in X^2 - V$, $(x, y) \in W \circ W$, that is, there exists a point $t \in X$ such that $(x, t) \in W$, $(t, y) \in W$. We denote by V' the subset of the cube X^3 formed of all the points (x, t, y) such that $(x, y) \in X^2 - V$ and $t \in X$. We note that V' is compact since $X^2 - V$ and X are compact.

For every $W \in \mathcal{U}$, let us denote by \widetilde{W} the set of all the points $(x, t, y) \in X^3$ such that $(x, y) \in X^2 - V$, $(x, t) \in W$, $(t, y) \in W$. Clearly, $\widetilde{W} \subset V'$; and, as seen above, the assumption that $W \circ W \subset V$ is false signifies that \widetilde{W} is not empty. It follows that the collection of sets \widetilde{W} , where $W \in \mathcal{U}$, can be taken as a base of a filter \mathcal{F} on V'. Making use of the compactness of V', we see that the filter \mathcal{F} has at least one accumulation point; let (a, h, b) be such a point.

We will assert that $a \leq h$. Indeed, let us suppose that $a \leq h$ is false. Since X is normally ordered by Theorem 3.5.1.6, we can determine an open increasing set P and an open decreasing set Q, such that $a \in P, h \in Q$ and $P \cap Q = \emptyset$. The topological space X is, moreover, normal and P is a neighborhood of the closed set i(a). We can, therefore, determine a closed neighborhood P' of i(a) such that $P' \subset P$. We note that P' is then a neighborhood of a. Similarly, we can determine a closed neighborhood Q'of h such that $Q' \subset Q$. The set $W = X^2 - P' \times Q'$ is open and contains G (since, if there existed a point (x, y) common to G and $P' \times Q'$, we should have $x \leq y, x \in P' \subset P, y \in Q' \subset Q$, and this would contradict the fact that P is increasing and disjoint from Q). This means that $W \in \mathcal{U}$ and, thus, $\widetilde{W} \in \mathcal{F}$. Furthermore, $P' \times Q' \times X$ is a neighborhood of (a, h, b) in X^3 which is disjoint from \widetilde{W} for, if (x, t, y) were a point belonging to this neighborhood, we would have $(x,t) \in X^2 - W$ whence $(x,t,y) \in X^3 - \widetilde{W}$. This fact contradicts the property of (a, h, b) to be an accumulation point of the filter \mathcal{F} . Consequently, $a \leq h$.

The relation $h \leq b$ is established in a corresponding manner. Combining the two inequalities obtained, we see that $a \leq b$ so that $(a, b) \in G \subset V$ and this contradicts the fact that $(a, h, b) \in V'$ (we recall that \mathcal{F} is a filter on V'). This contradiction shows that, given $V \in \mathcal{U}$, there exists $W \in \mathcal{U}$ such that $W \circ W \subset V$. Thus \mathcal{U} is a quasi-uniformity.

We will now prove that the quasi-uniformity \mathcal{U} determines the order whose graph is G. This is true since X^2 is Hausdorff space by Corollary 3.5.1.2 and since, in every Hausdorff space, every set (and, in particular the set G in the space X^2) is identical with the intersection of its neighborhoods.

As the last step, we will prove that the quasi-uniformity \mathcal{U} determines the uniformity \mathcal{U}^* . By virtue of the definitions of \mathcal{U} and \mathcal{U}^* , it is clear that every subset which is open according to the topology defined by \mathcal{U} is also open according to the topology defined by \mathcal{U}^* , that is, according to the topology originally given on X. As has already been seen, \mathcal{U} determines an order and not only a quasi-order. Thus the topology defined by \mathcal{U} is a Hausdorff topology. On the basis of a property of compact spaces which we have already used, we conclude that the two topologies defined above are identical. The uniqueness of the uniformity compatible with the topology of a compact Hausdorff space implies that the uniformity determined by \mathcal{U} is identical with the one determined by \mathcal{U}^* . The result is, thus, established.

Proposition 3.5.3.8 Every strongly locally compact T_2 -ordered space is a uniform ordered space.

Proof: Let (X, τ) be a strongly locally compact T_2 -ordered space. Then a partial order can be defined on X^* , such that (X^*, τ^*) is a one point T_2 ordered compactification of the space (X, τ) . According to Theorem 3.5.3.7 (X^*, τ^*) is a uniform ordered space. Therefore, the space (X, τ) is a uniform ordered space by Proposition 3.5.3.3.

3.5.4 Ordering completions of uniform ordered spaces

Definition 3.5.4.1 A partially ordered set (X, \leq) with uniformity \mathcal{U} is a nearly uniform ordered space in case \mathcal{U} is separated and there exists a quasi-uniform structure \mathcal{V} for X such that $G(\leq) \subset \cap \mathcal{V}$ and $\mathcal{V}^* = \mathcal{U}$.

Every uniform ordered space is a nearly uniform ordered space.

Example 3.5.4.1 This example shows that not every nearly uniform ordered space is a uniform ordered space.

Let $\mathbb{R}^- = \{r \in \mathbb{R} : r \leq 0\}$ and let $X = \mathbb{R} \times \mathbb{R}^-$. Let \mathcal{U} be the uniformity on X inherited from the usual uniformity on \mathbb{R}^2 . Then \mathcal{U} is separated. Let us define a binary relation L on X by: (x, y)L(u, v) if and only if (1) $y \neq 0$, $x \leq u$, and $y \leq v$, or (2) y = 0, v = 0 and $0 < x \leq u$, or (3) y = 0, v = 0, and $x \leq u \leq 0$. A straightforward case-by-case argument shows that L is a partial order on X. Let G(L) be the graph of L. Let us suppose that \mathcal{V} is a quasi-uniform structure for X such that $G(L) \subseteq \cap \mathcal{V}$ and $\mathcal{V}^* = \mathcal{U}$. Let $H \in \mathcal{V}$. Then there exists $F \in \mathcal{V}$ such that $F \circ F \subseteq H$. Since $\mathcal{V}^* = \mathcal{U}$, $F \in \mathcal{U}$. Therefore, there exists $\delta < 0$ such that $((-1,0), (-1,\delta)) \in F$. Since $F \in \mathcal{V}$, $G(L) \subseteq F$ and hence $((-1,\delta), (1,0)) \in F$. Therefore $((-1,0), (1,0)) \in F \circ F \subset H$. Thus $((-1,0), (1,0)) \in \cap \mathcal{V}$. But -1 < 0 < 1 and hence $((-1,0), (1,0)) \notin G(L)$. Therefore $\cap \mathcal{V} \nsubseteq G(L)$, i.e. (X,\mathcal{U}) is not a uniform ordered space. However, if we let \mathcal{V} be the quasi-uniformity on X inherited from the usual quasiuniformity on \mathbb{R}^2 , then clearly $\mathcal{V}^* = \mathcal{U}$ and $\cap \mathcal{V} \supseteq G(L)$. Therefore (X,\mathcal{U}) is a nearly uniform ordered space.

Proposition 3.5.4.1 Let (X, \leq) be a totally ordered space with a uniformity \mathcal{U} . Then (X, \mathcal{U}, \leq) is a uniform ordered space if and only if it is a nearly uniform ordered space.

Proof: Let \mathcal{V} be a quasi-uniform structure for X such that $\cap \mathcal{V} \supseteq G(\leqslant)$ and $\mathcal{V}^* = \mathcal{U}$. Let us suppose that $(x, y) \in \cap \mathcal{V} - G(\leqslant)$. Then $(y, x) \in G(\leqslant) \subseteq \cap \mathcal{V}$. Let $H \in \mathcal{U}$. Then $V \cap V^{-1} \subseteq H$ for some $V \in \mathcal{V}$. Since $(y, x) \in V$, $(x, y) \in V^{-1}$; thus $(x, y) \in H$. Therefore $(x, y) \in \cap \mathcal{U}$. But $(x, y) \notin G(\leqslant)$ and hence $(x, y) \notin \Delta$. Therefore, \mathcal{U} is not separated, which is a contradiction.

We will note the following result, which is straightforward to prove.

Proposition 3.5.4.2 Let (X, \mathcal{U}) be a separated uniform space. Let $(\mathbf{X}, \mathcal{U})$ be the completion of (X, \mathcal{U}) at \mathcal{U} . Let $\mathbf{x} \in \mathbf{X}$ and let $\{x_{\delta} : \delta \in D\}$ be a Cauchy net in X which converges to \mathbf{x} . If \mathcal{U}^s is the set of symmetric entourages of \mathcal{U} directed downwards, then there exists a Cauchy net $\{y_U : U \in \mathcal{U}^s\}$, with domain \mathcal{U}^s , such that $\{y_U\}$ converges to \mathbf{x} , and such that, as subsets of X, $\{y_U\} \subseteq \{x_\delta\}$. In particular, if J, $H \in \mathcal{U}^s$ are such that $J \subseteq H$, then $(\mathbf{x}, y_J) \in \mathbf{H}$.

Let (X, \mathcal{U}) be a nearly uniform ordered space, and let (X, \mathcal{U}) be the uniform completion of X at \mathcal{U} . Let us define a binary relation \leq on X as follows: $\mathbf{x} \leq \mathbf{y}$ if and only if for each Cauchy net $\{x_U : U \in \mathcal{U}^s\} \subseteq X$ converging to \mathbf{x} , there exists a Cauchy net $\{y_U : U \in \mathcal{U}^s\} \subseteq X$ converging to \mathbf{y} such that for all $U \in \mathcal{U}^s$, $x_U \leq y_U$. We call this relation the **strong order** on \mathbf{X} .

Let \mathcal{V} be a quasi-uniformity for X such that $\cap \mathcal{V} \supseteq G(\leqslant)$ and $\mathcal{V}^* = \mathcal{U}$. Let us define a binary relation $\leqslant_{\mathcal{V}}$ on \mathbf{X} by: $\mathbf{x} \leqslant_{\mathcal{V}} \mathbf{y}$ if and only if for each $V \in \mathcal{V}$, for each Cauchy net $\{x_U : U \in \mathcal{U}^s\} \subseteq X$ converging to \mathbf{x} , there exists a Cauchy net $\{y_U : U \in \mathcal{U}^s\} \subseteq X$ converging to \mathbf{y} such that for all $U \in \mathcal{U}^s$, $(x_U, y_U) \in V$. We call the relation $\leq_{\mathcal{V}}$ on \mathbf{X} the \mathcal{V} -order on \mathbf{X} . Clearly, if $\mathbf{x} \leq \mathbf{y}$, then $\mathbf{x} \leq_{\mathcal{V}} \mathbf{y}$ for every quasi-uniformity \mathcal{V} for X such that $\cap \mathcal{V} \supseteq G(\leq)$ and $\mathcal{V}^* = \mathcal{U}$.

Proposition 3.5.4.3 If (X, U) is a nearly uniform ordered space, then the strong order and the V-order are partial orders on X.

Proof: Let \mathcal{V} be a quasi-uniformity for X such that $\cap \mathcal{V} \supseteq G(\leqslant)$ and $\mathcal{V}^* = \mathcal{U}$. Clearly, $\mathbf{x} \leqslant \mathbf{x}$ and $\mathbf{x} \leqslant_{\mathcal{V}} \mathbf{x}$ for all $\mathbf{x} \in \mathbf{X}$.

Let us suppose that $\mathbf{x} \leq \mathbf{y} \ (\mathbf{x} \leq_{\mathcal{V}} \mathbf{y})$ and $\mathbf{y} \leq \mathbf{x} \ (\mathbf{y} \leq_{\mathcal{V}} \mathbf{x})$. By Proposition 3.5.4.2 there exists a Cauchy net $\{y_U : U \in \mathcal{U}^s\} \subseteq X$ which converges to \mathbf{y} . We will show that $\{y_U\}$ converges to \mathbf{x} . Let $W \in \mathcal{U}^s$. Let $H \in \mathcal{U}^s$ be such that $\mathbf{H} \circ \mathbf{H} \subseteq \mathbf{W}$, and let $V \in \mathcal{V}$ be such that $(V \circ V) \cap (V \circ V)^{-1} \subseteq H$. By definition of $\leq (\leq_{\mathcal{V}})$, there exist Cauchy nets $\{x_U : U \in \mathcal{U}^s\} \subseteq X$ converging to \mathbf{x} and $\{z_U : U \in \mathcal{U}^s\} \subseteq X$ converging to \mathbf{y} such that for all $U \in \mathcal{U}^s, y_U \leq$ $x_U \ ((y_U, x_U) \in V)$ and $x_U \leq z_U \ ((x_U, z_U) \in V)$. Hence $(y_U, x_U) \in G(\leq) \subseteq V$ $((y_U, x_U) \in V)$, and $(x_U, z_U) \in G(\leq) \subseteq V \ ((x_U, z_U) \in V)$. Since $\{y_U\}$ and $\{z_U\}$ are Cauchy nets converging to \mathbf{y} , there exists $K \in \mathcal{U}^s$ such that $K \subseteq H$ and if $J \subseteq K$, then $(y_J, z_J) \in V \cap V^{-1}$. Hence $(z_J, y_J) \in V$, and thus $(x_J, y_J) \in V \circ V$. Hence $(y_J, x_J) \in (V \circ V)^{-1} \cap V \subseteq H \subseteq H$. But by Proposition 3.5.4.2, $(\mathbf{x}, x_J) \in \mathbf{K} \subseteq \mathbf{H}$ and hence $(\mathbf{x}, y_J) \in \mathbf{H} \circ \mathbf{H} \subseteq \mathbf{W}$. Therefore, $\{y_U\}$ converges to \mathbf{x} and since \mathcal{U} is separated and $\{y_U\}$ also converges to $\mathbf{y}, \mathbf{x} = \mathbf{y}$.

Let us suppose that $\boldsymbol{x} \leq \boldsymbol{y}$ $(\boldsymbol{x} \leq_{\mathcal{V}} \boldsymbol{y})$ and $\boldsymbol{y} \leq \boldsymbol{z}$ $(\boldsymbol{y} \leq_{\mathcal{V}} \boldsymbol{z})$. (Let $V \in \mathcal{V}$, and let $W \in \mathcal{V}$ be such that $W \circ W \subseteq V$.) Let $\{x_U : U \in \mathcal{U}^s\} \subseteq X$ be a Cauchy net converging to \mathbf{x} . Then there exist Cauchy nets $\{y_U : U \in \mathcal{U}^s\} \subseteq$ X converging to \mathbf{y} and $\{z_U : U \in \mathcal{U}^s\} \subseteq X$ converging to \mathbf{z} such that for all $U \in \mathcal{U}^s$, $x_U \leq y_U((x_U, y_U) \in W)$ and $y_U \leq z_U$ $((y_U, z_U) \in W)$. Then for all $U \in \mathcal{U}^s$, $x_U \leq z_U$ $((x_U, z_U) \in W \circ W \subseteq V)$ and hence $\boldsymbol{x} \leq \boldsymbol{z}$ $(\boldsymbol{x} \leq_{\mathcal{V}} \boldsymbol{z})$.

Now we will show that if (X, \mathcal{U}, \leq) is a uniform ordered space, then the \mathcal{V} -order extends the order \leq on X, and that any \mathcal{V} -order (the strong order) on a nearly uniform ordered space makes (X, \mathcal{U}) a (nearly) uniform ordered space.

Proposition 3.5.4.4 Let (X, \mathcal{U}, \leq) be a nearly uniform space. Any \mathcal{V} -order on \mathbf{X} satisfies

$$G(\leqslant) \subseteq G(\leqslant_{\mathcal{V}}) \cap (X \times X) \,.$$

Proof: Let $x, y \in X$ be such that $x \leq y$. Let $V, W \in \mathcal{V}$ be such that $W \circ W \subseteq V$, and let us suppose that $\{x_U : U \in \mathcal{U}^s\} \subseteq X$ is a Cauchy net

converging to x. If $J \subseteq W \cap W^{-1}$, then $(x, x_J) \in W \cap W^{-1} \subseteq W^{-1}$. Since $(x, y) \in G(\leqslant) \subseteq W, (x_J, y) \in W \circ W \subseteq V$. Let

$$y_{\scriptscriptstyle U} = \left\{ \begin{array}{ll} x_{\scriptscriptstyle U}\,, & {\rm if} \ U \not\subseteq W \cap W^{-1}\,, \\ y\,, & {\rm if} \ U \subseteq W \cap W^{-1}\,. \end{array} \right.$$

Then $\{y_U\} \subseteq X$ is a Cauchy net converging to y and for all $U \in \mathcal{U}^s$, $(x_U, y_U) \in V$. Therefore, $x \leq_{\mathcal{V}} y$.

Proposition 3.5.4.5 Let $(X, \mathcal{U}, \leqslant)$ be a uniform ordered space, and let us suppose that \mathcal{V} is a quasi-uniformity for X such that $\cap \mathcal{V} = G(\leqslant)$ and $\mathcal{V}^* = \mathcal{U}$. Then the \mathcal{V} -order on \mathbf{X} extends the order \leqslant on X, i.e. $G(\leqslant) = G(\leqslant_{\mathcal{V}}) \cap (X \times X)$.

Proof: Let $x, y \in X$. By Proposition 3.5.4.4, if $x \leq y$, then $x \leq_{\mathcal{V}} y$. Let us suppose that $x \leq_{\mathcal{V}} y$. The net defined by $x_U = x$ for all $U \in \mathcal{U}^s$ is a Cauchy net converging to x. Let $V, W \in \mathcal{V}$ be such that $W \circ W \subseteq V$. Then there exists a net $\{y_U : U \in \mathcal{U}^s\} \subseteq X$ converging to y such that $(x, y_U) = (x_U, y_U) \in W$ for all $U \in \mathcal{U}^s$. If $J \subseteq W \cap W^{-1}$, then $(y, y_J) \in$ $W \cap W^{-1} \subseteq W^{-1}$. Thus $(x, y) \in W \circ W \subseteq V$. We have shown that $(x, y) \in \cap \mathcal{V}$ and hence $(x, y) \in G(\leq)$, i.e. $x \leq y$.

The next example shows that Proposition 3.5.4.4 (and therefore 3.5.4.5) does not necessarily hold for the strong order.

Example 3.5.4.2 In this example, we construct a uniform ordered space whose strong order does not extend the original order.

Let \mathbb{R} be the set of real numbers and let $X = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x = 0 \text{ or } y = 0\}$. Let \mathcal{U} be the uniformity on X inherited from the usual uniformity on $\mathbb{R} \times \mathbb{R}$. It is straightforward to show that an relation \leq defined on X by $(x, y) \leq (r, s)$ if and only if $x \leq r$ and $y \leq s$ is a partial order on X and that $(X, \mathcal{U}) = (X, \mathcal{U})$ is a uniform ordered space.

Now $(0,0) \leq (0,1)$, and the net $\{(0,1/n) : n = 1, 2, ...\}$ is a Cauchy net converging to (0,0). Let $\{x_U : U \in \mathcal{U}^s\} \subseteq \{(0,1/n)\}$ be a Cauchy net as in Proposition 3.5.4.2. If $\{y_U : U \in \mathcal{U}^s\}$ is a net satisfying $x_U \leq y_U$ for all $U \in \mathcal{U}^s$, then $y_U = (0, y'_U)$ for some $y'_U \in \mathbb{R}$ and for all $U \in \mathcal{U}^s$. Hence if $\{y_U\}$ converges, it must converge to (0, r) for some $r \in \mathbb{R}$. Therefore, there is no Cauchy net $\{y_U\}$ converging to (1,0) and satisfying $x_U \leq y_U$, i.e. (0,0) is not $\leq (1,0)$.

Let (X, \mathcal{U}, \leq) be a nearly uniform ordered space. Let \mathcal{V} be a quasiuniformity for X such that $G(\leq) \subseteq \cap \mathcal{V}$ and $\mathcal{V}^* = \mathcal{U}$. For $V \in \mathcal{V}$, let |V| be the subset of $X \times X$ consisting of all those (x, y) such that for each Cauchy net $\{x_U : U \in \mathcal{U}^s\} \subseteq X$ converging to x, there exists a Cauchy net $\{y_U : U \in \mathcal{U}^s\} \subseteq X$ converging to y such that $(x_U, y_U) \in V$ for all $U \in \mathcal{U}^s$. Let $|\mathcal{V}|$ be the filter on $X \times X$ generated by $\{|V| : V \in \mathcal{V}\}$.

Lemma 3.5.4.13 $|\mathcal{V}|$ is a quasi-uniform structure for X.

Proof: (QU_1) Since $\Delta \subseteq V$ for all $V \in \mathcal{V}$, $(\boldsymbol{x}, \boldsymbol{x}) \in |V|$ for all $\boldsymbol{x} \in \boldsymbol{X}$.

 (QU_2) Let $V, W \in \mathcal{V}$ be such that $W \circ W \subseteq V$. Let us suppose that $(\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{y}, \boldsymbol{z}) \in |W|$, and let $\{x_U : U \in \mathcal{U}^s\} \subseteq X$ be a Cauchy net converging to \boldsymbol{x} . Since $(\boldsymbol{x}, \boldsymbol{y}) \in |W|$ and $(\boldsymbol{y}, \boldsymbol{z}) \in |W|$, there exist Cauchy nets $\{y_U : U \in \mathcal{U}^s\} \subseteq X$ converging to \boldsymbol{y} and $\{z_U : U \in \mathcal{U}^s\} \subseteq X$ converging to \boldsymbol{z} such that for all $U \in \mathcal{U}^s, (x_U, y_U) \in W$ and $(y_U, z_U) \in W$. Thus $(x_U, z_U) \in W \circ W \subseteq V$ for all $U \in \mathcal{U}^s$. Therefore $(\boldsymbol{x}, \boldsymbol{z}) \in |V|$, i.e. $|W| \circ |W| \subseteq |V|$. Hence $|\mathcal{V}|$ is a quasi-uniformity for \boldsymbol{X} .

Lemma 3.5.4.14 $|\mathcal{V}|^* = \mathcal{U}$.

Proof: (If $A \in \mathcal{U}^s$, then A is the set of $(x, y) \in X \times X$ such that if $\{x_U : U \in \mathcal{U}^s\} \subseteq X$ converges to x and $\{y_U : U \in \mathcal{U}^s\} \subseteq X$ converges to y, then there exists $U \in \mathcal{U}^s$ such that for all $J, K \subseteq U, (x_J, y_K) \in A$. Also \mathcal{U} is the filter generated by $\{A : A \in \mathcal{U}^s\}$, and $\{A : A \in \mathcal{U}^s\} \subseteq \mathcal{U}^s$.)

Let $V, W \in \mathcal{V}$ be such that $W \circ W \subseteq V$, and let $H = V \cap V^{-1}$. We will first show that $|W| \cap |W|^{-1} \subseteq H$. Let $(\boldsymbol{x}, \boldsymbol{y}) \in |W| \cap |W|^{-1}$, and let us suppose that $\{\boldsymbol{x}_U : U \in \mathcal{U}^s\} \subseteq X$ and $\{\boldsymbol{y}_U : U \in \mathcal{U}^s\} \subseteq X$ are Cauchy nets converging to \boldsymbol{x} and \boldsymbol{y} respectively. Then there exists a Cauchy net $\{a_U : U \in \mathcal{U}^s\} \subseteq X$ converging to \boldsymbol{y} such that $(x_U, a_U) \in W$ for all $U \in \mathcal{U}^s$, and there exists $K \in \mathcal{U}^s$ such that for all $P, Q \subseteq K$, $(a_P, y_Q) \in W$. Hence $(x_P, y_Q) \in W \circ W \subseteq V$. Since $(\boldsymbol{x}, \boldsymbol{y}) \in |W|^{-1}$, there exists a Cauchy net $\{b_U : U \in \mathcal{U}^s\} \subseteq X$ converging to \boldsymbol{x} such that $(y_U, b_U) \in W$, i.e. $(b_U, y_U) \in$ W^{-1} , for all $U \in \mathcal{U}^s$. Since $\{b_U\}$ converges to \boldsymbol{x} , there exists $L \in \mathcal{U}^s$ such that for all $P, Q \subseteq L$, $(x_P, b_Q) \in W^{-1}$. Thus $(x_P, y_Q) \in W^{-1} \circ W^{-1} \subseteq V^{-1}$. Therefore, if $P, Q \subseteq L \cap K$, then $(x_P, y_Q) \in V \cap V^{-1}$, i.e. $(\boldsymbol{x}, \boldsymbol{y}) \in H$.

Let $V \in \mathcal{V}$, and let $H = V \cap V^{-1}$. We will next show that $H \subseteq |V|$. Let $(\boldsymbol{x}, \boldsymbol{y}) \in H$, let $\{x_U : U \in \mathcal{U}^s\} \subseteq X$ be a Cauchy net converging to \boldsymbol{x} , and let $\{z_U : U \in \mathcal{U}^s\} \subseteq X$ be a Cauchy net converging to \boldsymbol{y} . Let us suppose that $W \in \mathcal{U}^s$ is such that for all $J, K \subseteq W, (x_J, z_K) \in H \subseteq V$. Let us define

$$y_{\scriptscriptstyle U} = \left\{ \begin{array}{ll} x_{\scriptscriptstyle U}\,, & {\rm if} \ U \not\subseteq W\,, \\ z_{\scriptscriptstyle U}\,, & {\rm if} \ U \subseteq W\,. \end{array} \right.$$
Then, clearly, $\{y_U\} \subseteq X$ is a Cauchy net converging to \boldsymbol{y} such that $(x_U, y_U) \in V$ for all $U \in \mathcal{U}^s$. Therefore $(\boldsymbol{x}, \boldsymbol{y}) \in |V|$ and hence $\boldsymbol{H} \subseteq |V|$.

Now, if $A \in \mathcal{U}$, then there exists $V \in \mathcal{V}$ such that $H \subseteq A$, where $H = V \cap V^{-1}$, and hence, by the above, $A \supseteq |W| \cap |W|^{-1}$ for some $|W| \in |\mathcal{V}|$. Conversely, if $A \in |\mathcal{V}|^*$, then there exists $V \in \mathcal{V}$ such that $A \supseteq |V| \cap |V|^{-1}$. Let $H = V \cap V^{-1}$. Then, by the above, $|V| \supseteq H$, and thus, since H is symmetric, $|V| \cap |V|^{-1} \supseteq H$. Hence, $A \supseteq H$, and therefore, $|\mathcal{V}|^* = \mathcal{U}$.

Lemma 3.5.4.15 $G(\leq_{\mathcal{V}}) = \cap |\mathcal{V}|$.

Proof: The result follows immediately from the definitions.

Proposition 3.5.4.6 Let (X, \mathcal{U}, \leq) be a nearly uniform ordered space. Then any \mathcal{V} -order on X makes $(X, \mathcal{U}, \leq_{\mathcal{V}})$ a uniform ordered space.

Proof: By Lemmas 3.5.4.13, 3.5.4.14 and 3.5.4.15, $|\mathcal{V}|$ is a quasi-uniformity for X such that $G(\leq_{\mathcal{V}}) = \cap |\mathcal{V}|$ and $|\mathcal{V}|^* = \mathcal{U}$.

Proposition 3.5.4.7 Let (X, \mathcal{U}, \leq) be a nearly uniform ordered space. Then the strong order on X makes (X, \mathcal{U}, \leq) a nearly uniform ordered space.

Proof: Clearly, there exists an \mathcal{V} -order on X and clearly $G(\leq_{\mathcal{V}}) \supseteq G(\leqslant)$. Thus, by Lemmas 3.5.4.13, 3.5.4.14 and 3.5.4.15, $|\mathcal{V}|$ is a quasi-uniformity for X such that $\cap |\mathcal{V}| \supseteq \mathcal{G}(\leqslant)$ and $|\mathcal{V}|^* = \mathcal{U}$.

Let \mathcal{U} be a separated uniformity on the partially ordered set (X, \leq) . Let

 $\mathcal{F}(\mathcal{U}) = \{ V \in \mathcal{U} : \text{ there exist } V_1, V_2, \dots \in \mathcal{U} \text{ such that} \\ V_1 = V, \text{ and for all } n, \ G(\leqslant) \subseteq V_n \\ \text{and } V_{n+1} \circ V_{n+1} \subseteq V_n \}.$

Proposition 3.5.4.8 $\mathcal{F}(\mathcal{U})$ is a quasi-uniformity for X.

Proof: Clearly, $\Delta \subseteq V$ for all $V \in \mathcal{F}(\mathcal{U})$. If $V \in \mathcal{F}(\mathcal{U})$, then V_2 in the definition of $\mathcal{F}(\mathcal{U})$ is an element of $\mathcal{F}(\mathcal{U})$ such that $V_2 \circ V_2 \subseteq V$. To see that $\mathcal{F}(\mathcal{U})$ is a filter on $X \times X$, let us first note that clearly if $U \supseteq V \in \mathcal{F}(\mathcal{U})$, then $U \in \mathcal{F}(\mathcal{U})$. Furthermore, if $U, V \in \mathcal{F}(\mathcal{U})$, let us consider $U_1 \cap V_1$, $U_2 \cap V_2$,... Clearly, $U_1 \cap V_1 = U \cap V$. For any $n, U_n \cap V_n \in \mathcal{U}, U_n \cap V_n \supseteq G(\leqslant)$, and

 $(U_{n+1} \cap V_{n+1}) \circ (U_{n+1} \cap V_{n+1}) \subseteq U_{n+1} \circ U_{n+1} \subseteq U_n ,$ $(U_{n+1} \cap V_{n+1}) \circ (U_{n+1} \cap V_{n+1}) \subseteq V_{n+1} \circ V_{n+1} \subseteq V_n .$

Thus $(U_{n+1} \cap V_{n+1}) \circ (U_{n+1} \cap V_{n+1}) \subseteq U_n \cap V_n$.

Proposition 3.5.4.9 (X, \mathcal{U}, \leq) is a nearly uniform ordered space if and only if $\mathcal{F}(\mathcal{U})^* \supseteq \mathcal{U}$.

Proof: Since $\mathcal{F}(\mathcal{U}) \subseteq \mathcal{U}$, clearly $\mathcal{F}(\mathcal{U})^* \subseteq \mathcal{U}$. Thus, if $\mathcal{F}(\mathcal{U})^* \supseteq \mathcal{U}$, then $\mathcal{F}(\mathcal{U})^* = \mathcal{U}$. Therefore, $\mathcal{F}(\mathcal{U})$ is a quasi-uniformity for X such that $\cap \mathcal{F}(\mathcal{U}) \supseteq \mathcal{G}(\leqslant)$ and $\mathcal{F}(\mathcal{U})^* = \mathcal{U}$.

Conversely, let us suppose that S is a quasi-uniformity for X such that $\cap S \supseteq \mathcal{G}(\leqslant)$ and $S^* = \mathcal{U}$. Clearly, $S \subseteq \mathcal{F}(\mathcal{U})$. Hence, if $H \in \mathcal{U}$, then $H = U \cap V^{-1}$ for $U, V \in \mathcal{F}(\mathcal{U})$, i.e. $H \in \mathcal{F}(\mathcal{U})^*$.

Proposition 3.5.4.10 $(X, \mathcal{U}, \leqslant)$ is a uniform ordered space if and only if $\cap \mathcal{F}(\mathcal{U}) \subseteq \mathcal{G}(\leqslant)$ and $\mathcal{F}(\mathcal{U})^* \supseteq \mathcal{U}$.

Proof: Let S be a quasi-uniformity for X such that $\cap S = G(\leqslant)$ and $S^* = \mathcal{U}$. Clearly, $S \subseteq \mathcal{F}(\mathcal{U})$. Thus $\cap S \supseteq \cap \mathcal{F}(\mathcal{U})$, i.e. $G(\leqslant) \supseteq \cap \mathcal{F}(\mathcal{U})$. Now this proposition follows from Proposition 3.5.4.9.

Proposition 3.5.4.11 For every (nearly) uniform ordered space (X, U, \leq) , $\mathcal{F}(U)$ is the unique maximal quasi-uniformity for X satisfying

 $(\cap \mathcal{F}(\mathcal{U}) \supseteq \mathcal{G}(\leqslant))$, $\cap \mathcal{F}(\mathcal{U}) = \mathcal{G}(\leqslant)$ and $\mathcal{F}(\mathcal{U})^* = \mathcal{U}$.

Proof: If S is such a structure, then, clearly, $S \subseteq \mathcal{F}(\mathcal{U})$. Proposition 3.5.4.11 then follows from Propositions 3.5.4.8, 3.5.4.9 and 3.5.4.10 and their proofs.

Let X be a partially ordered set with a uniformity \mathcal{U} . We say that \mathcal{U} (or (X, \mathcal{U})) fulfills the **condition** (M) if

(M) for all
$$V \in \mathcal{U}$$
, there exists $W \in \mathcal{U}$
such that $W \circ G(\leqslant) \subseteq G(\leqslant) \circ V$.

Since $G(\leq) \circ (U \cap V) \subseteq (G(\leq) \circ U) \cap (G(\leq) \circ V)$ for all $U, V \in \mathcal{U}$, the set $\{G(\leq) \circ U : U \in \mathcal{U}\}$ is a filter base on $X \times X$. Let $\mathcal{S}(\mathcal{U})$ be the filter on $X \times X$ generated by $\{G(\leq) \circ U : U \in \mathcal{U}\}$.

Proposition 3.5.4.12 Let X be a partially ordered set with a uniformity \mathcal{U} . Then \mathcal{U} fulfills the condition (M) if and only if $\mathcal{F}(\mathcal{U}) = \mathcal{S}(\mathcal{U})$.

Proof: Let us suppose that \mathcal{U} fulfills the condition (M). If $H \in \mathcal{F}(\mathcal{U})$, then there exists $J \in \mathcal{F}(\mathcal{U})$ such that $J \circ J \subseteq H$. Since $G(\leqslant) \subseteq J$, $G(\leqslant) \circ J \subseteq$ $J \circ J \subseteq H$. Since $J \in \mathcal{U}$, this implies that $H \in \mathcal{S}(\mathcal{U})$. Thus $\mathcal{F}(\mathcal{U}) \subseteq \mathcal{S}(\mathcal{U})$. Conversely, let us suppose that $U \in \mathcal{U}$. By the condition (M), there exist $V_1, V_2, \ldots \in \mathcal{U}$ such that $V_1 = U$, and for all $n, V_{n+1} \circ V_{n+1} \subseteq V_n$ and $V_{n+1} \circ G(\leqslant) \subseteq G(\leqslant) \circ V_n$. Let us consider $W_n = G(\leqslant) \circ V_{2n-1}$ for $n = 1, 2, \ldots$ Then $W_1 = G(\leqslant) \circ V_1 = G(\leqslant) \circ U$. Furthermore, for all $n, G(\leqslant) \subseteq W_n$ and

$$W_{n+1} \circ W_{n+1} = G(\leqslant) \circ V_{2n+1} \circ G(\leqslant) \circ V_{2n+1} \subseteq G(\leqslant) \circ G(\leqslant) \circ V_{2n} \circ V_{2n} \subseteq G(\leqslant) \circ V_{2n-1} = W_n.$$

Therefore, $G(\leq) \circ U \in \mathcal{F}(\mathcal{U})$ and hence $\mathcal{S}(\mathcal{U}) \subseteq \mathcal{F}(\mathcal{U})$.

Let us suppose that $\mathcal{F}(\mathcal{U}) = \mathcal{S}(\mathcal{U})$. Let $U \in \mathcal{U}$. Then $G(\leqslant) \circ U \in \mathcal{F}(\mathcal{U})$ and hence there exist $V_1, V_2 \in \mathcal{F}(\mathcal{U})$ such that $V_1 \circ V_2 \subseteq G(\leqslant) \circ U$. Since $\mathcal{S}(\mathcal{U})$ is a filter base, there exist $J_1, J_2 \in \mathcal{U}$ such that $G(\leqslant) \circ J_1 \subseteq V_1$ and $G(\leqslant) \circ J_2 \subseteq V_2$. Thus

$$J_1 \circ G(\leqslant) \subseteq G(\leqslant) \circ J_1 \circ G(\leqslant) \circ J_2 \subseteq G(\leqslant) \circ U.$$

Therefore, \mathcal{U} fulfills the condition (M).

Proposition 3.5.4.13 Let (X, \mathcal{U}, \leq) be a nearly uniform ordered space. If \mathcal{U} fulfills the condition (M), then the strong order is equivalent to the $\mathcal{F}(\mathcal{U})$ -order.

Proof: Clearly, $G(\leqslant) \subseteq G(\leqslant_{\mathcal{F}(\mathcal{U})})$.

Conversely, let us suppose that $\boldsymbol{x} \leq_{\mathcal{F}(\mathcal{U})} \boldsymbol{y}$ and let $\{x_U : U \in \mathcal{U}^s\} \subseteq X$ be a Cauchy net converging to \boldsymbol{x} . Let $V \in \mathcal{U}^s$. Then there exists a Cauchy net $\{y_U : U \in \mathcal{U}^s\} \subseteq X$ converging to \boldsymbol{y} such that $(x_U, y_U) \in G(\leqslant) \circ V$ for all $U \in \mathcal{U}^s$. Hence there exists $a_V \in X$ such that $(x_V, a_V) \in G(\leqslant)$ and $(a_V, y_V) \in V$. Let us consider the net $\{a_V\} \subseteq X$. For all $V \in \mathcal{U}^s$, $x_V \leqslant a_V$, and $(a_V, y_V) \in V$. Let $V, V' \in \mathcal{U}^s$ be such that $V' \circ V' \subseteq V$. There exists $U \in \mathcal{U}^s$ such that for $J, K \subseteq U, (y_J, y_K) \in V'$. Thus, if $J, K \subseteq U \cap V'$, $(a_J, y_J) \in J \subseteq V'$ and $(y_J, y_K) \in V'$. Hence $(a_J, y_K) \in V$. Therefore $\{a_V\}$ is a Cauchy net converging to the same point to which $\{y_U\}$ converges; i.e. $\{a_V\} \subseteq X$ is a Cauchy net converging to \boldsymbol{y} . We conclude that $\boldsymbol{x} \leqslant \boldsymbol{y}$.

Corollary 3.5.4.1 Let (X, U) be a nearly uniform ordered space. If U fulfills the condition (M), then (X, U) with the strong order is a uniform ordered space.

Proof: The result follows from Propositions 3.5.4.7 and 3.5.4.13.

Corollary 3.5.4.2 Let (X, \mathcal{U}, \leq) be a uniform ordered space. If \mathcal{U} fulfills the condition (M), then the strong order on X extends the order on X, i.e. $G(\leq) = G(\leq) \cap (X \times X)$.

Proof: The result follows from Propositions 3.5.4.5 and 3.5.4.13.

Example 3.5.4.3 Not every uniform ordered space fulfills the condition (M): (X, \mathcal{U}) of Example 3.5.4.2 is a uniform ordered space whose strong order does not extend its original order. Therefore, by Corollary 3.5.4.2, (X, \mathcal{U}) does not satisfy the condition (M).

3.5.5 Proximity ordered spaces

Let X be a non-empty set and let δ be a quasi-proximity on X. If we set

 $x \leq y$ if and only if $\{x\}\delta\{y\}$,

it is easy to see that \leq is a quasi-order on X. We call \leq the **quasi-order** generated by δ . Further, if we define a relation δ^* on the power set PX of X by setting

 $A\delta^*B$ if and only if, whenever $\{A_i : i \in J_m\}$ is a finite cover of A and $\{B_j : j \in J_n\}$ is a finite cover of B, there follows $A_i\delta \cap \delta^{-1}B_j$ for some $(i, j) \in J_m \times J_n$,

then δ^* is a proximity on X. We call δ^* the **proximity generated by** the quasi-proximity δ .

Definition 3.5.5.1 An ordered triple (X, δ^*, \leq) consisting of a non-empty set X, a proximity δ^* and a quasi-order \leq , both defined on the set X, is called a **proximity quasi-ordered space** if there exists at least one quasiproximity δ on X which generates both δ^* and \leq . In this case, δ is called a **generating quasi-proximity**.

It is easily seen that, on a proximity quasi-ordered space $(X, \delta^*, \leq), \leq$ is a partial order if and only if δ^* is separated. In this case, we call (X, δ^*, \leq) a **proximity ordered space**.

Example 3.5.5.1 The set of real numbers with the usual order and the usual proximity is a proximity ordered space. A generating quasi-proximity is induced by the quasi-pseudo-metric defined by $d(x, y) = \max\{0, x - y\}$.

Example 3.5.5.2 Let (X, τ, \leq) be a completely regular quasi-ordered space. Then (X, δ^*, \leq) is a proximity quasi-ordered space, where δ^* is generated by the quasi-proximity δ which is defined by

 $A\delta B$ if and only if there exists a continuous increasing function $f: X \to [0,1]$ such that f(A) = 1 and f(B) = 0.

Example 3.5.5.3 Let (X, τ, \leq) be a compact ordered topological space. Then (X, δ^*, \leq) is a proximity ordered space, where δ^* is generated by the quasi-proximity

 $A\delta B$ if and only if $I(A) \cap D(B) \neq \emptyset$,

and where I(A)(D(B)) is the smallest increasing (decreasing) closed superset of A(B).

Lemma 3.5.5.16 Let (X, δ^*, \leq) be a proximity quasi-ordered space and let \mathcal{U} and \mathcal{L} be the upper and lower topologies of $(X, \tau_{\delta^*}, \leq)$. Then $\tau_{\delta} \subseteq \mathcal{U}$, $\tau_{\delta^{-1}} \subseteq \mathcal{L}$ and $\tau_{\delta} \vee \tau_{\delta^{-1}} = \mathcal{U} \vee \mathcal{L}$ for every generating quasi-proximity δ .

Proof: The set A is closed with respect to τ_{δ} if and only if it is of the form $A = A^{\delta} = \{x : x\delta A\}$. Also, A is then decreasing with respect to the quasi-order \leq given on X. Therefore open sets with respect to τ_{δ} are increasing. Similarly, sets open with respect to $\tau_{\delta^{-1}}$ are decreasing. Further, both τ_{δ} and $\tau_{\delta^{-1}}$ are coarser than τ_{δ^*} . Therefore $\tau_{\delta} \subseteq \mathcal{U}$ and $\tau_{\delta^{-1}} \subseteq \mathcal{L}$. Next, if δ^{**} denotes the finest quasi-proximity compatible with $\tau_{\delta} \vee \tau_{\delta^{-1}}$, then δ^{**} is finer than both δ and δ^{-1} , and hence finer than δ^* . This gives $\tau_{\delta^{**}} = \tau_{\delta} \vee \tau_{\delta^{-1}} \supseteq \tau_{\delta^*}$. Therefore $\tau_{\delta^*} = \tau_{\delta} \vee \tau_{\delta^{-1}} \supseteq \tau_{\delta^*}$.

Lemma 3.5.5.17 If (X, δ^*, \leq) is a proximity quasi-ordered space and if δ is a generating quasi-proximity, then

 $A\delta B$ if and only if $I(A)\delta D(B)$,

where I and D are defined in $(X, \tau_{\delta^*}, \leq)$.

Proof: This follows from the facts that

 $A\delta B$ if and only if $I^*(A)\delta D^*(B)$

and that $A \subseteq I(A) \subseteq I^*(A)$, $B \subseteq D(B) \subseteq D^*(B)$, where $I^*(A)$ denotes the $\tau_{\delta^{-1}}$ -closure of A and $D^*(B)$ denotes the τ_{δ} -closure of B.

Lemma 3.5.5.18 If (X, δ^*, \leq) is a proximity quasi-ordered space, then the topology τ_{δ^*} is convex and quasi-order is closed (the graph of quasi-order is closed in $(X \times X, \tau_{\delta^*} \times \tau_{\delta^*})$).

Proof: The proof follows from the Lemma 3.5.5.16.

Lemma 3.5.5.19 Let X be a non-empty set and let δ be a quasi-proximity on X. If $A\overline{\delta}B$, then there exists a function $f: X \to [0,1]$ with the following properties:

- (a) f is continuous with respect to the topology $\tau = \tau_{\delta} \vee \tau_{\delta^{-1}}$;
- (b) f is increasing with respect to the quasi-order \leq generated by δ ;
- (c) f(A) = 1, f(B) = 0.

Proof: If $A\overline{\delta}B$, then there exist sets P and Q such that $A\overline{\delta}X - P$, $P\overline{\delta}X - Q$ and $Q\overline{\delta}B$. Now $A\overline{\delta}X - P$ implies $A\overline{\delta}D(X - P)$, that is, $A \subseteq X - D(X - P) = i(P)$, where i(P) is the largest increasing open subset of P. Similarly, $P\overline{\delta}X - Q$ implies $I(P) \subseteq i(Q)$, and $Q\overline{\delta}B$ implies $I(Q) \subseteq X - B$. Thus we have

$$A \subseteq i(P) \subseteq I(P) \subseteq i(Q) \subseteq I(Q) \subseteq X - B$$
,

where *i* and *I* are defined in (X, τ, \leq) . This is similar to the main step in Nachbin's proof of Urysohn's lemma for ordered spaces. Proceeding as therein we obtain a function $f : X \to [0, 1]$, which is: (a) continuous with respect to the topology τ , (b) increasing with respect to the quasi-order \leq and (c) f(A) = 1, f(B) = 0.

Theorem 3.5.5.1 Let (X, δ^*, \leq) be a proximity quasi-ordered space. Then $(X, \tau_{\delta^*}, \leq)$ is a completely regular quasi-ordered space.

Proof: Let δ be a generating quasi-proximity on (X, δ^*, \leq) . Let $p \in X$ and let P be a τ_{δ^*} -neighborhood of p. Since $\tau_{\delta^*} = \tau_{\delta} \vee \tau_{\delta^{-1}}$, there exist sets $U \in \tau_{\delta}$ and $V \in \tau_{\delta^{-1}}$ such that $p \in U \cap V \subseteq P$. Clearly, $p\overline{\delta}X - U$ and therefore there exists a τ_{δ^*} -continuous increasing function $f : X \to [0, 1]$ such that f(p) = 1 and f(X - U) = 0. Similarly, $X - V\overline{\delta}p$ and therefore there exists a τ_{δ^*} -continuous decreasing function $g : X \to [0, 1]$ such that g(X - V) = 0 and g(p) = 1. Clearly, if $x \in X - P$ then either f(x) = 0 or g(x) = 0. Thus, the first condition for $(X, \tau_{\delta^*}, \leq)$ to be completely regular quasi-ordered is satisfied.

Next, let $x \leq y$. Since δ is a generating quasi-proximity, we have $x\overline{\delta}y$ and therefore, as above, there exists a τ_{δ^*} -continuous increasing function such that f(x) > f(y). This completes the proof. \clubsuit

Theorem 3.5.5.2 Let (X, τ, \leq) be a completely regular quasi-ordered space. Then there exists a proximity δ^* on X such that (X, δ^*, \leq) is a proximity quasi-ordered space and $\tau_{\delta^*} = \tau$.

Proof: We define a relation δ as follows:

 $A\delta B$ if and only if there exists a continuous increasing (decreasing) function $f: X \to [0,1]$ such that f(A) = 1, f(B) = 0(f(A) = 0, f(B) = 1, respectively).

We assert that δ is a quasi-proximity on X such that $x \leq y$ if and only if $x\delta y$ and that $\tau_{\delta} \vee \tau_{\delta^{-1}} = \tau$. To prove that δ is a quasi-proximity, it is sufficient to establish (QP_4) only. Let $A\overline{\delta}B$ and let $f: X \to [0,1]$ be a continuous decreasing function such that f(A) = 0, f(B) = 1. We set $E = \{x: 1/2 \leq f(x) \leq 1\}$ and let us define $g: [0,1] \to [0,1]$ by setting

$$g(y) = \begin{cases} 2y, & \text{for } y \in [0, 1/2], \\ 1, & \text{for } y \in [1/2, 1]. \end{cases}$$

Clearly, g is a continuous increasing function. Therefore $g \circ f : X \to [0, 1]$ is a continuous decreasing function such that $(g \circ f)(A) = 0$, $(g \circ f)(E) = 1$. Similarly, $g \circ (1 - f) : X \to [0, 1]$ is a continuous increasing function such that $g \circ (1 - f)(X - E) = 1$, $g \circ (1 - f)(B) = 0$.

Next, if $x\overline{\delta}y$, then, clearly, $x \notin y$. On the other hand, if $x \notin y$, then, by the definition of a completely regular quasi-ordered space, there exists a continuous increasing real-valued function f such that f(x) > f(y). If we set

$$k(z) = \frac{f(z) - f(y)}{f(x) - f(y)},$$

and $h(z) = \max\{\min(1, k(z)), \max(0, k(z))\}$, then $h : X \to [0, 1]$ is a continuous increasing function such that h(x) = 1, h(y) = 0, and therefore $x\overline{\delta}y$.

Finally, we will show that $\tau = \tau_{\delta} \vee \tau_{\delta^{-1}}$. Let $V \in \tau_{\delta}$ and let $x \in V$. Then $x\overline{\delta}X - V$. Let $f: X \to [0, 1]$ be a continuous increasing function such that f(x) = 1 and f(X - V) = 0. Clearly V is a τ -neighborhood of x and because this holds for each $x \in V$, V is a τ -open set. Therefore $\tau_{\delta} \subseteq \tau$. Similarly, $\tau_{\delta^{-1}} \subseteq \tau$.

Conversely, let $p \in X$ and let P be a τ -neighborhood of p. Since X is completely regular quasi-ordered, there exist two continuous functions $f, g: X \to [0, 1]$ such that f is increasing, g is decreasing, f(p) = g(p) = 1 and $\min\{f(x), g(x)\} = 0$ for all $x \in X - P$. It is clear from here that

 $f^{-1}((0,1])$ is a τ_{δ} -neighborhood of p, that $g^{-1}((0,1])$ is a $\tau_{\delta^{-1}}$ -neighborhood of p, and that $f^{-1}((0,1]) \cap g^{-1}((0,1]) \subseteq P$. Therefore P is a $\tau_{\delta} \vee \tau_{\delta^{-1}}$ neighborhood of p. Since p is arbitrary, it follows that every set open with respect to τ is also open with respect to $\tau_{\delta} \vee \tau_{\delta^{-1}}$. Thus $\tau_{\delta} \vee \tau_{\delta^{-1}} = \tau$ holds.

Clearly δ^* , the proximity generated by δ , meets all the requirements. This completes the proof.

Let (X, τ, \leq) be a topological space equipped with a quasi-order. Let δ be a quasi-proximity on X satisfying

$$(C_1) \ x \leq y \ \text{if and only if } x \delta y; (C_2) \ \tau_{\delta} \lor \tau_{\delta^{-1}} = \tau.$$

If δ^* is the proximity generated by δ , then (X, δ^*, \leq) becomes a proximity quasi-ordered space such that $\tau = \tau_{\delta^*}$. Thus, in order that a topological space equipped with a quasi-order be a proximizable quasi-ordered space, it is sufficient to have a quasi-proximity δ on X satisfying the conditions (C_1) and (C_2) given above. We call such a quasi-proximity a **compatible quasi-proximity** on (X, τ, \leq) .

Theorem 3.5.5.3 Let (X, τ, \leq) be a completely regular quasi-ordered space and let δ be defined by setting

$$A\delta B$$
 if and only if $I(A) \cap D(B) \neq \emptyset$.

Then δ is a compatible quasi-proximity if and only if X is normally quasiordered.

Proof: If the space is normally quasi-ordered, then $I(A) \cap D(B) = \emptyset$ if and only if there exists a continuous increasing function $f : X \to [0, 1]$ such that f(A) = 1, f(B) = 0, and therefore coincides with the quasiproximity defined in Theorem 3.5.5.2. Since the space is completely regular quasi-ordered, it follows from Theorem 3.5.5.2 that δ is a compatible quasiproximity.

Conversely, let δ be a compatible quasi-proximity. To prove that X is normally quasi-ordered, let A = I(A) and B = D(B) be such that $A \cap B = \emptyset$. Clearly, $A\overline{\delta}B$. Let E and F be such that $A\overline{\delta}X - E$, $X - F\overline{\delta}B$ and $E\overline{\delta}F$. Since δ is compatible, $A\overline{\delta}X - E$ implies $A\overline{\delta}D(X - E)$, that is $A \subseteq X - D(X - E) = i(E)$. Similarly, $X - F\overline{\delta}B$ implies $I(X - F)\overline{\delta}B$, that is $B \subseteq X - I(X - F) = d(F)$. Also, $E\overline{\delta}F$ implies $E \cap F = \emptyset$. Thus E and F are required neighborhoods of the sets A and B respectively. Hence X is a normally quasi-ordered space. \clubsuit It is well known that a compact Hausdorff space admits a unique compatible proximity. Here we generalize this result to compact ordered spaces.

Lemma 3.5.5.20 Let δ be a compatible quasi-proximity on (X, τ, \leq) . If A is compact and B is closed and decreasing, then $A \cap B = \emptyset$ implies $A\overline{\delta}B$.

Proof: Since B = D(B), therefore $A \cap B = \emptyset$ implies $a\overline{\delta}B$ whenever $a \in A$. This, together with the compatibility of δ , implies that for each a in A there exists an open set N_a such that $a\overline{\delta}X - N_a$ and $N_a\overline{\delta}B$. The family $\{N_a : a \in A\}$ is an open cover of a compact set A and therefore can be reduced to a finite subcover $\{N_i : i \in J_m\}$. Clearly, $A\overline{\delta}B$, because $A \subseteq \bigcup_{i=1}^m N_i$ and $N_i\overline{\delta}B$ for each $i \in J_m$.

Theorem 3.5.5.4 Every compact completely regular (in particular, Hausdorff) ordered space (X, τ, \leq) admits a unique compatible quasi-proximity, which is given by

 $A\delta B$ if and only if $I(A) \cap D(B) \neq \emptyset$.

Proof: By Theorem 3.5.5.3, δ is a compatible quasi-proximity. Also, δ is finer than any compatible quasi-proximity. Further, by Lemma 3.5.5.20, $I(A) \cap D(B) = \emptyset$ implies $I(A)\overline{\delta}'D(B)$, that is, $A\overline{\delta}'B$ for every compatible quasi-proximity δ' , because I(A) is compact. Thus δ is the only compatible quasi-proximity.

We will now introduce the concept of a proximity-order mapping, which will be used in the study of compactifications of a proximity ordered spaces.

Definition 3.5.5.2 Let $(X_i, \delta_i^*, \leq_i)$ (i=1,2) be two proximity quasi-ordered spaces. A mapping $f: X_1 \to X_2$ is said to be a **proximity ordered mapping** if there exists on X_i a generating quasi-proximity δ_i (i=1,2) such that $A\delta_1 B$ implies $f(A)\delta_2 f(B)$. The spaces $(X_1, \delta_1^*, \leq_1)$ and $(X_2, \delta_2^*, \leq_2)$ are said to be **proximally ordered isomorphic** if there exists a one-to-one mapping f of X_1 onto X_2 such that both f and f^{-1} are proximity ordered mappings. In this case f is said to be a **proximity ordered isomorphism**.

It is clear from the above definition that if $f: (X_1, \delta_1^*, \leq_1) \to (X_1, \delta_2^*, \leq_2)$ is a proximity ordered mapping, then $f: (X_1, \leq_1) \to (X_2, \leq_2)$ is increasing and $f: (X_1, \delta_1^*) \to (X_2, \delta_2^*)$ is a proximity mapping. Therefore we have

Theorem 3.5.5.5 If $f: (X_1, \delta_1^*, \leq_1) \to (X_2, \delta_2^*, \leq_2)$ is a proximity ordered mapping, then $f: (X_1, \tau_{\delta_1^*}, \leq_1) \to (X_2, \tau_{\delta_2^*}, \leq_2)$ is a continuous increasing mapping.

The following theorem, which gives a sufficient condition for the converse to hold, is again a generalization of a well-known result on proximity spaces.

Theorem 3.5.5.6 Every continuous increasing function from a compact ordered space to a proximity quasi-ordered space is a proximity ordered mapping.

Proof: If X is a compact ordered space, then the only compatible quasiproximity is given by

 $A\delta B$ if and only if $I(A) \cap D(B) \neq \emptyset$.

Therefore for every mapping $f : X \to (Y, \delta^*, \leq)$, $A\delta B$ implies $f(I(A)) \cap f(D(B)) \neq \emptyset$. If f is continuous and increasing, then $f(I(A)) \subseteq I(f(A))$, and $f(D(B)) \subseteq D(f(B))$. Therefore $A\delta B$ implies $I(f(A)) \cap D(f(B)) \neq \emptyset$, which, in turn, implies $I(A)\delta_2 D(f(B))$, that is, $f(A)\delta_2 f(B)$ for every compatible quasi-proximity δ_2 on (Y, δ^*, \leq) . Thus, for every continuous increasing mapping $f : X \to Y$ and for every compatible quasi-proximity δ_2 on (Y, δ^*, \leq) . Hence f is a proximity-order mapping.

3.5.6 Compactification of a proximity ordered spaces

In this subsection we will construct a compactification of a proximity ordered space. The construction is similar and, in fact, can be performed without resorting to the Leader's construction of the Smirnoff compactification. However, we make use of the latter to avoid manipulations and to simplify some proofs.

Let (X, δ^*, \leq) be a proximity ordered space and let δ be a quasi-proximity on X. Let \mathcal{X} be the set of all clusters in (X, δ^*) . For a subset $A \subseteq X$, let $\overline{\mathcal{A}} = \{\sigma \in \mathcal{X} : A \in \sigma\}$. If $\mathcal{P} \subseteq \mathcal{X}$, we say that A **absorbs** \mathcal{P} if $\mathcal{P} \subseteq \overline{\mathcal{A}}$, that is, A absorbs \mathcal{P} if and only if A is a member of every cluster in \mathcal{P} . Further, for $x \in X$, we let $f(x) = \sigma_x$, the point-cluster. On $P(\mathcal{X})$, let us define

 $\mathcal{P}\delta\mathcal{Q}$ if and only if $A\delta B$ whenever $\mathcal{P} \subseteq \overline{\mathcal{A}}$ and $\mathcal{Q} \subseteq \overline{\mathcal{B}}$; $\mathcal{P}\delta^*\mathcal{Q}$ if and only if $A\delta^*B$ whenever $\mathcal{P} \subseteq \overline{\mathcal{A}}$ and $\mathcal{Q} \subseteq \overline{\mathcal{B}}$.

Then (\mathcal{X}, δ^*) is the Smirnoff compactification of (X, δ^*) . First we will prove:

Lemma 3.5.6.21 The relation $\boldsymbol{\delta}$ is a quasi-proximity on \mathcal{X} . If $\boldsymbol{\delta}$ generates the proximity $\boldsymbol{\delta}^*$ on X, then the quasi-proximity $\boldsymbol{\delta}$ generates the proximity

 δ^* on \mathcal{X} . Further, if δ generates the partial order \leq on X, then the partial order \prec on \mathcal{X} generated by δ is such that X and $f(X) \subseteq \mathcal{X}$ are order isomorphic.

Proof: With a slight modification of arguments in the proof that δ^* is a proximity [238], it can be verified that δ is a quasi-proximity on \mathcal{X} . Let δ be a generating quasi-proximity on X and let δ generate the proximity δ^{**} on \mathcal{X} . To prove that δ^* and δ^{**} are identical, first we will observe that δ^{**} is the coarsest proximity finer than δ and δ^{-1} . Also, because δ^* is finer than both δ and δ^{-1} . Also, because δ^* is finer than δ^{**} . Next, let $\mathcal{P}\overline{\delta}^*\mathcal{Q}$ and let $A, B \subseteq X$ be such that $\mathcal{P} \subseteq \overline{\mathcal{A}}, \mathcal{Q} \subseteq \overline{\mathcal{B}}$ and $A\overline{\delta}^*\mathcal{B}$. Since δ^* is generated by δ , there exist covers $\{A_i : i \in J_m\}$, $\{B_j : j \in J_n\}$ of A and B such that $A_i\overline{\delta}\cap\delta^{-1}B_j$ for each $(i,j)\in J_m\times J_n$. Let us consider $\{\overline{\mathcal{A}}_i : i \in J_m\}$ which is a finite cover of \mathcal{P} and $\{\overline{\mathcal{B}}_j : j \in J_n\}$ which is a finite cover of \mathcal{Q} . As $(i,j)\in J_m\times J_n$ implies $A_i\overline{\delta}B_j$ or $A_i\overline{\delta}^{-1}B_j$, there follows that $\overline{\mathcal{A}}_i\overline{\delta}\cap\delta^{-1}\overline{\mathcal{B}}_j$, whenever $(i,j)\in J_m\times J_n$. Therefore $\mathcal{P}\overline{\delta}^{**}\mathcal{Q}$, the proximity generated by δ . Thus δ^{**} is finer than δ^* . Hence δ^* and δ^{**} are identical.

Let \prec denote the quasi-order generated by δ . As $\delta^*(=\delta^{**})$ is separated, \prec is a partial order. To prove that $x \leq y$ if and only if $f(x) \prec f(y)$, we need simply to show that $x\delta y$ implies $\sigma_x \delta \sigma_y$, since the opposite implication is trivial. Let $\sigma_x \in \overline{\mathcal{A}}$ and $\sigma_y \in \overline{\mathcal{B}}$. Then $A \in \sigma_x$ and $B \in \sigma_y$, that is, $x\delta^*A$ and $y\delta^*B$. Thus $A\delta x$ and $y\delta B$. But $x\delta y$ also holds. The three now taken together imply that $A\delta B$. Therefore $\sigma_x \delta \sigma_y$.

Lemma 3.5.6.22 A triple $(\mathcal{X}, \delta^*, \prec)$ is a compact ordered space such that (X, δ^*, \leqslant) is proximally ordered isomorphic to f(X). Also f(X) is dense in \mathcal{X} .

Proof: That $(\mathcal{X}, \delta^*, \prec)$ is compact and that f(X) is dense in \mathcal{X} , both follows from the fact that (\mathcal{X}, δ^*) is the Smirnoff compactification of (X, δ^*) . To show that \prec is a closed order, we let σ_1, σ_2 in \mathcal{X} be such that $\sigma_1 \prec \sigma_2$ is false, so that there exist $A \in \sigma_1$, $B \in \sigma_2$ such that $A\overline{\delta}B$. Let $E, F \subseteq X$ be such that $A\overline{\delta}E$, $F\overline{\delta}B$ and $X - E\overline{\delta}X - F$. If we consider $\overline{\mathcal{E}}, \overline{\mathcal{F}} \subseteq \mathcal{X}$, then $\sigma_1 \notin D(\overline{\mathcal{E}})$ and $\sigma_2 \notin I(\overline{\mathcal{F}})$. Also $X - E\overline{\delta}X - F$ implies $\overline{\mathcal{E}} \cup \overline{\mathcal{F}} = \mathcal{X}$. Therefore $\mathcal{X} - D(\overline{\mathcal{E}})$ is an increasing neighborhood of σ_1 which is disjoint from $\mathcal{X} - I(\overline{\mathcal{F}})$, a decreasing neighborhood of σ_2 . To prove that X and f(X) are proximally ordered isomorphic, we will show that $A\delta B$ if and only if $f(A)\delta f(B)$ for every generating quasi-proximity δ on X. First, let us observe that $f(B) \subseteq \overline{\mathcal{A}}$ if and only if $B \subseteq cl A(\tau_{\delta^*}$ -closure of A). The implication

$f(A)\boldsymbol{\delta}f(B) \Rightarrow A\delta B$

is obvious because $f(A) \subseteq \overline{A}$ and $f(B) \subseteq \overline{B}$. For the converse implication, let $A\delta B$, $f(A) \subseteq \overline{P}$ and $f(B) \subseteq \overline{Q}$. If possible, let $P\overline{\delta}Q$. Then by Lemma $3.5.5.17 \ I(P)\overline{\delta}D(Q)$ holds, which in turn implies $\operatorname{cl} P\overline{\delta}\operatorname{cl} Q$. But $A \subseteq \operatorname{cl} P$, $B \subseteq \operatorname{cl} Q$ and $A\delta B$ all taken together imply $\operatorname{cl} P\delta\operatorname{cl} Q$. The contradiction proves the assertion.

Lemma 3.5.6.23 Any proximity ordered isomorphism g of (X, δ^*, \leq) onto a dense subset of a compact ordered space (Y, δ_1^*, \leq_1) can be extended to a proximity-order isomorphism of $(\mathcal{X}, \delta^*, \prec)$ onto (Y, δ_1^*, \leq_1) .

Proof: Clearly g, which is a proximity isomorphism of (X, δ^*) onto a dense subset of a compact space (Y, δ_1^*) , can be extended to a proximity isomorphism $h : (\mathcal{X}, \delta^*)$ onto (Y, δ_1^*) defined as follows:

If $\sigma \in \mathcal{X}$, then there exists a unique cluster σ_1 which contains the image

$$g(\sigma) = \{g(A) : A \in \sigma\}$$
 of σ in Y .

Since Y is compact, there exists $y \in Y$ such that $\sigma_1 = \sigma_y$. We will denote $h(\sigma) = y$. We will prove that h defined in such a way is in fact a proximity ordered isomorphism of $(\mathcal{X}, \delta^*, \prec)$ onto (Y, δ_1^*, \leq_1) . Let δ_1 denote the unique compatible quasi-proximity on Y. We will show that $\mathcal{P}\delta \mathcal{Q}$ if and only if $h(\mathcal{P})\delta_1 h(\mathcal{Q})$ for every generating quasi-proximity δ . If $\mathcal{P}\delta\mathcal{Q}$, then $I(\mathcal{P}) \cap D(\mathcal{Q}) \neq \emptyset$. Let σ be in this intersection and let $h(\sigma) = y$. First we will claim that $h(\mathcal{P})\delta_1 y$. Let us assume the contrary and let E and F be subsets of Y such that $h(\mathcal{P})\overline{\delta}_1Y - E$, $Y - F\overline{\delta}_1y$ and $E\overline{\delta}_1F$. Now $h(\mathcal{P})\overline{\delta}_1Y - E$ implies that $g^{-1}(E)$ absorbs \mathcal{P} , i.e. $g^{-1}(E)$ belongs to each cluster in \mathcal{P} . Similarly, $Y - F\overline{\delta}_1 y$ implies $g^{-1}(F) \in \sigma$. Since $\mathcal{P}\delta\sigma$, we have $g^{-1}(E)\delta g^{-1}(F)$. But this is a contradiction because g is a proximity ordered mapping and $E\overline{\delta}_1 F$. Thus $h(\mathcal{P})\delta_1 y$. Similarly $y\delta_1 h(\mathcal{Q})$ and hence $h(\mathcal{P})\delta_1h(\mathcal{Q})$. Conversely, let $h(\mathcal{P})\delta_1h(\mathcal{Q})$. Since Y is compact ordered, there exists $y \in I(h(\mathcal{P})) \cap D(h(\mathcal{Q}))$. Let $\sigma = h^{-1}(y)$. If $A \in \sigma$ and B absorbs \mathcal{Q} , then (regarding X as a subset of Y) we have $A\delta h(\mathcal{Q})$ and $h(\mathcal{Q}) \subseteq \operatorname{cl} B \subseteq D(B)$. This implies that $A\delta B$, so that $\sigma \delta \mathcal{Q}$. Similarly, $\mathcal{P}\delta\sigma$ and therefore $\mathcal{P}\delta\mathcal{Q}$.

Combining Lemmas 3.5.6.21, 3.5.6.22 and 3.5.6.23, it follows that

Theorem 3.5.6.1 Every proximity ordered space (X, δ^*, \leq) is a dense subset of a unique (up to proximity ordered isomorphism) compact ordered space

 \mathcal{X} . Since \mathcal{X} has a unique compatible quasi-proximity, for subsets A, $B \subseteq X$, it follows that

 $A\delta B$ if and only if $I(A) \cap D(B) \neq \emptyset$ in \mathcal{X} ,

where δ is any generating quasi-proximity on X.

In the above theorem we call $(\mathcal{X}, \delta^*, \prec)$ the Smirnoff ordered compactification of (X, δ^*, \leq) .

Theorem 3.5.6.2 Every proximity-ordered mapping of a proximity ordered space $(X_1, \delta_1^*, \leq_1)$ onto another proximity ordered space $(X_2, \delta_2^*, \leq_2)$ can be extended uniquely to a continuous increasing mapping of the Smirnoff ordered compactification of X_1 onto the Smirnoff ordered compactification of X_2 .

Proof: Let g be a proximity ordered mapping of the proximity ordered space $(X_1, \delta_1^*, \leq_1)$ onto the proximity ordered space $(X_2, \delta_2^*, \leq_2)$. Since $g : (X_1, \delta_1^*) \to (X_2, \delta_2^*)$ is a proximity mapping, it can be extended to a continuous mapping $h : (\mathcal{X}_1, \boldsymbol{\delta}_1^*) \to (\mathcal{X}_2, \boldsymbol{\delta}_2^*)$, where h is defined by setting

 $h(\sigma_1) = \sigma_2 = \{ B \in P(X_2) : B\delta_2^* g(C) \text{ for all } C \in \sigma_1 \}.$

We will show that h is in fact a proximity ordered mapping and hence continuous and increasing. Let δ_i be a generating quasi-proximity on X_i (i = 1, 2) such that $A\delta_1 B$ implies $g(A)\delta_2 f(B)$. Now, let $\mathcal{P}\boldsymbol{\delta}_1 \mathcal{Q}$ and let Aabsorb $h(\mathcal{P})$ and let B absorb $h(\mathcal{Q})$. If possible, let $A\overline{\delta}_2 B$. Let $C, D \subseteq X_2$ be such that

$$A\overline{\delta}_2 X_2 - C$$
, $X_2 - D\overline{\delta}_2 B$ and $C\overline{\delta}_2 D$

Now A absorbs $h(\mathcal{P})$ and $A\overline{\delta}_2 X_2 - C$; therefore $X_2 - C$ belongs to no cluster in $h(\mathcal{P})$ and, as g is a proximity mapping, $g^{-1}(X_2 - C) = X_1 - g^{-1}(C)$ belongs to no cluster in \mathcal{P} , i.e. $g^{-1}(C)$ absorbs \mathcal{P} . Similarly, $g^{-1}(D)$ absorbs \mathcal{Q} . As $\mathcal{P}\boldsymbol{\delta}_1\mathcal{Q}$, we must have $g^{-1}(C)\delta_1g^{-1}(D)$, which is a contradiction because g is a proximity mapping and $C\overline{\delta}_2D$.

Further, g has a unique continuous extension; so it cannot have more than one continuous increasing extension, and hence the uniqueness of h follows.

Now we come to the result which relates the study of quasi-proximities to the study of compactifications of ordered spaces. Let X be an ordered space and let \mathcal{X}_1 and \mathcal{X}_2 be two compactifications of X. If we set $\mathcal{X}_1 \ge \mathcal{X}_2$ if and only if the identity mapping on X can be extended to a continuous increasing mapping of \mathcal{X}_1 onto \mathcal{X}_2 , then \geq is a partial order on the set of all compactificatons of X. Further, the set of quasi-proximities on X is also a partially ordered set. The following result follows directly from Theorems 3.5.5.4 and 3.5.6.2.

Theorem 3.5.6.3 Given a completely regular ordered space (X, τ, \leq) , the Smirnoff ordered compactification defines an order isomorphism of the set $\{\delta_i : i \in I\}$ of compatible quasi-proximities on X onto the set $\{\mathcal{X}_i : i \in I\}$ of the Smirnoff ordered compactifications of X.

We conclude this subsection by giving three results which are analogous of well-known results on proximity spaces and which can be proved using the Smirnoff compactification as constructed above.

Theorem 3.5.6.4 Let (X, δ^*, \leq) be a proximity ordered space. Then, for every generating quasi-proximity δ , there follows that

 $A\overline{\delta}B$ if and only if there exists a proximity ordered mapping $f: X \to [0,1]$ such that g(A) = 1, g(B) = 0.

Theorem 3.5.6.5 Let A be any subspace of a proximity ordered space (X, δ^*, \leq) and let $g : A \to [0, 1]$ be a proximity ordered mapping. Then g can be extended to a proximity ordered mapping $h : X \to [0, 1]$.

Proof: First we will apply Theorem 3.5.6.2 to extend g to $\tilde{g} : \mathcal{A} \to [0, 1]$. Next we will apply Corollary 3.5.1.4 to extend \tilde{g} to $\bar{h} : \mathcal{X} \to [0, 1]$. Now, using Theorem 3.5.5.6 the required mapping $h = \bar{h}|X$ follows.

Theorem 3.5.6.6 Let (X, τ, \leq) be a normally ordered space and let δ be a compatible quasi-proximity on X. Then the following assertions are equivalent:

(a) for subsets $A, B \subseteq X, I(A) \cap D(B) = \emptyset$ implies $A\overline{\delta}B$;

(b) every real-valued continuous increasing function on X is a proximity-order function.

Proof: The proof that (a) implies (b) is similar to that of Theorem 3.5.5.6. To prove that (b) implies (a), we apply Theorem 3.5.6.4

Historical and bibliographic notes

The concept of a topological quasi-ordered space was introduced by L. Nachbin in 1948 (see [230]). In the same year L. Nachbin introduced the concept of uniform ordered spaces (see [231], [232]). The systematic exposition of his results in theory of topological quasi-ordered and ordered spaces, uniform quasi-ordered and ordered spaces was given in his book [233]. Theorems 3.5.1.1, 3.5.1.3, 3.5.1.6, 3.5.3.1, 3.5.3.5, 3.5.3.6 and 3.5.3.7 were proved in that book. The theorem 3.5.1.2 was proved by Z. Semadeni and H. Zidenberg [282]. The notion of perfectly normally ordered topological space was introduced by the author in 1974 (see [72]). Theorems 3.5.1.4 and 3.5.1.5 were proved by the author in paper [75]. An order compactification was introduced first by L. Nachbin in 1948 (see [230]). T. McCallion first introduced the notion of an one point order compactification in paper [215]. In this paper he proved Theorems 3.5.2.3, 3.5.2.4, Proposition 3.5.2.2 and Theorem 3.5.2.5. Theorem 3.5.2.3 and Propositions 3.5.3.5, 3.5.3.6 and 3.5.3.7 were proved by L. Nachbin (see [233]). Proposition 3.5.3.8 was proved by the author (see [73]). The notion of nearly uniform ordered spaces was introduced by Redfield in paper [267]. All the results of subsection 5.4. were proved in that paper. M. K. Singal and S. Lal introduced the notion of proximity ordered space in 1976 (see [288]). All the results of subsections 5.5. and 5.6. were proved in paper [288] (see also [77]). Many other properties of topological ordered spaces, uniform ordered spaces and proximity ordered spaces can be found in [28], [29], [55], [78], [79], [81], [127], [199], [213], [214], [264] and [286].

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