UNIVERSITY OF NIŠ
FACULTY OF SCIENCES AND MATHEMATICS

Marija S. Cvetković

FIXED POINT THEOREMS OF PEROV TYPE

DOCTORAL DISSERTATION

Niš, 2017.

УНИВЕРЗИТЕТ У НИШУ
ПРИРОДНО-МАТЕМАТИЧКИ ФАКУЛТЕТ

Марија С. Цветковић

ФИКСНЕ ТАЧКЕ ЗА ПРЕСЛИКАВАЊА ПЕРОВОГ ТИПА

ДОКТОРСКА ДИСЕРТАЦИЈА

Ниш, 2017.

Data on Doctoral Dissertation

Doctoral Supervisor:	Vladimir Rakoćević, PhD, full professor at Faculty of Sciences and Mathematics, University of Niš, corresponding member of SASA
Title:	Fixed point theorems of Perov type
Abstract:	In this dissertation is introduced a new class of contractions in the setting of cone metric space, both solid and normal, by including an operator as a contractive constant. Some well-known fixed point theorems are improved and obtained results generalize, Banach, Perov, Ćirić and Fisher theorem, among others. Common fixed point problem for a pair or a sequence of mappings is studied from a different point of view. We also discuss Perov type results on partially ordered cone metric spaces and cone metric spaces with *w* distance. Wide range of applications is corroborated with numerous examples with special interest in Ulam's stability of functional equations.
Scientific Field:	Mathematics
Scientific Discipline:	Nonlinear Analysis, Functional Analysis
Key Words:	Fixed point, quasi-contraction, common fixed point, Perov theorem, *w* distance, Ulam's stability
UDC:	517.988(043.3)
CERIF Classification:	P130 Functions, differential equations P140 Series, Fourier analysis, functional analysis
Creative Commons License Type:	**CC BY-NC-ND**

Подаци о докторској дисертацији

Ментор:	Др Владимир Ракочевић, редовни професор, Природно-математички факултет, Универзитет у Нишу, дописни члан САНУ
Наслов:	Фиксне тачке за пресликавања Перовог типа
Резиме:	У оквиру истраживања обухваћених овом дисертацијом, разматрана је нова класа контрактивних пресликавања на конусном метричком простору којим се у контрктивни услов укључује оператор. У зависноти од посматраниог простора, наметнути су различити захтеви које оператор и/или пресликавање морају испунити да би се гарантовало постојање фиксне тачке и, у специјалним случајевима, њена јединственост.Представљени резултати уопштавају многе познате теореме о фиксној тачки попут Банахове, Ћирићеве, Фишерове и Перове. Представљени су и резултати на парцијално уређеним метричким просторима, те просторима са *w* дистанцом. Изучаван је и проблем заједничке фиксне тачке за пар и низ пресликавања. Широк спектар примене је демонстртиран кроз многобројне примере нарочито у решавању интегралних једначина и утврђивању Улам стабилности функционалних једначина.
Научна област:	Математика
Научна дисциплина:	Нелинеарна анализа, Функционална анализа
Кључне речи:	Фиксна тачка, Перова теорема, квазиконтракција, Улам стабилност, заједничка фиксна тачка, *w* дистанца
УДК:	517.988(043.3)
CERIF класификација:	P130 Functions, differential equations P140 Series, Fourier analysis, functional analysis
Тип лиценце Креативне заједнице:	

ПРИРОДНО - МАТЕМАТИЧКИ ФАКУЛТЕТ
НИШ

KEY WORDS DOCUMENTATION

Accession number, **ANO**:	
Identification number, **INO**:	
Document type, **DT**:	**monograph**
Type of record, **TR**:	**textual / graphic**
Contents code, **CC**:	**doctoral dissertation**
Author, **AU**:	**Marija S. Cvetković**
Mentor, **MN**:	**Vladimir Rakočević**
Title, **TI**:	**FIXED POINT THEOREMS OF PEROV TYPE**
Language of text, **LT**:	**Serbian**
Language of abstract, **LA**:	**English**
Country of publication, **CP**:	**Serbia**
Locality of publication, **LP**:	**Serbia**
Publication year, **PY**:	**2017.**
Publisher, **PB**:	**author's reprint**
Publication place, **PP**:	**Niš, Višegradska 33.**
Physical description, **PD**: (chapters/pages/ref./tables/pictures/graphs/appendixes)	**111 p. ; graphic representations**
Scientific field, **SF**:	**mathematics**
Scientific discipline, **SD**:	**mathematical analysis**
Subject/Key words, **S/KW**:	**nonlinear analysis, functional analysis,**
UC	517.988(043.3)
Holding data, **HD**:	**Library**
Note, **N**:	
Abstract, **AB**:	In this dissertation is introduced a new class of contractions in the setting of cone metric space, both solid and normal, by including an operator as a contractive constant. Some well-known fixed point theorems are improved and obtained results generalize, Banach, Perov, Ćirić and Fisher theorem, among others. Common fixed point problem for a pair or a sequence of mappings is studied from a different point of view. Wide range of applications is corroborated with numerous examples.
Accepted by the Scientific Board on, **ASB**:	
Defended on, **DE**:	
Defended Board, **DB**: President:	
Member:	
Member, Mentor:	

ПРИРОДНО - МАТЕМАТИЧКИ ФАКУЛТЕТ
НИШ

КЉУЧНА ДОКУМЕНТАЦИЈСКА ИНФОРМАЦИЈА

Редни број, **РБР**:	
Идентификациони број, **ИБР**:	
Тип документације, **ТД**:	монографска
Тип записа, **ТЗ**:	текстуални / графички
Врста рада, **ВР**:	докторска дисертација
Аутор, **АУ**:	Марија С. Цветковић
Ментор, **МН**:	Владимир Ракочевић
Наслов рада, **НР**:	ФИКСНЕ ТАЧКЕ ЗА ПРЕСЛИКАВАЊА ПЕРОВОГ ТИПА
Језик публикације, **ЈП**:	српски
Језик извода, **ЈИ**:	Српски
Земља публиковања, **ЗП**:	Србија
Уже географско подручје, **УГП**:	Србија
Година, **ГО**:	2017.
Издавач, **ИЗ**:	ауторски репринт
Место и адреса, **МА**:	Ниш, Вишеградска 33.
Физички опис рада, **ФО**: (поглавља/страна/ цитата/табела/слика/графика/прилога)	111 стр., граф. Прикази
Научна област, **НО**:	математика
Научна дисциплина, **НД**:	математичка анализа
Предметна одредница/Кључне речи, **ПО**:	нелинеарна анализа, функционална анализа
УДК	517.988(043.3)
Чува се, **ЧУ**:	библиотека
Важна напомена, **ВН**:	
Извод, **ИЗ**:	У оквиру истраживања обухваћених овом дисертацијом, разматрана је нова класа контрактивних пресликавања на конусном метричком простору којим се у контрктивни услов укључује оператор. Представљени резултати уопштавају многе познате теореме о фиксној тачки. Изучаван је и проблем заједничке фиксне тачке. Широк спектар примене је демонстртиран кроз многобројне примере.
Датум прихватања теме, **ДП**:	
Датум одбране, **ДО**:	
Чланови комисије, **КО**: Председник:	
Члан:	
Члан, ментор:	

Preface

This thesis is focused on the theory of operatorial contractions which arises as an extent of Perov fixed point theorem. The setting is a cone metric space, both solid and normal. Main idea was to, inspired by Perov contraction, obtain various fixed point results for a new class of contractions including positive linear operator with spectral radius less than one instead, of a contractive constant. Quasi-contraction and Fisher quasi-contraction could be also redefined in a sense of Perov type mappings and adequate theorems regarding existence of a fixed point are stated. We deal also with a new approach to Hardy-Rogers theorem in addition to common fixed point problem for pairs and sequence (family) of mappings. Omitting some requirements, such as linearity, is a line of study that should be followed in the future research. Theory is illustrated with important examples that accentuate originality, independence and applicability of collected results. Having in mind analogous results on metric or cone metric space (without operator as a constant), it is crucial to determine that results presented in this thesis could not be obtained from those previously published by any renormizaton or scalarization technique and, hence, represent a real improvement and generalization. This is substantiated with theoretical considerations and several examples demonstrating applications of our results in the case when well-known analogons are not applicable.

Finding a new approach to well-known and extensively studied problem in the functional analysis was the main motivation. Throughout history, even taking into account latest research, extensions of Banach theorem went in two directions. Some of the authors altered the set of distances, others just redefined same Banach condition but in the different surrounding. Only Perov's result gave a different overview and following those steps, but on a much wider class of spaces, we define Perov type contractions, prove some existence and uniqueness results along with numerical estimations that have impact on convergence, orbit, etc. It should be mentioned that used proof techniques combine operator theory, linear algebra and nonlinear analysis for the purpose of fixed point theory.

The dissertation is organized in five different chapters followed by Concluion, Bibiliography and Biography of the author. Starting with the introductory chapter 1, we gather short historical survey, basic concepts and notations needed to follow the storyline of the manuscript. Section 1.1 is a brief expose on the development of metric fixed point theory and famous fixed point theorems that will be studied in the following chapters (see sections 2.2 and 2.3). Defining cone metric space is what follows in section 1.2 along with simple auxiliary results determining the difference between solid and normal cone. We introduce the results of Perov in section 1.3 and make connections between cone metric space and generalized metric space in the sense of Perov. The end of this chapter is a short overview on some spectral properties of a bounded linear operator on Banach space.

Our main results are gathered in Chapter 2. Perov type fixed point theorem, both on solid and normal cone metric space, is the most important part of Section 2.1. Normal case is extension of Perov theorem since the operator (matrix in Perov case) is not necessarily positive. Theory is substantiated with several examples. The Perov condition is altered by adding a nonlinear operator in the sense of Berinde. Sections 2.2 and 2.3 analyze Perov type quasi-contraction and (p, q)-quasi-contraction pointing out to the possible future research. Omitting linearity request was important result in this research and therefore section 2.6 improves results of 2.1 and all published results on this type of mappings. Section 2.5 includes some new results of Perov type on partially ordered cone metric spaces that are not yet published. Moreover, this chapter contains several interesting applications. Important addendum to this chapter is Chapter 4 corroborating value of presented findings.

Common fixed point problem, as a special case of coincidence problem, is broadly studied due to important and valuable applications. We unify those results in Chapter 3. In the first part of the chapter, we extend and summarize results for the pair of mappings. Also, in 3.1, we define interesting properties in order to weaken commutativity condition. Section 3.2 suggests different approach in order to find common fixed point property for the sequence of mappings. Therein, as a special case, we additionally analyze pair of mappings apart from 3.1.

Of a great relevance is the Chapter 4. Perov theorem could originate from Banach theorem but just for the existence part since estimations regarding distance of the iterative sequence from fixed point are unrelated. Other part of 4.1 is justifying statements on normal cone metric space and certifying its value. We choose Du's scalarization method to examine relation between solid cone metric and metric space, and, as a consequence, our results and metric fixed point theorems. The fact that operator appears in a contractive condition plays the crucial role in obtaining preferable outcome-it is not possible to reduce theorems on solid cone metric space presented in this dissertation to Banach fixed point theorem and equivalent results.

Obtained results have a wide range of application including differential inclusions, solving operator, integral and differential equations, well-posedness of multiple problems and Ulam's stability of functional equations, among others. Overall, in every chapter you can find several examples implicating different kind of applications. Therefore, in Chapter 5 we cover only two implementations of Perov type results. Section 5.1 substantiates application in solving integral equations with a few examples representing different kind of integral equations. However, in 5.2 generalize Ulam's (or equivalently Ulam-Hyers or Ulam-Hyers-Rasias) stability of functional equations is an interesting topic and offers a lot of possibility for further research. It is interesting that many recently published Ulam's stability results using the fixed point techniques are direct corollaries of results in this thesis.

In the end we summarize presented result, emphasise their contribution and discuss further research. For further and more detailed information regarding some previous research or some results left over, see the reference list in chapter 6.

The content of this thesis is based on several published articles on this topic:

1. M. Cvetković, V. Rakočević, *Quasi-contraction of Perov type*, App. Math. Com., **237** (2014), 712-722.

2. M. Cvetković, V. Rakočević, *Exstensions of Perov theorem*, Carpathian J. Math., **31** (2015), 181-188.

3. M. Cvetković, V. Rakočević, *Fisher quasi-contraction of Perov type*, Nonlinear Convex. Anal., **16** (2015), 339-352.

4. M. Cvetković, V. Rakočević, *Common fixed point results for mappings of Perov type*, Math. Nach., **288** (2015) 1873-1890.

5. M. Cvetković, V. Rakočević, Billy E. Rhoades, *Fixed point theorems for contractive mappings of Perov type*, Nonlinear Convex. Anal., **16** (2015), 2117-2127.

6. M. Cvetković, V. Rakočević, *Fixed point of mappings of Perov type for w-cone distance*, Bul. Cl. Sci. Math. Nat. Sci. Math. **40** (2015), 57-71.

7. D. Ilić, M. Cvetković, Lj. Gajić and V. Rakočević, *F*ixed points of sequence of Ćirić generalized contractions of Perov type, Mediterr. J. Math., **13** (2016), 3921-3937.

8. M. Cvetković, *O*peratorial contractions on solid cone metric spaces, Nonlinear Convex. Anal., **17** (2016), 1399-1408.

and presented at three international and one national conference along with two guest lectures at Univeristy Babeş-Bolyai, Cluj-Napoca and UNSW, Sydney. Some papers were not included in the manuscript:

9. P. S. Stanimirović, D. Pappas, V. N. Katsikis, M. Cvetković, *Outer inverse restricted by a linear system*, Linear Multilinear Algebra, **63** (2015), 2461-2493.

10. X. Wang, H. Ma, M. Cvetković, *A Note on the Perturbation Bounds of W-weighted Drazin Inverse of Linear Operator in Banach Space*, Filomat, **13** (2017), 505-511.

11. M. Cvetković, E. Karapinar, V. Rakočević, *Some fixed point results on quasi-b-metric like spaces*, J. Inequal. Appl., **2015** (2015), 2015:374

On the other side, some included results are not yet published.

It is author's great pleasure to express sincere gratitude to all my friends and collaborators, among them prof. Lj. Gajić, prof. P. S. Stanimirović, prof. A. Petrusel, prof.

S. Radenović and especially prof. E. Karapinar for useful comments and discussions. I will stay grateful to my thesis advisor prof. Vladimir Rakočević for all his help, ideas, remarks, patient and successful guidance throughout this process. In the end, there is the beginning, so I am very thankful to have such amazing parents and enjoy a great amount of family's support, love and encouragement.

Contents

Chapter 1

Introduction

This chapter unifies all necessary notations, definitions and theorems to make this manuscript self-contained and to properly complement presented results. It includes many different areas of mathematics such as linear algebra, operator theory, nonlinear analysis and, the most significant, fixed point theory. For the convenience of a reader, we divide it in several sections.

First part gives a retrospective on some meaningful and well-known fixed point theorems on a complete metric space. Important section is related to the concept of cone metric space and w-cone distance and fixed point theorems with both solid and normal cones. Generalized metric space in the sense of Perov along with a Perov contraction is defined and discussed in the sequence. Additionally, some matrix properties are mentioned. Finally, some basic concepts and notations in operator theory are necessary in dealing with Perov type contraction and therefore included as a preliminary.

1.1 Metric fixed point theory

For a non-empty set X and a mapping $f : X \mapsto X$, x is called a fixed point of f if $f(x) = x$. Set of all fixed point for the mapping f is denoted with $Fix(f)$. An arbitrary mapping on X could have unique, many or none fixed point.

Example 1. (i) $f(x) = x$, $x \in \mathbb{R}$ has infinitely many fixed points;

(ii) $f(x) = x + 1$, $x \in \mathbb{N}$ has no fixed points;

(iii) $f(x) = \frac{x}{2}$, $x \in \mathbb{Q}$, has a unique fixed point.

This choice of examples underlines absence of correlation between continuity and fixed point. On the other hand, boundedness and linearity also have no effect on existence of a fixed point.
The easiest way to explain such amount of interest for this topic is analyzing the equation $F(x) = 0$ which easily transforms into the fixed point problem. Selected framework will be metric space.

Definition 1.1.1. *Let X be non-empty set and $d : X \times X \mapsto \mathbb{R}$ mapping such that:*

(d_1) $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$;

(d_2) $d(x, y) = d(y, x)$;

(d_3) $d(x, y) \leq d(x, z) + d(z, y)$.

Assuming that elementary facts about metric spaces are well-known, we would not go into the details.

In metric fixed point, theory we may say that everything starts and finishes with famous Banach fixed point theorem, also known as the contraction principle. It was first stated by famous mathematician Stefan Banach in [18] published in 1922. The proof given by Banach was based on defining an iterative sequence (sequence of successive approximations) and through decades was shortened and improved. The mapping f on a metric space X is named contraction if there exists some constant $q \in (0, 1)$ such that

$$d(f(x), f(y)) \leq qd(x, y),\ x, y \in X.$$

The constant q is known as the contractive constant. Clearly, every contraction is a non-expansive mapping. For any self-mapping we define a sequence (x_n), $x_n = f(x_{n-1})$, $n \in \mathbb{N}$, for arbitrary $x_0 \in X$. It is called a sequence of successive approximations or iterative sequence

Theorem 1.1.2. *([18]) Let (X, d) be a non-empty complete metric space with a contraction mapping $f : X \mapsto X$. Then f admits a unique fixed point in X and for any $x_0 \in X$ the iterative sequence (x_n) converges to the fixed point of f.*

Great part of Banach fixed point theorem's success is based on iterative sequence ad its convergence which found applications in numerical analysis. It also gives good upper bound for contractive constant q since if $q = 1$, fixed point do not necessary exist. (See Example 1 (*ii*))

Further on, research went in two separate ways-changing the contractive condition or changing the setting or even combining the both. Modification of the contractive condition is expressed through Ćirić and Fisher quasi-contractions, common and coupled fixed point problem, etc. Otherwise, metric space is replaced with partial metric, b-metric, cone metric space, space with ω distance and so on. In this section we remain interested only for results on a complete metric space and suggest to pay attention on two classical results that extend many different kinds of contractions.

Serbian mathematician Ljubomir Ćiric studied new type of mappings such that, for some $q \in (0, 1)$ and any $x, y \in X$:

$$d(f(x), f(y)) \leq q \max\Big\{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\Big\},$$

known as quasi-contraction or Ćirić quasi-contraction. In [39] in 1974. he gave the proof of a theorem:

Theorem 1.1.3. *If (X, d) is a complete metric space and $f : X \mapsto X$ a quasi–contraction, then it possesses a unique fixed point and the iterative sequence converges to the fixed point of f.*

Example that convinces us that the class of quasi-contraction is strict superset of contractions in presented in [39].

Fixed point in both Banach and Ćirić theorem, and many similar, is a limit of the iterative sequence. Therefore, f^n, $n \in \mathbb{N}$, has negligible impact on a fixed point. Having that in mind along with Ćiric's result, it proceeds different condition:

$$d(f^p(x), f^q y) \leq q \max \Big\{ d(f^r x, f^s y), d(f^r x, f^{r'} x), d(f^s y, f^{s'} y) \mid 0 \leq r, r' \leq p \quad \text{and} \quad 0 \leq s, s' \leq q \Big\}.$$

for some $p, q \in \mathbb{N}$ and any $x, y \in X$, determining a (p, q)-quasi-contraction. Fisher ([54]) proved that continuous (p, q)-quasi-contraction on a complete metric space possesses a unique fixed point. If $p = 1$ or $q = 1$, continuity is not necessary. Ćirić quasi-contraction is a special case for $p = q = 1$.

A multitude of fixed point theorems on metric spaces will be mentioned in the sequent, even though just those two will be extensively studied.

1.2 Cone metric space

The concept of a cone metric space (vector valued metric space, K-metric space) has a long history (see [67, 110, 129]) and first fixed point theorems in cone metric spaces were obtained by Schröder [122, 123] in 1956. Cone metric space may be considered as a generalization of metric space and it is focus of the research in metric fixed point theory last few decades (see, e.g., [2, 6, 19, 44, 57, 73, 79, 117, 112] for more details). Most authors give credit to Huang and Zhang ([67]), but it is not sufficiently mentioned that serbian mathematician Kurepa published the same idea much before.

Definition 1.2.1. *Let E be a real Banach space with a zero vector θ. A subset P of E is called a cone if:*

(i) *P is closed, nonempty and $P \neq \{\theta\}$;*

(ii) *$a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$;*

(iii) *$P \cap (-P) = \{\theta\}$.*

Given a cone $P \subseteq E$, the partial ordering $\preceq$ with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ denotes $y - x \in \operatorname{int} P$ where $\operatorname{int} P$ is the interior of P.

The cone P in a real Banach space E is called normal if

$$\inf\{\|x+y\| \mid x, y \in P \text{ and } \|x\| = \|y\| = 1\} > 0$$

or, equivalently, if there is a number $K > 0$ such that for all $x, y \in P$,

$$\theta \preceq x \preceq y \text{ implies } \|x\| \leq K \|y\|. \tag{1.1}$$

The least positive number satisfying (1.1) is called the normal constant of P. The cone P is called solid if $\operatorname{int} P \neq \emptyset$.

Definition 1.2.2. *Let X be a nonempty set, and let P be a cone on a real ordered Banach space E. Suppose that the mapping $d : X \times X \mapsto E$ satisfies:*

(d_1) $\theta \preceq d(x,y)$, for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;

(d_2) $d(x,y) = d(y,x)$, for all $x, y \in X$;

(d_3) $d(x,y) \preceq d(x,z) + d(z,y)$, for all $x, y, z \in X$.

Then d is called a cone metric on X and (X,d) is a cone metric space.

It is known that the class of cone metric spaces is bigger than the class of metric spaces.

Example 2. Let $E = l^1, P = \{(x_n)_{n\in\mathbb{N}} \in E \mid x_n \geq 0,\ n \in \mathbb{N}\}, (X, \rho)$ be a metric space and $d : X \times X \mapsto E$ defined by $d(x,y) = \left(\frac{\rho(x,y)}{2^n}\right)_{n\in\mathbb{N}}$. Then (X,d) is a cone metric space.

Example 3. Let $X = \mathbb{R}, E = \mathbb{R}^n$ and $P = \left\{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_i \geq 0,\ i = \overline{1,n}\right\}$. it is easy to see that $d : X \times X \mapsto E$ defined by $d(x,y) = (|x-y|, k_1|x-y|, \ldots, k_{n-1}|x-y|)$ is a cone metric on X, where $k_i \geq 0$ for $i = \overline{1, n-1}$.

Example 4. ([44]) Let $E = C^{(1)}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ on $P = \{x \in E \mid x(t) \geq 0 \text{ on } [0,1]\}$. Consider, for example,

$$x_n(t) = \frac{1 - \sin nt}{n+2} \quad \text{and} \quad y_n(t) = \frac{1 + \sin nt}{n+2},\ n \in \mathbb{N}.$$

Deducing $\|x_n\| = \|y_n\| = 1$ and $\|x_n + y_n\| = \frac{2}{n+2} \to 0$ as $n \to \infty$, we see that it is a non-normal cone.

Presumably, convergent and Cauchy sequence are naturally defined, Suppose that E is a Banach space, P is a solid cone in E, whenever it is not normal, and $\preceq$ is the partial order on E with respect to P.

Definition 1.2.3. *The sequence $(x_n) \subseteq X$ is convergent in X if there exists some $x \in X$ such that*

$$(\forall\, c \gg \theta)(\exists\, n_0 \in \mathbb{N})\, n \geq n_0 \implies d(x_n, x) \ll c.$$

We say that a sequence $(x_n) \subseteq X$ converges to $x \in X$ and denote that with $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, $n \to \infty$. Point x is called a limit of the sequence (x_n).

Definition 1.2.4. *The sequence $(x_n) \subseteq X$ is a Cauchy sequence if*

$$(\forall\, c \gg \theta)(\exists\, n_0 \in \mathbb{N})\, n, m \geq n_0 \implies d(x_n, x_m) \ll c.$$

Every convergent sequence is a Cauchy sequence, but reverse do not hold. If any Cauchy sequence in a cone metric space (X, d) is convergent, then X is a complete cone metric space.

As proved in [67], if P is a normal cone, even in the case $\operatorname{int} P = \emptyset$, then $(x_n) \subseteq X$ converges to $x \in X$ if and only if $d(x_n, x) \to \theta$, $n \to \infty$. Similarly, $(x_n) \subseteq X$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$, $n, m \to \infty$. Also, if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $d(x_n, y_n) \to d(x, y)$, $n \to \infty$. Let us emphasise that this equivalences do not hold if P is a non-normal cone.

The following properties are often used (particulary when dealing with cone metric spaces in which the cone need not to be normal):

($\mathbf{p}_1$) If $u \preceq v$ and $v \ll w$ then $u \ll w$.

($\mathbf{p}_2$) If $\theta \preceq u \ll c$ for each $c \in \operatorname{int} P$ then $u = \theta$.

($\mathbf{p}_3$) If $a \preceq b + c$ for each $c \in \operatorname{int} P$ then $a \preceq b$.

($\mathbf{p}_4$) If $\theta \preceq x \preceq y$, and $\lambda \geq 0$, then $\theta \preceq \lambda x \preceq \lambda y$.

($\mathbf{p}_5$) If $\theta \preceq x_n \preceq y_n$ for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, then $\theta \preceq x \preceq y$.

($\mathbf{p}_6$) If $\theta \preceq d(x_n, x) \preceq b_n$ and $b_n \to \theta$, then $x_n \to x$.

($\mathbf{p}_7$) If E is a real Banach space with a cone P and if $a \preceq \lambda a$, where $a \in P$ and $0 < \lambda < 1$, then $a = \theta$.

($\mathbf{p}_8$) If $c \in \operatorname{int} P$, $\theta \preceq a_n$ and $a_n \to \theta$, then there exists n_0 such that for all $n > n_0$ we have $a_n \ll c$.

From ($\mathbf{p}_8$) it follows that the sequence $\{x_n\}$ converges to $x \in X$ if $d(x_n, x) \to \theta$ as $n \to \infty$ and $\{x_n\}$ is a Cauchy sequence if $d(x_n, x_m) \to \theta$ as $n, m \to \infty$. In the situation with a non-normal cone we have only one part of Lemmas 1 and 4 from [67]. Also, in this case the fact that $d(x_n, y_n) \to d(x, y)$ if $x_n \to x$ and $y_n \to y$ is not applicable.
A mapping $f : X \mapsto X$ is a continuous mapping on X if for any $x \in X$ and a sequence $(x_n) \subseteq X$ such that $\lim_{n\to\infty} x_n = x$, it follows $\lim_{n\to\infty} f(x_n) = f(x)$.

For the purpose of Section 2.4, we collect some basic knowledge regarding ω-distance on cone metric space.

Function $G : X \to P$ is lower semi-continuous at $x \in X$ if for any $\varepsilon \gg \theta$, there is $n_0 \in \mathbb{N}$ such that

$$G(x) \preceq G(x_n) + \varepsilon, \quad \text{for all} \quad n \geq n_0, \tag{1.2}$$

whenever (x_n) is a sequence in X and $x_n \to x$, $n \to \infty$.

Definition 1.2.5 ([42]). *Let (X, d) be a cone metric space. Then a function $p : X \times X \to P$ is called a w-cone distance on X if the following conditions are satisfied:*

(w_1) *$p(x, z) \preceq p(x, y) + p(y, z)$, for any $x, y, z \in X$;*

(w_2) *For any* $x \in X$, $p(x,\cdot): X \to P$ *is lower semi-continuous;*

(w_3) *For any* ε *in* E *with* $\theta \ll \varepsilon$, *there is* δ *in* E *with* $\theta \ll \delta$, *such that* $p(z,x) \ll \delta$ *and* $p(z,y) \ll \delta$ *imply* $d(x,y) \ll \varepsilon$.

It is important to mention that every cone metric is w-cone distance and there exist w-cone distances such that underlying cone is not normal.

Lemma 1.2.6. *([42])Let* (X,d) *be a tvs-cone metric space and let* p *be a* w*-cone distance on* X. *Let* (x_n) *and* (y_n) *be sequences in* X, (α_n) *with* $\theta \preceq \alpha_n$, *and* (β_n) *with* $\theta \preceq \beta_n$, *be sequences in* E *converging to* θ, *and* $x,y,z \in X$. *Then:*

(i) If $p(x_n,y) \preceq \alpha_n$ *and* $p(x_n,z) \preceq \beta_n$ *for any* $n \in \mathbb{N}$, *then* $y = z$. *In particular, if* $p(x,y) = \theta$ *and* $p(x,z) = \theta$, *then* $y = z$.

(ii) If $p(x_n,y_n) \preceq \alpha_n$ *and* $p(x_n,z) \preceq \beta_n$ *for any* $n \in \mathbb{N}$, *then* (y_n) *converges to* z.

(iii) If $p(x_n,x_m) \preceq \alpha_n$ *for any* $n,m \in \mathbb{N}$ *with* $m > n$, *then* (x_n) *is a Cauchy sequence.*

(iv) If $p(y,x_n) \preceq \alpha_n$ *for any* $n \in \mathbb{N}$, *then* (x_n) *is a Cauchy sequence.*

1.3 Perov theorem

Russian mathematician A. I. Perov ([101]) defined generalized cone metric space by introducing a metric with values in $\mathbb{R}^n$. Then, this concept of metric space allowed him to define a new class of mappings, known as Perov contractions, which satisfy contractive condition similar to Banach's, but with a matrix $A \in \mathbb{R}^{n\times n}$ with nonnegative entries instead of a constant q.

Let X be a nonempty set and $n \in \mathbb{N}$.

Definition 1.3.1. *([101]) A mapping* $d: X \times X \mapsto \mathbb{R}^n$ *is called a vector-valued metric on* X *if the following statements are satisfied for all* $x,y,z \in X$.

(d_1) $d(x,y) \geq 0_n$ *and* $d(x,y) = 0_n \Leftrightarrow x = y$, *where* $0_n = (0,\dots,0) \in \mathbb{R}^n$;

(d_2) $d(x,y) = d(y,x)$;

(d_3) $d(x,y) \leq d(x,z) + d(z,y)$.

If $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n) \in \mathbb{R}^n$, then $x \leq y$ means that $x_i \leq y_i$, $i = \overline{1,n}$.

We denote by $\mathcal{M}_{n,n}$ the set of all $n \times n$ matrices, by $\mathcal{M}_{n,n}(\mathbb{R}^+)$ the set of all $n \times n$ matrices with nonnegative entries. We write Θ_n for the zero $n \times n$ matrix and I_n for the identity $n \times n$ matrix and further on we identify row and column vector in $\mathbb{R}^n$.

A matrix $A \in \mathcal{M}_{n,n}(\mathbb{R}^+)$ is said to be convergent to zero if $A^m \to \Theta_n$, as $m \to \infty$.

Theorem 1.3.2. *(Perov [101, 102]) Let* (X,d) *be a complete generalized metric space,* $f: X \mapsto X$ *and* $A \in \mathcal{M}_{n,n}(\mathbb{R}^+)$ *a matrix convergent to zero, such that*

$$d(f(x),f(y)) \leq A(d(x,y)), \quad x,y \in X.$$

Then:

(*i*) *f has a unique fixed point $z \in X$;*

(*ii*) *the sequence of successive approximations $x_n = f(x_{n-1}), n \in \mathbb{N}$, converges to z for any $x_0 \in X$;*

(*iii*) $d(x_n, z) \leq A^n(I_n - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N}$;

(*iv*) *if $g : X \mapsto X$ satisfies the condition $d(f(x), g(x)) \leq c$ for all $x \in X$ and some $c \in \mathbb{R}^n$, then by considering the sequence $y_n = g^n(x_0), n \in \mathbb{N}$, one has*

$$d(y_n, z) \leq (I_n - A)^{-1}(c) + A^n(I_n - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N}.$$

This result found main application in the area of differential equations ([102, 121, 109]).

Perov generalized metric space is obviously a kind of a normal cone metric space. Defined partial ordering determines a normal cone $P = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, i = \overline{1, n}\}$ on $\mathbb{R}^n$, with the normal constant $K = 1$. Evidently, $A(P) \subseteq P$ if and only if $A \in \mathcal{M}_{n,n}(\mathbb{R}^+)$. It appears possible to adjust and probably broadly modify Perov's idea on a concept of cone metric space. Preferably, we will get some existence results. Nevertheless, forcing the transfer of contractive condition on cone metric space would be possible for some operator A instead of a matrix.

1.4 Operator theory

Observe that with $\mathcal{B}(E)$ is denoted the set of all bounded linear operators on a Banach space E and with $\mathcal{L}(A)$ the set of all linear operators on E. As usual, $r(A)$ is a spectral radius of an operator $A \in \mathcal{B}(E)$,

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|A^n\|^{1/n}.$$

If $r(A) < 1$, then the series $\sum\limits_{n=0}^{\infty} A^n$ is absolutely convergent, $I - A$ is invertible in $\mathcal{B}(E)$ and

$$\sum_{n=0}^{\infty} A^n = (I - A)^{-1}.$$

Also, $r((I - A)^{-1}) \leq \frac{1}{1 - r(A)}$.

If $A, B \in \mathcal{B}(E)$ and $AB = BA$, then $r(AB) \leq r(A)r(B)$.
Furthermore, if $\|A\| < 1$, then $I - A$ is invertible and

$$\|(I - A)^{-1}\| \leq \frac{1}{1 - \|A\|}.$$

Chapter 2

Perov type theorems on cone metric spaces

This chapter gathers main results concerning Perov theorem. First part of extending Perov's result included changing the setting-instead of generalized space in the sense of Perov, we observe cone metric space. However, in that case, distance is a vector in Banach space, so the contractive matrix A must be omitted and our idea was to implement some operator on a Banach space instead. Regarding properties of the operator A we must discuss two different cases, if cone $P \subseteq E$ is solid or normal. Our initial assumption was to observe bounded linear operators with spectral radius less than 1. If the cone is solid, additional request would be that A is a positive (increasing) operator. Normal cone requires stronger norm inequality $K\|A\| < 1$, where K is a normal constant of observed cone metric space.

Another direction of extending Perov theorem involves change of the contractive condition. The focus was on the most general class of contractions such as Ćirić and Fisher quasi-contraction. Correlations between those results mutually were discussed along with links to analogous results on cone metric spaces with a constant instead of operator and their (equivalent) metric versions.

In the last section of this chapter, the goal is to get a wider class of contractive operators that would guarantee existence of a fixed point. The strongest request for operator A that significantly reduces the list of possible candidates, is linearity and it is successfully excluded in the main result of the last section.

Important part of this chapter are presented examples and comments that emphasise (non)existence of some connections between obtained results along with some possible applications.

2.1 Extensions of Perov theorem

In the setting of a cone metric space (X, d), distance is a vector in a Banach space E, and therefore contractive constant q from the well-known Banach contraction can be replaced with some operator $A : E \mapsto E$. Accordingly, for some $f : X \mapsto X$, the inequality

$$d(f(x), f(y)) \preceq A(d(x, y)),\ x, y \in X,$$

defines a new kind of contractions which we will name Perov type contractions. It remains to determine necessary and sufficient conditions for the operator A that will guarantee existence of a fixed point of Perov type contraction. Uniqueness of the fixed point will be also discussed. Taking into account previously stated Perov theorem, we may suppose that A should be positive operator on cone metric space and A^n should tend to zero operator when $n \to \infty$. For that reason, some auxiliary results are presented in the sequence.

Lemma 2.1.1. *Let (X, d) be a cone metric space. Suppose that x_n is a sequence in X and that b_n is a sequence in E. If $\theta \preceq d(x_n, x_m) \preceq b_n$ for $m > n$ and $b_n \to \theta, n \to \infty$, then (x_n) is a Cauchy sequence.*

Proof. For every $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $b_n \ll c, n > n_0$. It follows that $\theta \preceq d(x_n, x_m) \ll c, m > n > n_0$, i.e., (x_n) is a Cauchy sequence. $\square$

Discussing linear operators, it is important to emphasise that the class of positive and the class of increasing operators coincide. Remark that, without linearity, only inclusion remains.

Lemma 2.1.2. *Let E be Banach space, $P \subseteq E$ cone in E and $A : E \mapsto E$ a linear operator. The following conditions are equivalent:*
(i) A is increasing, i.e., $x \preceq y$ implies $A(x) \preceq A(y)$.
(ii) A is positive, i.e., $A(P) \subseteq P$.

Proof. If A is monotonically increasing and $p \in P$, then, by definitions, it follows $p \succeq \theta$ and $A(p) \succeq A(\theta) = \theta$. Thus, $A(p) \in P$, and $A(P) \subseteq P$.
To prove the other implication, let us assume that $A(P) \subseteq P$ and $x, y \in E$ are such that $x \preceq y$. Now $y - x \in P$, and so $A(y - x) \in P$. Thus $A(x) \preceq A(y)$. $\square$

The results of the following theorem apply to the cone metric spaces in the case when cone is not necessary normal, and Banach space should not be finite dimensional. This extends the results of Perov for matrices([101, 102]), and as a corollary generalizes Theorem 1 of Zima ([130]).

Theorem 2.1.3. *Let (X, d) be a complete solid cone metric space, $d : X \times X \mapsto E$, $f : X \mapsto X$, $A \in \mathcal{B}(E)$, with $r(A) < 1$ and $A(P) \subseteq P$, such that*

$$d(f(x), f(y)) \preceq Ad(x, y), \quad x, y \in X. \tag{2.1}$$

Then:

(i) f has a unique fixed point $z \in X$;

(ii) For any $x_0 \in X$ the sequence $x_n = f(x_{n-1})$, $n \in \mathbb{N}$ converges to z and

$$d(x_n, z) \preceq A^n(I - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N};$$

(iii) Suppose that $g : X \mapsto X$ satisfies the condition $d(f(x), g(x)) \preceq c$ for all $x \in X$ and some $c \in P$. Then if $y_n = g^n(x_0), n \in \mathbb{N}$,

$$d(y_n, z) \preceq (I - A)^{-1}(c) + A^n(I - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N}.$$

Proof. (i) For $n, m \in \mathbb{N}, m > n$, the inequality

$$\theta \preceq d(x_n, x_m) \preceq \sum_{i=n}^{m-1} A^i(d(x_0, x_1)) \preceq \sum_{i=n}^{\infty} A^i(d(x_0, x_1)),$$

along with $r(A) < 1$, implies

$$\| \sum_{i=n}^{\infty} A^i(d(x_0, x_1))\| \leq \sum_{i=n}^{\infty} \|A^i\|\|(d(x_0, x_1))\| \to 0,\ n \to \infty.$$

Thus $a_n = \sum_{i=n}^{\infty} A^i(d(x_0, x_1)) \to \theta$, $n \to \infty$, and by Lemma 2.1.1 $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is a complete cone metric space, there exists the limit $z \in X$ of sequence (x_n).

Let us prove that $f(z) = z$. Set $p = d(z, f(z))$, and suppose that $c \gg \theta$ and $\varepsilon \gg \theta$. Hence, there exists $n_0 \in \mathbb{N}$ such that

$$d(z, x_n) \ll c \text{ and} d(z, x_n) \ll \varepsilon \text{ for all } n \geq n_0.$$

Therefore, $p = d(z, f(z)) \preceq d(z, x_{n+1}) + d(x_{n+1}, f(z)) \preceq d(z, x_{n+1}) + A(d(z, x_n)) \preceq c + A(\varepsilon)$ for $n \geq n_0$. Thus, $p \preceq c + A(\varepsilon)$ for each $c \gg 0$, and so $p \preceq A(\varepsilon)$. Now, for $\varepsilon = \varepsilon/n$, $n = 1, 2, \ldots$, we get

$$\theta \preceq p \preceq A\left(\frac{\varepsilon}{n}\right) = \frac{A(\varepsilon)}{n},\ n = 1, 2, \ldots.$$

Because $\frac{A(\varepsilon)}{n} \to \theta$, $n \to \infty$, this shows that $p = \theta$. Consequently, $z = f(z)$.

If $f(y) = y$, for some $y \in X$, then $d(z, y) \preceq A(d(z, y))$. Thus, $d(z, y) \preceq A^n(d(z, y))$, for each $n \in \mathbb{N}$. Furthermore, $r(A) < 1$ implies

$$\|A^n(d(z, y)\| \leq \|A^n\|\|(d(z, y)\| \to 0,\ n \to \infty,$$

so, $d(z, y) = \theta$ and $z = y$.

(ii) By (i), for arbitrary $n \in \mathbb{N}$, we have

$$d(x_n, z) \preceq A(d(x_{n-1}, z)) \preceq \cdots \preceq A^n(d(x_0, z)). \tag{2.2}$$

On the other hand,

$$\begin{aligned} d(x_0, z) &\preceq d(x_0, x_n) + d(x_n, z) \\ &\preceq \sum_{i=0}^{n-1} d(x_i, x_{i+1}) + A^n(d(x_0, x_1)) + A^n(d(x_1, z)) \\ &\preceq \sum_{i=0}^{i=n} A^i(d(x_0, x_1)) + A^n(d(x_1, z)). \end{aligned}$$

Since $A^n(d(x_1,z)) \to \theta$, $n \to \infty$, we get

$$d(x_0,z) \leq \sum_{i=0}^{\infty} A^i(d(x_0,x_1)) = (I-A)^{-1}(d(x_0,x_1)),$$

and $d(x_n,z) \preceq A^n(I-A)^{-1}(d(x_0,x_1))$.

(iii) Let us remark that for any $n \in \mathbb{N}$, $d(y_n,z) \preceq d(y_n,x_n)+d(x_n,z)$, and (ii) imply

$$d(y_n,z) \preceq d(y_n,x_n) + A^n(I-A)^{-1}(d(x_0,x_1)).$$

Now

$$\begin{aligned}
d(y_n,x_n) &\preceq d(y_n,f(y_{n-1})) + d(f(y_{n-1}),x_n) \\
&\preceq c + A(d(y_{n-1},x_{n-1})) \\
&\preceq c + A\Big(d(y_{n-1},f(y_{n-2})) + d(f(y_{n-2}),x_{n-1})\Big) \\
&\preceq c + A(c) + A^2(d(y_{n-2},x_{n-2})) \\
&\preceq \ldots \preceq \sum_{i=0}^{n-1} A^i(c) \\
&\preceq (I-A)^{-1}(c),
\end{aligned}$$

implies (iii). $\square$

Research of the existence of fixed points of set-valued contractions in metric spaces were initiated by S. B. Nadler [95]. The following theorem is motivated by Nadler's results and generalizes the well-known Banach fixed point theorem in several ways. Furthermore, it is a generalization of the recent result Theorem 3.2 of Borkowski, Bugajewski and Zima ([25]) for a Banach space with a non-normal cone.

Theorem 2.1.4. *Let (X,d) be a complete cone metric space, $d : X \times X \mapsto E$, and let T be a set-valued d–Perov contractive mapping (i.e. there exists $A \in \mathcal{B}(E)$, such that $r(A) < 1$, $A(P) \subseteq P$ and for any $x_1,x_2 \in X$ and $y_1 \in Tx_1$ there is $y_2 \in Tx_2$ with $d(y_1,y_2) \preceq A(d(x_1,x_2))$) from X into itself such that for any $x \in X$, Tx is a nonempty closed subset of X. Then there exists $x_0 \in X$ such that $x_0 \in Tx_0$, i.e., x_0 is a fixed point of T.*

Proof. Suppose that $u_0 \in X$ and $u_1 \in Tu_0$. Then there exists $u_2 \in Tu_1$ such that $d(u_1,u_2) \preceq A(d(u_0,u_1))$. Thus, we have a sequence (u_n) in X such that $u_{n+1} \in Tu_n$ and $d(u_n,u_{n+1}) \preceq A(d(u_{n-1},u_n))$ for every $n \in \mathbb{N}$. For any $n \in \mathbb{N}$,

$$d(u_n,u_{n+1}) \preceq A(d(u_{n-1},u_n)) \preceq \ldots \preceq A^n(d(u_0,u_1)). \tag{2.3}$$

Assuming $m > n$,

$$\begin{aligned}
d(u_n,u_m) &\preceq d(u_n,u_{n+1}) + d(u_{n+1},u_{n+2}) + \ldots + d(u_{m-1},u_m) \\
&\preceq (A^n + A^{n+1} + \ldots + A^{m-1})(d(u_0,u_1)) \\
&\preceq A^n(I-A)^{-1}(d(u_0,u_1)) \to \theta,\ n \to \infty.
\end{aligned}$$

Thus, (u_n) is a Cauchy sequence in X there is some $v_0 \in X$ such that $u_n \to v_0$ as $n \to \infty$. Furthermore, for any $\varepsilon \gg \theta$ there exists $m_0 \in \mathbb{N}$ such that $d(u_m, v_0) \ll \varepsilon$, $m \geq m_0$. Thus, for $m \geq \max\{n, m_0\}$,

$$\begin{aligned} d(u_n, v_0) &\preceq d(u_n, u_m) + d(u_m, v_0) \\ &\preceq A^n(1-A)^{-1}(d(u_0, u_1)) + \varepsilon, \text{ for } n \geq 1. \end{aligned}$$

Hence,

$$d(u_n, v_0) \preceq A^n(1-A)^{-1}(d(u_0, u_1)), \text{ for } n \geq 1. \tag{2.4}$$

Let us define $w_n \in Tv_0$ such that $d(u_n, w_n) \preceq A(d(u_{n-1}, v_0))$, for $n \geq 1$. So, for any $n \in \mathbb{N}$,

$$d(u_n, w_n) \preceq A(d(u_{n-1}, v_0)) \preceq A^n(I-A)^{-1}(d(u_0, u_1)). \tag{2.5}$$

Now, we have

$$d(w_n, v_0) \preceq d(w_n, u_n) + d(u_n, v_0) \preceq 2A^n(I-A)^{-1}(d(u_0, u_1)).$$

Thus, (w_n) converges to v_0. Since Tv_0 is closed, we have $v_0 \in Tv_0$. $\square$

Example 5. Let $X = E$, $E = C[0,1]$ with the supremum norm and $P = \{x \in E \mid x(t) \geq 0,\ t \in [0,1]\}$. Let us define cone metric $d : X \times X \mapsto E$ by

$$d(f, g) = f + g, \text{ for } f \neq g;\ d(f, f) = 0, \quad f, g \in X.$$

If $T : X \mapsto X$ is defined by $T(f) = f/2$, $f \in X$, then

$$d(T(f), T(g)) \leq A(d(f, g)),\ f, g \in X,$$

where $A : E \mapsto E$, is a bounded linear operator defined by $A(f) = f/2$, $f \in E$. Clearly, $\|A\| = 1/2$, and all the assumptions from Theorem 2.1 are satisfied. Hence, T has a unique fixed point f, $f(t) = 0$, $t \in [0,1]$.

Remark 2.1.5. Let us remark that the initial assumption $A \in \mathcal{M}_{n,n}(\mathbb{R}_+)$, in Perov theorem, is unnecessary. This will be illustrated by the following example.

Example 6. Let

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ \frac{1}{4} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix},$$

$X = \left\{ \begin{bmatrix} x_1 \\ 1 \\ x_3 \end{bmatrix} \mid x \in \mathbb{R} \right\}$ and $f : X \mapsto X$, $f\left(\begin{bmatrix} x_1 \\ 1 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} \frac{x_1+1}{2} \\ 1 \\ \frac{x_3+2}{3} \end{bmatrix}$. Set $\|x\| = \max\{|x_1|, |x_2|, |x_3|\}$ for $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $x_i \in \mathbb{R}, i = 1, 2, 3.$

For arbitrary $x \in X$,

$$\begin{aligned}\|Ax\| &= \max\Big\{|\frac{1}{2}x_1 - \frac{1}{4}x_2|, |\frac{1}{4}x_1 - \frac{1}{2}x_2|, |\frac{1}{2}x_3|\Big\} \\ &\leq \max\Big\{\frac{1}{2}\|x\| + \frac{1}{4}\|x\|, \frac{1}{4}\|x\| + \frac{1}{2}\|x\|, \frac{1}{2}\|x\|\Big\} = \frac{3}{4}\|x\|.\end{aligned}$$

Thus, $\|A\| \leq \frac{3}{4}$. If $x = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $\|x\| = 1$, then $\|Ax\| = \frac{3}{4}$. Hence, $\|A\| = \frac{3}{4}$.

Evidently, $r(A) \leq \|A\| = 3/4$ and $d(f(x), f(y)) \preceq A(d(x,y))$, $x, y \in X$. Despite of $A(P) \nsubseteq P$, $(1,1,1)$ is a unique fixed point of f in X.

Based on the previous comments, we obtain the next result, where we do not suppose that $A(P) \subseteq P$.

Theorem 2.1.6. *Let (X, d) be a complete cone metric space, $d : X \times X \mapsto E$, P a normal cone with a normal constant K, $A \in \mathcal{B}(E)$ and $K\|A\| < 1$. If the condition (2.1) holds for a mapping $f : X \mapsto X$, then f has a unique fixed point $z \in X$ and the sequence $x_n = f(x_{n-1})$, $n \in \mathbb{N}$ converges to z for any $x_0 \in X$.*

Proof. Let $x_0 \in X$ be arbitrary, $x_n = f(x_{n-1})$, $n \in \mathbb{N}$. Inequality

$$d(x_n, x_{n+1}) \preceq A(d(x_{n-1}, x_n)),\ n \in \mathbb{N},$$

implies

$$\begin{aligned}\|d(x_n, x_{n+1})\| &\leq K\|A(d(x_{n-1}, x_n))\| \leq K\|A\|\|d(x_{n-1}, x_n)\| \\ &\leq K^2\|A\|^2\|d(x_{n-2}, x_{n-1})\| \leq \ldots \leq K^n\|A\|^n\|d(x_0, x_1)\|.\end{aligned}$$

If $n, m \in \mathbb{N}, n < m$, then

$$\|d(x_n, x_m)\| \leq \sum_{i=n}^{m-1} \|d(x_i, x_{i+1})\| \leq \sum_{i=n}^{m-1} K^i\|A\|^i\|d(x_0, x_1)\|.$$

Clearly, $K\|A\| < 1$, implies that the series $\sum\limits_{i=0}^{\infty} K^i\|A\|^i$ is convergent. Hence, $\|d(x_n, x_m)\| \to 0$, as $n, m \to \infty$. This shows that (x_n) is a Cauchy sequence, and there is $z \in X$ such that $\lim\limits_{n\to\infty} x_n = z$. Let us prove that $f(z) = z$. From $d(f(z), x_{n+1}) \preceq A(d(z, x_n))$, we get

$$\|d(f(z), x_{n+1})\| \leq K\|A(d(z, x_n))\| \leq K\|A\|\|d(z, x_n)\|.$$

Thus, $\lim\limits_{n\to\infty} x_n = f(z)$, and so $f(z) = z$.
It remains to show that z is a unique fixed point of f.
If $f(y) = y$, $y \in X$, then $d(z, y) = d(f(z), f(y)) \preceq A(d(z, y))$ it follows $\|d(z, y)\| \leq K\|A\|\|d(z, y)\|$. Now, $K\|A\| < 1$ implies $z = y$. □

Following the work of Berinde ([22], [23]), we investigate the existence of the fixed point for the class of Perov type weak contraction.

Theorem 2.1.7. *Let (X,d) be a complete cone metric space, $d : X \times X \mapsto E$, $f : X \mapsto X$, $A \in \mathcal{B}(E)$, with $r(A) < 1$ and $A(P) \subseteq P$, $B \in \mathcal{L}(E)$ with $B(P) \subseteq P$, such that*

$$d(f(x), f(y)) \preceq A(d(x,y)) + B(d(x, f(y))),\ x, y \in X. \tag{2.6}$$

Then

(i) $f : X \mapsto X$ has a fixed point in X and for any $x_0 \in X$ the sequence $x_n = f(x_{n-1})$, $n \in \mathbb{N}$ converges to a fixed point of f.

(ii) If additionally,

$$B \in \mathcal{B}(E) \text{ and } r(A+B) < 1, \tag{2.7}$$

or

$$d(f(x), f(y)) \preceq Ad(x,y) + B(d(x, f^{n_0}(x))),\ x, y \in X, \text{ for some } n_0 \in \mathbb{N}, \tag{2.8}$$

then f has a unique fixed point.

Proof. (i) For a arbitrary $x_0 \in X$ observe $x_n = f(x_{n-1})$, $n \in \mathbb{N}$. Since

$$\begin{aligned} d(x_n, x_{n+1}) &\preceq A(d(x_{n-1}, x_n)) + B(d(x_n, f(x_{n-1}))) = A(d(x_{n-1}, x_n)) \\ &\preceq A^2(d(x_{n-2}, x_{n-1})) \preceq \ldots \preceq A^n(d(x_0, x_1)), \end{aligned}$$

then, as in the proof of Theorem 2.1, we conclude that (x_n) converges to some $z \in X$.

Set $p = d(z, f(z))$, and suppose that $c \gg \theta$ and $\varepsilon \gg \theta$. Hence, there exists $n_0 \in \mathbb{N}$ such that $d(z, x_n) \ll c$ and $d(z, x_n) \ll \varepsilon$ for all $n \geq n_0$. Now

$$\begin{aligned} p = d(z, f(z)) &\preceq d(z, x_{n+1}) + d(x_{n+1}, f(z)) \\ &\preceq d(z, x_{n+1}) + A(d(z, x_n)) + B(d(z, x_{n+1})) \\ &\preceq c + A(\varepsilon) + B(\varepsilon),\ n \geq n_0. \end{aligned}$$

Thus, $p \preceq c + A(\varepsilon) + B(\varepsilon)$ for each $c \gg \theta$, and so $p \preceq A(\varepsilon) + B(\varepsilon)$. Now, for $\varepsilon = \varepsilon/n$, $n \in \mathbb{N}$ we get

$$\theta \preceq p \preceq A\left(\frac{\varepsilon}{n}\right) + B\left(\frac{\varepsilon}{n}\right) = \frac{A(\varepsilon)}{n} + \frac{B(\varepsilon)}{n},\ n \in \mathbb{N}.$$

Because $A(\varepsilon)/n + B(\varepsilon)/n \to \theta$, $n \to \infty$, this shows that $p = \theta$.

(ii) If $f(y) = y$, for some $y \in X$, then

$$d(z,y) \preceq A(d(z,y)) + B((d(z,y)) = (A+B)((d(z,y)).$$

Thus $d(z,y) \preceq (A+B)^n(d(z,y))$, for each $n \in \mathbb{N}$. Now, $r(A+B) < 1$ implies

$$\|(A+B)^n(d(z,y)\| \leq \|(A+B)^n\| \|(d(z,y)\| \to 0,\ n \to \infty,$$

and $z = y$. Observe that (2.8) implies $d(z,y) \preceq A(d(z,y))$, and so

$$d(z,y) \preceq A^n(d(z,y)),\ n \in \mathbb{N}.$$

The rest of the proof follows from the proof of Theorem 2.1 (i). $\square$

Remark 2.1.8. Let us notice that in the works of [45] and [129] the authors have studied (2.1) with more general approach where A is a nonlinear operator and $A(P) \subseteq P$. Their results are given for the case where cone P is a normal cone. For example, the "policeman lemma" is essential in their results (see p.p. 369 of [45]) while the policeman lemma is not true in the case when P is a non-normal cone. Furthermore, we do not suppose that $A(P) \subseteq P$ if cone P is normal. If A is a linear operator and obeys (2.1) then results in [129] are given under special assumptions on A and on a cone P (such that X is a sequentially complete (in the Weierstrass sense)) and in our results we do not need such assumptions. Thus, our results and results from ([45], [129]) are independent from each other.

The following two theorems generalize Theorem 1 of [14] and, consequently, Theorem 2 of [100].

Theorem 2.1.9. *Let (X, d) be a cone metric space, $P \subseteq E$ a cone and $T : X \mapsto X$. If there exists a point $z \in X$ such that $\overline{O(z)}$ is complete, $A \in \mathcal{B}(E)$ a positive operator with $r(A) < 1$, and*

$$d(Tx, Ty) \preceq A(d(x, y)), \text{ holds for any } x, y = T(x) \in O(z), \tag{2.9}$$

then $(T^n z)$ converges to some $u \in \overline{O(z)}$ and

$$d(T^n z, u) \preceq A^n(I - A)^{-1}(d(z, Tz)), \ n \in \mathbb{N}. \tag{2.10}$$

If (2.9) holds for any $x, y \in \overline{O(z)}$, then u is a fixed point of T.

Proof. First, we will show that $\{T^n z\}$ is a Cauchy sequence.
Since $d(T^n z, T^{n+1} z) \preceq A(d(T^{n-1} z, T^n z))$, and A is a positive operator by Lemma 2.1.2, it follows that

$$d(T^n z, T^{n+1} z) \preceq A^n(d(z, Tz)),$$

so, for $n, m \in \mathbb{N}$, $m > n$,

$$d(T^n z, T^m z) \preceq \sum_{i=n}^{m-1} d(T^i z, T^{i+1} z) \preceq \sum_{i=n}^{m-1} A^i(d(z, Tz)),$$

and, since $r(A) < 1$ and Lemma 2.1.1 holds, $(T^n z)$ is a Cauchy sequence. Because $\overline{O(z)}$ is complete, there exists an $u \in \overline{O(z)}$ such that $\lim\limits_{n\to\infty} T^n z = u$.
Let $n \in \mathbb{N}$ be arbitrary and $m \geq n$. Then,

$$\begin{aligned}
d(T^n z, u) &\preceq d(T^n z, T^m z) + d(T^m z, u) \\
&\preceq \sum_{i=n}^{m-1} A^i(d(z, Tz)) + d(T^m z, u) \\
&\preceq A^n \sum_{i=0}^{\infty} A^i(d(z, Tz)) + d(T^m z, u) \\
&= A^n(I - A)^{-1}(d(z, Tz)) + d(T^m z, u).
\end{aligned}$$

Taking the limit as $m \to \infty$ of the above inequality yields (2.10).
If (2.9) is true for $x, y \in \overline{O(z)}$, then

$$d(T^{n+1}z, Tu) \preceq A(d(T^n z, u)),$$

and $A(d(T^n z, u)) \to \theta, n \to \infty$, thus $(\mathbf{p}_6)$ implies $\lim_{n\to\infty} T^n z = Tu$. But $\lim_{n\to\infty} T^n z = u$ gives us that u is a fixed point of T. $\square$

In accordance with Example 6, positivity request could be omitted if P is a normal cone and we can modify the conditions of Theorem 2.1.9.

Theorem 2.1.10. *Let (X, d) be a cone metric space, $P \subseteq E$ a normal cone with a normal constant K and $T : X \mapsto X$. If there exists a point $z \in X$ such that $\overline{O(z)}$ is complete and $A \in \mathcal{B}(E)$ bounded linear operator, $K\|A\| < 1$, such that (2.9) holds, then $(T^n z)$ converges to some $u \in \overline{O(z)}$ and*

$$\|d(T^n z, u)\| \leq \frac{(K\|A\|)^n}{1 - K\|A\|} \|d(z, Tz)\|, \ n \in \mathbb{N}. \tag{2.11}$$

If (2.9) *holds for every $x, y \in \overline{O(z)}$, then u is a fixed point of T.*

Proof. Observe that (2.9) and the fact that P is a normal cone imply

$$\|d(T^n z, T^{n+1}z)\| \leq K\|A\|\|d(T^{n-1}z, T^n z)\|,$$

and, inductively, $\|d(T^n z, T^{n+1}z)\| \leq (K\|A\|)^n d(z, Tz)$ for every $n \in \mathbb{N}$. If $n, m \in \mathbb{N}$ and $m > n$, we have

$$\begin{aligned}
\|d(T^n z, T^m z)\| &\leq K\| \sum_{i=n-1}^{m-2} A(d(T^i z, T^{i+1}z))\| \\
&\leq K\|A\| \sum_{i=n-1}^{m-2} \|d(T^i z, T^{i+1}z)\| \\
&\leq K\|A\| \sum_{i=n-1}^{m-2} (K\|A\|)^i \|d(z, Tz)\| \\
&\leq \sum_{i=n}^{\infty} (K\|A\|)^i \|d(z, Tz)\| \\
&\leq \frac{(K\|A\|)^n}{1 - K\|A\|} \|d(z, Tz)\|. \qquad (2.12)
\end{aligned}$$

Because $K\|A\| < 1$, $(T^n z)$ is a Cauchy sequence and $\lim_{n\to\infty} T^n z = u$ for some $u \in \overline{O(z)}$.
Notice that, for any $n \in \mathbb{N}$,

$$\begin{aligned}
\|d(T^n z, u)\| &\leq K\|A(d(T^{n-1}z, T^{m-1}z))\| + K\|d(T^m z, u)\| \\
&\leq \frac{(K\|A\|)^n}{1 - K\|A\|} \|d(z, Tz)\| + K\|d(T^m z, u)\|.
\end{aligned}$$

Last inequality is obtained from (2.12) for any $m \in \mathbb{N}$ and, because $K\|d(T^m z, u)\| \to 0$, $m \to \infty$, obviously (2.10) is satisfied.
If we include $x, y \in \overline{O(z)}$ in the condition (2.11), then

$$d(T^n z, Tu) \preceq A(d(T^{n-1} z, u)), \ n \in \mathbb{N}.$$

and $d(T^{n-1} z, u) \to \theta$, $n \to \infty$, so $T^n z \to Tu$, $n \to \infty$. However, the limit of convergent sequence is unique, thus $Tu = u$ □

Evidently, Theorems 2.1.3, 2.1.6 and 2.1.7 can be obtained as a consequence of Theorem 2.1.9.

Corollary 2.1.11. *Let (X, d) be a complete cone metric space and $T : X \mapsto X$ a mapping satisfying*

$$d(Tx, Ty) \preceq A(d(Tx, x) + d(Ty, y)), \ x, y \in X, \tag{2.13}$$

for some positive operator $A \in \mathcal{B}(E)$ with $r(A) < \frac{1}{2}$. Then T has a unique fixed point $u \in X$ and $(T^n x)$ converges to u for any $x \in X$.

Proof. Since,

$$d(Tx, T^2 x) \preceq A(I - A)^{-1}(d(Tx, x))$$

and $A(I - A)^{-1}$ is a positive operator,

$$r(A(I - A)^{-1}) \leq r(A) r((I - A)^{-1}) \leq \frac{r(A)}{1 - r(A)} < 1,$$

condition (2.9) of the Theorem 2.1.9 holds. Hence, T has a fixed point $u \in X$. Uniqueness of the fixed point follows from (2.13). If $v \in X$ and $T(v) = v$, then $d(u, v) = d(Tu, Tv) \preceq A(d(Tu, u) + d(Tv, v)) = A(\theta) = \theta$. □

Corollary 2.1.12. *Let (X, d) be a complete cone metric space, P a normal cone with a normal constant K and $T : X \mapsto X$ a mapping satisfying (2.13) for some operator $A \in \mathcal{B}(E)$ with $K\|A\| < \frac{1}{2}$. Then T has a unique fixed point $u \in X$ and $(T^n x)$ converges to u for any $x \in X$.*

Proof. Obviously

$$\|d(Tx, T^2 x)\| \leq \frac{K\|A\|}{1 - K\|A\|} \|d(x, Tx)\|,$$

and $K\|A\|/(1 - K\|A\|) < 1$. Therefore, analogously to the proof of Theorem 2.1.10, it is easy to show that T has a fixed point and (2.13) implies uniqueness. □

Corollary 2.1.13. *Let (X, d) be a complete cone metric space and $T : X \mapsto X$ a mapping satisfying*

$$d(Tx, Ty) \preceq A(d(x, T^m z) + d(y, T^m z)), \tag{2.14}$$

for some $m \in \mathbb{N}$, $A \in \mathcal{B}(E)$ positive operator, $r(A) < 1$ and for all $x, y, z \in X$. Then the iterative sequence $(T^n x)$ converges to a unique fixed point of T for any $x \in X$.

Proof. If any $z \in X$ and $m \in \mathbb{N}$, set $x = T^{m-1}z$ and $y = T^m z$ in (2.14)

$$d(T^m z, T^{m+1}z) \preceq A(d(T^{m-1}z, T^m z)),$$

then

$$d(T^n z, T^{n+1}z) \preceq A(d(T^{n-1}z, T^n z)),\ n \geq m,$$

so, as in the proof of Corollary 2.1.11, T has a fixed point. Condition (2.14) gives uniqueness. □

Corollary 2.1.14. *Let (X, d) be a complete cone metric space, $P \subseteq E$ a normal cone with a normal constant K and $T : X \mapsto X$ a mapping satisfying (2.14) for some $m \in \mathbb{N}$, $A \in \mathcal{B}(E)$ such that $K\|A\| < 1$ and for all $x, y, z \in X$. Then the iterative sequence $(T^n x)$ converges to a unique fixed point of T for any $x \in X$.*

Proof. For any $z \in X$ and $m \in \mathbb{N}$, setting $x = T^{m-1}z$ and $y = T^m z$ in (2.14) gives

$$\|d(T^m z, T^{m+1}z)\| \leq K\|A\|\|d(T^{m-1}z, T^m z)\|,$$

and, by similar observations as in the proofs of Corollary 2.1.13 and Corollary 2.1.11, T has a unique fixed point in X. □

2.2 Perov type quasi-contraction

There are many different approaches to the problem of generalizing well-known Banach contractive condition, and most of them could be altered to suit Perov contraction. It is also important to mention that many of them are equivalent or imply each other and that is why we define Perov type quasi-contraction as one of the widest class of contractive mappings.

Ilić and Rakočević ([71]) introduced a quasi-contractive mapping on a normal cone metric spaces, and proved existence and uniqueness of a fixed point. Kadelburg, Radenović and Rakočević ([81]), without the normality condition, proved related results, but only in the case when contractive constant $q \in (0, 1/2)$. Later, Haghi, Rezapour and Shahzad ([118]) and also Gajić and Rakočević ([57]) gave proof of the same result without the additional normality assumption and for $q \in (0, 1)$ by providing two different proof techniques. For the more related results see ([58], [63], [88]).

Definition 2.2.1. *Let (X, d) be a cone metric space. A mapping $f : X \mapsto X$ such that for some bounded linear operator $A \in \mathcal{B}(E)$, $r(A) < 1$ and for each $x, y \in X$, there exists*

$$u \in C(f, x, y) \equiv \Big\{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\Big\},$$

such that

$$d(f(x), f(y)) \preceq A(u), \tag{2.15}$$

is called a quasi-contraction of Perov type.

If $f : X \mapsto X$, and $n \in \mathbb{N}$, set

$$O(x;n) = \Big\{x, f(x), f^2(x), ..., f^n(x)\Big\},$$

and

$$O(x;\infty) = \Big\{x, f(x), f^2(x), ...\Big\}.$$

Denote by $\delta(O(x,n)) = \max\{\|d(a,b)\| : a, b \in O(x,n)\}$, $n \in \mathbb{N}, x \in X$ and $\delta(O(x,\infty)) = \sup\limits_{n\in\mathbb{N}} \delta(O(x,n))$.

Theorem 2.2.2. *Let (X,d) be a complete solid cone metric space. If a mapping $f : X \mapsto X$ is a quasi-contraction and $A(P) \subseteq P$, then f has a unique fixed point and for any $x \in X$, the iterative sequence $(f^n(x))_{n\in\mathbb{N}}$ converges to the fixed point of f.*

Proof. We will prove the following two inequalities for arbitrary $x \in X$:

(i) $d(f^n(x), f(x)) \preceq (I-A)^{-1}A(d(f(x),x))$ $n \in \mathbb{N}$,

(ii) $d(f^n(x), x) \preceq (I-A)^{-1}(d(f(x),x))$ $n \in \mathbb{N}$.

Evidently, (i) is true for $n = 1$. Suppose that it is true for each $m \le n$.
Since

$$d(f^{n+1}(x), f(x)) \preceq A(u),$$

were

$$\begin{aligned} u \;\in\; & \Big\{ d(f^n(x), x), d(f^n(x), f(x)), d(x, f(x)), \\ & d(x, f^{n+1}(x)), d(f^n(x), f^{n+1}(x)) \Big\}, \end{aligned}$$

we have to consider the following five different cases.

(1) If $u = d(f^n(x), x)$, then

$$\begin{aligned} d(f^{n+1}(x), f(x)) &\preceq A(d(f^n(x), x)) \\ &\preceq A(d(f^n(x), f(x))) + A(d(f(x), x)) \\ &\preceq A(I-A)^{-1}A(d(f(x),x)) + A(d(f(x),x)) \\ &= [(A-I)+I](I-A)^{-1}A)(d(f(x),x)) + A(d(f(x),x)) \\ &= (I-A)^{-1}A(d(f(x),x)). \end{aligned}$$

(2) If $u = d(f^n(x), f(x))$, then $A(P) \subseteq P$ implies

$$\begin{aligned} d(f^{n+1}(x), f(x)) &\preceq A(d(f^n(x), f(x))) \\ &\preceq A(I-A)^{-1}A(d(f(x),x)) \\ &= [(A-I)+I](I-A)^{-1}A(d(f(x),x)) \\ &= -A(d(f(x),x)) + (I-A)^{-1}A(d(f(x),x)) \\ &\preceq (I-A)^{-1}A(d(f(x),x)). \end{aligned}$$

(3) Clearly, for $u = d(f(x), x)$, the inequality (i) holds.
(4) Suppose that $u = d(x, f^{n+1}(x))$, then

$$d(x, f^{n+1}(x)) \leq d(x, f(x)) + d(f(x), f^{n+1}(x))$$

and, since $A(P) \subseteq P$,

$$d(f^{n+1}(x), f(x)) \preceq A(d(x, f(x))) + A(d(f(x), f^{n+1}(x))).$$

Therefore,

$$d(f^{n+1}(x), f(x)) \preceq (I - A)^{-1} A(d(x, f(x))).$$

(5) If $u = d(f^n(x), f^{n+1}(x))$, then

$$d(f^{n+1}(x), f(x)) \preceq A(d(f^n(x), f^{n+1}(x)),$$

and because f is a quasi-contraction, there exist some $i, j \in \{0, 1, \ldots, n\}$ such that

$$d(f^n(x), f^{n+1}(x)) \preceq A^{n-1+i}(d(f(x), f^j(x))).$$

Thus,

$$\begin{array}{rcl} d(f^{n+1}(x), f(x)) & \preceq & A^{n+i}(d(f(x), f^j(x))) \\ & \preceq & A^{n+i}(I - A)^{-1} A(d(f(x), x)) \\ & = & (I - A)^{-1} A(d(f(x), x)) - \displaystyle\sum_{j=1}^{n+i} A^j(d(f(x), x)) \\ & \preceq & (I - A)^{-1} A(d(f(x), x)), \end{array}$$

unless $j = n + 1$.
If $j = n+1$, then $d(f^{n+1}(x), f(x)) \preceq A^{n+i}(d(f(x), f^{n+1}(x)))$ implies $d(f^{n+1}(x), f(x)) = \theta$.
Indeed, since $I - A^{n+i}$ is an invertible operator and $A^{n+i}(P) \subseteq P$, we have

$$d(f^{n+1}(x), f(x)) \preceq (I - A^{n+i})^{-1}(\theta) = \theta,$$

along with $d(f^{n+1}(x), f(x)) = \theta$.

Hence, using the method of the mathematical induction we have proved that the inequality (i) holds for each $n \in \mathbb{N}$.

The inequality (ii) proceeds from (i):

$$\begin{array}{rcl} d(f^n(x), x) & \preceq & d(f^n(x), f(x)) + d(f(x), x) \\ & \preceq & (I - A)^{-1} A(d(f(x), x)) + d(f(x), x) \\ & = & (I - A)^{-1}(d(f(x), x)), \ n \in \mathbb{N}. \end{array}$$

Let us prove that $(f^n(x))_{n \in \mathbb{N}}$ is a Cauchy sequence in X, thus it is convergent. Suppose that $n, m \in \mathbb{N}$, $m > n$. Mapping f is a quasi-contraction of Perov type, so there exist $i, j \in \mathbb{N}$ such that , $1 \leq i \leq n$, $1 \leq j \leq m$,

$$d(f^n(x), f^m(x)) \preceq A^{n-1}(d(f^i(x), f^j(x))).$$

By (i), this implies

$$d(f^n(x), f^m(x)) \preceq 2A^n(I-A)^{-1}(d(f(x), x)).$$

Since, $2A^n(I-A)^{-1}(d(f(x), x)) \to \theta$, $n \to \infty$, by Lemma 2.1.1, $(f^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Hence, there exists $z \in X$ such that $\lim_n f^n(x) = z$. Let us prove that z is a fixed point of f.
Suppose that $c \gg \theta$ and $\varepsilon \gg \theta$. Then there exists $n_0 \in \mathbb{N}$ such that

$$\begin{gathered} d(x^*, f^n(x)) \ll c,\ d(f^n(x), f^m(x)) \ll \varepsilon \\ \text{and } d(x^*, f^n(x)) \ll \varepsilon \text{ for all } n, m \geq n_0. \end{gathered} \tag{2.16}$$

Now, for any $n > n_0$,

$$\begin{aligned} d(x^*, f(z)) &\preceq d(z, f^n(x)) + d(f^n(x), f(z)) \\ &\preceq c + d(f^n(x), f(z)). \end{aligned} \tag{2.17}$$

Furthermore, because f is a quasi-contraction, we have

$$d(f^n(x), f(z)) \preceq A(u), \tag{2.18}$$

for some

$$u \in \Big\{ d(f^{n-1}(x), z), d(f^{n-1}(x), f^n(x)), d(f^{n-1}(x), f(z)),$$
$$d(z, f(z)), d(z, f^n(x)) \Big\}.$$

If

$$u \in \Big\{ d(f^{n-1}(x), z), d(f^{n-1}(x), f^n(x)), d(z, f^n(x)) \Big\},$$

for infinitely many $n > n_0$, then (2.16), (2.17) and (2.18) imply

$$d(z, f(z)) \preceq c + A(\varepsilon). \tag{2.19}$$

Because the inequality (2.19) is true for each $c \gg \theta$ we get

$$d(z, f(z)) \preceq A(\varepsilon). \tag{2.20}$$

If $u = d(f^{n-1}(x), f(z))$, then

$$d(f^{n-1}(x), f(z)) \preceq d(f^{n-1}(x), z) + d(z, f(z)),$$

and $A(P) \subseteq P$ imply

$$A(u) \preceq A(d(f^{n-1}(x), z)) + A(d(z, f(z))).$$

Now, by (2.16), (2.17) and (2.18) we have

$$d(z, f(z)) \preceq c + A(\varepsilon) + A(d(z, f(z))),$$

and, since $c \gg \theta$ is arbitrary,

$$(I - A)(d(z, f(z))) \preceq A(\varepsilon). \tag{2.21}$$

Thus, because $(I - A)^{-1}$ is increasing (2.21) implies

$$d(z, f(z)) \preceq (I - A)^{-1}A(\varepsilon). \tag{2.22}$$

Finally, in the case $u = d(z, f(z))$, (2.17) and (2.18) imply

$$d(z, f(z)) \preceq c + A(d(z, f(z))),$$

that is

$$(I - A)(d(z, f(z))) \preceq c. \tag{2.23}$$

Again, because $(I - A)^{-1}$ is increasing, (2.23) implies

$$d(z, f(z)) \preceq (I - A)^{-1}(c). \tag{2.24}$$

Now, by (2.20), (2.22) and (2.24), for $\varepsilon = \varepsilon/n$ and $c = c/n$, $n \in \mathbb{N}$, it follows, respectively,

$$\theta \preceq d(z, f(z)) \preceq A\left(\frac{\varepsilon}{n}\right) = \frac{A(\varepsilon)}{n} \to \theta,\ n \to \infty,$$

$$\theta \preceq d(z, f(z)) \preceq (I - A)^{-1}A\left(\frac{\varepsilon}{n}\right) = \frac{(I - A)^{-1}A(\varepsilon)}{n} \to \theta,\ n \to \infty,$$

and

$$\theta \preceq d(z, f(z)) \preceq (I - A)^{-1}A\left(\frac{c}{n}\right) = \frac{(I - A)^{-1}A(c)}{n} \to \theta,\ n \to \infty, .$$

Thus, $d(z, f(z) = \theta$, i.e., $f(z) = z$.
If y is a fixed point of f then

$$d(z, y) = d(f(z), f(y)) \preceq A(d(z, y)),$$

that is

$$(I - A)(d(z, y)) \preceq \theta \implies d(z, y) \preceq (I - A)^{-1}(\theta) = \theta, \tag{2.25}$$

so $z = y$. □

The presented results could be combined with P property presented in ([6]). It is said that the mapping f has the property P if $F(f) = F(f^n)$ for each $n \in \mathbb{N}$ (if it has no periodic points). From the proof of the previous theorem, we obtain as a corollary the extension of the known results Theorem 3.2 of [81] and Corollary 3.4 of [57].

Corollary 2.2.3. *Let (X, d) be a complete soilid cone metric space. Let $f : X \mapsto X$ be a quasi-contraction of Perov type with $A(P) \subseteq P$ and $\|A\| < \frac{1}{2}$. Then f has the property P.*

Proof. Theorem 2.2.2 implies that $F(f) \neq \emptyset$. Since, for each $n \in \mathbb{N}$, $F(f) \subseteq F(f^n)$ is always valid, only the reverse inclusion has to be proved. So, let $z \in F(f^n)$, $f^n z = z$. Theorem 2.2.2 (i) gives

$$d(z, f(z)) \preceq (I - A)^{-1} A(d(z, f(z))),$$

that is

$$d(z, f(z)) \preceq ((I - A)^{-1})^n A^n(d(z, f(z))),\ n \in \mathbb{N}.$$

Because $\|A\| < 1/2$, we get

$$\|((I - A)^{-1})^n A^n(d(z, f(z)))\| \leq \|((I - A)^{-1})^n\| \|A^n\| \|(d(z, f(z)))\| \to \theta, n \to \infty,$$

therefore $f(z) = z$. Now, clearly, we get $d(u, fu) = 0$, that is $fu = u$. □

Observe that the first part of Theorem 2.1.3 follows directly from Theorem 2.2.2.

Example 7. Let $X = [0, 3] \cup [4, 5]$ and $E = C^{(1)}[0, 1]$ be with a non-normal cone P as in Example 4. Let us define cone metric $d : X \times X \mapsto E$ by

$$d(x, y) = |x - y| \cdot \exp,\ x, y \in X.$$

If $f : X \mapsto X$ is defined by

$$f(x) = \begin{cases} 0, & \text{if} \quad x \in [0, 3], \\ 3, & \text{if} \quad x \in [4, 5], \end{cases}$$

then for each $x \in [4, 5]$ we have $d(x, f(x)) \leq 2 \cdot \exp$, $d(f(x), f^2(x)) = 3 \cdot \exp$. Thus, $d(f(x), f^2(x)) > d(x, f(x))$, and f does not obey the condition (2.1). Let us show that f obeys the condition (2.15).

It is enough to consider only the case $x \in [0, 3]$ and $y \in [4, 5]$. Now $d(f(x), f(y)) = 3 \exp$ and $d(y, f(x)) \geq 4 \cdot \exp$. Hence,

$$d(f(x), f(y)) = \frac{3}{4} \cdot 4 \cdot \exp \leq \frac{3}{4} \max\{x - f(y)|, |y - f(x)|\} \cdot \exp .$$

Thus, a mapping $f : X \mapsto X$ satisfies the condition (2.15), where $A : E \mapsto E$, is a bounded linear operator defined with $A(f) = (3/4)f$, $f \in E$. Clearly, $\|A\| = 3/4$, and all the assumptions from Theorem 2.2.2 are satisfied. Hence, f has a unique fixed point $x = 0 \in X$.

Based on the Example 6, we obtain the analogue result where we do not suppose $A(P) \subseteq P$.

Lemma 2.2.4. *Let (X, d) be a cone metric space, P a normal cone with a normal constant K, $f : X \mapsto X$ a quasi-contraction and $K\|A\| < 1$. Then, for every $x \in X$,*

(*i*) *for all $n \in \mathbb{N}$ there exists $i \in \{1, \ldots, n\}$ such that*

$$\delta(O(x, n)) = \|d(x, f^i(x))\|;$$

(ii) For arbitrary $n, n_0 \in \mathbb{N}$,

$$\delta(O(x,n)) \leq \frac{K}{1 - K^{n_0}\|A\|^{n_0}} \delta(O(x,n_0)) :$$

(iii) For any $n \in \mathbb{N}$,

$$\delta(O(x,\infty)) \leq \frac{K}{1 - K^{n}\|A\|^{n}} \delta(O(x,n)).$$

Proof. The next notations will be used to simplify the proof:

$$\delta_0 := \delta(O(x,n_0)),\ \delta_n := \delta(O(x,n)) \text{ and } \delta = \delta(O(x,\infty)).$$

(i) Choose arbitrary $x \in X$, $n \in \mathbb{N}$ and $1 \leq i < j \leq n$. Since f is a quasi-contraction, there exists

$$u \in \Big\{ d(f^{i-1}(x), f^{j-1}(x)), d(f^{i-1}(x), f^{i}(x)), d(f^{i-1}(x), f^{j}(x)),$$

$$d(f^{i}(x), f^{j-1}(x)), d(f^{j-1}(x), f^{j}(x)) \Big\} \subseteq O(x,n),$$

such that

$$d(f^i(x), f^j(x)) \preceq A(u).$$

Now,

$$\|d(f^i(x), f^j(x))\| \leq K\|A\|\|u\| \leq K\|A\|\delta_n < \delta_n.$$

Hence, $\delta_n = \|d(x, f^i(x))\|$ for some $i \in \{1, \ldots, n\}$.
(ii) If $n \leq n_0$, then $\frac{K}{1-K^{n_0}\|A\|^{n_0}} > 1$, and

$$\delta(O(x,n)) \leq \frac{K}{1 - K^{n_0}\|A\|^{n_0}} \delta(O(x,n_0)), \tag{2.26}$$

for every $x \in X$.
Thus, we may assume that $n > n_0$.
There exists $i, j \in \mathbb{N}$, $1 \leq i \leq n_0$ and $1 \leq j \leq n$ such that

$$\delta_0 = \|d(x, f^i(x))\| \text{ and } \delta_n = \|d(x, f^j(x))\|.$$

If $j \leq n_0$, then $\delta_n = \delta_0$ and (2.2) holds. Otherwise,

$$d(x, f^j(x)) \preceq d(x, f^{n_0}(x)) + d(f^{n_0}(x), f^j(x)).$$

Clearly,

$$d(f^{n_0}(x), f^j(x)) \preceq A\Big(u^{(1)}_{n_0,j}\Big),$$

where

$$u^{(1)}_{n_0,j} \in \Big\{ d(f^{n_0-1}(x), f^{j-1}(x)), d(f^{n_0-1}(x), f^{n_0}(x)),$$

$$d(f^{n_0-1}(x), f^{j}(x))), d(f^{n_0}(x), f^{j-1}(x)), d(f^{j-1}(x), f^{j}(x)) \Big\}.$$

Evidently,

$$\delta_n \leq K\delta_0 + K\|A\|\|u^{(1)}_{n_0,j}\|$$

and

$$u^{(1)}_{n_0,j} \preceq A\Big(u^{(2)}_{n_0,j} \Big),$$

where

$$u^{(2)}_{n_0,j} \in O\Big(f^{n_0-2}(x), j - n_0 + 2 \Big) \subseteq O(x,n).$$

Moreover,

$$\delta_n \leq K\delta_0 + K^2\|A\|^2\|u^{(2)}_{n_0,j}\|.$$

Continuing in the same manner, after $n_0 - 2$ more steps, we get

$$u^{(n_0-1)}_{n_0,j} \preceq A\Big(u^{(n_0)}_{n_0,j} \Big),$$

for

$$u^{(n_0)}_{n_0,j} \in O(x,n)$$

and

$$\delta_n \leq K\delta_0 + K^{n_0}\|A\|^{n_0}\delta_n.$$

As a conclusion, the inequality (2.2) holds for every $n, n_0 \in \mathbb{N}$.

(iii) By taking into account the definition of δ, (iii) follows directly from (ii). □

This auxiliary result contains estimation of $\delta(O(x,\infty))$.

Corollary 2.2.5. *Under the assumptions of Lemma 2.2.4,*

$$\delta(O(x,\infty)) \leq \frac{K}{1 - K\|A\|}\|d(x, f(x))\|, \ x \in X.$$

Theorem 2.2.6. *Let (X,d) be a complete cone metric space and P a normal cone with a normal constant K. If a mapping $f : X \mapsto X$ is a quasi contraction and $K\|A\| < 1$, then f has a unique fixed point $y \in X$ and for any $x \in X$, the iterative sequence $(f^n(x))$ converges to the fixed point of f.*

Proof. For arbitrary $x \in X$, let us define $x_n = f^n(x), n \in \mathbb{N}$. Furthermore, there exists

$$u^{(1)}_{n,n+1} \in \Big\{ d(f^{n-1}(x), f^n(x)), d(f^{n-1}(x), f^{n+1}(x)), d(f^n(x), f^{n+1}(x)), d(f^n(x), f^n(x)) \Big\},$$

such that

$$d(f^n(x), f^{n+1}(x)) \ \leq A(u^{(1)}_{n,n+1}),$$

implicating

$$\|d(f^n(x), f^{n+1}(x))\| \leq K\|A\|\|u^{(1)}_{n,n+1})\|.$$

There exists

$$\begin{aligned} u^{(2)}_{n,n+1} \in \ \Big\{ \ & d(f^{n-2}(x), f^{n-1}(x)), d(f^{n-2}(x), f^n(x)), d(f^{n-1}(x), f^n(x)), \\ & d(f^{n-2}(x), f^{n+1}(x)), d(f^n(x), f^{n+1}(x)), d(f^{n-1}(x), f^n(x)), \\ & d(f^{n-1}(x), f^{n+1}(x)), d(f^n(x), f^n(x)) \Big\}. \end{aligned}$$

such that

$$\|d(f^n(x), f^{n+1}(x))\| \leq K^2\|A\|^2\|u^{(2)}_{n,n+1}\|,$$

After $n-2$ more steps, we get $u^{(n)}_{n,n+1} \in O(x, \infty)$ such that

$$\|d(f^n(x), f^{n+1}(x))\| \leq K^n\|A\|^n\|u^{(n)}_{n,n+1})\| \leq K^n\|A\|^n\delta(O(x, \infty)).$$

Choose arbitrary $n, m \in \mathbb{N}, m > n$,

$$\|d(f^n(x), f^m(x))\| \leq \sum_{i=n}^{m-1} K^i\|A\|^i\delta(O(x, \infty)) \leq \delta(O(x, \infty)) \sum_{i=n}^{\infty}(K\|A\|)^i,$$

so $(f^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is a complete cone metric space, there exists $y \in X$ such that $\lim_n f^n(x) = y$. It remains to comment if y is a fixed point of f.

For some $n \in \mathbb{N}$, there exists

$$s_n \in \big\{d(f^n(x), y), d(f^{n+1}(x), y), d(f^{n+1}(x), f^n(x)), d(f(y), f^n(x)), d(f(y), y)\big\},$$

such that

$$d(y, f(y)) \preceq d(y, f^{n+1}(x)) + d(f^{n+1}(x), f(y)) \preceq d(y, f^{n+1}(x)) + A(s_n). \tag{2.27}$$

Consider subsequences $(s_{n,i})$, $i = \overline{1,5}$ of the sequence (s_n), such that all the elements of the sequence $\{s_{n,i}\}$, $i = \overline{1,5}$ are of the form, $d(f^n(x), y)$, $d(f^{n+1}(x), y)$, $d(f^n(x), f^{n+1}(x))$, $d(f^n(x), f(y))$ or $d(f(y), y)$, respectively.

It is clear that $\lim\limits_{n\to\infty} s_{n,i} = \theta$, $i = 1, 2, 3$, and $\lim\limits_{n\to\infty} s_{n,i} = d(y, f(y))$, $i = 4, 5$. Thus $\lim_n A(s_{n,i}) = 0$, $i = 1, 2, 3$, and $\lim\limits_{n\to\infty} A(s_{n,i}) = A(d(y, f(y)))$, $i = 4, 5$. Implementing (2.27), $f(y) = y$.

To prove the uniqueness of fixed point, let us suppose that there exist $z \in X$ such that $f(z) = z$. Then, $d(z, y) = d(f(z), f(y)) \preceq A(d(z, y))$, and so $\|d(z, y)\| \leq K\|A\|\|d(z, y)\|$, i.e., $y = z$.

□

2.3 Fisher quasi-contraction of Perov type

The question that rises looking at the theorems presented in previous sections is do only x and $f(x)$ have influence on existence of a fixed point. Obvious answer follows from the sequence of approximations that converges to a fixed point in any initial point $x \in X$. Having that in mind, it is important to somehow include more values from the orbit of f in the contractive condition. Combining that idea with the theorems in the Section 2.2, we introduce the concept of (p, q)-quasi-contraction of Perov type.

Similar contractive condition, but including maximum, was defined by Fisher in [54] along with some fixed point thorems. That is why is also known as Fisher quasi-contraction. In this section, we extend Fisher's results on cone metric spaces, but on the top of that, we incorporate operators in the sense of Perov type contraction.

Definition 2.3.1. *Let (X, d) be a cone metric space. A mapping $f : X \mapsto X$ such that for some $A \in \mathcal{B}(E)$, $r(A) < 1$ and for some fixed positive integers p and q and every $x, y \in X$, there exists*

$$u \in \mathrm{F}_f^{p,q}(x,y) \equiv \Big\{ d(f^{(}r)(x), f^{(}s)(y)), d(f^{(}r)(x), f^{r'}(x)), d(f^s(y), f^{s'}(y)) \mid$$
$$0 \le r, r' \le p \text{ and } 0 \le s, s' \le q \Big\}.$$

such that $d(f^p(x), f^q(y)) \preceq A(u)$, is called (p, q)-quasi–contraction (Fisher's quasi–contraction, F quasi-contraction) of Perov type.

This theorem extends the results of Perov for matrices, and also, as a corollary, generalize Theorem 1 of Zima ([130]) along with Theorem 2.2.2.

Theorem 2.3.2. *Let (X, d) be a complete cone metric space and P a solid cone. Suppose that the mapping $f : X \mapsto X$ is a $(p, q)-$quasi-contraction of Perov type, $A(P) \subseteq P$ and f continuous. Then f has a unique fixed point in X and for any $x \in X$, the iterative sequence $(f^n(x))$ converges to the fixed point.*

Proof. Without loss of generality, assume that $p \ge q$. If $x \in X$ be arbitrary and $\omega(x) = \sum_{0 \le i < p} d(f^i(x), f^p(x))$, we prove that

$$d(f^n(x), f^p(x)) \preceq (I - A)^{-1} A(\omega(x)),\ n \ge p. \tag{2.28}$$

Obviously, (2.28) is true for $n = p$. Suppose that it holds for $m \le n_0 - 1$, and observe $m = n_0 \ge p + 1$.
Because f is (p, q)- quasi-contraction, there exist some $i, j \in \mathbb{N}$, that

$$d(f^{n_0}(x), f^p(x)) \preceq A(d(f^i(x), f^j x)). \tag{2.29}$$

(1) If $i, j \le p$, then

$$\begin{aligned} d(f^{n_0}(x), f^p(x)) &\preceq A(d(f^i(x), f^p(x)) + d(f^p(x), f^j(x))) \\ &\preceq A(\omega(x)) \preceq (I - A)^{-1} A(\omega(x)). \end{aligned}$$

Remark that we have used that $i \neq j$ in this inference, but if $i = j$, (2.28) is fulfilled.

(2) If $p < i < n_0$, $j \leq p$ then (2.28) and (2.29) imply

$$\begin{aligned} d(f^{n_0}(x), f^p(x)) &\preceq A(d(f^i(x), f^p x)) + A(d(f^p(x), f^j(x))) \\ &\preceq A(I-A)^{-1}A(\omega(x)) + A(\omega(x)) \\ &= (I-A)^{-1}A(\omega(x)). \end{aligned}$$

(3) For $p < i < n_0$, $p < j < n_0$, we have

$$d(f^{n_0}x, f^p(x)) \preceq A^k(d(f^{i_0}(x), f^{j_0}(x))),$$

where $i_0 < p$ or $j_0 < p$ and $1 < k$.
Assume that at least $i_0 < p$.

$$\begin{aligned} d(f^{n_0}(x), f^p(x)) &\preceq A^k(d(f^{i_0}(x), f^p x)) + A^k(d(f^p(x), f^{j_0}(x))) \\ &\preceq A^k(\omega(x)) + A^k(I-A)^{-1}A(\omega(x)) \\ &\preceq (I-A)^{-1}A(\omega(x)), \end{aligned}$$

since $j_0 \leq j < n_0$, so the inequality (2.28) holds in this case.

(4) In the case $i = n_0$, $j \leq p$, the triangle inequality, $A(P) \subseteq P$ and (2.29) imply

$$\begin{aligned} d(f^{n_0}(x), f^p(x)) &\preceq A(d(f^{n_0}(x), f^p(x))) + A(d(f^p(x), f^j x)) \\ &\preceq A(d(f^{n_0}(x), f^p x)) + A(\omega(x)). \end{aligned}$$

Taking into account (3), (2.28) easily follows.

(5) Finally, consider $i = n_0$ and $p < j \leq n_0$.
If $j = n_0$, it follows $d(f^{n_0}(x), f^p(x)) \preceq A(\theta)$ and $d(f^{n_0}(x), f^p(x)) = \theta$.
Otherwise,

$$d(f^{n_0}(x), f^p(x)) \preceq A(d(f^j(x), f^{n_0}(x))) \tag{2.30}$$

and there exist $i_0 \leq j_0 \leq n_0$, $i_0 < p$ and some $k_0 > 1$ such that

$$d(f^j(x), f^{n_0}(x)) \preceq A^{k_0}(d(f^{i_0}(x), f^{j_0}(x))).$$

If $j_0 \leq p$, then (2.28) follows by the last inequality and (2.30). Notice that if $p < j_0 < n_0$, then

$$\begin{aligned} d(f^{n_0}(x), f^p(x)) &\preceq A^{1+k_0}(d(f^{i_0}(x), f^{j_0}(x))) \\ &\preceq A^{1+k_0}(d(f^{i_0}(x), f^p(x))) + A^{1+k_0}(d(f^p x, f^{j_0}(x))) \\ &\preceq A^{1+k_0}(\omega(x)) + A^{1+k_0}(I-A)^{-1}A(\omega(x)) \\ &= A^{1+k_0}(I-A)^{-1}(I-A+A)(\omega(x)) \\ &\preceq (I-A)^{-1}A(\omega(x)). \end{aligned} \tag{2.31}$$

But if $j_0 = n_0$, then

$$d(f^{n_0}(x), f^p(x)) \preceq A^{1+k_0}(d(f^{i_0}(x), f^p x)) + A^{1+k_0}(d(f^p(x), f^{n_0}(x))). \tag{2.32}$$

Then, for some $k_1 \geq 1$ and $i_1 \leq j_1 \leq n_0$, $i_1 < p$, $d(f^p(x), f^{n_0}(x)) \preceq A^{k_1}(d(f^{i_1}(x), f^{j_1}(x)))$, so by (2.32) we get

$$d(f^{n_0}(x), f^p(x)) \preceq A^{1+k_0}(d(f^{i_0}(x), f^p x)) + A^{1+k_0+k_1}(d(f^{i_1}(x), f^{j_1}(x))). \tag{2.33}$$

Again, if $j_1 < n_0$, as in (2.31), we have (2.28). Otherwise,

$$\begin{aligned} d(f^{n_0}(x), f^p(x)) \preceq\ & A^{1+k_0}(d(f^{i_0}(x), f^p x)) + A^{1+k_0+k_1}(d(f^{i_1}(x), f^p x)) \\ & + A^{1+k_0+2k_1}(d(f^{i_1}(x), f^{n_0}(x))). \end{aligned}$$

Hence, for arbitrary $n \in \mathbb{N}$,

$$\begin{aligned} d(f^{n_0}(x), f^p(x)) \preceq\ & A^{1+k_0}(d(f^{i_0}(x), f^p x)) \\ & + \sum_{m=1}^{n-1} A^{1+k_0+mk_1}(d(f^{i_1}(x), f^p x)) + A^{1+k_0+nk_1}(d(f^{i_1}(x), f^{n_0}(x))) \\ \preceq\ & \sum_{m=0}^{n-1} A^{1+k_0+mk_1} A(\omega(x)) + A^{1+k_0+nk_1}(d(f^{i_1}(x), f^{n_0}(x))) \\ \preceq\ & (I-A)^{-1} A^{1+k_0}(\omega(x)) + A^{1+k_0+nk_1}(d(f^{i_1}(x), f^{n_0}(x))) \\ \preceq\ & (I-A)^{-1} A(\omega(x)) + A^{1+k_0+nk_1}(d(f^{i_1}(x), f^{n_0}(x))). \end{aligned}$$

However, $A^{1+k_0+nk_1}(d(f^{i_1}(x), f^{n_0}(x))) \to \theta$, $n \to \infty$. For each $c \gg \theta$ there exists $n_c \in \mathbb{N}$ such that $A^{1+k_0+nk_1}(d(f^{i_1}(x), f^{n_0}(x))) \ll c$ for $n > n_c$, so

$$d(f^{n_0}(x), f^p(x)) \preceq (I-A)^{-1} A(\omega(x)) + c,\ c \gg \theta,$$

and $d(f^{n_0}(x), f^p(x)) \preceq (I-A)^{-1} A(\omega(x))$.
Therfore, (2.28) is true for any $n \in \mathbb{N}$. The inequality

$$\begin{aligned} d(f^n x, f^j(x)) \preceq\ & d(f^n x, f^p(x)) + d(f^p(x), f^j(x)) \\ \preceq\ & (I-A)^{-1} A(\omega(x)) + \omega(x) \\ =\ & (I-A)^{-1}(\omega(x)). \end{aligned}$$

proceeds from(2.28). Mapping f is a (p,q)-quasi-contraction, thus for $n > m \geq p$, $m = kp + r$, $0 \leq r < p$, $k \geq 1$,

$$d(f^n(x), f^m(x)) \preceq A^k(d(f^i(x), f^j(x))) \preceq A^k(I-A)^{-1}(\omega(x)),$$

where $0 \leq i \leq j \leq n$ and $i \leq p$.
Remark that $A^k(I-A)^{-1}(\omega(x)) \to \theta$, $k \to \infty\,(m \to \infty)$, $(f^n(x))$ is a Cauchy sequence in X and $z = \lim_{n\to\infty} f^n(x) \in X$ is a fixed point of f since f is a continuous it follows that $f(z) = z$. The uniqueness of z follows from the definition of a (p,q)-quasi-contraction.
□

In the particular case of Theorem 2.3.2 when $q = 1$ (or $p = 1$), the continuity of f is unnecessary (see [54]).

Theorem 2.3.3. *Let (X, d) be a complete cone metric space and P a solid cone. If the mapping $f : X \mapsto X$ is a $(p, 1)$-quasi-contraction of Perov type, $A(P) \subseteq P$, then f has a unique fixed point in X and for any $x \in X$, the iterative sequence $(f^n(x))$ converges to the fixed point of f.*

Proof. As in the proof of Theorem 2.3.2, the sequence $(f^n(x))$ is a Cauchy sequence for any $x \in X$ and has the limit z in X. For $n > p$,

$$\begin{aligned} d(z, f(z)) &\preceq d(z, f^n(x)) + d(f^n(x), f(z)) \\ &= d(z, f^n(x)) + d(f^p f^{n-p}(x), f(z)) \\ &\preceq d(z, f^n(x)) + A(u_n), \end{aligned}$$

where u_n belongs to the set

$$\left\{ d(f^r f^{n-p}(x), f(z)), d(f^r f^{n-p}(x), z), d(f^r f^{n-p}(x), f^{r'} f^{n-p}(x)), d(z, f(z)) : 0 \le r, r' \le p \right\}.$$

But

$$d(f^r f^{n-p}(x), f(z)) \le d(f^r f^{n-p}(x), z) + d(z, f(z)).$$

Since $\lim\limits_{n\to\infty} f^n(x) = z$, for each $c \gg 0$ choose n_0 such that $d(f^n x, z), d(f^n(x), f^m(x)) \ll c$, $n, m \ge n_0$. If $n > n_0 + p$, then

$$d(z, f(z)) \preceq c + A(d(z, f(z))) + A(c) \text{ for any } c \gg \theta.$$

Uniqueness of the fixed point z obviously follows. □

When $p = q = 1$, $(1, 1)$-quasi-contraction is Ćirić quasi-contraction and Theorem 2.2.2 is a consequence of Theorem 2.3.2.

Omitting positivity of operator A, we may state a new result in the case of normal cone metric space.

Theorem 2.3.4. *Let (X, d) be a complete cone metric space with a normal cone P and a normal constant K. Let the mapping $f : X \mapsto X$ be a continuous (p, q)-quasi-contraction of Perov type, $K\|A\| < 1$ Then f has a unique fixed point in X and for any $x \in X$, the iterative sequence $(f^n(x))$ converges to the fixed point of f.*

Proof. Assume that $p \ge q$ and for some $x \in X$ define

$$\eta(x) = \sum_{0 \le i < j \le p} \|d(f^i(x), f^j(x))\|,$$

then

$$\|d(f^n(x), f^p(x))\| \le \frac{K\|A\|}{1 - K\|A\|} \eta(x), \; n \ge p. \tag{2.34}$$

Obviously, the inequality (2.34) is true for $n = p$. Suppose that (2.34) holds for any $n \leq n_0 - 1$, $n_0 > p$. If $n = n_0$, then

$$d(f^{n_0}(x), f^p(x)) \preceq A(d(f^i(x), f^j(x))), \tag{2.35}$$

where $i, j \in \{0, \ldots, n_0\}$. Few different cases will be discussed.
(1) If $0 \leq i, j \leq p$, then

$$\|d(f^{n_0}(x), f^p(x))\| \leq K\|A\|\eta(x) \leq \frac{K\|A\|}{1 - K\|A\|}\eta(x).$$

(2) If $p < i < n_0$ and $j \leq p$ (analogously $i \leq p$, $p < j < n_0$), then the induction hypothesis and the triangle inequality imply

$$\begin{aligned} \|d(f^{n_0}(x), f^p(x))\| &\leq \frac{K^2\|A\|^2}{1 - K\|A\|}\eta(x) + K\|A\|\eta(x) \\ &= \frac{K\|A\|}{1 - K\|A\|}\eta(x). \end{aligned}$$

(3) Consider $p < i, j < n_0$. There exist $i_0, j_0 < n_0$, $i_0 < p$ such that

$$\|d(f^i(x), f^j(x))\| \leq (K\|A\|)^k\|d(f^{i_0}(x), f^{j_0}(x))\|$$

for some $k \geq 1$. The inequality (2.34) is satisfied if $i_0 = j_0$. If $j_0 \leq p$, then

$$\|d(f^{n_0}(x), f^p(x))\| \leq (K\|A\|)^{k+1}\eta(x) \leq \frac{K\|A\|}{1 - K\|A\|}\eta(x).$$

Otherwise,

$$\begin{aligned} \|d(f^{n_0}(x), f^p(x))\| &\leq (K\|A\|)^{k+1}\big(\|d(f^{i_0}(x), f^p(x))\| + \|d(f^p(x), f^{j_0}(x))\|\big) \\ &\leq (K\|A\|)^{k+1}\big(\eta(x) + \frac{K\|A\|}{1 - K\|A\|}\eta(x)\big) \\ &\leq \frac{K\|A\|}{1 - K\|A\|}\eta(x), \end{aligned}$$

because $K\|A\| < 1$.
(4) Assume that $i = n_0$. Inequality

$$\|d(f^{n_0}(x), f^p(x))\| \leq K\|A\|\|d(f^{n_0}(x), f^p(x))\| + K\|A\|\|d(f^p(x), f^j(x))\|,$$

leads to (2.34) when $j \leq p$.
Otherwise, if $p < j < n_0$, there exist some $i_0 \leq j_0 \leq n_0$, $i_0 < p$ and $k_0 \geq 1$ for which

$$\|d(f^{n_0}(x), f^j(x))\| \leq (K\|A\|)^{k_0}\|d(f^{i_0}(x), f^{j_0}(x))\|.$$

Evidently, for $j_0 \leq p$, (2.34) is fulfilled. Similarly as previously shown in the proof of Theorem 2.3.2, (2.34) holds if $j_0 < n_0$ by the induction hypothesis.

If $j_0 = n_0$, then again as in the proof of Theorem 2.3.3, there are some $i_1 \le j_1 \le n_0$, $i_1 < p$ and $k_1 \ge 1$ that satisfy $\|d(f^p(x), f^{n_0}(x))\| \le (K\|A\|)^{k_1}\|d(f^{i_1}(x), f^{j_1}(x))\|$. Then

$$\|d(f^{n_0}(x), f^p(x))\| \le (K\|A\|)^{1+k_0}\|d(f^{i_0}(x), f^p(x))\| + (K\|A\|)^{1+k_0+k_1}\|d(f^{i_1}(x), f^{j_1}(x))\|.$$

Again, if $j_1 < n_0$, (2.34) easily follows. If $j_1 = n_0$, then after $m-1$ more steps, we get

$$\begin{aligned}\|d(f^{n_0}(x), f^p(x))\| &\le (K\|A\|)^{1+k_0}\|d(f^{i_0}(x), f^p(x))\| \\ &+ \sum_{l=1}^{i=m-1}(K\|A\|)^{1+k_0+lk_1}\|d(f^{i_1}(x), f^p(x))\| \\ &+(K\|A\|)^{1+k_0+mk_1}\|d(f^p(x), f^{n_0}(x))\|.\end{aligned}$$

Therefore the inequality (2.34) is satisfied in this case. Hence, (1)-(5) imply that the inequality (2.34) holds for any $n \ge p$.

Let $n \ge m > 2p$, $m = (k+1)p + r$, $k \in \mathbb{N}, 0 \le r < p$. To estimate $d(f^n(x), f^m(x))$, observe $p \le i_{n,m} \le j_{n,m} \le n$ and $k \ge 1$ for which

$$\|d(f^n(x), f^m(x))\| \le (K\|A\|)^k\|d(f^{i_{n,m}}(x), f^{j_{n,m}}(x))\|.$$

Then

$$\|d(f^n(x), f^m(x))\| \le \frac{2(K\|A\|)^{k+1}}{1 - K\|A\|}\eta(x)$$

by (2.34). So, $(f^n(x))$ is a Cauchy sequence, thus $\lim\limits_{n\to\infty} f^n(x) = z$ for some $z \in X$. Since f is a continuous, $f(z) = z$. Obviously, z is a unique fixed point of f in X because $K\|A\| < 1$. $\square$

As in the solid case, if $p = 1$ or $q = 1$, the continuity condition may be excluded.

Theorem 2.3.5. *Let (X, d) be a complete cone metric space and P be a normal cone with a normal constant K. Suppose that the mapping $f : X \mapsto X$ is a $(p, 1)-$quasi-contraction of Perov type with $K\|A\| < 1$. Then f has a unique fixed point in X and for any $x \in X$, the iterative sequence $(f^n(x))$ converges to the fixed point.*

Proof. Let x be an arbitrary point in X. Then, as in the proof of Theorem 2.3.4, the sequence $(f^n(x))$ is a Cauchy sequence in the complete cone metric space X and so has a limit z in X. For $n > p$, we now have $d(f^n x, f(z)) \preceq A(u_n)$, where u_n belongs to the set

$$\Big\{d(f^r f^{n-p}(x), f(z)), d(f^r f^{n-p}(x), z), d(f^r f^{n-p}(x), f^{r'} f^{n-p}(x)), d(z, f(z)) : 0 \le r, r' \le p\Big\}.$$

But, recall that $\lim\limits_{n\to\infty} d(f^n(x), z) = \theta$ and $\lim\limits_{n,m\to\infty} d(f^n(x), f^m(x)) = \theta$. Since

$$d(z, f(z)) = \lim_{n\to\infty} d(f^n(x), f(z)) \preceq A(d(z, f(z)))$$

and P is a normal cone, therefore $f(z) = z$. Uniqueness goes analogously as in the previous proof. $\square$

Regarding the case $p = q = 1$, Theorem 2.2.6 is a consequence of this theorem on normal cone metric spaces. It also includes Ćirić quasi-contraction result on metric space due to the fact that metric space can be observed as normal cone metric space with a normal constant $K = 1$.
In order to express the significance of presented theorems, we add an interesting example.

Example 8. Let $E = C_R([0,1], \|\|_\infty)$ and $P = \{f \in E : f(t) \geq 0\}$. (X, ρ) a metric space and $d : X \times X \mapsto E$ defined by $d(x, y) = \rho(x, y)\varphi$, where $\varphi : [0,1] \mapsto R^+$ is continuous. Then (X, d) is a normal cone metric space and the normal constant of P is equal to $K = 1$.

To apply our results let us consider the solution of the equation $(I - Q)x = b$, where $b \in E$ is given, $I, Q \in \mathcal{B}(E)$ and $\|Q^2 - \alpha Q\| < 1 - \alpha$, $0 < \alpha < 1$ (see Theorem 3 of [126]). Let us take $X = E$, $\rho(x, y) = \|x - y\|$, and define $T : X \mapsto X$ by $T(x) = b + Q(x)$. Now, it is easy to see that T is continuous. If $x_n, x_0 \in X$ and $x_n \to x_0$, that is $d(x_n, x_0) \to 0$, then

$$\|d(T(x_n), T(x_0))\| = \|d(Q(x_n), Q(x_0))\| \leq \|Q\|\|d(x_n, x_0)\| \to 0$$

Let us remark that $T(T(x)) = b + Q(b) + Q^2(x)$, so $T^2(x) - T^2(y) = Q^2(x - y)$. Thus

$$\begin{aligned} T^2(x) - T^2(y) &= Q^2(x - y) \\ &= (Q^2 - \alpha Q)(x - y) + (\alpha Q)(x - y) \\ &= (Q^2 - \alpha Q)(x - y) + (\alpha T(x - y). \end{aligned}$$

Hence,

$$\|T^2(x) - T^2(y)\| \leq \|(Q^2 - \alpha Q)\|\|x - y\| + |\alpha|\|T(x - y)\|,$$

and

$$\|T^2(x) - T^2(y)\| \leq (\|(Q^2 - \alpha Q)\| + \alpha) \max\{\|x - y\|, \|T(x - y)\|\}.$$

It follows

$$\|T^2(x) - T^2(y)\|\varphi \leq (\|(Q^2 - \alpha Q)\| + \alpha) \max\{\|x - y\|\varphi, \|T(x - y)\|\varphi\}.$$

Finally we have

$$d(T^2(x), T^2(y)) \leq A(u),$$

where $v \in \{d(x, y), d(Tx, Ty)\}$ and $A \in \mathcal{B}(E)$ is defined by $A(v) = (\|(Q^2 - \alpha Q)\| + \alpha)v$, $v \in E$.

Because $0 \leq q = \|(Q^2 - \alpha Q)\| + \alpha < 1$ we can apply Theorem 2.3.4 to conclude that there is a unique $z \in E$ such that $T(z) = z$, i.e., $(I - Q)z = b$. Moreover, for any $x \in X$, the iterative sequence $(T^n x)$ converges to the fixed point z

2.4 Fixed point theorems for w-cone distance

The notion of w-distance was introduced in 1996 by Kada, Suzuki and Takahashi ([80]) with indications that it is more general concept than metric. They gave examples of

w-distance and improved Caristi's fixed point theorem ([29]), Eklands variationals principle ([50]) and the nonconvex minimization theorem according to Takahashi ([127]).
In [42], twenty years later, Ćirić, Lakzian and Rakočević generalized w-distance concept to the tvs-cone metric space where the underlying cone is in topological vector space instead of Banach space as in [67]. Therefore, they improved many results including [6], [66], [104] and [112] and established some unsolved problems.

We extend and improve Theorem 2 of [80] and Theorem 1 of [125], and give an estimation for a $w-$cone distance $p(x_n, z)$ of an approximate value x_n and a fixed point z.

Theorem 2.4.1. *Let (X, d) be a complete cone metric space with w-cone distance p on X. Suppose that for some increasing operator $A \in \mathcal{B}(E)$, $r(A) < 1$, a mapping $T : X \to X$ satisfies the following condition:*

$$p(Tx, T^2x) \preceq A\left(p(x, Tx)\right), \ \textit{for all } x \in X. \tag{2.36}$$

Assume that either of the following holds:

(i) If $y \neq Ty$, there exists $c \in \operatorname{int}(P)$, $c \neq \theta$, such that

$$c \ll p(x, y) + p(x, Tx), \ \textit{for all } x \in X;$$

(ii) T is continuous.

Then, there exists $z \in X$, such that $z = Tz$ and

$$p(T^n x, z) \preceq A^n \left(I - A\right)^{-1} \left(p(x, Tx)\right), \quad \textit{for} \quad n \in \mathbb{N}, \tag{2.37}$$

where $z = \lim\limits_{n\to\infty} T^n x$.

Moreover, if $y = Ty$ for some $y \in X$, then $p(y, y) = \theta$.

Proof. Let $x \in X$ be arbitrary and define a sequence (x_n) by $x_0 = x$, $x_n = T^n x$, for any $n \in \mathbb{N}$. Then from (2.36) we have, for any $n \in \mathbb{N}$,

$$\begin{aligned} p(x_n, x_{n+1}) &= p(Tx_{n-1}, Tx_n) \\ &\preceq A(p(x_{n-1}, x_n)) \preceq \cdots \preceq A^n(p(x, Tx)), \end{aligned} \tag{2.38}$$

since A is an increasing operator. Thus, if $m > n$, then from (w_1) and (2.38),

$$\begin{aligned} p(x_n, x_m) &\preceq \sum_{i=n}^{m-1} p(x_i, x_{i+1}) \\ &\preceq \sum_{i=n}^{m-1} A^i(p(x, Tx)) \\ &\preceq \sum_{i=n}^{\infty} A^i(p(x, Tx)) \\ &= A^n(I - A)^{-1}(p(x, Tx)). \end{aligned} \tag{2.39}$$

However $A^n(I-A)^{-1}(p(x,Tx)) \to \theta$, $n \to \infty$, so (x_n) is a Cauchy sequence in X by Lemma 1.2.6 and, because X is a complete, (x_n) converges to some $z \in X$.

We will prove that z is a fixed point of T by estimating $p(x_n, z)$. Since $x_n \to z$, as $n \to \infty$, from the lower semi-continuity of w distance, we have that for any $\varepsilon \gg \theta$, there is $n_0 \in \mathbb{N}$ such that for any $m \geq n_0$

$$p(x_n, z) \preceq p(x_n, x_m) + \varepsilon.$$

Therefore, for an arbitrary $n \in \mathbb{N}$, if we choose $m > \max\{n, n_0\}$, then, from (2.39), it follows that the inequality

$$p(x_n, z) \preceq A^n(I-A)^{-1}(p(x,Tx)) + \varepsilon$$

holds for any $\varepsilon \gg \theta$, i.e., (2.37) holds for any $n \in \mathbb{N}$.

Let us assume that (i) is satisfied and that $Tz \neq z$. Then, there exists $c \gg \theta$, $c \neq \theta$, such that

$$c \ll p(x,z) + p(x,Tx), \quad \text{for all} \quad x \in X. \tag{2.40}$$

Obviously, $A^n(I-A)^{-1}(p(x,Tx)) \to \theta$ and $A^n(p(x,Tx)) \to \theta$ as $n \to \infty$, so, from the definition of convergence and (p_5), there exists $n_1 \in \mathbb{N}$ such that

$$A^n(I-A)^{-1}(p(x,Tx)) \ll \frac{c}{3} \quad \text{and} \quad A^n(p(x,Tx)) \ll \frac{c}{3},$$

for any $n \geq n_1$. The last observation contradicts to (2.40) since, for any $n \geq n_1$ inequalities

$$\begin{aligned} c &\ll p(x_n, z) + p(x_n, x_{n+1}) \\ &\preceq A^n(I-A)^{-1}(p(x,Tx)) + A^n(p(x,Tx)) \\ &\ll \frac{2c}{3}, \end{aligned}$$

imply that $-c/3 \gg 0$, i.e., $c = \theta$. But, we have already assumed that $c \neq \theta$, hence $Tz = z$ in this case.

Otherwise, if T is a continuous, then, since $x_{n+1} = Tx_n \to Tz$, $n \to \infty$, by (i) of Lemma, we may conclude that $Tz = z$.

It remains to prove that if $Ty = y$, then $p(y,y) = \theta$. Obviously,

$$p(y,y) = p(Ty, T^2y) \preceq A(p(y,Ty)).$$

The operator $(I-A)^{-1} = \sum\limits_{n=0}^{\infty} A^n$ is an increasing linear operator and the last inequality gives us $p(y,y) \preceq (I-A)^{-1}(\theta) = \theta$, i.e., $p(y,y) = \theta$. □ □

Example 9. Let $X = E$, where E and P are defined as in Example 4. Let us define cone metric $d : X \times X \mapsto E$ for any $f, g \in X$ by

$$d(f,g) = \begin{cases} f+g, & f \neq g, \\ 0, & f = g. \end{cases}$$

If $T : X \mapsto X$ is defined by $T(f) = f/2$, $f \in X$, then

$$d(Tf, T^2 f) \preceq A(d(f, Tf)), \quad f \in X,$$

where $A : E \mapsto E$, is a bounded linear operator defined by $A(f) = f/2$, $f \in E$.

Clearly, $r(A) = \|A\| = 1/2$ and T is a continuous, thus all the assumptions from Theorem 2.4.1 are satisfied. Hence, T has a fixed point $f = 0 \in X$ and it is evidently an unique fixed point of T.

Corollary 2.4.2. *Let (X, d) be a complete cone metric space with $w-$cone distance p on X and $A \in \mathcal{B}(E)$ an increasing operator with spectral radius less than $1/2$. Suppose that the mapping $T : X \to X$ satisfies either (i) or (ii) of Theorem 2.4.1 and*

$$p(Tx, T^2 x) \preceq A(p(x, T^2 x)), \quad \textit{for all } x \in X.$$

Then, there exists $z \in X$, such that $z = Tz$ and if $y = Ty$, then $p(y, y) = \theta$.

Proof. If $x \in X$ is arbitrary, then

$$p(Tx, T^2 x) \preceq A(p(x, T^2 x)) \preceq A(p(x, Tx) + p(Tx, T^2 x)).$$

Hence,

$$p(Tx, T^2 x) \preceq A(I - A)^{-1}(p(x, Tx)).$$

Observe that

$$\begin{aligned} r(A(I-A)^{-1}) &\leq r(A) r\left((I-A)^{-1}\right) \\ &\leq \frac{r(A)}{1 - r(A)} < 1, \end{aligned}$$

and the condition (2.36) is satisfied. All the conclusions of this corollary follows directly from Theorem 2.4.1. □ □

If $T : X \to X$ and $F(T)$ is a set of all fixed points of T, then T has a property P if $F(T) = F(T^n)$ for each $n \in \mathbb{N}$. The following theorem extends and improves Theorem 2 of [6] and Theorem 12 of [42] for cone metric space.

Theorem 2.4.3. *Let (X, d) be a complete cone metric space with w-cone distance p on X. Suppose $T : X \to X$ satisfies the condition (2.36) for an increasing operator $A \in \mathcal{B}(E)$. If $r(A) < 1$, then T has property P.*

Proof. Obviously, $F(T) \subseteq F(T^n)$, $n \in \mathbb{N}$, so it remains to show that $Tz = z$ for any $z \in F(T^n)$ and arbitrary $n > 1$.

Remark that

$$p(T^i z, T^{i+1} z) = p(T^{kn+i} z, T^{kn+i+1} z) \preceq A^{kn+i}(p(z, Tz)), \quad k, i \in \mathbb{N},$$

allows us to determine that, because $A^{kn+i}(p(z, Tz)) \to 0$, as $k \to \infty$, when $r(A) < 1$, $p(T^i z, T^{i+1} z) = \theta$, $i \in \mathbb{N}$, and, furthermore, $Tz = T^n z = z$. □ □

Instead of observing contractive conditions on X, we observe only T-orbit $O(x,\infty)$ of an arbitrary element $x \in X$ where $O(x,\infty) = \{T^n x \mid n \in \mathbb{N}_0\}$.
Function $G : X \to P$ is a T-orbitally lower semi-continuous at x if for any $\varepsilon \gg \theta$, there is $n_0 \in \mathbb{N}$ such that (1.2) holds whenever $(x_n) \subseteq O(x;\infty)$ and $x_n \to x$, $n \to \infty$.

The following theorems implies some results of [66], [104], [67] and [112].

Theorem 2.4.4. *Let (X,d) be a complete cone metric space with w-cone distance p on X and $A \in \mathcal{B}(E)$ an increasing operator with spectral radius less than 1. Suppose that $T : X \to X$ and there exists an $x \in X$ such that*

$$p(Ty, T^2y) \preceq A(p(y,Ty)), \quad \textit{for all } y \in O(x,\infty).$$

Then,

(i) $\lim_{n\to\infty} T^n x = z$ *exists and*

$$p(T^n x, z) \preceq A^n (I-A)^{-1} (p(x,Tx)), \quad n \in \mathbb{N};$$

(ii) $p(z,Tz) = \theta$ *if and only if* $G(x) = p(x,Tx)$ *is T-orbitally lower semi-continuous at z.*

Proof. (i) First observation easily follows from the proof of Theorem 2.4.1.

(ii) If $p(z,Tz) = \theta$ then G is obviously T-orbitally lower semi-continuous at z. Otherwise, choose $\varepsilon \gg \theta$ arbitrary. There exists $n_1 \in \mathbb{N}$ such that

$$A^n (p(x,Tx)) \ll \frac{\varepsilon}{2}$$

for any $n \geq n_1$, and $n_2 \in \mathbb{N}$ such that

$$G(z) \preceq G(T^n x) + \frac{\varepsilon}{2}, \quad n \geq n_2.$$

Then, for $n \geq \max\{n_1, n_2\}$,

$$\begin{aligned} p(z,Tz) &\preceq p(T^n x, T^{n+1}x) + \frac{\varepsilon}{2} \\ &\preceq A^n(p(z,Tx)) + \frac{\varepsilon}{2} \ll \varepsilon. \end{aligned}$$

The last inequality holds for any $\varepsilon \gg \theta$, and by (p_2), $p(z,Tz) = \theta$. □ □

Theorem 2.4.5. *Let (X,d) be a complete cone metric space with w-cone distance p on X and $A \in \mathcal{B}(E)$ an increasing operator with spectral radius less than 1. Suppose that $T : X \to X$ is a p-contractive mapping of Perov type, i.e.,*

$$p(Tx, Ty) \preceq A\left(p(x,y)\right), \quad \textit{for all } x, y \in X.$$

Then, T has a unique fixed point $z \in X$, and $p(z,z) = \theta$.

Proof. From the proof of Theorem 2.4.1 we get that $T^n x \to z$ as $n \to \infty$, $Tz = z$ and $p(z,z) = \theta$.

If $Ty = y$, then

$$p(y,z) = p(Ty,Tz) \preceq A(p(y,z)) \implies p(y,z) \preceq (I-A)^{-1}(\theta) = \theta,$$

thus $p(y,z) = \theta$ and $p(z,z) = \theta$ imply, by (i) of Lemma 1.2.6, that $y = z$. □ □

Remark 2.4.6. De Pascale and De Pascale [103] used K-normed space to prove that Lou's fixed point theorem [93] in a space of continuous functions is equivalent to the Banach contraction principle with contractive constant replaced by bounded linear operator with spectral radius < 1. Observe that in [103] cone is normal, but we have investigated the case when cone is not normal [33]. It is interesting to to investigate possibility of extending Lou's theorem in the case when cone is not normal.

We state the similar results when cone metric space (X,d) is normal by replacing the condition $r(A) < 1$ and excluding the condition that the operator A is increasing, i.e., not demanding the condition $A(P) \subseteq P$.

Theorem 2.4.7. *Let (X,d) be a complete normal cone metric space with normal constant K and w-cone distance p on X. Suppose that for some operator $A \in \mathcal{B}(E)$, $K\|A\| < 1$, a mapping $T : X \to X$ satisfies the following condition:*

$$p(Tx, T^2x) \preceq A\left(p(x,Tx)\right), \quad \textit{for all } x \in X.$$

Assume that either of the following holds:

(i) If $y \neq Ty$, there exists $c > 0$, such that

$$c < \|p(x,y)\| + \|p(x,Tx)\|, \quad \textit{for all } x \in X;$$

(ii) T is continuous.

Then, there exists $z \in X$, such that $z = Tz$ and if $y = Ty$ for some $y \in X$, then $p(y,y) = \theta$.

Proof. Let $x \in X$ be an arbitrary and let us define a sequence (x_n), $x_0 = x$, $x_n = T^n x$, for any $n \in N$. Then,

$$\|p(x_n, x_{n+1})\| \le K\|A\|\|p(x_{n-1}, x_n)\| \le \cdots \le (K\|A\|)^n \|p(x,Tx)\|. \tag{2.41}$$

Thus, if $m > n$, then from (w_1) and (2.38),

$$\begin{aligned} \|p(x_n, x_m)\| &\le \sum_{i=n}^{m-1} (K\|A)^i \|p(x,Tx)\| \\ &\le \sum_{i=n}^{\infty} (K\|A\|)^i \|p(x,Tx)\| \\ &= \frac{(K\|A\|)^n}{1 - K\|A\|} \|p(x,Tx)\|. \end{aligned} \tag{2.42}$$

However,

$$\frac{(K\|A\|)^n}{1-K\|A\|}\|p(x,Tx)\| \to 0, \quad n\to\infty,$$

so (x_n) is a Cauchy sequence in X and, because X is complete, (x_n) converges to some $z\in X$.

From the lower semi-continuity of w distance, we have that for any $\varepsilon \gg \theta$, there is $n_0\in\mathbb{N}$ such that for any $m\geq n_0$

$$p(x_n,z)\preceq p(x_n,x_m)+\varepsilon.$$

Moreover, for arbitrary $n\in\mathbb{N}$, if we choose $m>n$, then from (2.42) it follows that the inequality

$$\|p(x_n,z)\|\leq \frac{(K\|A\|)^n}{1-K\|A\|}\|p(x,Tx)\|+K\|\varepsilon\|$$

holds for any $\varepsilon\gg\theta$, so, for $\varepsilon:=\varepsilon/n$, $n\in\mathbb{N}$,

$$\|p(x_n,z)\|\leq \frac{(K\|A\|)^n}{1-K\|A\|}\|p(x,Tx)\|.$$

Let us assume that (i) is satisfied and that $Tz\neq z$. Then, there exists $c>0$ such that

$$c<\|p(x,z)\|+\|p(x,Tx)\|, \quad \text{for all} \quad x\in X. \tag{2.43}$$

Then,

$$c<\frac{(K\|A\|)^n}{1-K\|A\|}\|p(x,Tx)\|+(K\|A\|)^n\|p(x,Tx)\|$$

for any $n\in\mathbb{N}$ and that is impossible since (2.43) holds.

Otherwise, if T is continuous, then, since $x_{n+1}=Tx_n\to Tz$, $n\to\infty$, it follows $Tz=z$.

It remains to prove that if $Ty=y$, then $p(y,y)=\theta$. Obviously,

$$\|p(y,y)\|=\|p(T^ny,T^{n+1}y)\|\leq (K\|A\|)^n\|p(y,Ty)\|, \quad n\in\mathbb{N},$$

implies $\|p(y,y)\|=0$. □ □

Corollary 2.4.8. *Let (X,d) be a complete normal cone metric space with normal constant K, w-cone distance p on X and $A\in\mathcal{B}(E)$ an operator, $K\|A\|<1/2$. Suppose that the mapping $T:X\to X$ satisfies either (i) or (ii) of Theorem 2.4.1 and*

$$p(Tx,T^2x)\preceq A(p(x,T^2x)), \quad \textit{for all } x\in X.$$

Then, there exists $z\in X$, such that $z=Tz$ and if $y=Ty$, then $p(y,y)=\theta$.

Proof. If $x\in X$ is arbitrary, then

$$\|p(Tx,T^2x)\|\leq K\|A\|\|p(x,Tx)\|+K\|A\|\|p(Tx,T^2x)\|.$$

Hence,

$$\|p(Tx,T^2x)\|\leq \frac{K\|A\|}{1-K\|A\|}\,\|p(x,Tx)\,.$$

and, since $K\|A\|/(1-K\|A\|)<1$ it directly follows by Theorem 2.4.7. □ □

Theorem 2.4.9. *Let (X, d) be a complete normal cone metric space with normal constant K and w-cone distance p on X. Suppose $T : X \to X$ satisfies the condition* (2.36) *for an operator $A \in \mathcal{B}(E)$. If $K\|A\| < 1$, then T has property P.*

Proof. As in the proof of previously stated theorem, it follows that, for any $z \in F(T^n)$ and $i \in \mathbb{N}$,

$$\|p(T^i z, T^{i+1} z)\| \leq (K\|A\|)^{kn+i} \|p(z, Tz)\| \to 0, \quad k \to \infty,$$

thus $Tz = T^n z = z$. $\square$ $\square$

The proofs of the following theorems follows similarly as in the case when cone metric space is not normal.

Theorem 2.4.10. *Let (X, d) be a complete normal cone metric space with normal constant K, w-cone distance p on X, $A \in \mathcal{B}(E)$ an operator such that $K\|A\| < 1$. Suppose $T : X \to X$ and there exists an $x \in X$ such that*

$$p(Ty, T^2 y) \preceq A(p(y, Ty)), \quad \text{for all } y \in O(x, \infty).$$

Then,

(i) $\lim_{n\to\infty} T^n x = z$ exists and

$$\|p(T^n x, z)\| \leq \frac{(K\|A\|)^n}{1 - K\|A\|} \|p(x, Tx)\| \quad \text{for } n \in \mathbb{N};$$

(ii) $p(z, Tz) = \theta$ if and only if $G(x) = p(x, Tx)$ is T-orbitally lower semi-continuous at z.

Theorem 2.4.11. *Let (X, d) be a complete normal cone metric space with normal constant K and w-cone distance p on X and $A \in \mathcal{B}(E)$ an with such that $K\|A\| < 1$. Suppose that $T : X \to X$ is a p-contractive mapping of Perov type, i.e.,*

$$p(Tx, Ty) \preceq A\left(p(x, y)\right), \quad \text{for all } x, y \in X.$$

Then, T has a unique fixed point $z \in X$, and $p(z, z) = \theta$.

2.5 Perov type theorems on partially ordered cone metric space

A. C. Ran and M. C. Reurings in [114] gave a fixed point theorem regarding contractions on partially ordered metric spaces but with contractive condition holding for only comparable element additionally assuming that a mapping is monotone. There were many extensions of this result concerning different type of spaces with partial order and extended contractive condition. There were many papers on the topic of fixed point in partially ordered cone metric spaces, but most of them in the case when underlying cone is normal. We investigate existence of a fixed point of a Perov type contraction but in partially ordered cone metric spaces.

Theorem 2.5.1. *([114]) Let $(X, \preceq)$ be a partially ordered set such that every pair $x, y \in X$ has a lower bound and an upper bound. Furthermore, let d be a metric on X such that (X, d) is a complete metric space. If $f : X \mapsto X$ is a continuous and monotone mapping such that*

(i) $(\exists\, 0 < c < 1)\, d(f(x), f(y)) \leq c d(x, y),$ *for all* $x \succeq y$,

(ii) $(\exists\, x_0 \in X)\, x_0 \preceq f(x_0)$ *or* $x_0 \succeq f(x_0)$,

then f has a unique fixed point z. Moreover, for every $x \in X$, $\lim\limits_{n\to\infty} f^n(x) = z$.

Since there were defined two structures, ordering and metric, on a set X, their compatibility should be discussed, as mentioned in [107]. Nevertheless compatibility easily follows from the condition (i), in the following, when dealing with partially ordered metric and partially ordered cone metric spaces, assume that defined metric, or cone metric, and ordering are compatible.

In [107] was stated the following result concerning existence of at least one fixed point, regardless uniqueness, by excluding condition of existence of a lower and an upper bound of any pair of points.

Theorem 2.5.2. *([107]) Let $(X, \preceq)$ be a partially ordered set, let d be a metric on X such that (X, d) is a complete metric space and the metric and ordered structure are compatible. Let $f : X \mapsto X$ is a continuous and monotone mapping such that*

(i) $(\exists\, 0 < c < 1)\, d(f(x), f(y)) \leq c d(x, y),$ *for all* $x \succeq y$,

(ii) $(\exists\, x_0 \in X)\, x_0 \preceq f(x_0)$ *or* $x_0 \succeq f(x_0)$,

then f has a fixed point $z \in X$ and for each $x \in X$ with $x \preceq x_0$ (or $x \succeq x_0$), the sequence $(f^n(x))$ of successive approximations of f starting from x converges to z.

Instead of cone metric space, we will investigate uniqueness and existence of a fixed point for Perov type contractions on partially ordered cone metric space assuming that contractive condition holds only for comparable points.
For a partially ordered cone metric space $(X, d, \preceq)$ we say that ordering and cone metric are compatible if for any sequences $(x_n), (y_n) \subseteq X$ convergent in X such that $x_n \preceq y_n$, $n \in \mathbb{N}$, it follows $\lim\limits_{n\to\infty} x_n \preceq \lim\limits_{n\to\infty} y_n$.
Points $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ and set of all comparable points of X will be denoted with $x_{\preceq}$.

Definition 2.5.3. *Let $(X, d, \preceq)$ be a partially ordered cone metric space. A mapping $f : X \mapsto X$ is continuous if*

$$(\forall\, (x_n) \subseteq X)\, x_n \to x \in X,\, n \to \infty \implies f(x_n) \to f(x),\, n \to \infty.$$

Let $O(x, f) = \{f^n(x) \mid n \in \mathbb{N}_0\}$ be the orbit of point x for a mapping f.

Definition 2.5.4. *Let $(X, d, \preceq)$ be a partially ordered cone metric space. A mapping $f : X \mapsto X$ is orbitally continuous if*

$$(\forall\, x \in X)(\forall\, (x_n) \subseteq O(x, f))\, x_n \to y \in X,\, n \to \infty \implies f(x_n) \to f(y),\, n \to \infty,$$

or, equivalently,

$$(\forall\, x \in X)\, f^{n(i)}(x) \to y \in X,\, i \to \infty \implies f^{n(i)+1}(x) \to f(y),\, i \to \infty,$$

Definition 2.5.5. *Let $(X, d, \leq)$ be a partially ordered cone metric space. A mapping $f : X \mapsto X$ is:*

(i) monotonically nondecreasing (order preserving) if $x \preceq y \implies f(x) \preceq f(y)$, $x, y \in X$;

(ii) monotonically nonincreasing (order reversing) if $x \preceq y \implies f(x) \succeq f(y)$, $x, y \in X$.

A mapping f is monotone if it is monotonically nondecreasing or nonincreasing map on X.

Theorem 2.5.6. *Let $(X, d, \preceq)$ be a partially ordered complete cone metric space with a solid cone P. If for a monotone and continuous mapping $f : X \mapsto X$ there exists $x_0 \in X$ such that*

$$x_0 \preceq f(x_0) \text{ or } x_0 \succeq f(x_0), \tag{2.44}$$

and an positive operator $A \in \mathcal{B}(E)$, $r(A) < 1$, satisfying

$$d(f(x), f(y)) \leq A(d(x, y)), \text{ for all } x \succeq y, \tag{2.45}$$

then f has a fixed point z. Moreover, for every $x \in X$ such that $x \succeq x_0$ or $x \preceq x_0$, the sequence of successive approximations $\{f^n(x)\}$ converges to z.

Proof. Choose x_0 as defined in (2.44) and observe a sequence of successive approximations $x_n = f(x_{n-1})$, $n \in \mathbb{N}$. Let us assume that f is monotonically nondecreasing mapping. The rest of the proof would go analogously in the other case. Remark that, thanks to the symmetry of the condition (2.45), it is applicable on any two comparable points $x, y \in X$.
If $x_0 \preceq f(x_0)$ or $x_0 \succeq f(x_0)$, then, by using monotonicity of the mapping f and principle of mathematical induction, we derive $x_n \preceq x_{n+1}$ or, respectively, $x_n \succeq x_{n+1}$, for any $n \in \mathbb{N}$. Since A is an positive operator,

$$d(x_n, x_{n+1}) \leq A(d(x_{n-1}, x_n)) \leq \ldots \leq A^n(d(x_0, x_1)),$$

along with triangle inequality, implies

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{m-1} A^i\left(d(x_0, x_1)\right) \\ &\leq \sum_{i=n}^{\infty} A^i\left(d(x_0, x_1)\right). \end{aligned}$$

As previously discussed, $\sum\limits_{i=1}^{\infty} A^i$ converges, thus $\lim\limits_{n\to\infty} \sum\limits_{i=n}^{\infty} A^i\left(d(x_0, x_1)\right) = \theta$ and (x_n) is a Cauchy sequence, so convergent in X. If $z = \lim\limits_{n\to\infty} x_n$, because f is continuous, then z is a fixed point of f, not necessarily unique.
If $x \in X$ is comparable with x_0, then since f is monotone, $f^n(x)$ and x_n are comparable, $n \in \mathbb{N}$, so by (2.45),

$$d(f^n(x), x_n) \leq A(f^{n-1}x, x_{n-1}) \leq \ldots \leq A^n\left(d(x, x_0)\right).$$

Moreover,

$$d(f^n(x), z) \leq d(f^n(x), x_n) + d(x_n, z) \leq A^n\left(d(x, x_0)\right) + d(x_n, z).$$

For any $c \gg 0$ there exists $n_0 \in \mathbb{N}$ chosen that the inequalities

$$d(x_n, z) \ll \frac{c}{2} \text{ and } A^n\left(d(x, x_0)\right) \ll \frac{c}{2},$$

hold for any $n \geq n_0$.
Hence,

$$d(f^n(x), z) \leq c,\ n \geq n_0$$

and arbitrariness of $c \in \operatorname{int}(P)$, lead to the conclusion $\lim\limits_{n\to\infty} f^n(x) = z$. □

We can state the similar result in the sense of Ran and Reurings, by including the hypothesis that each pair of points has an upper and a lower bound and obtain uniqueness.

Theorem 2.5.7. *Let $(X, d, \leq)$ be a partially ordered complete cone metric space such that for any $x, y \in X$ the set $\{x, y\}$ has a lower and an upper bound. If for a monotone and continuous mapping $f : X \mapsto X$ (2.44) and (2.45) hold for some $x_0 \in X$ and some positive operator $A \in \mathcal{B}(E)$ with $r(A) < 1$, then f has a unique fixed point z. Moreover, for every $x \in X$, the sequence of successive approximations $\{f^n(x)\}$ converges to z.*

Proof. From the proof of Theorem 2.5.6, an fixed point z of f is obtained and $\lim\limits_{n\to\infty} f^n(x) = z$ for any $x \succeq x_0$ or $x \preceq x_0$.
For arbitrary $x \in X$ denote with x_l a lower and with a x_u an upper bound of a set $\{x, x_0\}$. Then,

$$f^n(x_l) \preceq f^n(x) \preceq f^n(x_u) \text{ or } f^n(x_l) \succeq f^n(x) \succeq f^n(x_u),\ n \in \mathbb{N}.$$

However,

$$\begin{aligned} d(f^n(x), z) &\leq d(f^n(x), f^n(x_l)) + d(f^n(x_l), z) \\ &\leq A^n(d(x, x_l) + d(f^n(x_l), z). \end{aligned}$$

As $n \to \infty$, we obtain $\lim\limits_{n\to\infty} d(f^n(x), z) = 0$, thus $\lim\limits_{n\to\infty} f^n(x) = z$. Assume that $f(y) = y$, for some y in X. Due to the previously made observations the sequence $\{f^n(y)\}$ converges to z and since it is a constant sequence, $y = z$. □

Remark 2.5.8. By observing the proof of Theorem 2.5.7, it is evident that there is no need to request existence of both lower and upper bound. It is sufficient to ask that for any $x, y \in X$ there exists $z(x, y) \in X$ comparable with x and y. Since than, for $y = x_0$ and $z(x, x_0) = z$, $\lim_{n\to\infty} f^n(z) = z$, $f^n(z)$ and $f^n(x)$ are comparable for any $n \in \mathbb{N}$ and

$$\begin{aligned} d(f^n(x), z) &\leq d(f^n(x), f^n(z)) + d(f^n(z), z) \\ &\leq A^n(d(x, z) + d(f^n(x_l), z), \end{aligned}$$

lead to the same conclusion $\lim_{n\to\infty} f^n(x) = z$.

Remark 2.5.9. Theorem 2.5.1 ([114]) is a direct consequence of the Theorem 2.5.7 and Theorem 2.5.2 ([107]) follows from the Theorem 2.5.6 by observing a metric space as a specially kind of cone metric space with a cone $P = (0, \infty)$ and an operator $A \in \mathcal{B}(\mathbb{R})$ defined with $A(x) = cx$, $x \in \mathbb{R}$. Remark that A is a positive operator since $c > 0$ and that $r(A) = c < 1$, so all additional conditions of Theorems 2.5.6 and 2.5.7 are fulfilled.

By taking into the account the proofs of the Theorem 2.5.6 and Theorem 2.5.7, it is easy to conclude that continuity of function f could be replaced with the orbital continuity.

Theorem 2.5.10. *Let $(X, d, \leq)$ be a partially ordered complete cone metric space with a solid cone P. If for a monotone and orbitally continuous continuous mapping $f : X \mapsto X$ such that (2.44) and (2.45) hold for some $x_0 \in X$ and some positive operator $A \in \mathcal{B}(E)$ with $r(A) < 1$, then f has a fixed point z. Moreover, for every $x \in X$ such that $x \succeq x_0$ or $x \preceq x_0$, the sequence of successive approximations $(f^n(x))$ converges to z.*

Theorem 2.5.11. *Let $(X, d, \leq)$ be a partially ordered complete cone metric space with a solid cone such that for any $x, y \in X$ exists some $z(x, y) \in X$ comparable with both x and y. If for a monotone and orbitally continuous mapping $f : X \mapsto X$ such that (2.44) and (2.45) hold for some $x_0 \in X$ and some positive operator $A \in \mathcal{B}(E)$ with $r(A) < 1$, then f has a unique fixed point z. Moreover, for every $x \in X$, the sequence of successive approximations $(f^n(x))$ converges to z.*

Another way of weakening the conditions of previous theorems would be to, instead of monotonicity of a mapping f, require that for any to comparable points $x, y \in X$, $f(x)$ and $f(y)$ are also comparable.

Theorem 2.5.12. *Let $(X, d, \leq)$ be a partially ordered complete cone metric space with a solid cone P. If for a orbitally continuous continuous mapping $f : X \mapsto X$ such that*

$$(x \leq y \text{ or } y \leq x) \implies (f(x) \leq f(y) \text{ or } f(y) \leq f(x)), \tag{2.46}$$

(2.44) and (2.45) hold for some $x_0 \in X$ and some positive operator $A \in \mathcal{B}(E)$ with $r(A) < 1$, then f has a fixed point z. Moreover, for every $x \in X$ such that $x \succeq x_0$ or $x \preceq x_0$, the sequence of successive approximations $(f^n(x))$ converges to z.

Theorem 2.5.13. *Let $(X, d, \leq)$ be a partially ordered complete cone metric space such that for any $x, y \in X$ the set $\{x, y\}$ exists some $z(x, y) \in X$ comparable with both x and y. If for orbitally continuous mapping $f : X \mapsto X$ such that (2.46) is satisfied, (2.44) and (2.45) hold for some $x_0 \in X$ and some positive operator $A \in \mathcal{B}(E)$ with $r(A) < 1$, then f has a unique fixed point z. Moreover, for every $x \in X$, the sequence of successive approximations $(f^n(x))$ converges to z.*

Theorem 2.5.12 generalizes Theorem 4.7 of [107] and it do not require existence of a lower and an upper bound for any pair of points since comparability requirement in Theorem 2.5.12 is equivalent with existence one of the bounds, lower or upper.

In correlation with some results of J.J. Nieto and R. Rodriguez-Lopez ([96, 97]), we will prove that, instead of continuity or spherical continuity condition, we can observe the sequence oriented condition:
(C_1) If $(x_n) \subseteq X$ is a convergent sequence with comparable consecutive terms in respect with $\preceq$ and $\lim\limits_{n\to\infty} x_n = x$, then there exists a subsequence $\{x_{n_k}\} \subseteq (x_n)$ whose every term is comparable with the limit x.

Theorem 2.5.14. *Let $(X, d, \leq)$ be a partially ordered complete cone metric space with a solid cone P with (C_1) property, a mapping $f : X \mapsto X$ such that (2.46) is satisfied, (2.44) and (2.45) hold for some $x_0 \in X$ and some positive operator $A \in \mathcal{B}(E)$ with $r(A) < 1$, then f has a fixed point z. Moreover, for every $x \in X$ such that $x \succeq x_0$ or $x \preceq x_0$, the sequence of successive approximations $(f^n(x))$ converges to z.*

Proof. Define an iterative sequence (x_n) of f for $x_0 \in X$ which satisfies (2.45) providing that x_n and x_{n+1} are comparable because of (2.46). By applying (2.45) for any $n \in \mathbb{N}$, following inequalities are obtained:

$$d(x_n, x_{n+1}) \leq A^n(d(x_0, x_1)),$$

and, for any $n, m \in \mathbb{N}$ with $m \geq n$,

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{m-1} A^i\left(d(x_0, x_1)\right) \\ &\leq \sum_{i=n}^{\infty} A^i\left(d(x_0, x_1)\right). \end{aligned}$$

Since $\lim\limits_{n\to\infty} \sum\limits_{i=n}^{\infty} A^i\left(d(x_0, x_1)\right) = 0$, (x_n) is a Cauchy and convergent sequence in X. If $z = \lim\limits_{n\to\infty} x_n$, it remains to prove that $f(z) = z$ by estimating $d(z, f(z))$.

Due to (C_1), the sequence (x_n) generates the subsequence (x_{n_k}) whose every term is comparable with z. For some $c \gg \theta$, let $k_0 \in \mathbb{N}$ satisfies

$$d(z, x_n) \leq c, \ n \geq n_{k_0},$$

and for such k,

$$\begin{aligned} d(z, f(z)) &\leq d(z, x_{n_k+1}) + d(x_{n_k+1}, f(z)) \\ &\leq d(z, x_{n_k+1}) + A\left(d(x_{n_k}, z)\right) \\ &\leq c + A(c). \end{aligned}$$

According to previously made discussion, $d(z, f(z)) = 0$, i.e., $f(z) = z$.
Proof that iterative sequence of any point comparable with x_0 converges to z follows the lines of the proof of Theorem 2.5.6. □

Theorem 2.5.14 generalizes Theorems 4, 5 and 7 of [96] by choosing $A(x) = qx$, $x \in \mathbb{R}$, for some $q \in (0, 1)$, by adding the comparability request which guarantees a uniqueness of a fixed point.

The condition (C_1) could be replaced with:
(C_2) If a nondecreasing sequence $(x_n) \subseteq X$ converges to $x \in X$, then $x_n \leq x$ for any $n \in \mathbb{N}$.
or with
(C_3) If a nonincreasing sequence $(x_n) \subseteq X$ converges to $x \in X$, then $x_n \geq x$ for any $n \in \mathbb{N}$.
by adjusting some other requirements.

Theorem 2.5.15. *Let $(X, d, \leq)$ be a partially ordered complete cone metric space with a solid cone P with (C_2) property and $f : X \mapsto X$ a nondecreasing mapping such that (2.45) holds for some positive operator $A \in \mathcal{B}(E)$ with $r(A) < 1$. If $x_0 \preceq f(x_0)$ for some $x_0 \in X$, then f has a fixed point $z \in X$ and for every $x \in X$ such that $x \succeq x_0$ or $x \preceq x_0$, the sequence of successive approximations $\{f^n(x)\}$ converges to z.*

Proof. According to monotonicity of f, $x_n \preceq x_{n+1}$ for $x_n = f^n(x_0)$, , $n \in \mathbb{N}$ and as in previous proofs, (x_n) is a Cauchy and, thus convergent sequence with a limit z in X. Condition (C_2) allows us to make following estimation by applying (2.45)

$$d(z, f(z)) \leq d(z, x_{n+1}) + A(d(x_n, z)),\ n \in \mathbb{N}$$

and to conclude $f(z) = z$. Rest of the proof regarding iterative sequence's convergence is analogous to the proof of Theorem 2.5.6. □

Including (C_3) instead of (C_2) leads to the following result.

Theorem 2.5.16. *Let $(X, d, \leq)$ be a partially ordered complete cone metric space with a solid cone P with (C_3) property and $f : X \mapsto X$ a nondecreasing mapping such that (2.45) holds for some positive operator $A \in \mathcal{B}(E)$ with $r(A) < 1$. If $x_0 \succeq f(x_0)$ for some $x_0 \in X$, then f has a fixed point $z \in X$ and for every $x \in X$ such that $x \succeq x_0$ or $x \preceq x_0$, the sequence of successive approximations $\{f^n(x)\}$ converges to z.*

Proof. The proof is similar to the proof of Theorem 2.5.15 but includes a nonincreasing sequence of successive approximations (x_n) for $x_n = f^n(x_0)$, $n \in \mathbb{N}$. □

Theorem 2.5.15 extends Theorem 2.2 and Theorem 2.5.16 Theorem 2.4 of [97]. In those results sequence of successive approximations is not discussed. As previously mentioned uniqueness can be obtained by including comparability condition for each pair of points in X.

2.6 Nonlinear operatorial contractions

Operator A satisfying (2.1) is required to be bounded, linear and with spectral radius less than 1, thus A^n converges to zero, as $n \to \infty$, to obtain existence and uniqueness of a fixed point of mapping f. In the following results we will weaken this requirements primarily omitting the linearity condition.

Theorem 2.6.1. *Let (X, d) be complete cone metric space with a solid cone P and $f : X \mapsto X$ a continuous mapping. If there exists an increasing operator $A : E \mapsto E$ such that $\lim_{n\to\infty} A^n(e) = \theta$, $e \in E$, and, for any $x, y \in X$,*

$$d(f(x), f(y)) \preceq A(d(x, y)), \tag{2.47}$$

then a mapping f has a unique fixed point in X.

Proof. Define, for arbitrary $x_0 \in X$, an iterative sequence $x_n = f(x_{n-1})$, $n \in \mathbb{N}$. Then,

$$\theta \preceq d(x_n, x_{n+1}) \preceq A^n(d(x_0, x_1)),\ n \in \mathbb{N},$$

so $\lim_{n\to\infty} d(x_n, x_{n+1}) = \theta$.
For arbitrary $c \gg \theta$ choose $n_0 \in \mathbb{N}$ such that $A^n(c) \preceq \frac{c}{8}$, $n \geq n_0$, and $n_1 \in \mathbb{N}$ that $d(x_n, x_{n+1}) \prec \frac{c}{8n_0}$ for $n \geq n_1 n_0$. Observe a sequence $y_k = f^{kn_0}(x)$, $k \in \mathbb{N}$. Then

$$d(y_k, y_{k+1}) \preceq A^{kn_0}(d(x_0, x_{n_0})),\ k \in \mathbb{N},$$

and again $\lim_{n\to\infty} d(y_k, y_{k+1}) = \theta$. Hence, there exists $k_0 \in \mathbb{N}$ such that $d(y_k, y_{k+1}) \prec \frac{c}{8}$ holds also for any index greater than k_0 and choose such $k \geq n_1$.
Denote with S a closed ball $K[y_k, \frac{c}{4}] = \{x \in X \mid d(y_k, x) \preceq \frac{c}{4}\}$. It follows that $f^{n_0}(S) \subseteq S$ since, for any $x \in S$,

$$\begin{aligned} d(y_k, f^{n_0}(x)) &\preceq d(y_k, y_{k+1}) + d(y_{k+1}, f^{n_0}(x)) \\ &\preceq \frac{c}{8} + A^{n_0}(d(y_k, x)) \\ &\prec \frac{c}{4}. \end{aligned}$$

Moreover, $y_n \in S$ for any $n \geq k$.
If $m \geq kn_0$, let $m = qn_0 + r$ for some $q \geq k$ and $0 \leq r < n_0$, then the inequalities

$$\begin{aligned} d(y_k, x_m) &\preceq d(y_k, y_q) + d(y_q, x_m) \\ &\preceq d(y_k, y_q) + \sum_{i=qn_0}^{m-1} d(x_i, x_{i+1}) \\ &\preceq \frac{c}{4} + \sum_{i=qn_0}^{m-1} \frac{c}{8n_0} \\ &\preceq \frac{3c}{8}, \end{aligned}$$

lead to
$$d(x_n, x_m) \preceq d(x_n, y_k) + d(y_k, x_m) \preceq \frac{3c}{4} \prec c,\ n, m \geq kn_0,$$
with a conclusion that (x_n) is a Cauchy sequence in X and therefore convergent in X. Denote with $z \in X$ the limit of the sequence (x_n) and notice that z, along with $f^{mn_0}(z)$ for any $m \in \mathbb{N}$ is in S. Since the mapping f is continuous, $z = \lim_{n\to\infty} f^{n+1}(x_0) = f(\lim_{n\to\infty} f^n(x_0)) = f(z)$.

If $f(u) = u$, then
$$d(z, u) = d(f^n(z), f^n(u)) \preceq A^n(z, u),$$
along with $\lim_{n\to\infty} A^n(z, u) = \theta$ gives us $u = z$. Uniqueness also implies that $(f^n(x))$ converges to z for any $x \in X$ since first part of the proof induces that $(f^n(x))$ converges to the point with fixed point property. □

Remark 2.6.2. Comparing this theorem with Theorem 2.2.2, notice that A is not assumed to be linear. Continuity condition of f is implicitly requested in Perov theorem and Theorem 2.2.2. If $r(A) < 1$, then $\lim_{n\to\infty} \|A^n\|^{1/n} = 0$, moreover $\lim_{n\to\infty} \|A^n(e)\|^{1/n} = 0$, for any $e \in E$. Hence, this result generalizes theorem 2.2.2 and, as a consequence, Perov theorem.

Instead of requesting that $\lim_{n\to\infty} A^n(e) = \theta$, for any $e \in E$, it is enough to assume that for all $e \in P$.

Theorem 2.6.3. *Let (X, d) be complete cone metric space with a solid cone P and $f : X \mapsto X$ a continuous mapping. If there exists an increasing operator $A : E \mapsto E$ such that $\lim_{n\to\infty} A^n(e) = \theta$, $e \in P$, and (2.47) holds for any $x, y \in X$, then a mapping f has a unique fixed point in X.*

It is also possible to let $A\restriction_P : P \mapsto P$ be an increasing operator instead of $A : E \mapsto E$.

In the case that $A \in \mathcal{B}(E)$, it is equivalent for A to be increasing or positive. Boundedness do not have impact on this conclusion, only linearity. If A is a non linear operator, but increasing and satisfies (2.47), then for $x = y$, $\theta \preceq A(\theta)$ and for $x \in P$, $\theta \preceq A(\theta) \preceq A(x) \in P$, so A is a positive operator. On the other hand, positivity of A do not imply that A is increasing.

Example 10. Let E be a Banach space with a solid cone P and $c \in \operatorname{int}(P)$. Define an operator $A : E \mapsto E$ with
$$A(x) = \begin{cases} \frac{c}{2}, & x = \theta \\ \theta, & x \in E \setminus \{\theta\} \end{cases}.$$
Operator A is positive and $A^n(x) = \theta$, $x \in E$, for any $n \geq 2$, but it is not increasing.

Comparing requirements $r(A) < 1$ and $\lim_{n\to\infty} A^n(e) = \theta$, $e \in E$, for a bounded linear operator A, it is obvious that $r(A) < 1$ implies other condition, but reverse do not hold. Therefore, the condition $\lim_{n\to\infty} A^n(e) = \theta$, $e \in E$, (or $e \in P$) of Theorem 2.6.3 is less strict that corresponding condition of Theorem 2.2.2.

Example 11. Let c_0 be the set containing all sequences of real numbers convergent to zero equipped with supremum norm $\|\|_\infty$ and define $A : E \mapsto E$ with

$$A(x) = A(x_1, x_2, x_3, \ldots, x_n, \ldots) = (0, x_3, x_4, \ldots, x_{n+1}, \ldots), \quad x = (x_n) \in c_0.$$

Operator A is linear on Banach space $(c_0, \|\|_\infty)$ and also bounded since $\|Ax\|_\infty \leq \|x\|_\infty$. By choosing $e_3 = (0, 0, 1, 0, \ldots, 0, \ldots) \in c_0$, it follows $\|A\| = 1$.
For any $m \in \mathbb{N}$,

$$A^m(x) = A^m(x_1, x_2, x_3, \ldots) = (0, x_{m+2}, x_{m+3}, \ldots), \quad x = (x_n) \in c_0,$$

therefore, observing $e_{m+2} \in c_0$ with all zeros except one on $(m{+}2)$-nd place (i.e., $(e_{m+2})_n = \delta_{n,m+2}$, $n \in \mathbb{N}$), we obtain $\|A^m\| = 1$. Spectral radius of A is not less than 1, since $r(A) = \lim\limits_{m\to\infty} \|A^m\|^{\frac{1}{m}} = 1$.
However, for any $x \in c_0$, $\lim\limits_{m\to\infty} A^m(x) = \theta$ where θ is a zero sequence. For arbitrary $x \in c_0$ and $\varepsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \implies |x_n| < \frac{\varepsilon}{2},$$

implying $\|A^m(x)\|_\infty = \sup\limits_{n\geq m+2} |x_n| < \varepsilon$ for any $m \geq n_0 - 2$. Thus, $\lim\limits_{m\to\infty} A^m(x) = \theta$ despite of spectral radius.

Chapter 3

Common fixed point problems

Common fixed point problem refers to finding $x \in X$ such that it is a coincidence point for two self-mappings and, at the same time, fixed point of both of them. Meaning, for some $f, g : X \mapsto X$, we look for such $x \in X$ that $f(x) = g(x) = x$ and we say that x is a common fixed point of mappings f and g. Evidently, common fixed point even do not have to exist at all and, if it does exist, it is not necessarily unique as shown in the example.

Example 12. Let $f(x) = x$, $g(x) = x+1$, $x \in \mathbb{R}$, then common fixed point do not exist, but there is also no coincidence point for those two mappings.

If $f(x) = g(x) = x + 1$, $x \in \mathbb{R}$, then all real numbers are coincidence points, but still there is no common fixed point.

If $f(x) = x$, $g(x) = \frac{x}{2}$, $x \in \mathbb{R}$, then x=0 is a unique fixed point for f and g.

Every real number is a common fixed point for $f(x) = g(x) = x$, $x \in \mathbb{R}$.

Observe that, in a similar way, we may define this kind of examples on a cone metric space.

We have emphasised coincidence point problem since it is broadly researched topic in the fixed point theory.

Mean value of coupled fixed point problem is in wide area of application, especially in finding a solution of a system of two equations. This concept can be extended on a sequence or family of mappings. Therefore, this chapter is divided in three sections. Two sections summarize existence and uniqueness of a common fixed point for the pair or the sequence of mappings, respectively. In the third part of this chapter classical result of Hardy and Rogers ([64]) is transformed from the angle of common fixed point problem.

3.1 Fixed point for the pair of mappings

Instead of starting with Banach/Perov contraction, we will, relying on Section 2.2, begin from the much comprehensive condition.

Definition 3.1.1. *Let (X, d) be a cone metric space, and let $g, f : X \mapsto X$. Then, g is called a f- quasi-contraction of Perov type if for some operator $A \in \mathcal{B}(E)$, $r(A) < 1$ and* *for every $x, y \in X$, there exists*

$$u \in C(f; x, y) \equiv \Big\{d(f(x), f(y)), d(f(x), g(x)), d(f(x), g(y)), d(f(y), g(y)), d(f(y), g(x))\Big\}$$

such that

$$d(g(x), g(y)) \preceq A(u). \tag{3.1}$$

Remark that if $f = id_X$ is the identity map on X, and g satisfies (3.1), than g is called *quasi-contraction of Perov type.* If $f, g : X \mapsto X$, $g(X) \subseteq f(X)$ and $x_0 \in X$ arbitrary, then let $x_1 \in X$ be such that $g(x_0) = f(x_1)$. Having defined $x_n \in X$, let $x_{n+1} \in X$ be such that $g(x_n) = f(x_{n+1}) = y_n$, $n \in \mathbb{N}$.

Theorem 3.1.2. *Let (X, d) be a complete cone metric space with a solid cone P. Let $g, f : X \mapsto X$, f commutes with g, $g(X) \subseteq f(X)$, f or g is continuous and g is a f-quasi-contraction of Perov type, $A(P) \subseteq P$. Then f and g have a unique common fixed point z in X and for any $x_0 \in X$, the iterative sequence $(y_n)_{n\in\mathbb{N}}$ converges to some $y \in X$. In the case when f is continuous, then $z = g(y) = f(y)$, if g is continuous, then $z = y$.*

Proof. Let $x_0 \in X$ be arbitrary. We will prove two inequalities:

(i) $d(y_n, y_1) \preceq (I - A)^{-1}A(d(y_1, y_0))$, $n \in \mathbb{N}$,

(ii) $d(y_n, y_0) \preceq (I - A)^{-1}(d(y_1, y_0))$, $n \in \mathbb{N}$.

Evidently, (1) is true for $n = 1$. Suppose that it is fulfilled for each $m \leq n$.
Since $d(y_{n+1}, y_1) \preceq A(u)$, where

$$u \in \{d(y_n, y_0), d(y_n, y_1), d(y_0, y_1), d(y_0, y_{n+1}), d(y_n, y_{n+1})\},$$

we will discuss several different cases.
(1) If $u = d(y_n, y_0)$, then

$$\begin{aligned} d(y_{n+1}, y_1) &\preceq A(d(y_n, y_0)) \\ &\preceq A(d(y_n, y_1)) + A(d(y_1, y_0)) \\ &\preceq A(I - A)^{-1}A(d(y_1, y_0)) + A(d(y_1, y_0)) \\ &= A^2(I - A)^{-1}(d(y_1, y_0)) + A(I - A)(I - A)^{-1}(d(y_1, y_0)) \\ &= (I - A)^{-1}A(d(y_1, y_0)). \end{aligned}$$

(2) Presume $u = d(y_n, y_1)$, then

$$\begin{aligned} d(y_{n+1}, y_1) &\preceq A(d(y_n, y_1)) \\ &\preceq A(I - A)^{-1}A(d(y_1, y_0)) \\ &\preceq (I - (I - A))(I - A)^{-1}A(d(y_1, y_0)) \\ &\preceq (I - A)^{-1}A(d(y_1, y_0)). \end{aligned}$$

(3) Clearly, for $u = d(y_1, y_0)$, the inequality (1) is satisfied.
(4) Due to

$$d(y_0, y_{n+1}) \preceq d(y_0, y_1) + d(y_1, y_{n+1}),$$

and the fact that A is positive operator, it follows

$$d(y_{n+1}, y_1) \preceq A(d(y_0, y_1)) + A(d(y_1, y_{n+1})),$$

that points to

$$d(y_{n+1}, y_1) \preceq (I - A)^{-1} A(d(y_0, y_1)),$$

if $u = d(y_0, y_{n+1})$.
(5) If $d(y_{n+1}, y_1) \preceq A(d(y_n, y_{n+1}))$ and g is a f-quasi-contraction, we see that, for some $i \in \{0, 1, \ldots, n\}$, $j \in \{1, \ldots, n+1\}$, $d(y_n, y_{n+1}) \preceq A^{n-1+i}(d(y_1, y_j))$, .
The case where $j = n+1$, implies $d(y_{n+1}, y_1) = \theta$. Indeed, since $I - A^{n+i}$ is an invertible operator and $A^{n+i}(P) \subseteq P$, we see $d(y_{n+1}, y_1) = \theta$.
Otherwise, from the initial assumption,

$$\begin{aligned} d(y_{n+1}, y_1) &\preceq A^{n+i}(d(y_1, y_j))) \\ &\preceq A^{n+i}(I - A)^{-1} A(d(y_1, y_0)) \\ &\preceq (I - A)^{-1} A(d(y_1, y_0)). \end{aligned}$$

Thus, the inequality (i) holds for $n \in \mathbb{N}$ and inequality (ii) is obtained directly from (i):

$$\begin{aligned} d(y_n, y_0) &\preceq d(y_n, y_1) + d(y_1, y_0) \\ &\preceq (I - A)^{-1} A(d(y_1, y_0)) + d(y_1, y_0) \\ &= (I - A)^{-1}(d(y_1, y_0)),\ n \in \mathbb{N}. \end{aligned}$$

We will demonstrate that $(y_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in X, thus it is convergent. Suppose that $n, m \in \mathbb{N}$, $m > n$.
Mapping g is a f-quasi-contraction, so there exist $i, j \in \mathbb{N}$, $1 \leq i \leq n$, $1 \leq j \leq m$,

$$d(y_n, y_m) \preceq A^{n-1}(d(y_i, y_j)),$$

which brings us to

$$d(y_n, y_m) \preceq 2A^n (I - A)^{-1}(d(y_1, y_0)).$$

Since, $2A^n(I - A)^{-1}(d(y_1, y_0)) \to \theta$, $n \to \infty$, by Lemma 2.1.1, $(y_n)_{n \in \mathbb{N}}$ is a Cauchy sequence and there exists $y \in X$, $\lim\limits_{n \to \infty} y_n = y$.
First of all, we may infer that if f and g have a common fixed point, then it is unique. If $f(x) = g(x) = x$ and $f(y) = g(y) = y$, then

$$d(x, y) = d(g(x), g(y)) \preceq A(u),\ u \in \{\theta, d(x, y)\}. \tag{3.2}$$

Either way, it follows $x = y$.
Further on, let us observe two different cases, if f is continuous, or if g is continuous. Suppose that f is continuous. We need to corroborate that $g(y) = f(y)$ is a common fixed point of f and g.

The continuity of f leads to $\lim_{n\to\infty} f(y_n) = f(y)$. Additionally, f and g commute, indicating $g(y_n) = f(y_{n+1})$ and $\lim_{n\to\infty} g(y_n) = f(y)$.
For $c \gg \theta$ and $\varepsilon \gg \theta$ choose $n_0 \in n$ such that for each $m, n \geq n_0$ following inequalities hold:

$$d(f(y_n), f(y)) \ll c,\ d(f(y_n), f(y)) \ll \varepsilon \text{ and } d(f(y_n), f(y_m)) \ll \varepsilon. \tag{3.3}$$

For any $n > n_0 + 1$, observe the triangle inequality

$$d(g(y), f(y)) \preceq d(g(y), f(y_{n+1})) + d(f(y_{n+1}), f(y)) \tag{3.4}$$

Recall that g is a f-quasi-contraction of Perov type, so

$$d(g(y), f(y_{n+1})) = d(g(y), g(y_n)) \preceq A(u)$$

for some

$$u \in \{d(f(y), f(y_n)), d(f(y), f(y_{n+1})), d(f(y_n), f(y_{n+1})), d(f(y), g(y)), d(f(y_n), g(y))\}. \tag{3.5}$$

Form subsequences $(y_{n,i}) \subseteq (y_n)$ in a way that $d(g(y), f(y_{n,i})) \preceq A(u_{n,i})$ where $u_{n,i}$ is respectively chosen.
If any of subsequences $(y_{n,i})$, $i = \overline{1,3}$ is infinite, then (3.3), (3.4) and (3.5) along with

$$d(g(y), f(y)) \preceq A(\varepsilon) + c,$$

point to

$$d(g(y), f(y)) \preceq A(\varepsilon),\ \varepsilon \gg \theta. \tag{3.6}$$

In case $(y_{n,4})$ is infinite, then

$$d(g(y), f(y)) \preceq A(d(g(y), f(y))) + c,\ c \gg \theta,$$

we perceive

$$d(g(y), f(y)) \preceq A(d(g(y), f(y))). \tag{3.7}$$

Eventually, the case where $(y_{n,5})$ is the only infinite subsequence implies

$$\begin{aligned} d(g(y), f(y)) &\preceq A(d(f(y_n), g(y))) + d(f(y_{n+1}), f(y)) \\ &\preceq A(d(f(y_n), f(y))) + A(d(f(y), g(y))) + c \\ &\preceq A(d(f(y), g(y))) + A(\varepsilon) + c,\ c, \varepsilon \gg \theta. \end{aligned}$$

Further,

$$d(g(y), f(y)) \preceq A(d(g(y), f(y))) + A(\varepsilon),\ \varepsilon \gg \theta. \tag{3.8}$$

Each of the inequalities (3.6), (3.7) and (3.8) lead us to the conclusion

$$d(g(y), f(y)) = \theta \Leftrightarrow g(y) = f(y). \tag{3.9}$$

By choosing $\varepsilon = \frac{\varepsilon}{n}, n \in \mathbb{N}$, we get

$$d(g(y), f(y)) \preceq A(d(g(y), f(y))) \implies d(g(y), f(y)) \preceq \theta$$

and consequently (3.9).
Note that (3.9) and $fg = gf$ imply

$$g^2(y) = g(f(y)) = f(g(y)) = f^2(y). \tag{3.10}$$

Thus, it is sufficient to show $g(g(y)) = g(y)$. Select

$$u \in \{d(f(g(y)), f(y)), d(f(g(y)), g(y)), d(f(g(y)), g(g(y))), d(f(y), g(y)), d(f(y), g(g(y)))\}$$

that satisfies $d(g(g(y)), g(y)) \preceq A(u)$. From (3.9) and (3.10), we may conclude that $u \in \{\theta, d(g(g(y)), g(y))\}$ and $g(g(y)) = g(y)$. This equality combined with (3.10) determines this part of the proof.

Alternatively, consider that g is continuous. Then it will be demonstrated that y is a unique common fixed point of f and g.
Similarly as in first case, $\lim\limits_{n\to\infty} f(y_n) = \lim\limits_{n\to\infty} g(y_n) = g(y)$. For arbitrary $c \gg \theta$ and $\varepsilon \gg \theta$ choose $n_0 \in n$ such that for every $m, n \geq n_0$ following inequalities hold:

$$d(g(y), g(y_n)) \ll \frac{c}{2}, \varepsilon,\ d(y_n, y) \ll \frac{c}{2},\ d(y_n, y_m) \ll \varepsilon \text{ and } d(g(y_n), g(y_m)) \ll \varepsilon. \tag{3.11}$$

Applying the triangle inequality for $n > n_0$ we get

$$d(g(y), y) \preceq d(g(y), g(y_n)) + d(g(y_n), y_n) + d(y_n, y) \preceq c + d(g(y_n), y_n). \tag{3.12}$$

According to (3.1) choose u_n such that $d(g(y_n), y_n) \preceq A(u_n)$.
If $u_n \in \{d(g(y_{n-1}), g(y_n)), d(y_{n-1}, y_n)\}$ for infinitely many $n \in \mathbb{N}$, then (3.11) and (3.12) imply

$$d(g(y), y) \preceq c + A(\varepsilon),\ c, \varepsilon \gg \theta,$$

consequently

$$d(g(y), y) \preceq A(\varepsilon),\ \varepsilon \gg \theta. \tag{3.13}$$

Otherwise, choose $n > n_0$ such that (3.11) is fulfilled and $m \in \mathbb{N}$, $d(g(y_{n+m}), y_{n+m}) \preceq A(u_{n+m})$ where

$$u_{n+m} \in \{d(g(y_{n+m-1}), y_{n+m-1}), d(g(y_{n+m-1}), y_{n+m}), d(g(y_{n+m}), y_{n+m-1})\}.$$

Hence,

$$d(g(y_{n+m}), y_{n+m}) \preceq A^{m+k}(v_m),$$

for $v_m \in \{d(g(y_n), y_{n+i}), d(g(y_{n+j}), y_n)\}$ for some $0 \leq k, i, j \leq m$.
Relaying on

$$\begin{aligned} d(g(y_n), y_{n+i}) &\preceq d(g(y_n), y_n) + d(y_n, y_{n+i}) \\ &\preceq d(g(y_n), y_n) + \varepsilon, \end{aligned}$$

and

$$\begin{aligned} d(g(y_{n+j}), y_n) &\preceq d(g(y_{n+j}), g(y_n)) + d(g(y_n), y_n) \\ &\preceq \varepsilon + d(g(y_n), y_n), \end{aligned}$$

it proceeds

$$d(g(y_{n+m}), y_{n+m}) \preceq A^{m+k}(\varepsilon) + A^{m+k}(d(g(y_n), y_n)). \tag{3.14}$$

In addition,

$$\begin{aligned} d(g(y), y) &\preceq d(g(y), g(y_{n+m})) + d(g(y_{n+m}), y_{n+m}) + d(y_{n+m}, y) \\ &\preceq c + A^{m+k}(\varepsilon) + A^{m+k}(d(g(y_n), y_n)). \end{aligned} \tag{3.15}$$

leads to

$$d(g(y), y) \preceq c, \ c \gg \theta. \tag{3.16}$$

Conjointly, (3.15) and (3.16), for $\varepsilon = \frac{\varepsilon}{n}$ and $c = \frac{c}{n}$, $n = 1, 2, \ldots$, indicate

$$\theta \preceq d(g(y), y) \preceq A\left(\frac{\varepsilon}{n}\right) = \frac{A(\varepsilon)}{n} \to \theta, \quad n \to \infty,$$

$$\theta \preceq d(g(y), y) \preceq \frac{c}{n} \to \theta, \quad n \to \infty.$$

Hence, $g(y) = y$.
Notice that $\lim\limits_{n\to\infty} f(y_n) = \lim\limits_{n\to\infty} g(y_n) = y$ and for $c \gg \theta$, $\varepsilon \gg \theta$ there exists $n_1 \in \mathbb{N}$,

$$n, m \geq n_1 \Longrightarrow d(g(y_n), y) \ll c, \ d(g(y_n), y) \ll \varepsilon \text{ and } d(g(y_n), g(y_m)) \ll \varepsilon.$$

Due to the condition $g(X) \subseteq f(X)$, $(\exists z \in X)\ y = g(y) = f(z)$.
It remains to estimate distance between $g(z)$ and y. Choose $n > n_1$, then

$$\begin{aligned} d(g(z), y) &\preceq d(g(z), g(y_n)) + d(g(y_n), y) \\ &\preceq A(u_n) + c \end{aligned} \tag{3.17}$$

for

$$u_n \in \{d(y, g(y_{n-1})), d(y, g(z)), d(y, g(y_n)), d(g(y_{n-1}), g(z)), d(g(y_{n-1}), g(y_n))\}.$$

If $u_n \in \{d(y, g(y_{n-1})), d(y, g(y_n)), d(g(y_{n-1}), g(y_n))\}$ for infinitely many $n > n_1$ then

$$d(g(z), y) \preceq A(\varepsilon) + c, \quad \overset{\text{for any } c \gg \theta}{\Longrightarrow} \quad d(g(z), y) \preceq A(\varepsilon). \tag{3.18}$$

In the case that $u_n = d(g(z), y)$ for infinitely many $n > n_1$,

$$d(g(z), y) \preceq (I - A)^{-1}(c), \ c \gg \theta. \tag{3.19}$$

Otherwise, observe $n > n_1$ such that $d(g(z), g(y_{n+m})) \preceq A(g(z), g(y_{n+m-1}))$ for every $m \in \mathbb{N}$. Accordingly,

$$\begin{aligned} d(g(z), y) &\preceq d(g(z), g(y_{n+m})) + d(g(y_{n+m}), y) \\ &\preceq A^m(d(g(z), g(y_n))) + c. \end{aligned}$$

If $m \to \infty$, then $d(g(z), y) \preceq c$. Again, on a similar way as in the proof $g(y) = y$, from the last observation, (3.18) and (3.19) we conclude that $g(z)$ and y coincide which brings us to $f(y) = f(g(z)) = g(f(z)) = g(y) = y$. □

Theorem 3.1.2 for $f(x) = x$, $x \in X$ implies Theorem 2.2.2. Obviously, the result of Ćirić ([39]) follows from this corollary. Also as direct consequences of Theorem 3.1.2 we get generalizations of Jungck's ([77]) and Das and Naik's ([43]) results on non-normal cone metric spaces and generalizations of Ilić and Rakočević's results ([70]).

Corollary 3.1.3. *Let (X, d) be a complete cone metric space with a solid cone P. Let f be a continuous self-mapping on X and g be any self-mapping on X that commutes with f. If f and g satisfy $g(X) \subseteq f(X)$ and there exists a positive operator $A \in \mathcal{B}(E)$ with $r(A) < 1$ such that for any $x, y \in X$*

$$d(g(x), g(y)) \preceq A(d(f(x), f(y))). \tag{3.20}$$

Then f and g have a unique common fixed point.

Proof. If f and g satisfy (3.20), then (3.1) is evidently satisfied for $u = d(f(x), f(y))$. From Theorem 3.1.2 result of Corollary directly follows. □

Corollary 3.1.4. *Let (X, d) be a complete cone metric space with a solid cone P. Let f be a continuous self-mapping on X and g be any self-mapping on X that commutes with f. If f and g satisfy $g(X) \subseteq f(X)$ and there exists a positive operator $A \in \mathcal{B}(E)$ with $r(A) < 1$ such that for any $x, y \in X$*

$$d(g(x), g(y)) \preceq A(M_d(x, y)), \tag{3.21}$$

for some

$$M_d(x, y) \in \Big\{d(f(x), f(y)), d(f(x), g(x)), d(f(x), g(y)), d(f(y), g(y)), d(f(y), g(x))\Big\}.$$

Then f and g have a unique common fixed point.

Theorem 3.1.5. *Let (X, d) be a complete cone metric space with a solid cone P, $f : X \mapsto X$, f^2 continuous, $g : f(X) \mapsto X$ such that $g(f(X)) \subseteq f^2(X)$ and $f(g(x)) = g(f(x))$, $x \in f(X)$. Assume that there exists a positive operator $A \in \mathcal{B}(E)$ with $r(A) < 1$, such that (3.1) is satisfied for every $x, y \in f(X)$, then f and g have a unique common fixed point in $f(X)$.*

Proof. As in the proof of Theorem 3.1.2, define sequence $(y_n)_{n\in\mathbb{N}}$ for arbitrary $x_0 \in f(X)$ such that $f(x_{n+1}) = g(x_n) = y_n$, $n \in \mathbb{N}$. Perceive that $f(y_n) = g(y_{n-1}) = z_n$ determines yet another sequence $(z_n)_{n\in\mathbb{N}}$.
Because

$$d(z_n, z_m) \preceq A(u)$$

where

$$u \in \{d(z_{n-1}, z_{m-1}), d(z_{n-1}, z_n), d(z_{n-1}, z_m), d(z_{m-1}, z_n), d(z_{m-1}, z_m)\}.$$

Similarly as in the proof of Theorem 3.1.2, it follows that (z_n) is a Cauchy sequence. There exists $z \in X$, $\lim_{n\to\infty} z_n = z \in X$.
We will prove that $f^2(z) = g(f(z)) = u$ and that u is a unique common fixed point of f

and g in X.
Since $f^2(z_{n+1}) = g(f(z_n))$, then $d(g(f(z_n)), g(f(z))) \preceq A(u_n)$ where

$$\begin{aligned} u_n \in & \{d(g(f(z_{n-1})), f^2(z)), d(g(f(z_{n-1})), g(f(z_n))), d(g(f(z_{n-1})), g(f(z)), \\ & d(f^2(z), g(f(z_n))), d(f^2(z), g(f(z)))\}. \end{aligned}$$

The inequality

$$\begin{aligned} d(f^2(z), g(f(z))) & \preceq d(f^2(z), f^2(z_{n+1})) + d(g(f(z_n)), g(f(z))) \\ & \preceq d(f^2(z), f^2(z_{n+1})) + A(u_n), \end{aligned}$$

along with $\lim_{n\to\infty} f^2(z_n) = \lim_{n\to\infty} g(f(z_n)) = f^2(z)$, after similar discussion as in previous theorems, gives us $f^2(z) = g(f(z))$.

Remark that $f^3(z) = f(g(f(z))) = g(f^2(z))$ allows us to prove only $f^3(z) = f^2(z)$. But recall that g is a f-quasi-contraction, so $d(f^3(z), f^2(z)) = d(g(f^2(z)), g(f(z))) \preceq A(u)$, where $u \in \{d(f^3(z), f^2(z)), \theta\}$.

Anyway, $f^2(z)$ is a fixed point of f. As in the proof of Theorem 3.1.2, we have that if u is a common fixed point of f and g then, since $u \in f(X)$, $d(z, u) \preceq A(d(z, u))$, thus $u = z$. $\square$

Let $x \in X$ be arbitrary, $(y_n)_{n\in\mathbb{N}}$ as defined above. If we denote $O_{g,f}(x, n) = \{y_0, \dots, y_n\}$, $O_{g,f}(x, \infty) = \{y_0, \dots, y_n \dots\}$, $\delta = \max\{\|d(y_i, y_j)\| \mid i, j \in \mathbb{N}_0\}$ and $\delta_n = \max\{\|d(y_i, y_j)\| \mid 0 \le i, j \le n\}$, $n \in \mathbb{N}$, then we can state the following result.

Lemma 3.1.6. *Let (X, d) be a cone metric space, P a normal cone with a normal constant K, $g : X \mapsto X$ a f-quasi-contraction and $K\|A\| < 1$. Then, for every $x \in X$,*

(i) for each $n \in \mathbb{N}$ there exists $i \in \{1, \dots, n\}$ such that

$$\delta_n = \|d(y_0, y_i)\|;$$

(ii) For arbitrary $n, n_0 \in \mathbb{N}$,

$$\delta_n \le \frac{K}{1 - K^{n_0}\|A\|^{n_0}} \delta_{n_0};$$

(iii) For each $n \in \mathbb{N}$,

$$\delta \le \frac{K}{1 - K^n\|A\|^n} \delta_n.$$

Proof. (i) For $1 \le i \le j \le n$, $d(y_i, y_j) \preceq A(u_{i,j}^{(1)})$ where

$$u_{i,j}^{(1)} \in \{d(y_{i-1}, y_{j-1}), d(y_{i-1}, y_i), d(y_{i-1}, y_j), d(y_{j-1}, y_i), d(y_{j-1}, y_j)\},$$

so, $\|d(y_i, y_j)\| \le K\|A\|\|u_{i,j}^{(1)}\| \le K\|A\|\delta_n$. Since $K\|A\| < 1$, $\delta_n = \|d(y_0, y_i)\|$ for some $i \in \{1, \dots, n\}$.

(*ii*) If $n \le n_0$, then the inequality

$$\delta_n \le \frac{K}{1 - K^{n_0}\|A\|^{n_0}}\delta_{n_0} \tag{3.22}$$

evidently holds cause $\frac{K}{1-K^{n_0}\|A\|^{n_0}} > 1$. Thus, we may assume $n > n_0$.
There exists $i, j \in \mathbb{N}$, $1 \le i \le n_0$ and $1 \le j \le n$ such that

$$\delta_0 = \|d(y_0, y_i)\| \text{ and } \delta_n = \|d(y_0, y_j)\|.$$

If $j \le n_0$, then $\delta_n = \delta_0$. Otherwise, $d(y_0, y_j) \preceq d(y_0, y_{n_0}) + d(y_{n_0}, y_j)$.
Clearly, $d(y_{n_0}, y_j) \preceq A\Big(u^{(1)}_{n_0,j}\Big)$, where

$$u^{(1)}_{n_0,j} \in \Big\{d(y_{n_0-1}, y_{j-1}), d(y_{n_0-1}, y_{n_0}), d(y_{n_0-1}, y_j), d(y_{n_0}, y_{j-1}), d(y_{j-1}, y_j)\Big\}.$$

Moreover, the inequality

$$\delta_n \le K\delta_0 + K\|A\|\|u^{(1)}_{n_0,j}\|$$

jointly with

$$u^{(1)}_{n_0,j} \preceq A\Big(u^{(2)}_{n_0,j}\Big),$$

where

$$u^{(2)}_{n_0,j} \in O\Big(y_{n_0-2}, j - n_0 + 2\Big) \subseteq O_{g,f}(x, n)$$

implies

$$\delta_n \le K\delta_0 + K^2\|A\|^2\|u^{(2)}_{n_0,j}\|.$$

Continuing in the same way, after $n_0 - 2$ more steps, we get

$$u^{(n_0-1)}_{n_0,j} \preceq A\Big(u^{(n_0)}_{n_0,j}\Big),\ u^{(n_0)}_{n_0,j} \in O_{g,f}(x, n)$$

and

$$\delta_n \le K\delta_0 + K^{n_0}\|A\|^{n_0}\delta_n.$$

Hence, the inequality (3.22) holds for every $n, n_0 \in \mathbb{N}$.

(*iii*) Considering the definition of δ, (*iii*) follows directly from (*ii*). □

Based on made estimations, we can state new result regarding diameter of an orbit.

Corollary 3.1.7. *Under the assumptions of Lemma 3.1.6 we have*

$$\delta \le \frac{K}{1 - K\|A\|}\|d(y_0, y_1))\|, \quad x \in X.$$

Theorem 3.1.8. *Let* (X, d) *be a complete cone metric space and* P *a normal cone with a normal constant* K*. If* $g, f : X \mapsto X$*,* f *commutes with* g*,* $g(X) \subseteq f(X)$*,* f *or* g *is continuous and satisfy* (3.1) *with* $K\|A\| < 1$*, then* f *and* g *have a unique common fixed point in* X*.*

Proof. If the fixed point exists it is unique thanks to (3.2) and $K\|A\| < 1$.
As in the proof of Lemma 3.1.6 (i), $d(y_n, y_{n+1}) \preceq A(u^{(1)}_{n,n+1})$

$$u^{(1)}_{n,n+1} \in \{d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_{n-1}, y_{n+1}), d(y_n, y_n), d(y_n, y_{n+1})\},$$

and $\|d(y_n, y_{n+1})\| \leq K\|A\|\|u^{(1)}_{n,n+1}\|$.
Further on, there exists

$$\begin{aligned} u^{(2)}_{n,n+1} \in \; & \{d(y_{n-2}, y_{n-1}), d(y_{n-2}, y_n), d(y_{n-1}, y_{n-1}), d(y_{n-1}, y_n), \\ & d(y_{n-2}, y_{n+1}), d(y_n, y_{n+1}), d(y_{n-1}, y_{n+1})\} \end{aligned}$$

such that $\|d(y_n, y_{n+1})\| \leq K^2\|A\|^2\|u^{(2)}_{n,n+1}\|$, $u^{(2)}_{n,n+1} \in O_{g,f}(x, n+1) \subseteq O_{g,f}(x, \infty)$. Applying the same procedure $n-2$ more times, we get

$$\|d(y_n, y_{n+1})\| \leq K^n\|A\|^n\|u^{(n)}_{n,n+1}\| \leq K^n\|A\|^n\delta.$$

Therefore,

$$d(y_n, y_m) \preceq \sum_{i=n}^{m-1} d(y_i, y_{i+1}) \preceq \sum_{i=n}^{m-1} A(u^{(1)}_{i,i+1}), \; m > n,$$

implies

$$\|d(y_n, y_m)\| \leq \sum_{i=n}^{m-1} K\|A\|\|u^{(1)}_{i,i+1}\| \leq \ldots \leq \sum_{i=n}^{m-1}(K\|A\|)^i\delta.$$

Since $\delta < \infty$ and $K\|A\| < 1$, $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in a complete cone metric space X, thus $(\exists y \in X) \lim\limits_{n\to\infty} y_n = y$.
If g is continuous, then $\lim\limits_{n\to\infty} d(g(y_n), g(y)) = \lim\limits_{n\to\infty} d(f(y_n), g(y)) = \theta$, $\lim\limits_{n\to\infty} d(g(y_n), y_n) = d(g(y), y)$ and $d(g(y_n), y_n) \preceq A(u_{n,i})$ for some $i \in \{1, \ldots, 5\}$ where $u_{n,i}$ is $d(g(y_{n-1}), y_{n-1})$, $d(g(y_{n-1}), y_n)$, $d(y_{n-1}, g(y_n))$, $d(g(y_{n-1}), g(y_n))$, $d(y_{n-1}, y_n)$, $i = \overline{1,5}$, respectively.
Obviously $\lim\limits_{n\to\infty} u_{n,i} = d(g(y), y)$, $i = \overline{1,3}$ and $\lim\limits_{n\to\infty} u_{n,i} = \theta$ if $i = 4, 5$, however $g(y) = y$.
Recall that $g(X) \subseteq f(X)$ and consider $z \in X$ such that $y = g(y) = f(z)$. Then $d(g(z), y_n) \preceq A(u)$ where

$$u \in \{d(g(y), y_{n-1}), d(g(y), g(z)), d(g(y), y_n), d(y_{n-1}, g(z)), d(y_{n-1}, y_n)\}. \tag{3.23}$$

Furthermore, from $d(g(z), y) = \lim\limits_{n\to\infty} d(g(z), y_n)$, (3.23) and

$$\begin{aligned} & \lim_{n\to\infty} d(g(y), y_{n-1}) = \lim_{n\to\infty} d(g(y), y_n) = d(g(y), y) = \theta \\ & \lim_{n\to\infty} d(y_{n-1}, g(z)) = d(y, g(z)); \\ & \lim_{n\to\infty} d(y_{n-1}, y_n) = \theta, \end{aligned}$$

it follows $g(z) = y$. On the other hand, $f(y) = f(g(z)) = g(f(z)) = g(y) = y$. Hence, in this case, y is a common fixed point of f and g.
Similarly, we get that $g(y) = f(y)$ is a common fixed point of mappings f and g

if f is continuous. Then $\lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} f(y_n) = f(y)$ and $\lim_{n\to\infty} d(g(y), f(y)) = \lim_{n\to\infty} d(g(y), g(y_n))$. Again, because g is a f-quasi-contraction, we can make a conclusion that $g(y) = f(y)$.
Equation $g^2(y) = f^2(y) = g(f(y)) = f(g(y))$ shows that it is sufficient to prove $g(g(y)) = g(y)$. Evidently, $d(g(g(y)), g(y)) \preceq A(u)$, for some

$$\begin{aligned} u \in & \{d(f(g(y)), f(y)), d(f(g(y)), g(y)), d(f(g(y)), g(g(y))), \\ & d(f(y), g(y)), d(f(y), g(g(y)))\}. \end{aligned}$$

Even though $g(y) = f(y)$ and $f(g(y)) = g(g(y))$, so $u \in \{\theta, d(g(g(y)), g(y))\}$ and $g(g(y)) = g(y)$. □

Fixed point problem may be observed as a special case of common fixed point problem when one of the mappings is identity map on X. Regarding that, if g is identity mapping on X, and f has all the properties requested by Theorem 3.1.2, then it is obvious that this assumptions are equivalent to the statement of Theorem 2.2.2. Particulary, Theorem 2.2.2 is corollary of this common fixed point result. Furthermore, all obtained assessments hold. Having that in mind, some inequalities about iterative sequence and diameter of an orbit of Perov type quasi-contraction f are gathered.

Corollary 3.1.9. *Let (X, d) be a cone metric space, P a normal cone with a normal constant K, $f : X \mapsto X$ a quasi-contraction and $K\|A\| < 1$. Then, for every $x \in X$, $n, m \in \mathbb{N}$, $m > n$, the following inequalities hold:*

(i) $\|d(y_n, y_{n+1})\| \leq (K\|A\|)^n \delta_{n+1} \leq (K\|A\|)^n \delta$;

(ii) $\|d(y_n, y_m)\| \leq \frac{(K\|A\|)^n}{1-K\|A\|}\delta$,

where $g(x) = f(x_1)$ and $g(x_n) = f(x_{n+1}) = y_n$, for each $n \in \mathbb{N}$.

As we made some comment before, Ćirić quasi-contraction is a special type of Perov type quasi-contraction on normal cone metric space (metric space), so we can make similar assessment for this kind of mapping regarding its fixed point, orbit and the iterative sequence.

Analogously, as for Theorem 3.1.2, we can formulate proposition that combines statements similar to Corollaries 3.1.3 and 3.1.4 but in the setting of normal cone metric space.

Corollary 3.1.10. *Let (X, d) be a complete cone metric space and P be a normal cone. Let $f, g : X \mapsto X$, g commutes with f, f continuous and $g(X) \subseteq f(X)$. If there exists $A \in \mathcal{B}(E)$, $K\|A\| < 1$ such that condition (3.20) is satisfied for all $x, y \in X$ or (3.21) is satisfied for all $x, y \in X$, then g and f have a unique common fixed point in X.*

Theorem 3.1.11. *Let (X, d) be a complete cone metric space and P be a normal cone. Let $f : X \mapsto X$, f^2 is continuous, $g : f(X) \mapsto X$ be such that $g(f(X)) \subseteq f^2(X)$ and $f(g(x)) = g(f(x))$, $x \in f(X)$. If (3.1) is satisfied for every $x \in f(X)$, $A \in \mathcal{B}(E)$, $K\|A\| < 1$, then f and g have a unique common fixed point in $f(X)$.*

Proof. As in the proof of Theorem 3.1.5, define a sequence $(z_n)_{n\in\mathbb{N}}$ for arbitrary $x_0 \in f(X)$. If f and g have a common fixed point then it is evidently unique. Due to Corollary 3.1.9, $\delta < \infty$ (Lemma 3.1.6) and

$$\|d(z_n, z_m)\| \leq \frac{(K\|A\|)^n}{1 - K\|A\|}\delta,$$

(z_n) is a Cauchy sequence in (X, d) and $\lim_{n\to\infty} z_n = z$ for some $z \in X$.
We will prove that $f^2(z) = g(f(z)) = u$ and that u is a unique common fixed point of f and g in X.
Since $f^2(z_{n-1}) = g(f(z_n))$, then $d(g(f(z_n)), g(f(z))) \preceq A(u_n)$ where

$$\begin{aligned} u_n \in\ & \{d(f^2(z_n), f^2(z)), d(f^2(z_n)), g(f(z_n))), d(f^2(z_n), g(f(z)), \\ & d(f^2(z), g(f(z_n))), d(f^2(z), g(f(z)))\}. \end{aligned} \tag{3.24}$$

Consider

$$d(f^2(z), g(f(z))) \preceq d(f^2(z), f^2(z_{n-1})) + d(g(f(z_n)), g(f(z)))$$

along with (3.24) and note that $\lim_{n\to\infty} d(f^2(z), f^2(z_n)) = \theta$.
If $u_n = d(f^2(z), g(f(z)))$ for infinitely many $n \in \mathbb{N}$, then clearly $f^2(z) = g(f(z))$.
If $u_n \in \{d(f^2(z_n), f^2(z)), d(f^2(z_n)), g(f(z_n))), d(f^2(z), g(f(z_n)))\}$ for infinitely many $n \in \mathbb{N}$, then, since $\lim_{n\to\infty} u_n = \theta$ we have $d(f^2(z), g(f(z))) = \theta$, i.e., $f^2(z) = g(f(z))$.
Ultimately, if $u_n = d(f^2(z_n), g(f(z))$ for infinitely many $n \in \mathbb{N}$, then as $n \to \infty$ we get

$$d(f^2(z), g(f(z))) \preceq A(d(f^2(z), g(f(z))),$$

thus $f^2(z) = g(f(z))$. If we denote that point with u it is enough to prove that $g(u) = u$ because $f(u) = f(g(f(z))) = g(f^2(z)) = g(u)$. Finally,

$$d(g(u), u) = d(g(f^2(z)), g(f(z))) \preceq A(d(f^3(z), f^2(z))) = A(d(g(u), u)),$$

and $g(u) = u$. Analogously to the proof of the Theorem 3.1.8 and Theorem (3.1.5), u is a unique common fixed point of f and g. □

Since the previous theorems contain as necessary condition commutativity of the mappings, some extensions of these results were inspired by the goal of replacing the commutativity with some weaker conditions. That way were introduced weak commutativity, R-commutativity, weak compatibility and other. We will include some results using the weak compatibility condition.
Compatible mappings, but in the case of metric space, were introduced by Jungck [77]. Those concepts were extended on a cone metric spaces in [82], [128] and [19].

Definition 3.1.12. *A pair of self-mappings on a cone metric space (X, d) is said to be compatible if for arbitrary sequence (x_n) in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t \in X$, and for arbitrary $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $d(f(g(x_n)), g(f(x_n))) \ll c$ wherever $n \geq n_0$. It is said to be weakly compatible if $f(x) = g(x)$ implies $f(g(x)) = g(f(x))$.*

Lemma 3.1.13. *([82]) If the pair of a self-mappings (f, g) on a cone metric space (X, d) is compatible, then it is also weakly compatible.*

In the Example 2.4. of [82] it has been shown that weak compatibility doesn't imply the compatibility neither in normal, nor in non-normal cone metric spaces.
Definition of the property (E.A.) is used to replace strict conditions of commutativity in addition with weak compatibility and it has been introduced by Aamri and Moutawakil [1] in 2002. It has been modified also by Kadelburg at al. in [82] for the setting of cone metric space.

Definition 3.1.14. *A pair of self-mappings on a cone metric space (X, d) has a (E.A.) property if there exists a sequence $(x_n) \subseteq X$ such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t \in X$.*

Obviously, every non-compatible pair of mappings has the property (E.A). Proceeding example shows that the class of the pairs of self-mappings that have property (E.A.) contains some compatible pair of mappings.

Example 13. Let $X = [0, 1]$ with a usual metric, $d(x, y) = |x-y|$, $x, y \in X$. If $f(x) = x$, $x \in X$ and

$$g(x) = \begin{cases} 1, & x = 0 \\ 0, & x \in (0, 1] \end{cases},$$

then only for all sequences $(x_n) \subseteq (0, 1]$, $x_n \to 0$, $n \to \infty$ (or with finitely many $x_n = 0$), $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) (= 0)$. But $f(g(x_n)) = 0$ and $g(f(x_n)) = 0$, so (f, g) is a pair of compatible mappings.

Theorem 3.1.15. *Let (X, d) be a cone metric space with a solid cone P, $f : X \mapsto X$ and $g : X \mapsto X$ be two weakly compatible mappings such that*

(i) (f, g) satisfy property (E.A.);

(ii) g is a f-quasi-contraction;

If $f(X)$ is a closed subspace of X or $g(X)$ is closed and $g(X) \subseteq f(X)$, then f and g have a unique common fixed point.

Proof. Observe a sequence $(x_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = x$. Assume that $f(X)$ is a closed subspace of X. The proof goes similarly in the other case.
Then $x = f(a)$ for some $a \in X$. Let us estimate $d(g(a), f(a))$.
Choose arbitrary $c, \varepsilon \in \operatorname{int}(P)$ and $n_0 \in \mathbb{N}$ such that

$$d(g(x_n), f(a)) \ll \varepsilon,\ d(g(x_n), f(a)) \ll \frac{c}{2},\ d(f(x_n), f(a)) \ll \frac{c}{2} \quad \text{for} \quad n \geq n_0. \tag{3.25}$$

Then the inequality

$$d(g(a), f(a)) \preceq d(g(a), g(x_n)) + \varepsilon,\ n \geq n_0 \tag{3.26}$$

follows by triangle inequality and (3.25). Remark that, further on, we consider $n > n_0$. Since g is a f-quasi-contraction, then $d(g(a), g(x_n)) \preceq A(u_n)$ where

$$u_n \in \{d(f(a), f(x_n)), d(g(a), f(a)), d(g(x_n), f(x_n)), d(g(a), f(x_n)), d(f(a), g(x_n))\}. \tag{3.27}$$

Equations (3.25), (3.26) and (3.27,) along with some triangle inequalities, now imply that one of the following inequalities is satisfied:

(1) $d(g(a), f(a)) \preceq \frac{1}{2}A(c) + \varepsilon$;

(2) $d(g(a), f(a)) \preceq A(d(g(a), f(a))) + \varepsilon$;

(3) $d(g(a), f(a)) \preceq A(c) + \varepsilon$;

(4) $d(g(a), f(a)) \preceq A(d(g(a), f(a))) + \frac{1}{2}A(c) + \varepsilon$;

(5) $d(g(a), f(a)) \preceq \frac{1}{2}A(c) + \varepsilon$.

However, each of these inequalities holds for any $\varepsilon \gg \theta$. If we additionally observe $c := \frac{c}{n}$, $n \in \mathbb{N}$, it proceeds

$$d(g(a), f(a)) \preceq \theta \quad \text{or} \quad d(g(a), f(a)) \preceq A(d(g(a), f(a))).$$

Second inequality is equivalent to the first because $(I - A)^{-1}$ is a positive linear operator and a is a coincidence point of f and g.
It remains to prove $g(g(a)) = g(a)$ since then, by weakly compatibility of f and g, $g(f(a)) = g(g(a)) = f(g(a)) = f(f(a))$.
Note that $d(g(a), g(g(a))) \preceq A(u)$ for some $u \in \{d(g(a), g(g(a))), \theta\}$. Hence, $g(g(a)) = g(a)$ and, by previous observations, $f(g(a)) = g(a)$ i.e. $f(a) = g(a)$ is a common fixed point of f and g.
Uniqueness obviously follows since g is a f-quasi-contraction.
In the case that $g(X)$ is a closed subspace of X and $g(X) \subseteq f(X)$, we have $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} g(x_n) = f(b) = g(a)$ for some $a, b \in X$, thus the proof is analogous. □

Same proof technique can be used to justify the next two theorems, therefore we present them without any proof.

Theorem 3.1.16. *Let (X, d) be a cone metric space with a solid cone P, $f, g : X \mapsto X$ weakly compatible mappings that satisfy property (E.A.). If there exists positive operator $A \in \mathcal{B}(E)$ with $r(A) < 1$ such that for each $x, y \in X$*

$$d(g(x), g(y)) \preceq A(u)$$

where

$$u \in \left\{d(f(x), f(y)), d(f(x), g(x)), d(f(y), g(y)), \frac{d(g(x), f(y)) + d(g(y), f(x))}{2}\right\}.$$

If $f(X)$ is a closed or $g(X) \subseteq f(X)$ is a closed subspace of X, then f and g have a unique common fixed point.

Theorem 3.1.17. *Let (X,d) be a cone metric space with a solid cone P, $f,g : X \mapsto X$ weakly compatible mappings that satisfy property (E.A.). If there exists positive operator $A \in \mathcal{B}(E)$ with $r(A) < 1$ such that for each $x,y \in X$*

$$d(g(x),g(y)) \preceq A(u)$$

where

$$u \in \left\{ d(f(x),f(y)), \frac{d(f(x),g(x)) + d(f(y),g(y))}{2}, \frac{d(g(x),f(y)) + d(g(y),f(x))}{2} \right\}.$$

If $f(X)$ is a closed or $g(X) \subseteq f(X)$ is a closed subspace of X, then f and g have a unique common fixed point.

Note that these three conditions are not related in the setting of cone metric space since they do not have to be even comparable.

3.2 Sequence of mappings

Along with common fixed point problem for a pair of mappings, we discuss on existence of a common fixed point for a sequence of mappings. Goal of this section is to determine conditions for existence and uniqueness of the common fixed point for some sequence of mappings. All presented results can be transferred to the family of mappings on cone metric space, and consequently, on metric space.

Two kind of contractive conditions are analyzed depending of the fact is there some mean mapping that relates to all other members of a sequence or not. We also pay attention on some kind of delay contractive conditions including a convergent series of constants.

Theorem 3.2.1. *Let (X,d) be a complete solid cone metric space and $(T_n)_{n\in\mathbb{N}_0}$ a sequence of self-mappings on X such that*

$$d(T_0x, T_ny) \preceq A(u_n), \text{ for all } x,y \in X,\ n \in \mathbb{N},$$

where

$$u_n \in D_{x,y} = \left\{ d(x,y), d(x,T_0x), d(y,T_ny), \frac{1}{2}\left(d(x,T_ny) + d(T_0x,y)\right) \right\},$$

for an increasing operator $A \in \mathcal{B}(E)$. If $r(A) < 1$, then there exists a unique $z \in X$ such that $T_nz = z$, $n \in \mathbb{N}_0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ converges to z for arbitrary $x_0 \in X$ where

$$x_{2n-1} = T_0x_{2n-2} \text{ and } x_{2n} = T_nx_{2n-1},\ n \in \mathbb{N}. \tag{3.28}$$

Moreover, z is a unique fixed point of T_n, $n \in \mathbb{N}_0$.

Proof. For an arbitrary $x_0 \in X$ observe sequence (x_n) defined as in (3.28) and denote D_{x_n,x_m} with $D_{n,m}$. Then for some $u_n \in D_{2n,2n-1}$,

$$d(x_{2n+1}, x_{2n}) \preceq A(u_n),$$

where $D_{2n,2n-1} = \left\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), \frac{1}{2}d(x_{2n-1}, x_{2n+1})\right\}$.
Similarly, for $u_{n+1} \in D_{2n,2n+1}$,

$$d(x_{2n+1}, x_{2n+2}) \preceq A(u_{n+1}),$$

where $D_{2n,2n+1} = \left\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{1}{2}d(x_{2n}, x_{2n+2})\right\}$.
Hence,

$$d(x_n, x_{n+1}) \preceq A(u)$$

for $u \in \left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2}d(x_{n-1}, x_{n+1})\right\}$, $n \in \mathbb{N}$.

We will consider few different cases.

(1) If $d(x_n, x_{n+1}) \preceq A(d(x_n, x_{n+1}))$, then $d(x_n, x_{n+1}) \preceq (I - A)^{-1}(\theta) = \theta$, i.e. $d(x_n, x_{n+1}) = \theta$. Remark that positivity of $(I - A)^{-1}$ must be used to obtain this inequality.

(2) Assuming $d(x_n, x_{n+1}) \preceq \frac{1}{2}A(d(x_{n-1}, x_{n+1}))$, then,

$$d(x_n, x_{n+1}) \preceq \left(I - \frac{A}{2}\right)^{-1} \frac{A}{2} \left(d(x_{n-1}, x_n)\right). \tag{3.29}$$

Further on,

$$r\left(\left(I - \frac{A}{2}\right)^{-1} \frac{A}{2}\right) \leq \frac{1}{1 - \frac{r(A)}{2}} \frac{r(A)}{2} = \frac{r(A)}{2 - r(A)} < 1,$$

and $\left(I - \frac{A}{2}\right)^{-1} \frac{A}{2}$ is an increasing operator.

(3) Ultimately, if $u = d(x_{n-1}, x_n)$, then $d(x_n, x_{n+1}) \preceq A(d(x_{n-1}, x_n))$.
If $A_n \in \left\{A, \left(I - \frac{A}{2}\right)^{-1} \frac{A}{2}\right\}$, then, from previous argumentation (1)-(3), it proceeds

$$d(x_n, x_{n+1}) \preceq A_n(d(x_{n-1}, x_n)), \ n \in \mathbb{N},$$

and A_n is an increasing operator with spectral radius less than 1. Continuing in the same manner, after $n - 1$ more steps, the inequaity

$$d(x_n, x_{n+1}) \preceq \left(\prod_{i=1}^{n} A_i\right) (d(x_0, x_1)) \tag{3.30}$$

is obtained. Denote with B an increasing operator $\frac{1}{2}\left(I - \frac{A}{2}\right)^{-1}$ and select $n, m \in \mathbb{N}$ such that $m > n$. Due to (3.30), it follows

$$d(x_n, x_m) \preceq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \preceq \sum_{i=n}^{m-1} A^i B^{j_i}(d(x_0, x_1))$$

where $0 \leq j_i \leq i$ and $j_i \leq j_{i+1}$, $i \in \mathbb{N}$.
If the sequence j_n is bounded, then $\sum\limits_{n=1}^{\infty} A^n B^{j_n}$ evidently converges and

$$\sum_{i=n}^{m-1} A^i B^{j_i}(d(x_0, x_1)) \to \theta, \ n, m \to \infty.$$

Consequently, (x_n) is a Cauchy sequence.
Otherwise, if $j_n \to \infty$, $n \to \infty$, then $\sum\limits_{n=1}^{\infty} B^{j_n}$ converges since

$$\begin{aligned} r\left(\left(I - \frac{A}{2}\right)^{-1}\right) &= r\left(I + \frac{A}{2}\left(I - \frac{A}{2}\right)^{-1}\right) \\ &\leq 1 + r\left(\frac{A}{2}\left(I - \frac{A}{2}\right)^{-1}\right) \\ &\leq 1 + r\left(\frac{A}{2}\right) r\left(\left(I - \frac{A}{2}\right)^{-1}\right), \end{aligned}$$

thus

$$r(B) = r\left(\frac{1}{2}\left(I - \frac{A}{2}\right)^{-1}\right) \leq \frac{1}{2}\frac{1}{1 - r\left(\frac{A}{2}\right)} = \frac{1}{2 - r(A)} < 1,$$

Moreover, $\sum\limits_{k=0}^{\infty} B^k$ converges and $\left(\sum\limits_{k=0}^{\infty} B^k\right) - B^j$ is an increasing operator for any $j \in \mathbb{N}_0$,

$$d(x_n, x_m) \preceq \sum_{k=0}^{\infty} B^k \sum_{i=n}^{\infty} A^i(d(x_0, x_1)).$$

Thanks to $\sum\limits_{i=n}^{\infty} A^i \to \Theta$, as $n \to \infty$, and Lemma 2.1.1, (x_n) is a Cauchy sequence in a complete cone metric space X.

In either way, there exists $\lim\limits_{n\to\infty} x_n = z$ for some $z \in X$.
For any $\varepsilon \gg \theta$, choose $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $d(x_n, z) \ll \frac{\varepsilon}{2}$ and $d(x_n, x_m) \ll \frac{\varepsilon}{2}$. If $n \geq n_0$, estimate $d(z, T_0 z)$:

$$d(z, T_0 z) \preceq d(z, x_{2n}) + d(x_{2n}, T_0 z) \preceq \frac{\varepsilon}{2} + A(u),$$

where

$$u \in \left\{d(z, x_{2n-1}), d(z, T_0 z), d(x_{2n-1}, x_{2n}), \frac{1}{2}\left(d(z, x_{2n}) + d(x_{2n-1}, T_0 z)\right)\right\}.$$

For $u \in \{(z, x_{2n-1}), d(x_{2n-1}, x_{2n})\}$, it follows

$$d(z, T_0 z) \preceq \frac{\varepsilon}{2} + A\left(\frac{\varepsilon}{2}\right). \tag{3.31}$$

If $d(z, T_0 z) \preceq \frac{\varepsilon}{2} + A(d(z, T_0 z))$, then

$$d(z, T_0 z) \preceq (I - A)^{-1} \left(\frac{\varepsilon}{2}\right). \tag{3.32}$$

Otherwise,

$$\begin{aligned} d(x_{2n}, T_0 z) &\preceq A\left(\frac{\varepsilon}{4}\right) + \frac{A}{2}\left(d(x_{2n-1}, T_0 z)\right) \\ &\preceq A\left(\frac{\varepsilon}{2}\right) + \frac{A}{2}\left(d(x_{2n}, T_0 z)\right), \end{aligned}$$

gives $d(x_{2n}, T_0 z) \preceq \left(I - \frac{A}{2}\right)^{-1} A\left(\frac{\varepsilon}{2}\right)$ and

$$d(z, T_0 z) \preceq \frac{\varepsilon}{2} + \left(I - \frac{A}{2}\right)^{-1} A\left(\frac{\varepsilon}{2}\right). \tag{3.33}$$

Anyway, as $\varepsilon := \frac{\varepsilon}{n}$, for $n \in \mathbb{N}$, in (3.31), (3.32) and (3.33), respectively, we get $T_0 z = z$. Analogously, $T_n z = z$ for any $n \in \mathbb{N}$. It remains to prove that z is uniquely determined. Assume that $T_n y = y$ for all $n \in \mathbb{N}_0$, then

$$d(z, y) = d(T_0 z, T_n y) \preceq A(u)$$

for $u \in \{d(z, y), \theta\}$.
This shows that z is a unique common fixed point of the sequence (T_n).
If $T_{n_0} y = y$ for some $n_0 \in \mathbb{N}_0$, then

$$d(T_0 y, y) = d(T_0 y, T_{n_0} y) \preceq A(u), \text{ where } u \in \left\{0, d(y, T_0 y), \frac{1}{2} d(y, T_0 y)\right\},$$

so $d(T_0 y, y) \preceq A(d(y, T_0 y))$ gives $T_0 y = y$. Nevertheless, if $T_0 y = y$ for some $y \in X$, then, for any $n \in \mathbb{N}$

$$d(y, T_n y) = d(T_0 y, T_n y) \preceq A(u) \text{ where } u \in \left\{0, d(y, T_n y), \frac{1}{2} d(y, T_n y)\right\}.$$

Therefore, $T_n y = y$ for all $n \in \mathbb{N}_0$, thus z is unique fixed point for any T_n, $n \in \mathbb{N}_0$. □

Let us recall that the following results was proved by Ćirić in [38].

Theorem 3.2.2. *Let $(T_n \mid n \in \mathbb{N}_0)$ be a sequence of mappings on a complete metric space (X, d). If for some $q \in (0, 1)$*

$$d(T_0 x, T_n x) \leq q \max\left\{d(x, y), d(x, T_0 x), d(y, T_n y), \frac{1}{2}\left(d(x, T_n y) + d(y, T_0 x)\right)\right\}$$

holds for each $n \in \mathbb{N}_0$ and all $x, y \in X$, then there exists a unique point $u \in X$ such that $T_n u = u$ for each n=0,1,2,... and for arbitrary $x_0 \in X$ a sequence

$$x_0, x_1 = T_0 x_0, x_2 = T_1 x_1, x_3 = T_0 x_2, \ldots, x_{2n-1} = T_0 x_{2n-2}, x_{2n} = T_n x_{2n-1}, \ldots$$

converges to u.

It is possible to state several results for common fixed point of sequence of mappings on normal cone metric space under slightly modified conditions. First difference is that K^2 figures instead of just K and positivity is excluded.

Theorem 3.2.3. *Let (X, d) be a complete cone metric space, P a normal cone with a normal constant K and $(T_n)_{n\in\mathbb{N}_0}$ a sequence of self-mappings on X such that*

$$d(T_0x, T_ny) \preceq A(u), \text{ for all } x, y \in X,\ n \in \mathbb{N},$$

where

$$u \in D_{x,y} = \left\{ d(x,y), d(x,T_0x), d(y,T_ny), \frac{1}{2}\left(d(x,T_ny) + d(T_0x,y)\right) \right\},$$

for some operator $A \in \mathcal{B}(E)$. If $K^2\|A\| < 1$, then there exists a unique $z \in X$ such that $T_nz = z$, $n \in \mathbb{N}_0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ defined in (3.28) converges to z for any $x_0 \in X$.

Proof. Let $x_0 \in X$ be arbitrary and (x_n) defined as in (3.28). By considering different cases as in the proof of Theorem 3.2.1, we get

$$\|d(x_n, x_{n+1})\| \le b\|d(x_{n-1}, x_n)\|$$

where $b \in \left\{K\|A\|, \frac{K^2\|A\|}{2-K^2\|A\|}\right\}$. Without loss of generality, let us assume that $b = \max\left\{K\|A\|, \frac{K^2\|A\|}{2-K^2\|A\|}\right\}$. Continuing in the same manner, it follows

$$\|d(x_n, x_{n+1})\| \le b^n\|d(x_0, x_1)\|.$$

Choose $n, m \in \mathbb{N}$ such that $m \ge n$. Since

$$\|d(x_n, x_m)\| \le \sum_{i=n}^{m-1} b^i\|d(x_0, x_1)\|,$$

and both of the series $\sum\limits_{n=1}^{\infty}(K\|A\|)^n$ and $\sum\limits_{n=1}^{\infty}\left(\frac{K^2\|A\|}{2-K^2\|A\|}\right)^n$ converge, (x_n) is a Cauchy sequence and $\lim\limits_{n\to\infty} x_n = z$ for some $z \in X$.
Moreover,

$$d(T_0z, z) \preceq d(T_0z, x_{2n}) + d(x_{2n}, z) \preceq A(u_n) + d(x_{2n}, z)$$

where

$$u_n \in \left\{ d(z, x_{2n-1}), d(z, T_0z), d(x_{2n-1}, x_{2n}), \frac{1}{2}\left(d(z, x_{2n}) + d(x_{2n-1}, T_0z)\right) \right\}.$$

Choose $\varepsilon > \theta$ arbitrary and $n_0 \in \mathbb{N}$ such that $\|d(z, x_n)\|, \|d(x_n, x_m)\| < \frac{\varepsilon}{2K}$ for any $n, m \ge n_0$. Let $n \ge n_0$.
If $u_n \in \{d(z, x_{2n-1}), d(x_{2n-1}, x_{2n})\}$, then

$$\|d(T_0z, z)\| < K\|A\|\frac{\varepsilon}{2K} + \frac{\varepsilon}{2} < \varepsilon, \tag{3.34}$$

since $\|A\| < 1$. If $u_n = d(z, T_0 z)$, then

$$\|d(z, T_0 z)\| < \frac{\varepsilon}{2\left(1 - K\|A\|\right)}. \tag{3.35}$$

In the last case, $u_n = \frac{1}{2}\left(d(z, x_{2n}) + d(x_{2n-1}, T_0 z)\right)$, it follows

$$\begin{aligned}\|d(T_0 z, x_{2n})\| &\leq \frac{K\|A\|}{2}\left(\|d(z, x_{2n})\| + \|d(x_{2n-1}, T_0 z)\|\right) \\ &\leq \frac{\|A\|\varepsilon}{4} + \frac{K^2\|A\|}{2}\left(\|d(x_{2n-1}, x_{2n})\| + \|d(x_{2n}, T_0 z)\|\right),\end{aligned}$$

and

$$\|d(T_0 z, x_{2n})\| \leq \frac{(1+K)\|A\|}{2(2 - K^2\|A\|)}\varepsilon.$$

Consequently,

$$\|d(z, T_0 z)\| \leq \frac{(2+K)\|A\|}{2(2 - K^2\|A\|)}\varepsilon. \tag{3.36}$$

Positive constant ε was arbitrary, so the inequalities (3.34), (3.35) and (3.36) prove that $T_0 z = z$. After similar estimations as in Theorem 3.2.1, $T_n z = z$ for any $n \in \mathbb{N}$. Uniqueness also follows analogously. $\square$

Theorem 3.2.1 and Theorem 3.2.3 are obvious generalizations of Theorem 1 of [34] if $Ax = qx$, $x \in E$ and, additionally, Theorem 2.2.6.

It is a reasonable choice to transfer this result of a sequence of mappings on a more particular case, pair of mappings. Therefore, it is a new kind of common fixed point problem.

Corollary 3.2.4. *Let (X, d) be a complete solid cone metric space and $S, T : X \mapsto X$ mappings such that for some increasing operator $A \in \mathcal{B}(E)$*

$$d(Tx, Sy) \preceq A(u), \text{ for all } x, y \in X,$$

where

$$u \in D_{x,y} = \left\{d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2}\left(d(x, Sy) + d(Tx, y)\right)\right\}.$$

If $r(A) < 1$, then both T and S have a unique fixed point and it is a common fixed point for T and S.

Corollary 3.2.5. *Let (X, d) be a complete cone metric space, P a normal cone with a normal constant K and $T, S : X \mapsto X$ mappings such that for some operator $A \in \mathcal{B}(E)$*

$$d(Tx, Sy) \preceq A(u), \text{ for all } x, y \in X,$$

where

$$u \in D_{x,y} = \left\{d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2}\left(d(x, Sy) + d(Tx, y)\right)\right\}.$$

If $K^2\|A\| < 1$, then both T and S have a unique fixed point and it is a common fixed point for T and S.

Observe that, in both corollaries, a sequence (x_n), where $x_{2n-1} = Tx_{2n-2}$ and $x_{2n} = Sx_{2n-1}$, $n \in \mathbb{N}$, converges to a uniquely determined fixed point z for an arbitrary $x_0 \in X$.

Corollary 3.2.6. *Let (X, d) be a complete solid cone metric space and $(T_n)_{n\in\mathbb{N}_0}$ a sequence of self-mappings on X. If there exists some $T_0 : X \mapsto X$ such that for each $i \in \mathbb{N}$ there exist $k_n, m_n \in \mathbb{N}$ such that*

$$d(T_0^{k_n}x, T_n^{m_n}y) \preceq A(u),$$

for all $x, y \in X$, where

$$u \in D_n = \left\{ d(x,y), d(x, T_0^{k_n}x), d(y, T_n^{m_n}y), \frac{1}{2}\left(d(x, T_n^{m_n}y) + d(y, T_0^{k_n}x)\right) \right\},$$

where $A \in \mathcal{B}(E)$ is an increasing operator and $r(A) < 1$, then each T_n, $n \in \mathbb{N}_0$, has a unique fixed point in X and it is a common fixed point for all T_n, $n \in \mathbb{N}_0$.

Proof. If we apply Corollary 3.2.4 for mappings $T_0^{k_n}$ and $T_n^{m_n}$ where T_n is an arbitrary, we get that $T_0^{k_n}z_n = z_n$ for unique $z_n \in X$, $n \in \mathbb{N}$. But,

$$T_0^{k_n}(T_0z_n) = T_0(T_0^{k_n}z_n) = T_0z_n,$$

so $T_0z_n = z_n$. For any $j \in \mathbb{N}$, z_n is also a fixed point of mapping $T_0^{k_j}$, thus $z := z_n = z_j$. Obviously, $T_nz = z$ and it is satisfied for any $n \in \mathbb{N}_0$ and z is a unique point with that property. □

Corollary 3.2.7. *Let (X, d) be a complete cone metric space, P a normal cone with a normal constant K and $(T_n)_{n\in n_0}$ a sequence of self-mappings on X. If there exists some $T_0 : X \mapsto X$ such that for each $n \in \mathbb{N}$ there exist $k_n, m_n \in \mathbb{N}$ such that*

$$d(T_0^{k_n}x, T_n^{m_n}y) \preceq A(u) \text{ for } x, y \in X,$$

where

$$u \in D_n = \left\{ d(x,y), d(x, T_0^{k_n}x), d(y, T_n^{m_n}y), \frac{1}{2}\left(d(x, T_n^{m_n}y) + d(y, T_0^{k_n}x)\right) \right\},$$

operator $A \in \mathcal{B}(E)$ and $K^2\|A\| < 1$, then each T_n has a unique fixed point in X and it is a common fixed point for all T_n, $n \in \mathbb{N}_0$.

As we commented before, the same results could be easily obtained for a family of self-mappings and the proof is analogous. When dealing with a sequence of self-mappings, many researches followed the course of determining existence of a "limit" fixed point, in other words, $z \in X$ such that $\lim\limits_{n\to\infty} T_nz = z$. This do not presume that z is fixed point for any of mappings. This results find application in numerical analysis and justifying convergence of defined iterative methods. On the top of that, existence of this kind of point could direct us on possible fixed point, and looking from application angle, to some solution of equation (functional, integral, differential).

Theorem 3.2.8. *Let (X, d) be a complete solid cone metric space and $(T_n)_{n\in\mathbb{N}_0}$ a sequence of self-mappings on X such that*

$$d(T_n x, T_{n+1} y) \preceq A(u_n) + a_n,$$

for all $x, y \in X$ and $n \in \mathbb{N}$, where

$$u_n \in D_{x,y} = \left\{ d(x,y), d(x, T_n x), d(y, T_{n+1} y), \frac{1}{2}\left(d(x, T_{n+1} y) + d(T_n x, y)\right) \right\},$$

$a_n \geq 0$ and array $\sum_{n=1}^{\infty} a_n$ is convergent. If $A \in \mathcal{B}(E)$ is an increasing operator and $r(A) < 1$, then there exists a unique $z \in X$ such that $\lim_{n\to\infty} T_n z = z$. Furthermore, the sequence $(x_n)_{n\in\mathbb{N}_0}$ where $x_n = T_n x_{n-1}$, $n \in \mathbb{N}$, converges to z for any $x_0 \in X$.

Proof. Let $x_0 \in X$ be an arbitrary point and $x_n = T_n x_{n-1}$ for $n \in \mathbb{N}$.

$$d(x_n, x_{n+1}) \preceq A(u_n) + a_n$$

for

$$u_n \in D_{n,n+1} = \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2} d(x_{n-1}, x_{n+1}) \right\}.$$

Setting $u_n = \frac{1}{2} d(x_{n-1}, x_{n+1})$, we get

$$d(x_n, x_{n+1}) \preceq \left(I - \frac{A}{2}\right)^{-1} \frac{A}{2}(d(x_{n-1}, x_n)) + \left(I - \frac{A}{2}\right)^{-1} (a_n).$$

Assuming that $B_n \in \{A, \left(I - \frac{A}{2}\right)^{-1} \frac{A}{2}\}$, obviously $r(B_n) < 1$ and B_n is an increasing operator, then

$$d(x_n, x_{n+1}) \preceq B_n(d(x_{n-1}, x_n)) + \left(I - \frac{A}{2}\right)^{-1} (a_n),$$

because $\left(I - \frac{A}{2}\right)^{-1} (a_n) \succeq a_n$.

Continuing in the same manner, after $n-1$ more steps, we get

$$\begin{aligned} d(x_n, x_{n+1}) \preceq & \left(\prod_{i=1}^{n} B_i\right)(d(x_0, x_1)) \\ & + \left(I - \frac{A}{2}\right)^{-1} \left(\left(\prod_{i=2}^{n} B_i\right)(a_1) + \ldots + B_n(a_{n-1}) + a_n \right). \end{aligned}$$

Being so, the inequality

$$\begin{aligned} d(x_n, x_m) \preceq & \sum_{i=n}^{m-1} \left(A^i \frac{\left(I - \frac{A}{2}\right)^{j_i}}{2^{j_i}} \right)(d(x_0, x_1)) \\ & + \left(I - \frac{A}{2}\right)^{-1} \sum_{i=n}^{m-1} \left(A^i \frac{\left(I - \frac{A}{2}\right)^{k_i}}{2^{k_i}} \right)(w), \end{aligned}$$

where $0 \leq j_i \leq i$, $j_i \leq j_{i+1}$, $k_i \in \{j_i, j_i - 1\}$ depending of B_i, $i = \overline{n, m-1}$, and $w = \sum_{n=1}^{\infty} a_n$. As in the proof of Theorem 3.2.1 it follows that (x_n) is a Cauchy sequence, and $\lim_{n\to\infty} x_n = z$ for some $z \in X$.
Let us assume that for arbitrary $y_0 \in X$, $y_n = T_n y_{n-1}$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} y_n = w$ and estimate $d(x_n, y_{n+1})$ based on

$$d(z,w) \preceq d(z,x_n) + d(x_n, y_{n+1}) + d(y_{n+1}, w),\ n \in \mathbb{N} \tag{3.37}$$

For arbitrary $\varepsilon \gg 0$, choose $n_0 \in \mathbb{N}$ with the property

$$d(x_n, x_m) \ll \frac{\varepsilon}{2},\ d(x_n, z) \ll \frac{\varepsilon}{2},\ d(y_n, y_m) \ll \frac{\varepsilon}{2} \text{ and } d(y_n, w) \ll \frac{\varepsilon}{2},$$

for any $m \geq n \geq n_0$. Fix $n > n_0$. Then, $d(x_n, y_{n+1}) \preceq A(v_n) + a_n$ where

$$v_n \in \left\{ d(x_{n-1}, y_n), \frac{\varepsilon}{2}, \frac{1}{2}\left(d(x_{n-1}, y_{n+1}) + d(x_n, y_n)\right) \right\}.$$

Consider that $v_n = d(x_{n-1}, y_n) \preceq d(x_{n-1}, x_n) + d(x_n, y_{n+1}) + d(y_{n+1}, y_n)$ implies

$$d(x_n, y_{n+1}) \preceq A(\varepsilon) + A(d(x_n, y_{n+1})) + a_n,$$

along with

$$d(x_n, y_{n+1}) \preceq (I - A)^{-1} A(\varepsilon) + (I - A)^{-1}(a_n).$$

With the assumption $v_n = \frac{1}{2}\left(d(x_{n-1}, y_{n+1}) + d(x_n, y_n)\right)$, we get

$$d(x_n, y_{n+1}) \preceq A\left(\frac{\varepsilon}{2}\right) + A(d(x_n, y_{n+1})) + a_n,$$

and

$$d(x_n, y_{n+1}) \preceq (I - A)^{-1} A\left(\frac{\varepsilon}{2}\right) + (I - A)^{-1}(a_n).$$

With respect to previous discussion and (3.37), without depending on the choice of v_n, it follows

$$d(z,w) \preceq \varepsilon + (I - A)^{-1} A(\varepsilon) + (I - A)^{-1}(a_n),$$

because $a_n \preceq (I - A)^{-1}(a_n)$ and $(I - A)^{-1} A\left(\frac{\varepsilon}{2}\right), A\left(\frac{\varepsilon}{2}\right) \preceq (I - A)^{-1} A(\varepsilon)$.
However, since $a_n \to \theta$ as $n \to \infty$, by taking $\varepsilon \to \theta$, equality of z and w is obtained.
It remains to prove that $\lim_{n\to\infty} T_n z = z$.
The inequality

$$d(z, T_n z) \preceq (I - A)^{-1}\left(\frac{\varepsilon}{2} + a_n\right)$$

is attained in accordance with

$$d(z, T_{n+1} z) \preceq d(z, x_n) + d(x_n, T_{n+1} z) \preceq \frac{\varepsilon}{2} + w_n,$$

for

$$w_n \in \left\{ A\left(\frac{\varepsilon}{2}\right) + a_n, A\left(d(z, T_{n+1}z)\right) + a_n, \right.$$
$$\left. \left(I - \frac{A}{2}\right)^{-1} A\left(\frac{\varepsilon}{2}\right) + \left(I - \frac{A}{2}\right)^{-1} (a_n) \right\}.$$

Consequently, by $(\mathbf{p}_2)$, $\lim\limits_{n\to\infty} T_n z = z$.

Remark that if $\lim\limits_{n\to\infty} T_n u = u$, then exists $m_0 \in \mathbb{N}$,

$$d(z, T_m z),\ d(T_n z, T_m z),\ d(u, T_m u),\ d(T_n u, T_m u) \ll \frac{\varepsilon}{2}, \tag{3.38}$$

while $n, m \geq m_0$. Select $m > m_0$, and perceive

$$d(z, u) \preceq d(z, T_m z) + d(T_m z, T_{m+1} u) + d(T_{m+1} u, u) \preceq \varepsilon + d(T_m z, T_{m+1} u).$$

In addition,

$$d(T_m z, T_{m+1} u) \preceq \begin{cases} A(d(z, u)) + a_m, & s_m = d(z, u) \\ A\left(\frac{\varepsilon}{2}\right) + a_m, & s_m \in \{d(z, T_m z), d(u, T_{m+1} u)\} \\ (I - A)^{-1}\left(A\left(\frac{\varepsilon}{2}\right) + a_n\right), & s_m = \frac{1}{2}\left(d(z, T_{m+1} u) + d(u, T_m z)\right) \end{cases}$$

and (3.38) allow us to determine, based on arbitrariness of ε and $\lim\limits_{n\to\infty} a_n = \theta$, u must be equal to z, a unique point in X with the property $\lim\limits_{n\to\infty} T_n z = z$. □

Theorem 3.2.9. *Let (X, d) be a complete cone metric space, P a normal cone with a normal constant K and $(T_n)_{n\in\mathbb{N}}$ a sequence of self-mappings on X such that*

$$d(T_n x, T_{n+1} y) \preceq A(u) + a_n, \text{ for all } x, y \in X,\ n \in \mathbb{N},$$

where

$$u \in D_{x,y} = \left\{ d(x, y), d(x, T_n x), d(y, T_{n+1} y), \frac{1}{2}\left(d(x, T_{n+1} y) + d(T_n x, y)\right) \right\}$$

and array $\sum\limits_{n=1}^{\infty} \|a_n\|$ is convergent. If $A \in \mathcal{B}(E)$ and $K^2\|A\| < 1$, then there exists a unique $z \in X$ such that $\lim\limits_{n\to\infty} T_n z = z$.

Proof. Proofs goes similarly as in Theorem 3.2.3 and Theorem 3.2.8.
First, estimate $\|d(x_n, x_{n+1})\|$ according to

$$d(x_n, x_{n+1}) \preceq A(u_n) + a_n,$$

for $u_n \in \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2} d(x_{n-1}, y_{n+1}) \right\}$. Denote $\|d(x_{n-1}, x_n)\|$ with δ_n. One of the inequalities

$$\|d(x_n, x_{n+1})\| \leq \begin{cases} K\|A\|\delta_n + \|a_n\|, & u_n = d(x_{n-1}, x_n) \\ \frac{\|a_n\|}{1 - K\|A\|}, & u_n = d(x_n, x_{n+1}) \\ \frac{1}{2 - K^2\|A\|}\left(K^2\|A\|\delta_n + 2\|a_n\|\right), & u_n = \frac{1}{2} d(x_{n-1}, y_{n+1}) \end{cases}$$

holds for any $n \in \mathbb{N}$. For $b = \max\left\{K\|A\|, \frac{K^2\|A\|}{2-K^2\|A\|}\right\}$, we get

$$\|d(x_n, x_{n+1})\| \leq b^n \|d(x_0, x_1) + \frac{1}{1-K\|A\|}\sum_{i=0}^{n-1} b^i \|a_{n-i}\|.$$

Consequently, $x_n \to z$, $n \to \infty$, for some $z \in X$.
If $\lim\limits_{n\to\infty} y_n = w$ for some $\omega \in X$, then for arbitrary $\varepsilon \gg 0$, as in the previous proof but with $\frac{\varepsilon}{2K}$ estimations instead of $\frac{\varepsilon}{2}$, we get

$$\|d(z,w)\| \leq \begin{cases} \varepsilon + \frac{\|A\|\varepsilon}{2} + K\|a_n\|, & v_n \in \{d(x_{n-1}, x_n), d(y_n, y_{n+1})\} \\ \varepsilon + \frac{\|A\|\varepsilon}{1-K\|A\|} + \frac{K\|a_n\|}{1-K\|A\|}, & v_n = d(x_{n-1}, y_n) \\ \varepsilon + \frac{K|A\|}{(1-K\|A\|)}\left(\frac{\varepsilon}{2} + K\|a_n\|\right), & v_n = \frac{1}{2}d(x_{n-1}, y_{n+1}) \end{cases}$$

Hence, z is a uniquely determined as a limit and analogously as in the proof of Theorem 3.2.8, $\lim\limits_{n\to\infty} T_n z = z$. □

Example 14. Let X be $C[0,1]$, set of real continuous functions on a closed interval $[0,1]$ and $P \subseteq X$ a cone defined with

$$x \in P \Leftrightarrow x(t) \geq 0 \text{ for all } t \in [0,1].$$

Then, $d(x,y) = |x-y|$, $x, y \in X$, is a cone metric on X. If $f \in X$ is chosen arbitrary, then for $0 < L < 2$, the sequence of mappings $T_n : X \mapsto X$, $n \in \mathbb{N}$, is defined as follows:

$$(T_n x)(t) = \frac{1}{n} f(t) + \int_0^t Lx(\sqrt{s})ds,\ t \in [0,1], n \in \mathbb{N}.$$

Evidently,

$$\begin{aligned} d(T_{n+1}x, T_n y)(t) &= |(T_{n+1}x)(t) - (T_n y)(t)| \\ &\leq \left(\frac{1}{n} - \frac{1}{n+1}\right)|f(t)| + \int_0^t L\left|x(\sqrt{s}) - y(\sqrt{s})\right| ds \\ &= \frac{1}{n^2+n}|f(t)| + (Ad(x,y))(t), \end{aligned}$$

where

$$(Ax)(t) = \int_0^t Lx(\sqrt{s})ds,\ t \in [0,1],$$

is a bounded operator in $\mathcal{B}(X)$.
However, since cone P is defined as above, it follows

$$d(T_{n+1}x, T_n y) \preceq \frac{1}{n^2+n}|f| + A(d(x,y)).$$

Zima proved in [130] that the spectral radius of operator A is $\frac{L}{2}$, thus less than 1 and, evidently, A is an increasing operator. If $a_n = \frac{1}{n^2+n}|f|$, then $\sum\limits_{n=1}^{\infty} a_n$ converges in X. Hence, we may apply Theorem 3.2.8 on a sequence of mappings (T_n) and conclude that there exists $g \in X$ such that $\lim\limits_{n\to\infty} T_n g = g$. Notice that g will be zero function.

3.3 Hardy-Rogers Theorem

In [64] (see also [119]), authors consider a mapping $f : X \mapsto X$ on a complete metric space X such that for each $x, y \in X$

$$d(f(x), f(y)) \leq a_1 d(x, y) + a_2 d(x, f(x)) + a_3 d(y, f(y)) + a_4 d(x, f(y)) + a_5 d(y, f(x)),$$

where $a_i \geq 0$, $i = \overline{1,5}$. It was proved that if, additionally, $\sum_{i=1}^{5} a_i < 1$, then f has a unique fixed point in X and iti is known as Hardy-Rogers theorem.

Note that Hardy-Rogers theorem on metric space is a corollary of Ćirić quasi–contraction with $q = \sum_{i=1}^{5} a_i < 1$. But on a cone metric space we can not make this correlation since these two vectors do not have to be comparable. Therefore, we focus on a result of [79] where is given a generalization of this theorem on a complete cone metric space but for a pair of self-mappings with some restraints. Assuming property (E.A.) and weak-compatibility, Kadelburg and al. in [82] improved this result and gave the proof of the following theorem

Theorem 3.3.1. *Let (X, d) be a cone metric space and let (f, g) be a weakly compatible pair of self-mappings on X satisfying condition (E.A). Suppose that there exist nonnegative scalars a_i, $i = \overline{1,5}$ such that $\sum_{i=1}^{5} a_i < 1$ and that for each $x, y \in X$,*

$$\begin{aligned} d(g(x), g(y)) \prec\ & a_1 d(f(x), f(y)) + a_2 d(f(x), g(x)) + a_3 d(f(y), g(y)) \\ & + a_4 (d(f(x), g(y)) + a_5 d(f(y), g(x)). \end{aligned} \tag{3.39}$$

If $g(X) \subseteq f(X)$ and at least one of $f(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique common fixed point.

Obviously, these conditions are less strict than in [79], Theorem 2.8. We will prove the generalization of this result considering bounded linear operators A_i, $i = \overline{1,5}$, instead of scalars.

Theorem 3.3.2. *Let (X, d) be a cone metric space with a solid cone P, and let (f, g) be a weakly compatible pair of self-mappings on X satisfying condition (E.A). Suppose that there exist positive bounded linear operators $A_i \in \mathcal{B}(E)$, $i = \overline{1,5}$, such that the inequality*

$$\begin{aligned} d(g(x), g(y)) \preceq\ & A_1(d(f(x), f(y))) + A_2(d(f(x), g(x))) + A_3(d(f(y), g(y))) \\ +\ & A_4(d(f(x), g(y))) + A_5(d(f(y), g(x))) \end{aligned} \tag{3.40}$$

holds for all $x, y \in X$. If $f(X)$ is a closed subspace of X or $g(X) \subseteq f(X)$ and $g(X)$ is a closed subspace of X, $r(A_1 + A_4 + A_5) < 1$ and one of the conditions

(i) $r(A_3 + A_4) < 1$;

(ii) $r(A_2 + A_5) < 1$;

(iii) $r(A_2 + A_3 + A_4 + A_5) < 2$,

is satisfied, then f and g have a unique common fixed point.

Proof. Since f and g have the (E.A.) property, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = f(a)$ for some $a \in X$. The last conclusion follows from the assumptions of closedness of $f(X)$ or equivalently $g(X) \subseteq f(X)$.
Considering the distance between $f(a)$ and $g(a)$,

$$d(f(a), g(a)) \preceq d(f(a), g(x_n)) + d(g(x_n), g(a)). \tag{3.41}$$

For arbitrary $c \gg \theta$ and $\varepsilon \gg \theta$, choose $n_0 \in \mathbb{N}$ such that

$$d(g(x_n), f(a)) \ll c, d(g(x_n), f(a)) \ll \frac{\varepsilon}{5} \text{ and } d(f(x_n), f(a)) \ll \frac{\varepsilon}{5},\ n \geq n_0. \tag{3.42}$$

Let us estimate $d(g(x_n), g(a))$, $n > n_0$ under the assumption (i).

$$\begin{aligned} d(g(x_n), g(a)) &\preceq A_1(d(f(x_n), f(a))) + A_2(d(f(x_n), g(x_n))) + A_3(d(f(a), g(a))) \\ &\quad + A_4(d(f(x_n), g(a))) + A_5(d(f(a), g(x_n))) \\ &\preceq (A_1 + 2A_2 + A_4 + A_5)\left(\frac{\varepsilon}{5}\right) + (A_3 + A_4)(d(g(a), f(a))). \end{aligned}$$

By (3.41) and (3.42), now follows

$$d(g(a), f(a)) \preceq c + (A_1 + 2A_2 + A_4 + A_5)(\frac{\varepsilon}{5}) + (A_3 + A_4)(d(g(a), f(a))).$$

This inequality holds for any $c \gg \theta$ leading to

$$d(g(a), f(a)) \preceq (A_3 + A_4)(d(g(a), f(a))).$$

But (i) implies that $I - A_3 - A_4$ is an invertible operator, and $(I - A_3 - A_4)^{-1}$ is a positive operator, and taking that into account $g(a)$ and $f(a)$ coincide.
Obviously, if (3.40) is satisfied, then

$$\begin{aligned} d(g(y), g(x)) &\preceq A_1(d(f(x), f(y))) + A_2(d(f(y), g(y))) + A_3(d(f(x), g(x))) \\ &\quad + A_4(d(f(y), g(x))) + A_5(d(f(x), g(y))) \end{aligned} \tag{3.43}$$

and if we assume that (ii) holds, same as in the first case, $g(a) = f(a)$.
Combining (3.40) and (3.41), we conclude that

$$\begin{aligned} d(g(x), g(y)) &\preceq A_1(d(f(x), f(y))) + \frac{A_2 + A_3}{2}(d(f(x), g(x)) + d(f(y), g(y))) \\ &+ \frac{A_4 + A_5}{2}(d(f(x), g(y)) + d(f(y), g(x))) \end{aligned}$$

holds for each $x, y \in X$. If $r(A_2 + A_3 + A_4 + A_5) < 2$, then again, after some consideration, we get $g(a) = f(a)$.
Furthermore, f and g are weakly compatible, so $g(g(a)) = g(f(a)) = f(g(a)) = f(f(a))$.
On the other hand,

$$\begin{aligned} d(g(a), g(g(a))) &\preceq A_1(d(f(a), f(g(a)))) + A_2(d(f(a), g(a))) + A_3(d(f(g(a)), g(g(a)))) \\ &\quad + A_4(d(f(a), g(g(a)))) + A_5(d(f(g(a)), g(a))) \\ &\preceq (A_1 + A_4 + A_5)(d(g(a), g(g(a)))). \end{aligned}$$

Analogously, $(I - A_1 - A_4 - A_5)$ is an invertible operator, with a positive inverse, hence $g(g(a)) = g(a)$. Therefore, $g(a)$ is a common fixed point for the pair of mappings (f, g). If b is a common fixed point of f and g, then

$$d(g(a), b) = d(g(g(a)), g(b)) \preceq (A_1 + A_4 + A_5)(d(g(a), b)),$$

and additionally $b = g(a)$. Further, $g(a)$ is the unique common fixed point of f and g. $\square$

Evidently, this theorem generalizes Theorem 3.3.1 and the results in [82] by taking $A_i = a_i I$, $i = \overline{1, 5}$, but instead of $<$ we have $\leq$ and one of the nonnegative scalars a_2 and a_3 can be chosen voluntarily if $a_1 + a_4 + a_5 < 1$ and $a_3 < 1 - a_4$ or $a_2 < 1 - a_5$. Also, instead of completeness condition, it is enough to assume that $f(X)$ or $g(X)$ is a closed subspace of X.

Chapter 4

Comparison between metric fixed point and Perov type results

When publishing some scientific results, the most important question is the proof of novelty. In the time of mass publishing and overpowering quantity over quality, it is significant to pay attention to this kind of problem and to substantiate originality of results presented in previous chapters.

Some links between cone metric and metrics spaces are made in both solid and normal case. The question is could results on cone metric space be derived from well-known metric fixed point theorems via any of those methods such as scalarization, renormization and so on.

We will show that existence of a fixed point in Perov theorem follows from Banach fixed point theorem but even though, estimations presented in Perov theorem do not and they are most valuable for applications. When discussing cone metric space, we will show that, thanks to operatorial characteristics of A, this do not hold and our Perov type results are independent from similar metric or cone metric (comstant type) theorems.

Since attempts of obtaining cone metric from metric are different in the solid and normal case, this chapter will be divided in two sections while the first section will also deal with Perov theorem on generalized metric space.

4.1 Perov theorem and normal cone metric space

First part of this section is dedicated to Perov theorem on generalized metric space and how it is related to Banach fixed point theorem. Further on, we focus on Perov type theorems on a normal cone metric space having in mind behavior of operatorial constant A through all made renormizations. Many efforts are made in the attempt of reduction any cone metric space to a metric space. There were several papers ([15, 52, 73]) studying relations between cone metric spaces in general, and especially normal cone metric spaces, on one, and metric spaces on the other side. Recent results in cone metric fixed point theory established some relation between b-metric spaces and normal cone metric spaces.

Definition 4.1.1. *Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d : X \times X \to [0, +\infty)$ is said to be a b-metric if for all $x, y, z \in X$ the following* *conditions are satisfied:*

(b_1) $d(x, y) = 0$ *if and only if* $x = y$*;*

(b_2) $d(x, y) = d(y, x)$*;*

(b_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) *is called a b-metric space (with constant* s*).*

Definitions of Cauchy and convergent sequence in a b-metric space, as well as completeness, go analogously as in a metric space.

Let (X, d) be a cone metric space, P a normal cone with a normal constant K. Define a function $D : X \times X \mapsto \mathbb{R}$,

$$D(x, y) = \|d(x, y)\|, \quad x, y \in X \tag{4.1}$$

Theorem 4.1.2. *A function D defined in (4.1) is a b-metric on X with a constant K.*

Proof. Let $x, y, z \in X$ be arbitrary points. From the definition of norm and (d_1) it easily follows that (b_1) holds. D is also a symmetric function since it directly follows from the symmetry of the norm. From the fact that d is a metric on X, (d_3) and since (X, d) is a normal cone metric space, we have

$$D(x, y) = \|d(x, y)\| \leq K \left(\|d(x, z)\| + \|d(z, y)\|\right) = K \left(D(x, z) + D(z, y)\right).$$

Thus, (X, D) is a b-metric space. □

If the normal constant K is equal to 1, then (X, D) is a metric space.

Let us recall, if (X, d) is a complete normal cone metric space, (x_n) is Cauchy sequence in (X, d) if and only if $\lim\limits_{n,m\to\infty} \|d(x_n, x_m)\| = 0$ and $\lim\limits_{n\to\infty} x_n = x$ if and only if $\lim\limits_{n\to\infty} \|d(x_n, x)\| = 0$. Therefore, we may state the following corollary.

Theorem 4.1.3. (X, d) *is a complete cone metric space, P a normal cone with a normal constant K and D an b-metric defined as in (4.1) if and only if (X, D) is a complete b-metric space.*

Proof. Choose an arbitrary D-Cauchy sequence $(x_n) \subseteq (X, D)$. From the definition of metric D, we may conclude that

$$\lim_{n,m\to\infty} D(x_n, x_m) = \lim_{n,m\to\infty} \|d(x_n, x_m)\| = 0,$$

thus $\lim\limits_{n,m\to\infty} d(x_n, x_m) = \theta$, i.e., (x_n) is a d-Cauchy sequence. There exists some $x \in X$ such that $\lim\limits_{n\to\infty} x_n = x$ in the sense of cone metric d. Observe that, since P is a normal cone,

$$\lim_{n\to\infty} x_n = x \quad \Leftrightarrow \quad \lim_{n\to\infty} d(x_n, x) = \theta \quad \Leftrightarrow \quad \lim_{n\to\infty} D(x_n, x) = 0.$$

Therefore, (x_n) converges to x with respect to b-metric D and, since (x_n) was an arbitrary Cauchy sequence in (X, D), (X, D) is a complete b-metric space. □

Since the previous comments are derived on several equivalences, we may conclude that the statement of the theorem could include if and only if, i.e., (X, d) is complete metric space if and only if (X, D) is complete.

We will give another proof of the generalization of Perov fixed point theorem in the setting of normal cone metric space, only for the existence part.

Theorem 4.1.4. *Let (X, d) be a complete cone metric space, P a normal cone with a normal constant K and $f : X \mapsto X$ a self-mapping. If there exists an operator $A \in \mathcal{B}(E)$ such that $K\|A\| < 1$, for all $x, y \in X$,*

$$d(f(x), f(y)) \preceq A(d(x, y)), \tag{4.2}$$

then f has a unique fixed point in X.

Proof. From the condition (4.2) and the fact that P is a normal cone, it proceeds

$$D(f(x), f(y)) = \|d(f(x), f(y))\| \leq K\|A(d(x, y))\| \leq K\|A\|D(x, y),\ \ x, y \in X,$$

and f is a contraction on a b-metric space and the existence of an unique fixed point follows by the generalization of Banach fixed point theorem in b-metric space presented in [38]. □

Observe that we can obtain the same result from Banach fixed point theorem (on complete metric spaces) by renorming, as presented in [59].

Theorem 4.1.5. *Let (X, d) be a cone metric space, $P \subseteq E$ a normal cone with a normal constant K where $(E, \|\|)$ is a Banach space. Then:*

(i) A function $\|\cdot\|_1 : E \mapsto \mathbb{R}$ defined with

$$\|x\|_1 = \inf\{\|u\| \mid x \preceq u\} + \inf\{\|v\| \mid v \preceq x\},\ \ x \in E,$$

is a norm on E.

(ii) Norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent norms on X.

(iii) If we observe P as a cone in Banach space $(E, \|\cdot\|_1)$, then (X, d) is a normal cone metric space with a normal constant equal to 1.

Proof. (i) Obviously, $\|x\|_1 \geq 0$ and

$$\|x\|_1 = 0 \Leftrightarrow\ (\forall\, \varepsilon > 0)\, (\exists\, v \preceq x \preceq u)\ \|u\| + \|v\| < \varepsilon.$$

Observing only $\varepsilon = \frac{1}{n}$ and conjoint u_n, v_n, $n \in \mathbb{N}$, it follows $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} v_n = \theta$ since policemen lemma holds in normal cone metric spaces, $x = \theta$. Trivially, $\|\theta\|_1 = 0$.
If $q \in \mathbb{R} \setminus \{0\}$ is arbitrary, then

$$\begin{aligned}
\|qx\|_1 &= \inf\{\|u\| \mid qx \preceq u\} + \inf\{\|v\| \mid v \preceq qx\} \\
&= \inf\{\|qu\| \mid x \preceq u\} + \inf\{\|qv\| \mid v \preceq x\} \\
&= |q|\|x\|_1.
\end{aligned}$$

If $q = 0$, then $\|0x\|_1 = \|\theta\|_1 = 0 = 0\|x\|_1$.
For some $x, y \in X$, select $u_x \preceq x \preceq v_x$ and $u_y \preceq y \preceq v_y$, then

$$\|x + y\|_1 \leq \|u_x + u_y\| + \|v_x + v_y\| \leq \|u_x\| + \|u_y\| + \|v_x\| + \|v_y\|,$$

and by taking an infimum for all such u and v, it points to the triangle inequality for $\|\cdot\|_1$. *(ii)* It is easy to see that $\|x\|_1 \leq 2\|x\|$. Assuming that there exist no m such that $m\|x\| \leq \|x\|_1$ for all x, we can define a sequence (x_n) of vectors with a norm 1 such that $\frac{1}{n}\|x_n\| > \|x_n\|_1$, $n \in \mathbb{N}$. Notice some $u_n \leq x_n \leq v_n$ such that $\frac{1}{n}\|x_n\| > \|u_n\| + \|v_n\|$. As a consequence, since we are discussing normal cone metric, $\lim\limits_{n\to\infty} \|u_n\| = \lim\limits_{n\to\infty} \|v_n\| = \lim\limits_{n\to\infty} \|x_n\| = \theta$ gives us that (x_n) tends to θ, as $n \to \infty$, which contradicts our assumption regarding norm.

(iii) Let $x, y \in E$ be such that $\theta \preceq x \preceq y$. Therefore, $\{u \mid y \preceq u\} \subseteq \{u \mid x \preceq u\}$ and $\|x\|_1 \leq \|y\|_1$. The normal constant is never less than 1, so $K = 1$. □

The equivalence of the norms allows us to determine the relation between $\|A\|$ and $\|A\|_1$.

Remark 4.1.6. Based on the previously made observations regarding renorminization of a normal cone with a normal constant K and Theorem 2.1.6, we may conclude that existence of the unique fixed point Perov type contractions (including extended and more general contractive conditions) on normal cone metric spaces could be derived from analogous results on metric spaces.

Focusing on just first two statements of Perov theorem, we may state the following result:

Theorem 4.1.7. *Perov theorem (i) is a consequence of a Banach fixed point theorem.*

Proof. Notice that generalized metric space introduced by Perov is a type of normal cone metric space.

If $P = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0,\ i = \overline{1,n}\}$, then P evidently determines a cone in a Banach space $\mathbb{R}^n$ with supremum norm, $\|x\| = \max\limits_{i=1^n} |x_i|$, and $x \leq y$ if and only if $x_i \leq y_i$, $i = \overline{1,n}$. Since $\theta \leq x \leq y$, for $\theta = (0, 0, \ldots, 0)$, $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, implies $0 \leq x_i \leq y_i$, $i = \overline{1,n}$, then $\|x\| = \max\limits_{i=\overline{1,n}} |x_i| \leq \max\limits_{i=\overline{1,n}} |y_i| = \|y\|$ and P is a normal cone with a normal constant $K = 1$.

By taking into the account results of Theorem 4.1.4, it follows that for any generalized metric space (X, d) in the sense of Perov, the appropriate b-metric space (X, D) is a metric space.

Assume that the requirements of Perov theorem are fulfilled for some $A \in \mathcal{M}_{m,m}(\mathbb{R}^+)$ such that $A^n \to \Theta_m$, as $n \to \infty$. Since a matrix A converges to the zero matrix, then $\|A^n\| \to 0$, $n \to \infty$. Choose $n_0 \in \mathbb{N}$ such that $\|A^n\| < 1$ for any $n \geq n_0$. For such n,

$$d(f^n(x), f^n y) \preceq A^n(d(x, y)), \quad x, y \in \mathbb{R}^m,$$

and

$$D(f^n(x), f^n y) \leq \|A^n\| D(x, y), \quad x, y \in \mathbb{R}^m. \tag{4.3}$$

If we apply Banach contraction principle for f^n and $q = \|A^n\| < 1$, f^n has a unique fixed point z in X. Since $f^n(f(z)) = f(z)$, it must be $f(z) = z$. If $fu = u$ for some $u \in X$, then $f^n u = u$, so $u = z$.

Hence, Perov theorem is a direct consequence of Banach contraction principle.

It is easy to observe that the iterative sequence (x_n) is a Cauchy sequence, thus convergent, and since $(f^{nk}(x))_{k\in\mathbb{N}}$ converges to z by Banach fixed point theorem, (ii) holds. □

Remark 4.1.8. On the other hand, if $n = 1$, then generalized metric space is a metric space and a positive matrix $A = [q]$ tends to zero if and only if $q < 1$. Thus, Banach contraction principle is a Perov fixed point theorem for $n = 1$. However, remarks regarding distance presented in (iii) and (iv) (easily observed if we take $g = f$) could not be derived directly from Banach contraction principle since the inequality (4.3) do not imply (iii).

Example 15. Define a mapping $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ with $f(x) = (\frac{x_1}{2} + x_2, \frac{x_2}{2})$, $x = (x_1, x_2) \in \mathbb{R}^2$. Let

$$A = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{bmatrix},$$

then $\lim\limits_{n\to\infty} A^n = \Theta_2$ and

$$d(f(x), f(y)) \preceq A(d(x, y)),\ x, y \in \mathbb{R}^2.$$

Since $\|A\| = 1$, $D(f(x), f(y)) \leq D(x, y)$ and if $x = (0, 0)$, $y = (0, 1)$, it follows that f is not a contraction in $(\mathbb{R}^2, D)$, but it is a Perov contraction and based on Perov theorem it possesses a unique fixed point $(0, 0)$.

From the proof of Theorem 4.1.7 and the previous example, we may notice correlation between Perov theorem and well-known consequence of Banach theorem.

Corollary 4.1.9. *Let (X, d) be a complete metric space, $f : X \mapsto X$ a mapping. If*

$$d(f^n(x), f^n(y)) \leq q d(x, y), \quad x, y \in X,$$

for some $n \in \mathbb{N}$ and $q \in [0, 1)$, then f has a unique fixed point in X.

The following example shows that Perov type theorems including requirement $r(A) < 1$ could not be derived directly from Banach theorem. For additional comments see Section 5.1.

Example 16. Let c_0 be the set containing all sequences of real numbers convergent to zero equipped with supremum norm $\|\|_\infty$ and define $A : E \mapsto E$ with

$$A(x) = A(x_1, x_2, x_3, \ldots, x_n, \ldots) = (0, x_3, \frac{x_4}{2}, \ldots, \frac{x_{n+1}}{2}, \ldots), \quad x = (x_n) \in c_0.$$

Operator A is linear on Banach space $(c_0, \|\|_\infty)$ and also bounded since $\|Ax\|_\infty \leq \|x\|_\infty$. By choosing $e_3 = (0, 0, 1, 0, \ldots, 0, \ldots) \in c_0$, it follows $\|A\| = 1$ by taking into account

previous inequality.
For any $m \in \mathbb{N}$,

$$A^m(x) = A^m(x_1, x_2, x_3, \ldots) = (0, \frac{x_{m+2}}{2^{m-1}}, \frac{x_{m+3}}{2^m}, \ldots), \quad x = (x_n) \in c_0,$$

therefore, observing $e_{m+2} \in c_0$ with all zeros except one on (m+2)-nd place (i.e., $(e_{m+2})_n = \delta_{n,m+2}$, $n \in \mathbb{N}$), we obtain $\|A^m\| = \frac{1}{2^{m-1}}$. Spectral radius of A is $\frac{1}{2}$, A is a positive operator, so all the conditions of Theorem 2.1 are satisfied since

$$d(A(x), A(y)) \preceq A(d(x, y)), \; x, t \in c_0,$$

where $\leq$ is usual partial ordering on c_0, i.e. $x_n \leq y_n$, $n \in \mathbb{N}$, determining a normal cone and $d : c_0 \times c_0 \mapsto c_0$ defined by $d(x, y)(n) = |x(n) - y(n)|$, $n \in \mathbb{N}$ is a cone metric.
On the other hand, since normal constant and $\|A\|$ are equal to 1, norm inequality implies

$$D(A(x), A(y)) \leq D(x, y),$$

thus Banach theorem is not applicable (let $x = \theta$ and $y = e_3$).

We may also assume that $K = 1$ due to the renormization and the invariance of spectral radius in renormized space. It is important to notice that $r(A) < 1$ implies $\|A^n\| < 1$ starting from some $n \in \mathbb{N}$, so instead of Banach theorem, we should consider Consequence 4.1.9.

If the inequality (2.1) holds, then, since A is an increasing operator,

$$d(f^n(x), f^n(y)) \preceq A^n(d(x, y)),$$

thus,

$$D(f^n(x), f^n(y)) \leq \|A^n\|(d(x, y)),$$

and existence and uniqueness of a fixed point for a mapping f follows directly from Corollary 4.1.9.

In Example 15 f^3 is a contraction in induced metric space, and in Example 16 f^2.

As presented in [33], the requirement that A contains only positive entries, as stated in Perov theorem, could be removed thanks to the normality of the defined cone in generalized metric space. This could be explained also by the fact that, from the definition of matrix norm, only absolute value of matrix entries has impact on the norm value. So Perov type theorems are applicable, regardless of the positivity of matrix elements, if all entries are between -1 and 1. Perov fixed point theorem found application in solving various systems of differential equations. But, in some cases like [121], it is possible to replace it with the Corollary 4.1.9. The following example shows possible application of Theorem 2.6.1 in solving integral equations and obtained results improve previous results regarding existence and uniqueness of this type of equations.

Example 17. Let $x \in C[0, 1]$ and $K \in C\left([0, 1] \times [0, 1] \times \mathbb{R}\right)$, where $C([0, 1])$ is the set of all continuous functions $f : [0, 1] \mapsto \mathbb{R}$. Consider an integral equation

$$x(t) = \int_0^t K(t, s, x(s))\mathrm{d}s, \; t \in [0, 1]. \tag{4.4}$$

Define on $C[0,1]$ supremum norm, $\|x\| = \sup\limits_{0\leq t\leq 1} |x(t)|$, and a mapping $d : C[0,1] \times C[0,1] \mapsto C[0,1]$ with

$$d(x,y) = |x-y|,\ x,y \in C[0,1],$$

where $|x-y|(t) = |(x-y)(t)| = |x(t)-y(t)|$, $t \in [0,1]$. It is easy to check that $(C[0,1], d)$ is a cone metric space while ordering on $C[0,1]$ is induced by a solid cone $P = \{x \in C[0,1] \mid x(t) \geq 0,\ t \in [0,1]\} \subseteq C[0,1]$. Assume that there exists some nonnegative function $\alpha : [0,1] \times [0,1] \mapsto \mathbb{R}$ such that

$$\int_0^1 \alpha(t,s)ds \leq 1,\ t \in [0,1],$$

and

$$|K(t,s,x(s)) - K(t,s,y(s))| \leq \alpha(t,s)A(|x-y|)(t),\ x,y \in C[0,1],\ t \in [0,1],$$

for some positive and increasing operator $A : C[0,1] \mapsto C[0,1]$ such that $\lim\limits_{n\to\infty} A^n(x) = \theta$, $x \in C[0,1]$. (θ is a zero function with a domain $[0,1]$)
Then a mapping $f : C[0,1] \mapsto C[0,1]$ defined with a $f(x)(t) = \int\limits_0^t K(t,s,x(s))\mathrm{ds}$, $x \in C[0,1]$, $t \in [0,1]$, has a unique fixed point in $C[0,1]$. Therefore, integral equation (4.4) has a unique solution.

This assertion would follow as a direct consequence of Theorem 2.6.1 since, for any $t \in [0,1]$,

$$\begin{aligned}
d(F(x),F(y))(t) &= \left|\int_0^t (K(t,s,x(s)) - K(t,s,y(s)))\,\mathrm{ds}\right| \\
&\leq \int_0^t |K(t,s,x(s)) - K(t,s,y(s))|\mathrm{ds} \\
&\leq \int_0^t \alpha(t,s)A(|x-y|)(t)\mathrm{ds} \\
&\leq A(|x-y|)(t)\int_0^1 \alpha(t,s)\mathrm{ds} \\
&\leq A(|x-y|)(t),
\end{aligned}$$

indicating $d(F(x),F(y)) \preceq A(d(x,y))$.

By adding this inequality to the previous assumptions, all conditions of Theorem 2.6.1 are fulfilled. A mapping F has a unique fixed point in $C[0,1]$ and every iterative sequence converges to the fixed point which also represents a unique solution of integral equation (4.4).

Emphasise that this requirements are less strict than usual ones assuming that

$$|K(t,s,x) - K(t,s,y)| \le q|x-y|, \text{ for some } q \in [0,1),$$

which is obtained for $\alpha(t,s) = 1$, $t,s \in [0,1]$ and $A(x) = qx$, $x \in C[0,1]$.

4.2 Du's scalarization method

Solid cone metric space can not be renormized as presented in Section 4.1 since there is no link between partial ordering and defined norm. In other words, $x \preceq y$ do not lead to any general relation among $\|x\|$ and $\|y\|$. The problem of equivalence of solid cone metric spaces and metric spaces is extensively researched topic, but just few of those presented significant result ([12, 15, **?**, 47]).

Implementation of those scalarizations led to devaluation of many published results on cone metric spaces since it is shown that they are equivalent to appropriate metric theorems. That is why this section is important in emphasising novelty of results presented in this dissertation and its independence from, if existing, some similar metric (cone metric) theorems.

Among different approaches, we will focus on Du's scalarization method.

Du ([46]) studied the equivalence of vectorial versions of fixed point theorems in generalized cone metric spaces and scalar versions of fixed point theorems in (general) metric spaces (in usual sense). He has shown that the Banach contraction principles in general metric spaces and in TVS-cone metric spaces are equivalent. His theorems also extend some results in Huang and Zhang ([67]), Rezapour and Hamlbarani ([117]) and others.

We state Theorem 2.1 of [46] regarding metrizability of solid cone metric space.

Theorem 4.2.1. *Let (X,d) be a solid cone metric space, $e \in \operatorname{int} P$ and ξ_e is defined by*

$$\xi_e(u) = \inf\{q \in \mathbb{R} \mid u \in qe - P\},$$

for each $u \in E$. Then $d_\xi = \xi_e \circ d$ is a metric on X .

Let $f : X \mapsto X$ be such that Banach contractive condition holds for some $q \in (0,1)$. Then, applying Lemma 1.1 of [46], we have

$$d_\xi(f(x), f(y)) \le q d_\xi(x,y), \quad \text{holds for any} \quad x,y \in X. \tag{4.5}$$

Hence, generalizations of Banach contraction principle directly follows as a consequence of Banach theorem on metric space. However, if f satisfies (2.47), restricted with an additional conditions for A, we cannot conclude that there exists some $q \in (0,1)$ such that (4.5) is satisfied, and Theorem 2.1.3 could not be derived from Banach theorem. Despite that, Banach fixed point theorem on cone metric space is equivalent to classical result of Banach. Analogously, Ćirić's quasi-contraction of Perov type is not equivalent of its metric version and, along with Fisher quasi-contraction, improves and extends results of [39], [119].

In a similar manner we may discuss Theorem 2.1.10 as a local version of Banach theorem.

Let $T : X \mapsto X$ be such that there exists a point $z \in X$ for which $\overline{O(z)}$ is complete, and a $q \in (0, 1)$ such that

$$d(Tx, Ty) \leq qd(x, y) \quad \text{holds for any} \quad x, y = T(x) \in O(z). \tag{4.6}$$

Then, applying Lemma 1.1 of [46], we have

$$d_\xi(Tx, Ty) \leq qd_\xi(x, y), \quad \text{holds for any} \quad x, y = T(x) \in O(z). \tag{4.7}$$

Hence, Theorem 1 of [14] directly follows from Park's result by Theorem 2 of [100]. However, if T satisfies (2.9), restricted with a linear bounded mapping, we cannot conclude that there exists some $q \in (0, 1)$ such that (4.7) is satisfied, and so Theorem 2.1.9 cannot be derived from Park's result. Therefore Theorem 2.1.9 indeed improves the corresponding result of [100].

Deducing related conclusions for other Perov type results of this thesis, we see that they could not originate from some metric reductions. For some more recent results see [76], [92], [91]. Another approach to this problem is pointing out to some examples like Example 16 which show that Theorem 2.1.3 can be applied even in the case when Banach theorem (equivalently on cone metric space) can not. In order to illustrate that difference between ours and some well-known metric and cone metric results, we present several examples in the Chapter 5.

Chapter 5

Applications

Expressing importance of some research study is, preferably, connected with a wide range of possible applications. Throughout whole manuscript appear different examples having two main purposes. One is evidently defining area of application, but the second one concerns independence from similar conclusions in that area of expertise. In order to achieve that goal, separately from other chapters, we collect some examples that should speak in favor deductions made in Section 4.2.

Famous Serbian mathematician, Kurepa said that every mathematical problem could be reduced on some fixed point problem. This is just one way to express influence this branch of mathematics. It appears that one of the reasons why is hidden in huge class of applications. The accent is on solving equations, including differential, integral, operator and so on, along with systems and some delay problems.

Regarding theorems and corollaries throughout this manuscript, it appears that they can be applied even in the case when regular fixed point theorems with contractive constant could not. Valuable part of those statements are estimations regarding distance between fixed point, sequence of iterative approximation or diameter of an orbit, etc.

First section contain several integral equations which apply different Perov type theorems even though can not be solved as usual. Well-posedness and Ulam's stability of some functional equations are the topic of the last section concerning use of Perov type contraction in some recently published papers. Wide range of application is amplified with several examples throughout various chapters such as Examples 8,11,14, 16 and 17.

5.1 Integral equations

Etimations obtained by Perov theorem and generalized metric are better than by using usual metric spaces and some well-known theorems. In [108] coupled fixed point problem on Banach space was analyzed and, implementation of various metric and vector-valued metric in the sense of Perov, lead to the conclusion that results obtained by Perov theorem are better and unify other results. The comparison is made for Schauder, Krasnoselskii, Leray-Schauder and Perov theorem. We will discuss results obtained by Banach fixed point theorem and compare them in the case of metric space.

Example 18. If (X, d) is a complete metric space and $T_i : X \times X \mapsto X$, $i = 1, 2$, solution of a system

$$\begin{aligned} T_1(x,y) &= x \\ T_2(x,y) &= y, \end{aligned} \tag{5.1}$$

is a fixed point of a mapping $T : X \times X \mapsto X \times X$ defined with

$$T(x,y) = (T_1(x,y), T_2(x,y)), \ x, y \in X.$$

To apply Banach theorem, T should be a contraction on $X \times X$. Let D be a metric on $X \times X$ induced by d, then

$$D(T(x,y), T(u,v)) \leq qD((x,y),(u,v)), \ (x,y),(u,v) \in X \times X,$$

for some $q \in (0,1)$.
If $D((x,y),(u,v)) = d(x,y) + d(u,v)$, $(x,y),(u,v) \in X \times X$, then

$$d(T_1(x,y), T_1(u,v)) + d(T_2(x,y), T_2(u,v)) \leq q(d(x,y) + d(u,v)), \tag{5.2}$$

for any $(x,y),(u,v) \in X \times X$, because of

$$d(T_i(x,y), T_i(u,v)) \leq \frac{q}{2}(d(x,y) + d(u,v)), \ i = 1,2, \tag{5.3}$$

holds for any $(x,y),(u,v) \in X \times X$.
On the other hand, if Perov theorem would be applied, T_1 and T_2 should be such that

$$d(T_i(x,y), T_i(u,v)) \leq a_i d(x,u) + b_i d(y,v), \ (x,y),(u,v) \in X \times X, i = 1,2,$$

for some nonnegative $a_i, b_i \geq 0$, $i = 1,2$, and a matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

convergent to zero. This means that $r(A) < 1$ or, equivalently,

$$a_1 + b_2 + \sqrt{-2a_1b_2 + 4a_2b_1 + {a_1}^2 + {b_2}^2} < 2.$$

Considering (5.3), $\max\{a_1, a_2\}$, $\max\{b_1, b_2\}$ should be less than $\frac{1}{2}$, or in view of (5.2), $\max\{a_1, a_2\} + \max\{b_1, b_2\} < 1$. Anyway, this result is more strict than $r(A) < 1$.

If

$$A = \begin{bmatrix} \frac{2}{3} & \frac{1}{9} \\ \frac{1}{9} & \frac{2}{3} \end{bmatrix},$$

then $r(A) = \frac{7}{9}$, but neither of the inequalities (5.2) and (5.3) is satisfied.

Observe that this problem could characterize as common fixed point problem. Let $f(x,y) = (T_1(x,y), y)$ and $g(x,y) = (x, T_2(x,y))$, $x, y \in X$.

Example 19. Let (X_i, d_i), $i = \overline{1,m}$ be some complete metric spaces and define a generalized metric d on their Cartesian product $X = \prod_{i=1}^{m} X_i$ with

$$d(x,y) = \begin{bmatrix} d_1(x_1,y_1) \\ d_2(x_2,y_2) \\ \vdots \\ d_m(x_m,y_m) \end{bmatrix},$$

for $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m) \in X$. As previously discussed, (X, d) is, as generalized metric space, also a normal cone metric space with a normal constant $K = 1$. Let (Y, τ) be a Hausdorff topological space and $f = (f_1, f_2) : X \times Y \mapsto X \times Y$ an operator.Theorem 2.1 of [121] states that if f is continuous, (Y, τ) has a fixed point property (i.e., every continuous mapping $g : Y \mapsto Y$ has a fixed point) and there exists a matrix $S \in R^{m\times m}$ convergent to zero matrix such that

$$d(f_1(u,y), f_1(v,y)) \leq S(d(u,v)), \quad u,v \in X, y \in Y, \tag{5.4}$$

then f has a fixed point. Uniqueness is not guaranteed because of contractive condition based on the first coordinate.
Instead of using Perov theorem, as presented in [121], observe that, since $S^n \to \Theta$, $n \to \infty$, then there exists some $n \in \mathbb{N}$ such that $\|S^n\| = q < 1$, where assumed norm is the supremum norm. For such n, (5.4) holds and

$$d(f_1^n(u,y), f_1^n(v,y)) \leq S^n\left(d(u,v)\right), \quad u,v \in X, y \in Y,$$

so

$$d_\infty(f_1^n(u,y), f_1^n(v,y)) \leq q d_\infty(u,v), \quad u,v \in X, y \in Y,$$

where $d_\infty : X \times X \mapsto \mathbb{R}$ is a maximum metric defined with

$$d(u,v) = \max_{i=\overline{1,m}} d_i(u_i, v_i), \ u,v \in X.$$

Hence, Consequence 4.1.9 guarantees unique fixed point z of a mapping $f_1^n(,y) : X \mapsto X$ for any $y \in Y$. As in the proof of Theorem 4.1.7, z is also unique fixed point of $f_1(,y) : X \mapsto X$ for a fixed $y \in Y$. The rest of the proof would follow analogously as in [121].
As stated in this paper, Y could be any compact convex subset of a Banach space. This results is applied in solving systems of functional-differential equations such as:

$$\begin{aligned} x(t) &= \int_0^1 K(t,s,x(s),y(s))ds + g(t), \quad t \in [0,1], \\ y(t) &= \int_0^1 H(t,s,x(s),y(s),y(y(s)))ds, \quad t \in [0,1], \end{aligned}$$

where $x \in X$ and $y \in Y$, $K \in C\left([0,1] \times [0,1] \times \mathbb{R}^m \times [0,1], \mathbb{R}^m\right)$, $g \in C\left([0,1], \mathbb{R}^m\right)$ and $H \in C\left([0,1] \times [0,1] \times \mathbb{R}^m \times [0,1] \times [0,1], \mathbb{R}\right)$, Under assumptions that codomain of H is contained in $[0,1]$, that H is a first coordinate Lipschitzian mapping with a constant L and K is a Perov generalized contraction, this system has at least one solution in $X \times Y$ for $X = C([0,1], \mathbb{R}^m) = \prod_{i=1}^{m} X_i$, $X_i = C[0,1]$, $i = \overline{1,m}$ and Y set of all Lipschitzian mappings on $C([0,1],[0,1])$ with a constant L. Observe that we could not use Banach theorem instead of Perov to obtain this conclusion due to the contractive condition for K.

Similar class of integral equations has already appeared in Example 14 but as a common fixed point problem for the sequence of mappings. Here it is stated in order to show one more significant application of Perov type fixed point theorem although Banach theorem forces some other conclusions.

Example 20. Let E be $C[0,1]$ with supremum norm and usually defined cone $P \subseteq E$, $X = E$ and metric $d(x,y) = |x-y|$, $x, y \in X$, a cone metric on X. In the first part we deal with solving integral equation

$$\int_0^t x(\sqrt{t})dt = x(t),\ t \in [0,1]$$

. Define, $f, A : X \mapsto X$ with

$$f(x)(t) = A(x)(t) = \int_0^t x(\sqrt{t})dt\, t \in [0,1].$$

Operator A is increasing, bounded and linear with the spectral radius $\frac{1}{2}$ and

$$\begin{aligned} d(f(x), f(y)) &= |\int_0^t \left(x(\sqrt{t}) - y(\sqrt{t})\right) dx| \\ &\leq \int_0^t |x(\sqrt{t}) - y(\sqrt{t})| dx \\ &= A(d(x,y)). \end{aligned}$$

Theorem 2.1.3 guarantees existence and uniqueness of fixed point. Despite of that, if D is a metric arising from norm, $x(t) = 2$, $y(t) = 0$, $t \in [0,1]$, then

$$D(f(x), f(y)) = 2 = D(x,y),$$

meaning that f is not a contraction.

This integral equation can be redefined by choosing some $g \in X$ in a way that

$$g(t) + \int_0^t x(\sqrt{t})dt = x(t),\ t \in [0,1],$$

or depending on some constant $L \in (0,2)$

$$g(t) + \int_0^t \frac{2}{L} x(\sqrt{t})dt = x(t),\ t \in [0,1].$$

Definition of A implicates that $r(A) = \frac{2}{L}$.

5.2 Ulam's stability

Polish mathematician Stanislaw Ulam, posed in 1940 a question concerning the stability of group homomorphisms. However, that question was the beginning of this stability problem for functional equations that is in the focus of research last decades. One year later, D. H. Hyers ([68]) made the first significant breakthrough. He gave a partially affirmative answer to the question of Ulam for additive mappings on Banach space. Since then, a large number of papers have been published in connection with various generalizations of Ulams problem and Hyerss theorem. Almost forty years after, in 1978., T. M. Rassias ([115]) made first bigger progress. He succeeded in extending Hyerss theorem for mappings between Banach spaces by considering an unbounded Cauchy difference subject to a continuity condition upon the mapping. He was the first to prove the stability of the linear mapping and to initiate research in this field. The influence of all three mathematicians was immeasurable, and that is why proposed stability problem is known as Ulam-Hyers-Rassias stability, sometimes in different arrangement, or Ulam's stability.

Ulam observed δ-homomorphism between two groups G_1 and G_2, assuming that G_2 is equipped with metric d, and a mapping $f : G_1 \mapsto G_2$ such that, for all $x, y \in G_1$,

$$d(f(xy), f(x)f(y)) \leq \delta.$$

The question was which assumptions should be forced on f if the last inequality proceeds existence of some homomorphism close to f.

However, stability question applies on functional and differential equations (including more variables and differential equations with a delay). In order to discuss application of Perov type result, we will skip preliminaries regarding metric space, and pass along to the cone metric and accordingly introduced generalization of Ulam's stability.

Definition 5.2.1. *([120]) Let (X, d) and (Y, ρ) be two cone metric spaces, $f, g : X \mapsto Y$ mappings. The coincidence equation $f(x) = g(x)$ is Ulam's stable if there exists a linear increasing operator $\psi : E \mapsto E$ such that for any $\varepsilon \gg \theta$ and each x fulfilling the inequality $\rho(f(x), g(x)) \preceq \varepsilon$, exists some solution z of the coincidence equation such that $d(x, z) \preceq \psi(\varepsilon)$.*

Regarding this property, original and important is $\psi(x) = \lambda x$, for some positive λ. Particular case of the coincidence equation is fixed point problem for $X = Y$, choosing $g(x) = x$, $x \in X$.

Definition 5.2.2. *Let (X, d) be a solid cone metric space, $f : X \mapsto X$ and $\psi : P \to P$ a nondecreasing function such that $\psi(\theta) = \theta$. The equation $f(x) = x$ is Ulam's stable with respect to ψ if, for any $\varepsilon \gg \theta$, and y such that $d(f(y), y) \preceq \varepsilon$ there exists some solution z of this equation such that*

$$d(z, y) \preceq \psi(\varepsilon).$$

Theorem 5.2.3. *If (X, d) is a solid cone metric space and a mapping $f : X \mapsto X$ satisfies condition (2.1) for some increasing operator $A \in \mathcal{B}(E)$ with spectral radius less than 1, than the equation $f(x) = x$ is Ulam's stable.*

Proof. Due to Theorem 2.1.3, f has a unique fixed point $z \in X$. Accordingly,

$$\begin{aligned} d(x,z) &\leq d(x,f(x)) + d(f(x),f(z)) \\ &\leq d(x,f(x)) + A(d(x,z)) \\ &\leq (I-A)^{-1}(\varepsilon). \end{aligned}$$

Taking $\psi = (I-A)^{-1}$, since it is nondecreasing linear function, the equation is Ulam's stable. □

Theorem 5.2.4. *If (X,d) is a solid cone metric space and a continuous mapping $f : X \mapsto X$ satisfies condition (2.1) for some increasing operator $A : P \mapsto P$ such that $\lim\limits_{n\to\infty} A^n(e) = \theta$, for any $e \in P$, than the equation $f(x) = x$ is Ulam's stable.*

Proof. In order to estimate $d(x,z)$, where z is a unique fixed point guaranteed by Theorem 2.6.1, choose arbitrary ε and $n \in \mathbb{N}$ such that $A^n(d(x,z)) \preceq (I-A)^{-1}(\varepsilon)$.
Having in mind (2.1),

$$\begin{aligned} d(x,z) &\leq d(x,f^n(x)) + d(f^n(x),f^n(z)) \\ &\leq (I-A^n)(I-A)^{-1}(d(x,f(x))) + A^n(d(x,z)) \\ &\leq 2(I-A)^{-1}(\varepsilon). \end{aligned}$$

Certainly, this implies Ulam's stability. □

Concerning recent results in this area ([17],[28]), there are obvious tendencies to incorporate fixed points results and, in that way, obtain Ulam's stability of functional, operator, differential or integral equations of higher order or with several variables. Many of those results are obtainable from Perov type theorems included in this thesis and without any extensive proof or complicated proof approach despite of what presented in [27].

Theorem 5.2.5. *Let S be a nonempty set, let (X,d) be a complete metric space, $k \in \mathbb{N}$, $f_i : S \mapsto S$, $L_i : S \to \mathbb{R}_+$, $i = \overline{1,k}$, and $\Lambda : {\mathbb{R}_+}^S \mapsto {\mathbb{R}_+}^S$ given by*

$$(\Lambda(\delta))\,(t) = \sum_{i=1}^{k} L_i(t)\delta(f_i(t)), \quad t \in S. \tag{5.5}$$

If operator $\mathcal{T} : X^S \mapsto X^S$ satisfying the inequality

$$\Delta(\mathcal{T}(u),\mathcal{T}(v))(t) \leq \Lambda(\Delta(u,v))(t), \quad u,v \in X^S,\ t \in S,$$

and functions $g \in X^S$ and $\varepsilon \in \mathbb{R}^S$ such that

$$\Delta(\mathcal{T}(g),g)(t) \leq \varepsilon(t), \quad t \in \mathbb{R}^+,$$

and

$$\sum_{n=1}^{\infty} \Lambda^n(\varepsilon(t))\sigma(t) < \infty, \quad t \in S,$$

then for every $t \in S$ the limit $\lim\limits_{n\to\infty}(\mathcal{T}^n(g))(t) = f(t)$ exists and the function $f \in X^S$ defined in this way, is a unique fixed point of $\mathcal{T}$ with

$$\Delta(g,f)(t) \leq \sigma(t), \quad t \in S$$

Proof. Observe that $\Delta : \left(X^S\right)^2 \mapsto r_+^S$ defined with

$$\Delta(u,v)(t) = d(u(t), v(t)), \quad u, v \in X^S,\ t \in S,$$

is just an example of a cone metric and operator Λ has properties of operator A of Theorem 2.6.1 so this result can be obtained as a direct consequence of Theorem 5.2.5. $\square$

Hence, we see that this result points out to Ulam's stability of wide class of functional equations and therefore, could be applied in biology, economy, etc, par example in SIS infection model of constant population or model of economic monopoly with constant output.
Another approach could be looking at Ulam's stability of differential equations with delay like in [99], and in a similar way, we may apply presented results in this and previous sections to assure Ulam's stability.

Preferably, we would add well-posedness problem to the content of this section. Well-posed problem is introduced by J. Hadamard and it unites three important requests: existence of a solution, uniqueness and that solutions's behavior is continuously dependent on initial data, that, in his opinion, every mathematical model should have. Not well-posed problems are known as ill-posed.

Definition 5.2.6. *Let (X,d) be a cone metric space and $f : X \mapsto X$ a mapping. The fixed point problem $f(x) = x$ is well-posed if $Fix(f) = \{z\} \subseteq X$ and if $\lim\limits_{n\to\infty} d(x_n, f(x_n)) = \theta$, $(x_n) \in X$, then $\lim\limits_{n\to\infty} x_n = z$.*

Similarly to what follows, we can obtained well-posedness for most of Perov type results on solid cone metric space presented in Sections 2.1-2.3. To justify this assertion, we prove that fixed point problem studied in Theorem 2.1.3.

Theorem 5.2.7. *Fixed point problem for Perov type contraction on a complete solid cone metric space is well-posed.*

Proof. Take $(x_n) \subseteq X$ such that $\lim\limits_{n\to\infty} d(x_n, f(x_n)) = \theta$, $\varepsilon \gg \theta$ and $n_0 \in \mathbb{N}$ determined that $d(x_n, f(x_n)) \preceq \varepsilon$ when $n \geq n_0$. Furthermore,

$$\begin{aligned} d(x_n, z) &\leq d(x_n, f(x_n)) + d(f(x_n), z) \\ &\leq \varepsilon + A(d(x_n, z)) \\ &= (I - A)^{-1}(\varepsilon). \end{aligned}$$

Choice of ε along with properties of solid cone, justifies conclusion $\lim\limits_{n\to\infty} x_n = z$. $\square$

Chapter 6

Conclusion

The purpose of this chapter is to give a short overview on presented results, their significance, novelty and applications because that determines overall contribution of this dissertation. Since all correlations are already explained, we will not go into the details. As previously mentioned, main results are collected in Chapter 2. In the first part we extend and improve results of Banach ([18]), Perov ([101, 102]), Berinde ([22, 23]) among others. We failed to emphasise that Banach theorem and obvious corollaries on a cone metric spaces are consequence of Perov type theorems extending ([1, 6, 14, 48, 49, 67]). Most important difference is including operator in a contractive condition and giving sufficient conditions in order to obtain fixed point. Perov type quasi-contraction is an extension of Ćirić quasi-contraction and in this section we unify and improve many results from [40, 91, 92, 93, 104]. The main theorem of [57] is a corollary of Theorem 2.2.2. In a same manner, we see that several theorems presented in [38, 54, 56] were the motivation for the progress made in the third section 2.3. Looking at this problem on partially ordered cone metric space, there is an obvious link to [11, 96, 97, 114] and results presented therein.
Third chapter directs our research to common fixed point problem and rounds up a great amount of different results concerning common fixed point problem. Various approaches and techniques led to obtaining many results presented in [2, 3, 5, 10, 13, 16, 21, 64, 78, 79, 82, 85, 128].
Last two chapters, 4 and 5, show the progress made in this dissertation. Despite many examples, most of them regarding integral equations, we should pay attention to Ulam's stability section 5.2 from a different research angle that one presented in articles [17, 27, 28].

The future research could split in two directions, obtaining new theoretical results and discussing their impact on already published results, and looking for new areas of applications in mathematics, but also in other sciences. Some questions regarding cone metric and metric spaces still stay open, so it would be interesting to see the influence of these findings.

Bibliography

[1] M. Aamri and D. El Moutawakil, *Common fixed points under contarctive conditions in symmetric spaces*, Appl. Math. E-Notes **3** (2003), 156–162.

[2] M. Abbas and G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl. **341** (2008), 416–420.

[3] M. Abbas and T. Nazir, *Common fixed point of a power graphic contraction pair in partial metric spaces endowed with a graph*, Fixed Point Theory Appl. **2013** (2013), 2013:20

[4] M. Abbas, V. Rakočević and A. Hussain, *Proximal cyclic contraction of Perov type on regular cone metric spaces*, J. Adv. Math. Stud. Vol. **9** (2016), 65–71.

[5] M. Abbas, V. Rakočević and A. Iqbal, *Coincidence and common fixed points of Perov type generalized Ćirić−contraction mappings*, Mediterr. J. Math. **13** (2016), 3537-3555.

[6] M. Abbas and B. E. Rhoades, *Fixed and periodic point results in cone metric spaces*, Appl. Math. Lett. **22** (2009), 511-515.

[7] S. M. A. Aleomraninejad, Sh. Rezapour and N. Shahzad, *Some fixed point results on a metric space with a graph*, Topol. Appl. **159** (2012), 659–663.

[8] M. A. Alghamdi, S.H. Alnafei, S. Radenović and N. Shahzad, *Fixed point theorems for convex contraction mappings on cone metric spaces*, Math. Comput. Model. **54** (2011), 2020–2026.

[9] M. A. Al-Thagafi and N. Shahzad, *Convergence and existence for best proximiti points*, Nonlinear Anal. **5** (2009), 3665–3671.

[10] I. Altun, B. Damjanović and D. Đorić, *Fixed point and common fixed point theorems on ordered cone metric spaces*, Appl. Math. Lett. **23** (2010), 310–316.

[11] I. Altun and G. Durmaz, *Some fixed point theorems on ordered cone metric spaces*, Rend. Circ. Mat. Palermo **58** (2009), 319-325.

[12] A. Amini-Harandi and M. Fakhar, *Fixed point thoery in cone metric spaces obtained via the scalarization method*, Comput. Math. Appl. **59** (2010), 3529-3534.

[13] M. Arshad, A. Azam and P. Vetro, *Some common fixed point results in cone metric spaces*, Fixed Point Theorey Appl. **2009** (2009), 2009:493965

[14] M. Asadi, S. M. Vazepour, V. Rakočević and B. E. Rhoades, *Fixed point theorems for contractive mapping in cone metric space*, Math. Commun. **16** (2011), 147-155.

[15] M, Asadi, B. E. Rhoades and H. Soleimani, *Some notes on the paper The equivalence of cone metric spaces and metric spaces*, Fixed Point Theory Appl. **2012** (2012), 2012:87

[16] A. Azam and M. Arshad, *Common fixed points of generalized contractive maps in cone metric spaces*, B. Iran. Math. Soc. **35** (2009), 255-264.

[17] R. Badora, J. Brzdek *Fixed points of a mapping and HyersUlam stability*, J. Math.Anal.Appl. **413** (2014), 450-457.

[18] S. Banach, *Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales*, Fund. Math. **3** (1922), 133–181.

[19] C. D. Bari, P. Vetro, *φ-Pairs and Common Fixed Points in Cone Metric Space*, Rend. Circ. Mat. Palermo **57** (2008), 279–285.

[20] S. S. Basha, *Best proximity point theorems generalizing the contraction principle*, Nonlinear Anal. **74** (2011), 5844–5850.

[21] I. Beg, A. Azam and M. Arshad, *Common fixed points for maps on topological vector space valued cone metric spaces*, Int. J. Math. Math. Sci. **2009** (2009), Article ID 560264

[22] V. Berinde, *On the approximation of fixed points of weak contractive mappings*, Carpatian J. Math. **19** (2003), 7–22.

[23] V. Berinde, *Approximating fixed points of weak contractions using Picard iteration*, Nonlinear Anal. Forum **9** (2004), 43–53.

[24] F. Bojor, *Fixed point theorems for Reich type contractions on metric spaces with graph*, Nonlinear Anal. **75** (2012), 3895–3901.

[25] M. Borkowski, D. Bugajewski and M. Zima, *On some fixed-point theorems for generalized contractions and their perturbations*, J. Math. Anal. Appl. **367** (2010), 464–475.

[26] V.W. Bryant, *A remark on a fixed point theorem for iterated mappings,* Amer. Math. Monthly **75** (1968), 399–400.

[27] N. Brillouët-Bellout, J. Brzdęk and K. Ciepeliński, *On ome recent developments in Ulam's stability*, Abstr. Appl. Anal. **2012** (2012), Article ID 716936

[28] J. Brzdęk and K. Ciepeliński, *Remarks on the HyersUlam stability of some systems of functional equations*, Appl. Math. Comput. **219** (2012), 4096-4105.

[29] J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Am. Math. Soc. **215** (1976), 241-251.

[30] S. K. Chatterjea, *Fixed point theorems*, C. R. Acad. Bulg. Sci. **25** (1972), 727-730.

[31] M. Cvetković, *O*peratorial contractions on solid cone metric spaces, Nonlinear Convex. Anal. **17** (2016), 1399-1408.

[32] M. Cvetković and V. Rakočević, *Quasi-contraction of Perov Type*, Appl. Math. Comput. **235** (2014), 712-722.

[33] M. Cvetković and V. Rakočević, *Exstensions of Perov theorem*, Carpathian J. Math. **31** (2015), 181–188.

[34] M. Cvetković and V. Rakočević, *Fisher quasi-contraction of Perov type*, J. Nonlinear Convex. Anal. **16** (2015), 339–352.

[35] M. Cvetković and V. Rakočević, *Common fixed point results for mappings of Perov type*, Math. Nach. **288** (2015), 1873–1890.

[36] M. Cvetković and V. Rakočević, *Fixed point of mappings of Perov type for w-cone distance*, Bul. Cl. Sci. Math. Nat. Sci. Math. **40** (2015), 57-71.

[37] M. Cvetković, V. Rakočević nad B. E. Rhoades, , *Fixed point theorems for contractive mappings of Perov type*, Nonlinear Convex. Anal. **16** (2015), 2117-2127.

[38] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostrav. **1** (1993), 5–11.

[39] Lj. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Sc., **45** (1974), 267–273.

[40] Lj. B. Ćirić, *Generalized contractions and fixed point theorems*, Publ. L'Inst. Math. **26** (1971), 19–26.

[41] Lj. B. Ćirić, *On a family of contractive maps and fixed points* , Publ. L'Inst. Math. **17** (1974), 45–51.

[42] Lj. Ćirić, H. Lakzian and V. Rakočević *Fixed point theorems for w-cone distance contraction mappings in tvs-cone metric spaces*, Fixed Point Theory Appl. **2012** (2012), 2012:3

[43] K. M. Das and K. V. Naik, *Common fixed point theorems for commuting maps on a metric space,* Proc. Amer. Math. Soc. **77** (1979), 369–373.

[44] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, 1985.

[45] E. De Pascale, G. Marino and P. Pietromala, *The use of E-metric spaces in the search for fixed points*, Matematiche **48** (1993), 367–376.

[46] W. Du, *A note on cone metric fixed point theory and its equivalence*, Nonlinear Anal. **72** (2010), 2259–2261.

[47] W. Du and E. Karapınar, *A note on cone b-metric and its related results: generalizations or equivalence?*, Fixed Point Theory Appl. **2013** (2013) 2013:210

[48] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. **37** (1962), 74–79.

[49] M. Edelstein, *An extension of Banach's contraction principle*, Proc. Amer. Math. Soc. **12** (1961), 7–10.

[50] I. Ekelend, *Nonconvex minimization problems*, Bull. Am. Math. Soc., **1** (1979), 443–474.

[51] A. A. Eldered and P. Veeramani, *Existence and convergence of best proximity points*, J. Math. Anal. Appl. **323** (2006), 1001–1006.

[52] Z. Ercan, *On the end of the cone metric spaces*, Topol. Appl. **166** (2014), 10-14.

[53] A. D. Filip, *Perov's fixed point theorem for multivalued mappings in generalized Kasahara spaces*, Stud. Univ. Babes-Bolyai Math. **56** (2011), 19-28.

[54] B. Fisher, *Quasi-contractions on metric spaces*, Proc. Amer. Math. Soc. **75** (1979), 321–325.

[55] B. Fisher, *A fixed point theorem*, Math. Mag. **48** (1975), 223–225.

[56] B. Fisher, *Four mappings with a common fixed point*, J. Univ. Kuwait Sci. **8** (1981), 131-139.

[57] L. Gajić and V. Rakočević, *Quasi-contractions on a non-normal cone metric space*, Funct. Anal. Appl. **46** (2012), 62-65.

[58] Lj. Gajić, D. Ilić and V. Rakočević, *On Ćirić maps with a generalized contractive iterate at a point and Fisher's quasi-contractions in cone metric spaces*, Appl. Math. Comput. **216** (2010), 2240–2247.

[59] D. Guo, Y. L. Cho and J. Zhu, *Partial ordering methods in nonlinear problems*, Nova Science Publishers, 2004.

[60] L. Guran, *Multivalued Perov-type theorems in generalized metric spaces*, Surv. Math. Appl. **4** (2009), 89–97.

[61] R. H. Haghi and Sh. Rezapour, *Fixed points of multifunctions on regular cone metric spaces*, Expo. Math. **28** (2010), 71–77.

[62] R. H. Haghi, V. Rakočević, S. Rezapour and N. Shahzad, *Best proximity results in regular cone metric spaces*, Rend. Circ. Mat. Palermo **60** (2011), 323–327.

[63] R.H. Haghi, Sh. Rezapour and N. Shahzad, *On fixed points of quasi-contraction type multifunctions*, Appl. Math. Lett. **25** (2012), 843–846.

[64] G.E. Hardy and T. D. Rogers, *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull. **16** (1973), 201–206.

[65] Y. M. He and H. J. Wang, *Fractal image decoding based on extended fixed-point theorem*, Proc. Int. Conf. Mach. Learn. Cybern. **2006** (2006), 4160-4163.

[66] T. L. Hicks and B. E. Rhoades, *A Banach type fixed-point theorem*, Math. Jpon. **24** (1979), 327– 330.

[67] L.G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **332** (2007), 1468–1476.

[68] D. H. Hyers,*On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222-224.

[69] D. Ilić, M. Cvetković, Lj. Gajić and V. Rakočević, *F*ixed points of sequence of iri generalized contractions of Perov type, Mediterr. J. Math. **13** (2016), 3921-3937.

[70] D. Ilić and V. Rakočević, *Common fixed points for maps on cone metric space*, J. Math. Anal. Appl. **341** (2008), 876–882.

[71] D. Ilić and V. Rakočević, *Quasi-contraction on cone metric space*, Appl. Math. Lett. **22** (2009), 728–731.

[72] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc. **136** (2007), 1359-1373.

[73] S. Janković, Z. Kadelburg and S. Radenović, *On cone metric spaces: A survey*, Nonlinear Anal. **74** (2011), 2591-2601.

[74] S. Janković, Z. Kadelburg, S. Radenović and B. E. Rhoades, *Assad-Kirk-Type fixed point theorems for a pair of non self mappings on cone metric spaces*, Fixed Point Theory Appl. **2009** (2009), 2009:761086

[75] G. S. Jeong and B. E. Rhoades, *Some remarks for improving fixed point theorems for more than two maps*, Indian J. Pure Appl.Math. **28** (1997), 1177-1196.

[76] S. Jiang and Z. Li, *Extensions of Banach contraction principle to partial cone metric spaces over a non-normal solid cone*, Fixed Point Theory Appl. **2013** (2013), 2013:250

[77] G. Jungck, *Commuting maps and fixed points*, Amer. Math. Monthly **83** (1976), 261-263.

[78] G. Jungck, *Common fixed points for noncontinuous nonself maps on non-metric spaces*, Far East J. Math. Sci. **4** (1996), 199-215.

[79] G. Jungck, S. Radenović, S. Radojević and V. Rakočević, *Common fixed point theorems for weakly compatible pairs on cone metric spaces*, Fixed Point Theory Appl. **2009** (2009), 2009: 643840

[80] O. Kada, T. Suzuki and W. Takahashi, *Nonconvex minimization theorems and fixed point theorems in complete metric spaces*, Math. Jpon. **44** (1996), 381–591.

[81] Z. Kadelburg, S. Radenović and V. Rakočević, *Remarks on "Quasi-contraction on a cone metric space"*, Appl. Math. Lett. **22** (2009), 1674–1679.

[82] Z. Kadelburg, S. Radenović and B. Rosić, *Strict contractive conditions and common fixed point theorems in cone metric spaces*, Fixed Point Theory Appl. **2009** (2009), 2009:173838

[83] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. **10** (1968), 71–76.

[84] E. Karapinar, *Some nonunique fixed point theorems of Ćirić type on cone metric spaces*, Abstr. Appl. Anal. **2010** (2010), Article ID 123094

[85] E.Karapinar, *Couple fixed point theorems for nonlinear contractions in cone metric spaces*, Comput. Math. Appl. **59(12)** (2010), 3656–3668.

[86] E.Karapinar, *Fixed point theorems in cone Banach spaces*, Fixed Point Theory Appl. **2009** (2009), 2009:609281

[87] W. A. Kirk, P. S. Srinivasan and P. Veeramani, *Fixed points for mappings satisfying cyclical contractive conditions*, Fixed Point Theory **4** (2003), 79–89.

[88] H. Kunze, D. La Torre, F. Mendivil and E. R. Vrscay, *Generalized fractal transforms and self-similar objects in cone metric spaces*, Comput. Math. Appl. **64** (2012), 1761–1769.

[89] Đ. Kurepa, *Tableaux ramifies d'ensembles, Espaces pseudo-distancis*, C. R. Math. Acad. Sci. Paris **198** (1934), 1563–1565.

[90] K. Leibovic, *The principle of contraction mapping in nonlinear and adaptive control systems*, IEEE Trans. Autom. Control **9** (1964), 393-398.

[91] H. Liu and S. Xu, *Fixed point theorems of quasicontractions on cone metric spaces with banach algebras*, Abstr. Appl. Anal. **2013** (2013), Article ID 187348.

[92] H. Liu and S. Xu, *Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings*, Fixed Point Theory Appl. **2013** (2013), 2013:320.

[93] B. Lou, *Fixed points for operators in a space of continuous functions and applications*, Proc. Amer. Math. Soc. **127** (1999), 2259-2264.

[94] G. A. Medrano-Cerda, *A fixed point formulation to parameter estimation problems*, Proc. IEEE Conf. Decis. Control **26** (1987), 1914–1915.

[95] S.B. Nadler Jr., *Multi-valued contraction mappings*, Pacific J. Math. **30** (1969), 475–487.

[96] J.J. Nieto and R. Rodriguez-Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Oder, **2**2 (2005), 223–239.

[97] J.J. Nieto and R. Rodriguez-Lopez, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta Math. Sin., **2**2 (2005), 223–239.

[98] J. O. Olaleru, *Some generalizations of fixed point theorems in cone metric spaces*, Fixed Point Theory Appl. **2008** (2008), 2009:657914

[99] D. Otrocol and V. Ilea, *lam stability for a delay differential equation*, Open Math. **11** (2013), 1296-1303.

[100] S. Park, *A unified approach to fixed points of contractive maps*, J. Korean Math. Soc. **16** (1980), 95–106.

[101] A.I. Perov, *On Cauchy problem for a system of ordinary diferential equations*, (in Russian), Priblizhen. Metody Reshen. Difer. Uravn. **2** (1964), 115-134.

[102] A.I. Perov and A.V. Kibenko, *On a certain general method for investigation of boundary value problems* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **30** (1966), 249-264.

[103] E. de Pascale and L. de Pascale, *Fixed points for some non-obviously contractive operators*, Proc. Amer. Math. Soc. **130** (2002), 3249-3254.

[104] H.K. Pathak and N. Shahzad, *Fixed point results for generalized quasicontraction mappings in abstract metric spaces*, Nonlinear Anal. **71** (2009), 6068–6076.

[105] M. A. Petric, *Some remarks concerning C̀irič-Reich-Rus operators*, Creat. Math. Inform. **18** (2009), 188–193.

[106] A. Petrusel, *Vector-valued metrics in fixed point theory*, Babes-Bolyai University, Cluj-Napoca, Faculty of Mathematics and Computer Science, 2012.

[107] A. Petrusel and I.A. Rus, *Fixed point theorems in ordered L-spaces*, Proc. A.M.S. **134** (2005), 411-418.

[108] R. Precup, *The role of matrices that are convergent to zero in the study of semilinear operator systems*, Math. Comput. Modelling **49** (2009), 703–708.

[109] R. Precup and A. Viorel, *Existance results for systems of nonlinear evolution equations*, Int. J. Pure Appl. Math. **2** (2008), 199-206.

[110] P. D. Proinov, *A unified theory of cone metric spaces and its applications to the fixed point theory*, Fixed Point Theory Appl. **2013** (2013), 2013:103

[111] S. Radenović and B. E. Rhoades, *Fixed point theorem for two non-self mappings in cone metric spaces*, Comput. Math. Appl. **57** (2009), 1701–1707.

[112] P. Raja and S. M. Vaezpour, *Some extensions of Banach's contraction principle in complete cone metric spaces*, Fixed Point Theory Appl. **2008** (2008), 2008: 768294

[113] W. Rakowski, *The fixed point theorem in computing magnetostatic fields in a nonlinear medium*, IEEE Trans. Magn. **18** (1982), 391-392.

[114] A. C. M. Ran and M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc. **132** (2004), 1435-1443.

[115] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297300.

[116] S. Reich, *Some remarks concerning contraction mappings*, Canad. Math. Bull. , **1**4 (1971), 121–124.

[117] Sh. Rezapour and R. Hamlbarani, *Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings"*, J. Math. Anal. Appl. **345** (2008), 719–724.

[118] Sh. Rezapour, R.H. Haghi and N. Shahzad, *Some notes on fixed points of quasi-contraction maps*, Appl. Math. Lett. **23** (2010), 498–502.

[119] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. **226** (1977), 257–290.

[120] I. A. Rus, *Chapter 23: Ulam stability of the operatorial equations, Functional equations in mathematical analysis*, Springer Optimization and Its Applications, Springer, New York, **52** (2012), 3358.

[121] I. A. Rus and M-A. Şerban, *Some existance results for systems of operatorial equations*, Bull. Math. Soc. Sci. Math. Roumanie **57** (2014), 101-108.

[122] J. Schröder, *Das Iterationsverfahren bei allgemeinerem Abstandsbegriff*, Math. Z. **66** (1956), 111–116.

[123] J. Schröoder, *Nichtlineare Majoranten beim Verfahren der schrittweisen Näherung*, Arch. Math. **7** (1956), 471–484.

[124] M. Stanić, A. Cvetković, S. Simić and S. Dimitrijević, *Common fixed point under contractive condition of Ćirić type on cone metric type spaces*, Fixed Point Theory Appl. **2012** (2012), 2012:35

[125] T. Suzuki, *Several fixed point theorems in complete metric spaces*, Yokohama Math. J. **44** (1997), 61-72.

[126] A. Szilárd, *A note on Perov's fixed point theorem*, Fixed Point Theory **4** (2003), 105–108.

[127] W. Takahashi, *A convexity in metric space and nonexpansive mappings*, I. Kodai Math. Sem. Pep. **22** (1970), 142–149.

[128] P. Vetro, *Common fixed points in cone metric spaces*, Rend. Circ. Mat. Palermo **56** (2007), 464-468.

[129] P. P. Zabreiko, *K-metric and K-normed spaces: survey,* Collect. Math **48** (1997), 825-859.

[130] M. Zima, *A certain fixed point theorem and its applications to integral-functional equations*, Bull. Austral. Math. Soc. **46** (1992), 179–186.

Biography

Marija Cvetković was born on October 4th, 1989. in Surdulica, Serbia. She finished her primary and secondary education, elementary school "Branko Radičević" and grammar school "Jovan Skerlić", in Vladičin Han, as valedictorian in 2004. an 2007.. During that period she won several prizes on national and international level problem-solving competitions in Mathematics and Physics. In 2007. she started studies of Mathematics at the Department of Mathematics, the Faculty of Sciences and Mathematics, University of Niš, where she earned a bachelors degree in 2010., and in continuance, a masters degree in 2012. with average 10,00 and Silver medallion of University of Niš in 2013. as the best graduate student from the Faculty of Sciences and Mathematics. She graduated defending a master thesis "Matrix inequalities" and enrolled PhD studies at the same faculty. During the studies, she participated on many international courses organized by DAAD and BEST. From 2007. she is an assistant at lecturer at mathematics programme in Petnica Science Center.

Starting from 2012. she is employed at the Faculty of Sciences and Mathematics, one year as a teaching associate, after that as a teaching assistant. She teaches Linear algebra, Introduction to Topology on undergraduate studies, and Mathematical Logic (2012-2016.), Algebraic Topology, Fixed Point Theory and Application, Approximation Theory and Quadrature Formulas, Set Theory on graduate level. She is participating as a researcher on a project: Problems in nonlinear analysis, operator theory, topology and applications, No.174205 supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia. Since 2013. she teaches children talented for Mathematics in VII and VIII grade in a special mathematical class within "Svetozar Marković" Grammar School, Niš.

As the author and coauthor, she published eleven research papers in international mathematical journals and one in journal of Serbian Academy of Science and Arts. She took part on several conferences, congresses and workshops, and communicated presented results on four occasions: at FPTA 2015. in Istanbul, EMC in 2016., Berlin, and in the same year CUTS in Cluj-Napoca and XIX Geometrical Seminar, Zlatibor. She was a guest lecturer at the Univerisity Babeş-Bolyai, Cluj-Napocca, Romania and University of New South Wales, Sydney, Australia. She is a reviewer in nine international journals.