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PRODUCTS OF INTEGRAL-TYPE AND COMPOSITION OPERATORS FROM GENERALLY WEIGHTED BLOCH SPACE TO $F(p, q, s)$ SPACE

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Abstract

Products of integral-type and composition operators has been recently introduced by Li and Stević and studied in a series of their papers. In this note we study the boundedness and compactness of these products from generally weighted Bloch space to $F(p, q, s)$ space, where $0 < p, s < \infty$, $q > -2$, $\alpha > 0$.

1 Introduction and preliminaries

Let D be the unit disc on the complex plane and φ a holomorphic self-map of D. Denote by $H(D)$ the space of all holomorphic functions on D and $dA(z) = \frac{1}{\pi}dxdy = \frac{1}{\pi}rdrd\theta$ the normalized Lebesgue area measure.

The space of analytic functions on D such that

$$\|f\|_{B_{\log}} = |f(0)| + \sup_{z\in D} |f'(z)|(1 - |z|^2) \log \frac{2}{1 - |z|^2} < \infty$$

is called the logarithmic Bloch space $B_{\log} = B_{\log}(D)$.

$B_{\log}$ space was probably first appeared in the study of the boundedness of the Hankel operators on the Bergman space

$$A^1 = \left\{ f \in H(D) : \int_D |f(z)|dA(z) < \infty \right\}$$

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and the Hardy space H^1, respectively. For more details see [1], [2], [18] and [27]. In [27], Yoneda, among others, studied the composition operators on $B_{\log}$ space. The space $B_{\log}$ was recently extended in [3], where was introduced, so called, the iterated logarithmic space $B_{\log_k}$ for the case of the unit ball in C^n.

In [4] and [5] was introduced the space $B^{\alpha}_{\log}$, so called, the generally weighted Bloch space as the space of all analytic functions on D such that

$$\|f\|_{B^{\alpha}_{\log}} = |f(0)| + \sup_{z\in D} |f'(z)|(1-|z|^2)^{\alpha} \log \frac{2}{1-|z|^2} < \infty.$$

The space of analytic functions on D such that

$$\|f\|^p_{F(p,q,s)} = \sup_{a\in D} \int_D |f'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) < \infty$$

is called $F(p,q,s)$ space, where $0 < p, s < \infty$, $-2 < q < \infty$, $a \in D$, $g(z,a) = \log \frac{1}{|\phi_a(z)|}$ is the Green's function and $\phi_a(z) = \frac{a-z}{1-\bar{a}z}$. The $F(p,q,s)$ space is a Banach space with the norm $|f(0)| + \|f\|_{F(p,q,s)}$ ([28]). Some results related to the space can be also found in [9] and [17].

Let $\phi \in H(D)$, for $f \in H(D)$, the integral-type operators I_ϕ and J_ϕ are respectively defined by

$$I_\phi f(z) = \int_0^z f'(\zeta)\phi(\zeta)d\zeta, \qquad J_\phi f(z) = \int_0^z f(\zeta)\phi'(\zeta)d\zeta, \quad z \in D.$$

The importance of the operators I_ϕ and J_ϕ comes from the fact that

$$I_\phi f(z) + J_\phi f(z) = M_\phi f(z) - f(0)\phi(0), \quad z \in D,$$

where M_ϕ is the multiplication operator

$$(M_\phi f)(z) = \phi(z)f(z), \quad f \in H(D), \quad z \in D.$$

In [16], Pommerenke introduced the operator J_ϕ and showed that $J_\phi : H^2 \to H^2$ is bounded if and only if $\phi \in BMOA$. For some information on the operators I_ϕ and J_ϕ and their n-dimensional extensions, see, for example [6, 7, 8, 9, 13, 14, 16, 18, 19, 20, 21, 22, 24, 25, 26] as well as the related references therein.

Let $g \in H(D)$ and φ be a holomorphic self-map of D. Products of integral and composition operators on $H(D)$ were introduced by S. Li and S. Stević (see [6], [11], [12], [15], as well as closely related operators in [10] and [23]) as follows

$$C_\varphi J_\phi f(z) = \int_0^{\varphi(z)} f(\zeta)\phi'(\zeta)d\zeta, \quad J_\phi C_\varphi f(z) = \int_0^z f(\varphi(\zeta))\phi'(\zeta)d\zeta;$$

$$C_\varphi I_\phi(f)(z) = \int_0^{\varphi(z)} f'(\zeta)\phi(\zeta)d\zeta, \quad I_\phi C_\varphi(f)(z) = \int_0^{z} (f\circ\varphi)'(\zeta)\phi(\zeta)d\zeta.$$

Note that when $\varphi(z) = z$ and g is g' these operators reduce to the integral operator introduced in [16]. For the case of the unit ball the operator $C_\varphi J_g$ has been recently extended by Stević in [25] (see also [24], [26]).

In this article, we characterize the boundedness and compactness of the products of integral-type and composition operators from generally weighted Bloch space to $F(p, q, s)$ space on the unit disk.

Throughout the remainder of this paper C will denote a positive constant independent of functions, the exact value of which may vary from one appearance to the next.

2 Auxiliary results

In this part, we introduce some lemmas which will be needed in our proof of the theorems.

First, the following Lemma 2.1 can be found in [4].

Lemma 2.1 Let $f \in B^\alpha_{\log}$ and $z \in D$, then
(a) For $0 < \alpha < 1$, $|f(z)| \leq (1 + \frac{1}{(1-\alpha)\log 2})\|f\|_{B^\alpha_{\log}}$;
(b) For $\alpha = 1$, $|f(z)| \leq \frac{\log\frac{4}{1-|z|^2}}{\log 2}\|f\|_{B^\alpha_{\log}}$;
(c) For $\alpha > 1$, $|f(z)| \leq (1 + \frac{2^{\alpha-1}}{(\alpha-1)\log 2})\frac{1}{(1-|z|^2)^{\alpha-1}}\|f\|_{B^\alpha_{\log}}$.

Lemma 2.2 ([5]) There exist $f,\ g \in B^\alpha_{\log}$ such that

$$|f'(z)| + |g'(z)| \geq \frac{C}{(1-|z|)^\alpha \log\frac{2}{1-|z|}}.$$

The following characterization of compactness can be proved in a standard way (see, e.g., the proofs of the corresponding lemmas in [8, 20, 22]). We will give a proof of this result for a benefit of the reader.

Lemma 2.3 Assume that φ is a holomorphic self-map of D and $\alpha > 0$. Then $C_\varphi I_\phi$ (or $I_\phi C_\varphi$) : $B^\alpha_{\log} \to F(p, q, s)$ is compact if and only if for any bounded sequence $(f_j)_{j\in N}$ in $B^\alpha_{\log}$, when $f_j \to 0$ uniformly on compact subsets of D, then $\|C_\varphi I_\phi f_j\|_{F(p,q,s)} \to 0$ as $j \to \infty$.

Proof Assume that $C_\varphi I_\phi : B^\alpha_{\log} \to F(p, q, s)$ is compact and that $(f_j)_{j\in N}$ is a bounded sequence in $B^\alpha_{\log}$ such that $f_j \to 0$ uniformly on compact subsets of D as $j \to \infty$. By the compactness of $C_\varphi I_\phi$, we have that the sequence $(C_\varphi I_\phi f_j)_{j\in N}$ has a subsequence $(C_\varphi I_\phi f_{j_m})_{m\in N}$ which converges to an $f \in F(p,q,s)$. By Lemma 2.1 and obvious inequality $|f(0)| \le \|f\|_{B^\alpha_{\log}}$, it follows that for any compact $K \subset D$, there is a $C_K \ge 0$ such that

$$|C_\varphi I_\phi f_{j_m}(z) - f(z)| \le C_K \|C_\varphi I_\phi f_{j_m} - f\|_{F(p,q,s)}, \text{ for every } z \in K.$$

This implies that $C_\varphi I_\phi f_{j_m}(z) - f(z) \to 0$ uniformly on compact subsets of D as $m \to \infty$. Since $f_{j_m} \to 0$ on compact subsets of D, by the definition of the operator $C_\varphi I_\phi$, it is easy to see that for each $z \in D$, $\lim_{m\to\infty} C_\varphi I_\phi f_{j_m}(z) = 0$. Hence $f \equiv 0$. Since $(f_j)_{j\in N}$ is an arbitrary sequence, we obtain that $C_\varphi I_\phi f_j \to 0$ in $F(p,q,s)$ as $j \to \infty$.

Conversely, let $\{h_j\}$ be any sequence in the ball $K_M = B_{B^\alpha_{\log}}(0,M)$ (centered at zero with the radius M) of the space $B^\alpha_{\log}$. Since $\sup_{j\in N} \|h_j\|_{B^\alpha_{\log}} \le M < \infty$, by Lemma 2.1, $\{h_j\}_{j\in N}$ is uniformly bounded on compact subsets of D and hence normal by Montel's theorem. Hence we may extract a subsequence $\{h_{j_m}\}_{m\in N}$ which converges uniformly on compact subsets of D to some $h \in H(D)$, moreover $h \in B^\alpha_{\log}$ and $\|h\|_{B^\alpha_{\log}} \le M$. It follows that $(h_{j_m} - h)_{m\in N}$ is such that $\|h_{j_m} - h\|_{B^\alpha_{\log}} \le 2M < \infty$ and converges to zero on compact subsets of D as $m \to \infty$. By the hypothesis, we have that $C_\varphi I_\phi h_{j_m} \to C_\varphi I_\phi h$ in $F(p,q,s)$. Thus the set $C_\varphi I_\phi(K)$ is relatively compact. Hence $C_\varphi I_\phi : B^\alpha_{\log} \to F(p,q,s)$ is compact. The proof for the operator $I_\phi C_\varphi : B^\alpha_{\log} \to F(p,q,s)$ is similar. Hence we omit it.

3 The boundedness and compactness of $C_\varphi I_\phi : B^\alpha_{\log} \to F(p,q,s)$

In this section, we will investigate the boundedness and compactness of the products of integral-type and composition operators $C_\varphi I_\phi$ ($I_\phi C_\varphi$) from generally weighted Bloch space to $F(p,q,s)$ space.

Theorem 3.1 Let $0 < p, s < \infty$, $-2 < q < \infty$, $\alpha > 0$. If φ is a holomorphic self-map of the unit disc, then $C_\varphi I_\phi : B^\alpha_{\log} \to F(p,q,s)$ is bounded if and only if

$$\sup_{a\in D} \int_D \frac{|\phi(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^q g^s(z,a)}{(1-|\varphi(z)|^2)^{p\alpha} (\log \frac{2}{1-|\varphi(z)|^2})^p} dA(z) < \infty. \tag{3.1}$$

Proof For any $f \in B^{\alpha}_{\log}$,

$$\begin{aligned}
& \sup_{a\in D}\int_D |(C_\varphi I_\phi f)'(z)|^p(1-|z|^2)^q g^s(z,a)dA(z) \\
= & \sup_{a\in D}\int_D |f'(\varphi(z))|^p(1-|\varphi(z)|^2)^{p\alpha}\times \\
& \times(\log\frac{2}{1-|\varphi(z)|^2})^p\frac{|\phi(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^q g^s(z,a)}{(1-|\varphi(z)|^2)^{p\alpha}(\log\frac{2}{1-|\varphi(z)|^2})^p}dA(z) \qquad (3.2)\\
\le & \|f\|^p_{B^{\alpha}_{\log}}\cdot\sup_{a\in D}\int_D \frac{|\phi(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^q g^s(z,a)}{(1-|\varphi(z)|^2)^{p\alpha}(\log\frac{2}{1-|\varphi(z)|^2})^p}dA(z).
\end{aligned}$$

By (3.1), then $C_\varphi I_\phi f \in F(p,q,s)$, thus $C_\varphi I_\phi : B^{\alpha}_{\log} \to F(p,q,s)$ is bounded.

Conversely, we assume that $C_\varphi I_\phi : B^{\alpha}_{\log} \to F(p,q,s)$ is bounded, for f, $h \in B^{\alpha}_{\log}$, $C_\varphi I_\phi f$, $C_\varphi I_\phi h \in F(p,q,s)$.

By Lemma 2.2, there exist $f, h \in B^{\alpha}_{\log}$ such that

$$|f'(z)| + |h'(z)| \ge \frac{C}{(1-|z|)^{\alpha}\log\frac{2}{1-|z|}}.$$

Then

$$\begin{aligned}
\infty &> \sup_{a\in D}\int_D 2^p\{|(C_\varphi I_\phi f)'(z)|^p + |(C_\varphi I_\phi h)'(z)|^p\}(1-|z|^2)^q g^s(z,a)dA(z) \\
&\ge \sup_{a\in D}\int_D \{|(C_\varphi I_\phi f)'(z)| + |(C_\varphi I_\phi h)'(z)|\}^p(1-|z|^2)^q g^s(z,a)dA(z) \\
&= \sup_{a\in D}\int_D \{|f'(\varphi(z))| + |h'(\varphi(z))|\}^p|\phi(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^q g^s(z,a)dA(z) \\
&\ge C\sup_{a\in D}\int_D \frac{|\phi(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^q g^s(z,a)}{(1-|\varphi(z)|^2)^{p\alpha}(\log\frac{2}{1-|\varphi(z)|^2})^p}dA(z).
\end{aligned}$$

Then (3.1) holds.

Theorem 3.2 Let $0 < p, s < \infty$, $-2 < q < \infty$, $\alpha > 0$. If φ is a holomorphic self-map of the unit disc, then $C_\varphi I_\phi : B^{\alpha}_{\log} \to F(p,q,s)$ is compact if and only if

$$\sup_{a\in D}\int_D |\phi(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^q g^s(z,a)dA(z) < \infty. \qquad (3.3)$$

and

$$\lim_{r\to 1}\sup_{a\in D}\int_{\{|\varphi(z)|>r\}} \frac{|\phi(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^q g^s(z,a)}{(1-|\varphi(z)|^2)^{p\alpha}(\log\frac{2}{1-|\varphi(z)|^2})^p}dA(z) = 0. \qquad (3.4)$$

Proof First we assume that $C_\varphi I_\phi : B^\alpha_{\log} \to F(p,q,s)$ is compact. Let $f_0(z) \equiv z$, then $C_\varphi I_\phi(f_0) \in F(p,q,s)$, then (3.2) holds by the definition of $C_\varphi I_\phi$.

Since $\|\frac{z^n}{n}\|_{B^\alpha_{\log}} \le C$ and $\frac{z^n}{n} \to 0$ as $n \to \infty$, locally uniformly on the unit disc, then by the compactness of $C_\varphi I_\phi$, it follows that $\|C_\varphi I_\phi(z^n)\|_{F(p,q,s)} \to 0$, as $n \to \infty$.

This means that for each $r \in (0,1)$, and for every $\varepsilon > 0$ there is $n_0 \in N$ such that

$$r^{p(n_0-1)} \sup_{a\in D} \int_{\{|\varphi(z)|>r\}} |\phi(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) < \varepsilon.$$

If we choose $r \ge 2^{-\frac{1}{p(n_0-1)}}$, then

$$\sup_{a\in D} \int_{\{|\varphi(z)|>r\}} |\phi(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) < 2\varepsilon. \tag{3.5}$$

Let now f be a function such that $\|f\|_{B^\alpha_{\log}} \le 1$. We consider the functions $f_t(z) = f(tz)$, $t \in (0,1)$. By the compactness of $C_\varphi I_\phi$ we get that for all $\varepsilon > 0$ there exists $t_0 \in (0,1)$ such that for all $t > t_0$,

$$\sup_{a\in D} \int_{\{|\varphi(z)|>r\}} |f'(\varphi(z)) - f_t'(\varphi(z))|^p |\phi(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) < \varepsilon.$$

Then we fix t, by (3.4)

$$\begin{aligned}
&\sup_{a\in D} \int_{\{|\varphi(z)|>r\}} |f'(\varphi(z))|^p |\phi(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) \\
\le\ & 2^p \sup_{a\in D} \int_{\{|\varphi(z)|>r\}} |f'(\varphi(z)) - f_t'(\varphi(z))|^p |\phi(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) \\
&+2^p \sup_{a\in D} \int_{\{|\varphi(z)|>r\}} |f_t'(\varphi(z))|^p |\phi(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) \\
\le\ & 2^p\varepsilon + 2^p \|f_t'\|^p_{H^\infty} \sup_{a\in D} \int_{\{|\varphi(z)|>r\}} |\phi(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) \\
\le\ & 2^{p+1}\varepsilon(1 + \|f_t\|^p_{H^\infty}).
\end{aligned} \tag{3.6}$$

By (3.4) and (3.5), for each $\|f\|_{B^\alpha_{\log}} \le 1$ and $\varepsilon > 0$, there exists δ depending on f, ε, such that for $r \in [\delta, 1)$,

$$\sup_{a\in D} \int_{\{|\varphi(z)|>r\}} |f'(\varphi(z))|^p |\phi(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) < \varepsilon. \tag{3.7}$$

Since $C_\varphi I_\phi$ is compact, it maps the unit ball of $B^\alpha_{\log}$ to a relative compact subset of $F(p,q,s)$. Thus for each $\varepsilon > 0$ there exists a finite collection of functions $f_1, f_2, \ldots, f_N$ in the unit ball of $B^\alpha_{\log}$, such that for each $\|f\|_{B^\alpha_{\log}} \le 1$ there is a $k \in \{1, 2, \ldots, N\}$ with

$$\sup_{a\in D}\int_{\{|\varphi(z)|>r\}} |f'(\varphi(z)) - f'_k(\varphi(z))|^p|\phi(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^q g^s(z,a)dA(z) < \varepsilon.$$

By (3.6), we get that for $\delta = \max_{1\le k\le N}\delta(f_k,\varepsilon)$ and $r\in[\delta,1)$,

$$\sup_{a\in D}\int_{\{|\varphi(z)|>r\}} |f'_k(\varphi(z))|^p|\phi(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^q g^s(z,a)dA(z) < \varepsilon.$$

Thus we get that

$$\sup_{\|f\|_{B^\alpha_{\log}}\le 1}\sup_{a\in D}\int_{\{|\varphi(z)|>r\}} |f'(\varphi(z))|^p|\phi(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^q g^s(z,a)dA(z) < 2\varepsilon.$$

From this and by Lemma 2.2, there are functions $h_1, h_2 \in B^\alpha_{\log}$ such that

$$\begin{aligned} & \sup_{a\in D}\int_{\{|\varphi(z)|>r\}} \frac{|\phi(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^q g^s(z,a)}{(1-|\varphi(z)|^2)^{p\alpha}(\log\frac{2}{1-|\varphi(z)|^2})^p}dA(z) \\ \le\ & \int_{\{|\varphi(z)|>r\}} |h'_1(\varphi(z))|^p|\phi(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^q g^s(z,a)dA(z) \\ & +\int_{\{|\varphi(z)|>r\}} |h'_2(\varphi(z))|^p|\phi(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^q g^s(z,a)dA(z) \\ <\ & C\varepsilon \end{aligned}$$

Hence (3.3) holds.

Conversely, we assume that (3.2) and (3.3) holds. Let $\{f_n\}_{n\in N}$ be a sequence of functions in the unit ball of $B^\alpha_{\log}$, such that $f_n \to 0$ as $n\to\infty$, uniformly on the compact subsets of the unit disc.

Let $r\in(0,1)$, then

$$\begin{aligned} & \|C_\varphi I_\phi f_n\|^p_{F(p,q,s)} \le 2^p|C_\varphi I_\phi f_n(0)|^p \\ & +2^p\sup_{a\in D}\int_{\{|\varphi(z)|\le r\}} |f'_n(\varphi(z))|^p|\phi(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^q g^s(z,a)dA(z) \\ & +2^p\sup_{a\in D}\int_{\{|\varphi(z)|>r\}} |f'_n(\varphi(z))|^p|\phi(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^q g^s(z,a)dA(z) \\ =\ & 2^pI_1 + 2^pI_2 + 2^pI_3. \end{aligned}$$

Since $f_n \to 0$ as $n \to \infty$, uniformly on compacts of D, then $I_1 \to 0$ as $n \to \infty$ and for each $\varepsilon > 0$ there is $n_0 \in N$ such that for each $n > n_0$,

$$I_2 \leq \varepsilon \sup_{a \in D} \int_{\{|\varphi(z)| \leq r\}} |\phi(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) < C\varepsilon,$$

$$I_3 \leq \|f_n\|_{B^\alpha_{\log}} \sup_{a \in D} \int_{\{|\varphi(z)| > r\}} \frac{|\phi(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^q g^s(z,a)}{(1-|\varphi(z)|^2)^{p\alpha} (\log \frac{2}{1-|\varphi(z)|^2})^p} dA(z).$$

By (3.3), then for every $n > n_0$ and every $\varepsilon > 0$, there exists r_0 such that for every $r > r_0$, $I_3 < \varepsilon$. Thus $\|C_\varphi I_\phi f_n\|_{F(p,q,s)} \to 0$ as $n \to \infty$. By Lemma 2.3 the compactness of the operator $C_\varphi I_\phi : B^\alpha_{\log} \to F(p,q,s)$ follows.

Similarly, we can obtain the following results on the operator $I_\phi C_\varphi : B^\alpha_{\log} \to F(p,q,s)$. We omit their proofs.

Theorem 3.3 Let $0 < p, s < \infty$, $-2 < q < \infty$, $\alpha > 0$. If φ is a holomorphic self-map of the unit disc, then $I_\phi C_\varphi : B^\alpha_{\log} \to F(p,q,s)$ is bounded if and only if

$$\sup_{a \in D} \int_D \frac{|\phi(z)|^p |\varphi'(z)|^p (1-|z|^2)^q g^s(z,a)}{(1-|\varphi(z)|^2)^{p\alpha} (\log \frac{2}{1-|\varphi(z)|^2})^p} dA(z) < \infty. \tag{3.8}$$

Theorem 3.4 Let $0 < p, s < \infty$, $-2 < q < \infty$, $\alpha > 0$. If φ is a holomorphic self-map of the unit disc, then $I_\phi C_\varphi : B^\alpha_{\log} \to F(p,q,s)$ is compact if and only if

$$\sup_{a \in D} \int_D |\phi(z)|^p |\varphi'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) < \infty. \tag{3.9}$$

and

$$\lim_{r \to 1} \sup_{a \in D} \int_{\{|\varphi(z)| > r\}} \frac{|\phi(z)|^p |\varphi'(z)|^p (1-|z|^2)^q g^s(z,a)}{(1-|\varphi(z)|^2)^{p\alpha} (\log \frac{2}{1-|\varphi(z)|^2})^p} dA(z) = 0. \tag{3.10}$$

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