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SOME CONDITIONS UNDER WHICH SUBSEQUENTIAL CONVERGENCE FOLLOWS FROM (A, m) SUMMABILITY

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Abstract

In this paper, we obtain some conditions on a sequence under which subsequential convergence [Math. Morav. 5, 19-56 (2001; Zbl 1047.40005)] of the sequence follows from its (A, m) summability.

1 Introduction

Dik [1] introduced the concept of subsequential convergence, a generalization of ordinary convergence, and proved several Tauberian type theorems to obtain subsequential convergence of a sequence out of its Abel summability. Later a number of authors including Çanak et al.[3], Çanak and Totur [5] have investigated conditions under which Abel summability implies subsequential convergence of a sequence. Purpose of this work is to obtain some conditions on a sequence under which subsequential convergence of the sequence follows from its (A, m) summability.

Now we give some definitions and notations.

Let $u = (u_n)$ be a sequence of real numbers. Define

$$\sigma_n^{(m)}(u) = \begin{cases} \frac{1}{n+1} \sum_{j=0}^{n} \sigma_j^{(m-1)}(u), & m \geq 1 \\ u_n, & m = 0 \end{cases}$$

for each integer $m \geq 0$ and for all nonnegative integers n. A sequence (u_n) is said to be Cesàro summable to s if $\sigma_n^{(1)}(u) \to s$ as $n \to \infty$. A sequence (u_n) is said to be (A, m) summable to s if $(1 - x) \sum_{n=0}^{\infty} \sigma_n^{(m)}(u) x^n \to s$ as $x \to 1^-$ and we write $u_n \to s$ (A, m). If $m = 0$, then (A, m) summability reduces to Abel summability. It is clear that $u_n \to s$ $(A, 0)$ implies $u_n \to s$ (A, m) for each integer $m \geq 1$.

Abel's theorem states that if $u_n \to s$ as $n \to \infty$, then $u_n \to s\,(A, 0)$. The converse is not true, for the sequence (u_n) defined by $u_n = \sum_{k=0}^{n} (-1)^k$ is not

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convergent but it is Abel summable to $1/2$. However, the converse of Abel's theorem is valid under some conditions called Tauberian conditions. Any theorem which states that convergence of sequence follows from a summability method and some Tauberian condition is said to be Tauberian theorem. If the Tauberian conditions for the oscillatory behavior of (u_n) are considerably weakened, then we may not obtain convergence of (u_n) out of Abel summability of (u_n), but we obtain a deeper insight into the structure of (u_n). Several results of this kind are given in [1] and suggest a new kind of convergence, namely the subsequential convergence.

A sequence $u = (u_n)$ is subsequentially convergent [1] if there exists a finite interval $I(u)$ such that all accumulation points of $u = (u_n)$ are in $I(u)$ and every point of $I(u)$ is an accumulation point of $u = (u_n)$. Equivalently, for every $r \in I(u)$ there exists a subsequence $(u_{n(r)})$ of (u_n) such that $\lim_{n(r)} u_{n(r)} = r$. An example of subsequentially convergent sequence is $(\sin(\log n))$.

Let $\mathcal{L}$ be any linear space of real sequences and $\mathcal{A}$ be a subclass of $\mathcal{L}$. If

$$u_n = \alpha_n + \sum_{k=1}^{n} \frac{\alpha_k}{k}, \tag{1}$$

for some $\alpha = (\alpha_n) \in \mathcal{A}$, we say that the sequence (u_n) is regularly generated by the sequence (α_n) and (α_n) is called a generator of (u_n).

The Kronecker identity

$$u_n - \sigma_n^{(1)}(u) = V_n^{(0)}(\Delta u), \tag{2}$$

where $V_n^{(0)}(\Delta u) = \frac{1}{n+1} \sum_{k=0}^{n} k \Delta u_k$, is well known and used in the various steps of proofs. Since arithmetic means of (u_n) can be also expressed in the form

$$\sigma_n^{(1)}(u) = u_0 + \sum_{k=1}^{n} \frac{V_k^{(0)}(\Delta u)}{k},$$

we may rewrite (2) as

$$u_n = V_n^{(0)}(\Delta u) + \sum_{k=1}^{n} \frac{V_k^{(0)}(\Delta u)}{k} + u_0.$$

A sequence (u_n) is slowly oscillating [1, 2] if

$$\lim_{\lambda \to 1^+} \overline{\lim_{n}} \max_{n+1 \le k \le [\lambda n]} |u_k - u_n| = 0,$$

where $[\lambda n]$ denotes the integer part of λn.

Dik [2] proved that a sequence (u_n) is slowly oscillating if and only if the generator of (u_n) is slowly oscillating and bounded.

The following definition is a generalization of the concept of slow oscillation.

A sequence (u_n) is moderately oscillating [1, 2] if for $\lambda > 1$,

$$\overline{\lim_{n}} \max_{n+1 \le k \le [\lambda n]} |u_k - u_n| < \infty,$$

where $[\lambda n]$ denotes the integer part of λn.

Dik [2] proved that the generator of $u = (u_n)$ is bounded and $(\sigma_n^{(1)}(u))$ is slowly oscillating for a moderately oscillating sequence (u_n).

The fact that the bounded slowly oscillating sequences are subsequentially convergent follows from the definition of subsequential convergence. The sequence $(\sin(\sum_{k=1}^{n} \frac{\log k}{k}))$ is subsequential convergent, but not slowly oscillating. It is clear from the definition of subsequential convergence that subsequential convergence implies boundedness. However, the converse is not generally true. For instance, the sequence $((-1)^n)$ is bounded, but not subsequential convergent.

The theorem which reveals that the converse is true under some additional condition is obtained by Dik [1] as follows.

Theorem 1.1. *[1] Let (u_n) be a bounded sequence. If $u_n - u_{n-1} = o(1)$, then (u_n) is subsequentially convergent.*

The classical control modulo of the oscillatory behavior of (u_n) is denoted by $\omega_n^{(0)}(u) = n\Delta u_n$, where $\Delta u_n = u_n - u_{n-1}$ and $u_{-1} = 0$. The general control modulo of the oscillatory behavior of integer order $m \geq 1$ of a sequence (u_n) is defined inductively in [1, 2] by $\omega_n^{(m)}(u) = \omega_n^{(m-1)}(u) - \sigma_n^{(1)}(\omega^{(m-1)}(u))$.

For a sequence (u_n) and for each integer $m \geq 1$ define

$$(n\Delta)_m u_n = n\Delta((n\Delta)_{m-1}u_n),$$

where $(n\Delta)_0 u_n = u_n$ and $(n\Delta)_1 u_n = n\Delta u_n$. Define $\Delta^2 u_n = \Delta(\Delta u_n)$ for a sequence (u_n). It is proved in [4] that for each integer $m \geq 1$, $\omega_n^{(m)}(u) = (n\Delta)_m V_n^{(m-1)}(\Delta u)$, where $V_n^{(m-1)}(\Delta u) = \sigma_n^{(1)}(V^{(m-2)}(\Delta u))$.

We write $u_n = O(1)$ to mean that (u_n) is a bounded sequence, and $u_n = o(1)$ to mean that (u_n) converges to zero.

2 Auxiliary Results

Throughout this section for any real sequence $\alpha = (\alpha_n)$, let $\mu = (\mu_n)$ be defined by $\mu_n = (n+1)\alpha_n$.

We will now provide some important results used in the next section.

By the following Lemma it is shown that the generator of $\Delta\mu = (\Delta\mu_n)$ is the classical control modulo of $\alpha = (\alpha_n)$.

Lemma 2.1. *The generator of $\Delta\mu = (\Delta\mu_n)$ is $\omega^{(0)}(\alpha) = (\omega_n^{(0)}(\alpha))$.*

Proof. Since $\mu_n = (n+1)\alpha_n$, then

$$\Delta\mu_n = \alpha_n + n\Delta\alpha_n. \tag{3}$$

Taking the arithmetic means of both sides of (3), we have

$$\sigma_n^{(1)}(\Delta\mu) = \sigma_n^{(1)}(\alpha_n + n\Delta\alpha_n).$$

Replacing u_n by $\Delta\mu_n$ in (2) we obtain that

$$\Delta\mu_n - \sigma_n^{(1)}(\Delta\mu) = V_n^{(0)}(\Delta^2\mu). \tag{4}$$

From (3) and (4) we get $V_n^{(0)}(\Delta^2\mu) = \omega_n^{(0)}(\alpha)$. This completes the proof. □

The following Lemma shows that the generator of arithmetic means of the sequence whose generator is $(\Delta\mu_n)$ is $\alpha = (\alpha_n)$.

Lemma 2.2. *If the generator of $x = (x_n)$ is $\Delta\mu_n$, then the generator of $\sigma^{(1)}(x) = (\sigma_n^{(1)}(x))$ is $\alpha = (\alpha_n)$.*

Proof. By (2) we have $x_n = V_n^{(0)}(\Delta x) + \sum_{k=1}^{n} \frac{V_k^{(0)}(\Delta x)}{k}$. Since $V_n^{(0)}(\Delta x) = \Delta\mu_n$ by hypothesis, the identity above can be expressed as

$$x_n = \Delta\mu_n + y_n, \tag{5}$$

where $y = (y_n) = (\sum_{k=1}^{n} \frac{\Delta\mu_k}{k})$. Taking the arithmetic means of both sides of (5) we obtain that

$$\sigma_n^{(1)}(x) = \sigma_n^{(1)}(\Delta\mu) + \sigma_n^{(1)}(y).$$

Then we have

$$\sigma_n^{(1)}(x) = \alpha_n + \sigma_n^{(1)}(y) = \alpha_n + \sigma_n^{(1)}(z) + \sigma_n^{(1)}(\alpha), \tag{6}$$

where $z = (z_n) = (\sum_{k=1}^{n} \frac{\alpha_k}{k})$. By (2) applied to the sequence $\sum_{k=1}^{n} \frac{\alpha_k}{k}$ we have

$$z_n - \sigma_n^{(1)}(z) = V_n^{(0)}(\Delta(z)) = \sigma_n^{(1)}(\alpha). \tag{7}$$

Taking into consideration (6) and (7), we have $\sigma_n^{(1)}(x) = \alpha_n + \sum_{k=1}^{n} \frac{\alpha_k}{k}$. This completes the proof. □

Lemma 2.3. *[5] For each integer $m \geq 1$ and for all nonnegative integers n,*

$$\omega_n^{(m)}(u) = \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} n\Delta V_n^{(j)}(\Delta u)$$

where $\binom{m-1}{j} = \dfrac{(m-1)(m-2)...(m-j)}{j!}$.

Theorem 2.4. *[5] Let $u_n \to s\,(A,0)$. If $(\sigma_n^{(1)}(\omega^{(m)}(u)))$ is slowly oscillating for some integer $m \geq 0$, then $u_n \to s$ as $n \to \infty$.*

Since sequence of arithmetic means of a slowly oscillating sequence is slowly oscillating, we have the following corollary.

Corollary 2.5. *[5] Let $u_n \to s\,(A,0)$. If $(\omega_n^{(m)}(u))$ is slowly oscillating for some integer $m \geq 0$, then $u_n \to s$ as $n \to \infty$.*

Corollary 2.6. *[5] Let $u_n \to s\,(A,0)$. If $(\omega_n^{(m)}(u))$ is bounded for some integer $m \geq 0$, then $u_n \to s$ as $n \to \infty$.*

3 Results

Denote by $\mathcal{M}$ and $\mathcal{S}$ the space of moderately oscillating sequences and slowly oscillating sequences, respectively.

Throughout this section it will be assumed that $\mu = (\mu_n)$ is defined by $\mu_n = (n+1)\omega_n^{(m)}(u)$.

Theorem 3.1. *Let $u_n \to s\,(A, m)$ for some $m \geq 2$. If the generator of $(\sigma_n^{(m)}(\Delta\mu))$ is moderately oscillating and $\Delta\sigma_n^{(m-2)}(u) = o(1)$, then $(\sigma_n^{(m-2)}(u))$ is subsequentially convergent.*

Proof. It follows by Lemma 2.1 that $V_n^{(0)}(\Delta^2\mu) = n\Delta\omega_n^{(m)}(u)$. Thus we have

$$(V_n^{(0)}(\Delta\sigma^{(m)}(\Delta^2\mu))) = (\sigma_n^{(m-1)}(\omega^{(m+1)}(u))) \in \mathcal{M}, \tag{8}$$

and

$$(\sigma_n^{(m)}(\omega^{(m+1)}(u))) = (\omega_n^{(m+1)}(\sigma^{(m)}(u))) \in \mathcal{S}. \tag{9}$$

We here notice that the statement (u_n) is (A, m) summable to s is equivalent to the statement that $(\sigma_n^{(m)}(u))$ is Abel summable to s. Using (9) we obtain by Corollary 2.5

$$\sigma_n^{(m)}(u) \to s,\ n \to \infty. \tag{10}$$

This means that $(\sigma_n^{(m-1)}(u))$ is Cesàro summable to s. Since every Cesàro summable sequence is Abel summable, $(\sigma_n^{(m-1)}(u))$ is Abel summable to s. Using (8) we obtain by Theorem 2.4

$$\sigma_n^{(m-1)}(u) \to s,\ n \to \infty. \tag{11}$$

Since the generator of a moderately oscillating sequence is bounded, it follows from (8) that

$$V_n^{(0)}(\sigma^{(m-1)}(\omega^{(m+1)}(u))) = \sigma_n^{(m-1)}(\omega^{(m+2)}(u)) = O(1). \tag{12}$$

By Lemma 2.3, we have

$$\sigma_n^{(m-1)}(\omega^{(m+2)}(u)) = \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} n\Delta V_n^{(j+m-1)}(\Delta u). \tag{13}$$

It follows by (11) that

$$V_n^{(m-1)}(\Delta u) = o(1). \tag{14}$$

Therefore $n\Delta V_n^{(m-1)}(\Delta u) = O(1)$ by (13). From (14) and the identity

$$n\Delta V_n^{(m-1)}(\Delta u) = V_n^{(m-2)}(\Delta u) - V_n^{(m-1)}(\Delta u),$$

we obtain $(V_n^{(m-2)}(\Delta u)) = O(1)$. It follows by identity

$$\sigma_n^{(m-2)}(u) - \sigma_n^{(m-1)}(u) = V_n^{(m-2)}(\Delta u)$$

that $\sigma_n^{(m-2)}(\Delta u) = O(1)$. Since $\Delta\sigma_n^{(m-2)}(u) = o(1)$, we see by Theorem 1.1 applied to the sequence $(\sigma_n^{(m-2)}(u))$ that $(\sigma_n^{(m-2)}(u))$ is subsequentially convergent. $\square$

Theorem 3.2. *Let $u_n \to s\,(A,m)$ for some $m \geq 2$. If the sequence whose generator is $(\sigma_n^{(m)}(\Delta\mu))$ is moderately oscillating and $\Delta\sigma_n^{(m-2)}(u) = o(1)$, then $(\sigma_n^{(m-2)}(u))$ is subsequentially convergent.*

Proof. By hypothesis there exists a sequence $x = (x_n)$ in $\mathcal{M}$ such that $\Delta\mu_n = V_n^{(0)}(\Delta x)$. Thus

$$\sigma_n^{(1)}(\Delta\mu) = \sigma_n^{(1)}(V^{(0)}(\Delta x)) = V_n^{(1)}(\Delta x) = V_n^{(0)}(\Delta\sigma^{(1)}(x)).$$

Let $W_n^{(m)}(u) := \omega_n^{(m)}(u) + \sum_{k=1}^{n} \frac{\omega_k^{(m)}(u)}{k}$. It follows by Lemma 2.1 and Lemma 2.2 that

$$\sigma_n^{(1)}(\Delta\mu) = V_n^{(0)}\left(\Delta\left(W^{(m)}(u)\right)\right) = \omega_n^{(m)}(u). \tag{15}$$

If taking arithmetic means of both sides in (15) is repeated $(m-1)$ times, we have

$$\sigma_n^{(m)}(\Delta\mu) = V_n^{(0)}(\Delta\sigma^{(m-1)}(W^{(m)})) = \sigma_n^{(m-1)}(\omega^{(m)}(u)).$$

It follows from the identity above that

$$(\sigma_n^{(m-1)}(W^{(m)}(u))) \in \mathcal{M}.$$

Since the generator of the moderately oscillating sequence is bounded, we obtain that

$$\sigma_n^{(m-1)}(\omega^{(m)}(u)) = \omega_n^{(m)}(\sigma^{(m-1)}(u)) = O(1). \tag{16}$$

By the fact that arithmetic means of a bounded sequence is bounded, we obtain

$$\sigma_n^{(m)}(\omega^{(m)}(u)) = \omega_n^{(m)}(\sigma^{(m)}(u)) = O(1). \tag{17}$$

We here notice that the statement (u_n) is (A,m) summable to s is equivalent to the statement that $(\sigma_n^{(m)}(u))$ is Abel summable to s. Using (17) we obtain by Corollary 2.6

$$\sigma_n^{(m)}(u) \to s,\ n \to \infty. \tag{18}$$

This means that $(\sigma_n^{(m-1)}(u))$ is Cesàro summable to s. Since every Cesàro summable sequence is Abel summable, $(\sigma_n^{(m-1)}(u))$ is Abel summable to s. Using (16) we obtain by Corollary 2.6

$$\sigma_n^{(m-1)}(u) \to s,\ n \to \infty. \tag{19}$$

By Lemma 2.3,

$$\sigma_n^{(m-1)}(\omega^{(m)}(u)) = \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} n\Delta V_n^{(j+m-1)}(\Delta u). \tag{20}$$

Since $\sigma_n^{(m-1)}(u) \to s$ as $n \to \infty$, then

$$V_n^{(m-1)}(\Delta u) = o(1). \tag{21}$$

Therefore $n\Delta V_n^{(m-1)}(\Delta u) = O(1)$ by (20). From (21) and the identity

$$n\Delta V_n^{(m-1)}(\Delta u) = V_n^{(m-2)}(\Delta u) - V_n^{(m-1)}(\Delta u),$$

we obtain $(V_n^{(m-2)}(\Delta u)) = O(1)$. It follows by identity

$$\sigma_n^{(m-2)}(u) - \sigma_n^{(m-1)}(u) = V_n^{(m-2)}(\Delta u)$$

that $\sigma_n^{(m-2)}(\Delta u) = O(1)$. Since $\Delta\sigma_n^{(m-2)}(u) = o(1)$, we see by Theorem 1.1 applied to the sequence $(\sigma_n^{(m-2)}(u))$ that $(\sigma_n^{(m-2)}(u))$ is subsequentially convergent. □

Theorem 3.3. *Let $u_n \to s\,(A, m)$ for some $m \geq 2$. If the sequence whose generator is $(V_n^{(m)}(\Delta^2\mu))$ is moderately oscillating and $\Delta\sigma_n^{(m-2)}(u) = o(1)$, then $(\sigma_n^{(m-2)}(u))$ is subsequentially convergent.*

Proof. Using (2) we have $V_n^{(m)}(\Delta^2\mu) = V_n^{(0)}(\Delta\sigma^{(m)}(\Delta\mu))$. The proof is completed by the fact that $(\sigma_n^{(m)}(\Delta\mu)) = (\sigma_n^{(m-1)}(\omega^{(m)}(u))) \in \mathcal{M}$. The rest of the proof is as in Theorem 3.1. □

Same conclusions can be obtained if the class $\mathcal{M}$ in which the generators of $(\sigma_n^{(m)}(\Delta\mu))$ belong for each integer $m \geq 0$ is replaced by the class $\mathcal{S}$ or the class of bounded sequences in Theorem 3.1 through Theorem 3.3.

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