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QUASICONFORMAL HARMONIC MAPPINGS AND CLOSE-TO-CONVEX DOMAINS

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Abstract

Let $f = h + \overline{g}$ be a univalent sense preserving harmonic mapping of the unit disk $\mathbb{U}$ onto a convex domain Ω. It is proved that: for every a such that $|a| < 1$ (resp. $|a| = 1$) the mapping $f_a = h + a\overline{g}$ is an $|a|$ quasiconformal (a univalent) close-to-convex harmonic mapping. This gives an answer to a question posed by Chuaqui and Hernández (J. Math. Anal. Appl. (2007)).

1 Introduction and notation

A planar harmonic mapping is a complex-valued harmonic function $w = f(z)$, $z = x + iy$, defined on some domain $\Omega \subset \mathbb{C}$. When Ω is simply connected, the mapping has a canonical decomposition $f = h + \overline{g}$, where h and g are analytic in Ω. Since the Jacobian of f is given by $|h'|^2 - |g'|^2$, by Lewy's theorem ([14]), it is locally univalent and sense preserving if and only if $|g'| < |h'|$, or equivalently if $h'(z) \neq 0$ and the second dilatation $\mu = \frac{g'}{h'}$ has the property $|\mu(z)| < 1$ in Ω. A univalent harmonic mapping is called $k-$quasiconformal $(0 \leq k < 1)$ if $|\mu(z)| \leq k$. For the general definition of quasiconformal mappings see [1]. Following the first pioneering work by O. Martio ([16]), the class of quasiconformal harmonic mappings (QCH) has been extensively studied by various authors in the papers [6], [7], [8], [17], [13], [10], [18], [15].

In this short note, by using some results of Clunie and Sheil-Small ([4]) we improve a result by Chuaqui and Hernández ([2]) and answer a question posed there. In addition, for a given harmonic diffeomorphism (quasiconformal harmonic mapping) we produce a large class of harmonic diffeomorphisms (quasiconformal harmonic mappings). The main result (Theorem 2.1) can be considered as a partial extension of the fundamental theorem of Choquet-Rado-Kneser ([3] and [5]) which states that: if Ω is a bounded convex domain and $\phi : S^1 \to \partial\Omega$ is a homeomorphism, then the harmonic extension f of ϕ to the unit disk $\mathbb{U}$ is univalent.

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Following Kaplan ([9]), an analytic mapping $f : \mathbb{U} \to \mathbb{U}$ is called close-to-convex if there exists a univalent convex function ϕ defined in $\mathbb{U}$ such that

$$\mathrm{Re}\, \frac{f'(z)}{\phi'(z)} > 0, \quad z \in \mathbb{U}.$$

A domain Ω is close-to-convex if $\mathbb{C} \setminus \Omega$ can be represented as a union of non crossing half-lines. Let f be analytic in $\mathbb{U}$. Then f is close-to-convex if and only if f is univalent and $f(\mathbb{U})$ is a close-to-convex domain. It is evident that for $F(z) = f \circ \phi^{-1}$, $F'(z) = \frac{f'(z)}{\phi'(z)}$. Therefore if f is close-to-convex, then according to Lemma 2.3 F is univalent; that is f is also univalent.

A harmonic mapping $f : \mathbb{U} \to \mathbb{C}$ is close-to-convex if it is injective and $f(\mathbb{U})$ is a close-to-convex domain.

2 The main result

The aim of this paper is to prove the following theorem.

Theorem 2.1. *Let $f = h + \overline{g}$ be a univalent sense preserving harmonic mapping of the unit disk $\mathbb{U}$ onto a convex domain Ω. Then for every a such that $|a| < 1$ (resp. $|a| = 1$) the mapping $f_a = h + a\overline{g}$ is an $|a|$ quasiconformal close-to-convex harmonic mapping ((resp. a univalent) close-to-convex harmonic mapping).*

Theorem 2.1 gives an answer to the question posed by Chuaqui and Hernández in [2], where they proved Theorem 2.1 for every a such that $|a| = 1$ (see [2, Theorem 3], the convex case) under the condition $|\mu(z)| = |\frac{g'}{h'}| < \frac{1}{3}$, and asked if this is the best possible condition. It is shown that, no restriction is needed on the dilatation μ. This result can be considered as an extension of Choquet-Rado-Kneser theorem mentioned in the introduction of this paper.

The proof of Theorem 2.1 depends on the following proposition which we prove for the sake of completeness.

Proposition 2.2. *[4] If $f = h + \overline{g} : \mathbb{U} \to \Omega$ is a univalent sense preserving harmonic mapping of the unit disk onto a convex domain Ω, then*

(i) for every $\varepsilon \in \overline{\mathbb{U}}$ the mapping

$$F_\varepsilon = h(z) + \varepsilon g(z) \tag{2.1}$$

is close-to-convex;

(ii) for every $z_1, z_2 \in \mathbb{U}$, $z_1 \neq z_2$

$$|g(z_1) - g(z_2)| < |h(z_1) - h(z_2)|. \tag{2.2}$$

Let us first prove the following lemma.

Lemma 2.3. *If f is an analytic mapping defined in a convex domain Ω, such that for some $\alpha \in [0, 1]$*

$$\alpha \mathrm{Re}\, f'(z) + (1 - \alpha) \mathrm{Im}\, f'(z) > 0, \quad z \in \Omega,$$

then f is univalent sense preserving.

Proof. For $z_1, z_2 \in \Omega$, $z_1 \neq z_2$, we have

$$f(z_2) - f(z_1) = \int_{z_1}^{z_2} f'(z)\,dz = (z_2 - z_1)\int_0^1 f'(z_1 + t(z_2 - z_1))dt.$$

Therefore

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1} = \int_0^1 f'(z_1 + t(z_2 - z_1))dt.$$

Since

$$\alpha a + (1-\alpha)b \leq (\alpha^2 + (1-\alpha)^2)^{1/2}(a^2 + b^2)^{1/2} \leq \left(a^2 + b^2\right)^{1/2},$$

it follows that

$$\begin{aligned} &|\frac{f(z_2) - f(z_1)}{z_2 - z_1}| \\ &\geq \int_0^1 \alpha \mathrm{Re}\, f'(z_1 + t(z_2 - z_1)) + (1-\alpha)\mathrm{Im}\, f'(z_1 + t(z_2 - z_1))dt > 0. \end{aligned}$$

This implies that f is univalent. □

Proof of Proposition 2.2. Since f is convex, then for every $\varepsilon = e^{i\varphi}$ the mapping $e^{-i\varphi/2} f = e^{-i\varphi/2} h + \overline{e^{i\varphi/2} g}$ is convex. On the other hand

$$G_\varepsilon := e^{-i\varphi/2} h - e^{i\varphi/2} g = e^{-i\varphi/2} f - 2\mathrm{Re}(e^{i\varphi/2} g).$$

It follows that $G_\varepsilon(\mathbb{U})$ is convex in direction of real axis. Now we show that G_ε is injective. Since f is univalent, it follows that

$$G_\varepsilon \circ f^{-1}(w) = e^{-i\varphi/2} w + p(w),$$

where p is a real function. Let $w_1 = e^{i\varphi/2}\omega_1$ and $w_2 = e^{i\varphi/2}\omega_2$ be two points such that

$$w_1 + e^{i\varphi/2} p(w_1) = w_2 + e^{i\varphi/2} p(w_2),$$

i.e.

$$\omega_1 + q(\omega_1) = \omega_2 + q(\omega_2),$$

where $q(\omega) = p(e^{i\varphi/2}\omega)$. It follows that

$$\mathrm{Im}\,\omega_1 = \mathrm{Im}\,\omega_2 = v_0 \tag{2.3}$$

and

$$\mathrm{Re}\,\omega_1 + q(\omega_1) = \mathrm{Re}\,\omega_2 + q(\omega_2).$$

According to Lewy's theorem

$$G'_\varepsilon := e^{-i\varphi/2} h' - e^{i\varphi/2} g' \neq 0.$$

Therefore $G_\varepsilon \circ f^{-1}$ is locally univalent. Write $\omega = u + iv$. Then the function $u \to u + q(u + iv_0)$ is locally univalent. Since it is a real function, it follows that it is univalent. As

$$u_1 + q(u_1 + iv_0) = u_2 + q(u_2 + iv_0),$$

it follows that $u_1 = u_2$ and consequently $w_1 = w_2$.

This means that G_ε is convex in direction of real axis and univalent. In particular $F_\varepsilon = h + \varepsilon g$ is close-to-convex for $|\varepsilon| = 1$. According to a Kaplan's theorem ([9, Eqs. (16′)]); this is equivalent to the fact that for $0 < r < 1$ and $\theta_1 < \theta_2 < \theta_1 + 2\pi$

$$\arg\left(h'(re^{i\theta_1}) + \varepsilon g'(re^{i\theta_1})\right) - \arg\left(h'(re^{i\theta_2}) + \varepsilon g'(re^{i\theta_2})\right) \le \pi + \theta_2 - \theta_1. \tag{2.4}$$

As the expression on the left-side of (2.4) is well-defined harmonic function in ε for $|\varepsilon| < 1$ (because $h'(w) \neq 0$ and $\mathrm{Re}\,(1 + \varepsilon \frac{g'(w)}{h'(w)}) > 0$), according to the maximum principle the inequality (2.4) continues to hold when $|\varepsilon| \le 1$. We proved that F_ε, $|\varepsilon| \le 1$, is close-to-convex. According to the introduction and Lemma 2.3 it is univalent.

To prove (ii) we argue by contradiction. Assume there exist an A: $|A| \ge 1$ and $z_1, z_2 \in \mathbb{U}$, $z_1 \neq z_2$ such that

$$\frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} = A.$$

Hence for $\varepsilon = -1/A$ we have

$$h(z_1) - h(z_2) + \varepsilon(g(z_1) - g(z_2)) = 0.$$

This contradicts (i).

□

Proof of Theorem 2.1. Assume that f_a is not univalent. Then for some distinct points $z_1, z_2 \in \mathbb{U}$

$$f_a(z_1) = f_a(z_2).$$

It follows that,

$$\overline{h(z_1) - h(z_2)} = \overline{a}(g(z_2) - g(z_1)).$$

This contradicts (2.2). The second dilatation μ_a of f_a is equal to $\overline{a}\mu$. Thus f_a is $|a|$ quasiconformal if $|a| < 1$.

To continue we need the following lemma.

Lemma 2.4. *[4, Lemma 5.15] Suppose G and H are analytic in $\mathbb{U}$ with $|G'(0)| < |H'(0)|$ and that $H + \varepsilon G$ is close-to-convex for $|\varepsilon| = 1$. Then $H + \overline{G}$ is harmonic close-to-convex.*

First of all $|a||g'(0)| < |h'(0)|$. By Proposition 2.2 $h + \varepsilon \overline{a} g$ is close-to-convex for every $|\varepsilon| = 1$. Therefore $f_a = h + a\overline{g}$ is close-to-convex. □

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