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SUBORDINATION AND SUPERORDINATION RESULTS FOR THE FAMILY OF JUNG-KIM-SRIVASTAVA INTEGRAL OPERATORS

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Abstract

In this paper, we derive some subordination and superordination results associated with the family of Jung-Kim-Srivastava integral operators defined on the space of meromorphic functions. Several sandwich-type results are also obtained.

1 Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \tag{1.1}$$

which are *analytic* in the *punctured* open unit disk

$$\mathbb{U}^* := \{z : \ z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\} =: \mathbb{U} \backslash \{0\}.$$

Let $\mathcal{H}$ be the linear space of all analytic functions in $\mathbb{U}$. For a positive integer number n and $a \in \mathbb{C}$, we let

$$\mathcal{H}[a, n] := \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

Denote by Q the set of all functions f that are analytic and injective on $\overline{\mathbb{U}} \backslash E(f)$, where

$$E(f) = \left\{\varepsilon \in \partial\mathbb{U} : \lim_{z \to \varepsilon} f(z) = \infty\right\},$$

and are such that $f'(\varepsilon) \neq 0$ for $\varepsilon \in \partial\mathbb{U} \backslash E(f)$.

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Let $f,\ g \in \Sigma$, where f is given by (1.1) and g is defined by

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k.$$

Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) := \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k =: (g * f)(z).$$

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g\big(\omega(z)\big) \qquad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $h,\ \kappa \in \mathcal{H}$ and let

$$\phi(r, s, t; z) : \ \mathbb{C}^3 \times \mathbb{U} \longrightarrow \mathbb{C}.$$

If h and $\phi\left(h(z), zh'(z), z^2h''(z); z\right)$ are univalent and h satisfies the second-order superordination

$$\kappa(z) \prec \phi\left(h(z), zh'(z), z^2h''(z); z\right), \tag{1.2}$$

then h is a solution of the differential superordination (1.2). Note that if f is subordinate to g, then g is superordinate to f. An analytic function q is called a subordinant if $q \prec h$ for all h satisfying (1.2). An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant.

Analogous to the integral operator defined by Jung *et al.* [10], Lashin [11] recently introduced and investigated the integral operator

$$\mathcal{Q}_{\alpha,\beta} : \Sigma \longrightarrow \Sigma$$

defined, in terms of the familiar Gamma function, by

$$\begin{aligned}\mathcal{Q}_{\alpha,\beta}f(z) &= \frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)\Gamma(\alpha)}\frac{1}{z^{\beta+1}}\int_0^z t^\beta\left(1-\frac{t}{z}\right)^{\alpha-1}f(t)dt \\ &= \frac{1}{z}+\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)}\sum_{k=1}^{\infty}\frac{\Gamma(k+\beta+1)}{\Gamma(k+\beta+\alpha+1)}a_k z^k \qquad (\alpha>0;\ \beta>0;\ z\in\mathbb{U}^*).\end{aligned} \tag{1.3}$$

By setting

$$f_{\alpha,\beta}(z) := \frac{1}{z}+\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\sum_{k=1}^{\infty}\frac{\Gamma(k+\beta+\alpha+1)}{\Gamma(k+\beta+1)}z^k \quad (\alpha>0;\ \beta>0;\ z\in\mathbb{U}^*), \tag{1.4}$$

we define a new function $f^\lambda_{\alpha,\beta}(z)$ in terms of the Hadamard product (or convolution)

$$f_{\alpha,\beta}(z) * f^\lambda_{\alpha,\beta}(z) = \frac{1}{z(1-z)^\lambda} \qquad (\alpha>0;\ \beta>0;\ \lambda>0;\ z\in\mathbb{U}^*). \tag{1.5}$$

Then, motivated essentially by the operator $\mathcal{Q}_{\alpha,\beta}$, Wang *et al.* [21] introduced the operator

$$\mathcal{Q}^\lambda_{\alpha,\beta}:\ \Sigma \longrightarrow \Sigma,$$

which is defined as

$$\begin{aligned}\mathcal{Q}^\lambda_{\alpha,\beta}f(z) :&= f^\lambda_{\alpha,\beta}(z) * f(z) \\ &= \frac{1}{z}+\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)}\sum_{k=1}^{\infty}\frac{(\lambda)_{k+1}}{(k+1)!}\frac{\Gamma(k+\beta+1)}{\Gamma(k+\beta+\alpha+1)}a_k z^k \quad (z\in\mathbb{U}^*;\ f\in\Sigma),\end{aligned} \tag{1.6}$$

where (and throughout this paper unless otherwise mentioned) the parameters α, β and λ are constrained as follows:

$$\alpha>0;\ \beta>0 \quad \text{and} \quad \lambda>0,$$

and $(\lambda)_k$ is the Pochhammer symbol defined by

$$(\lambda)_k := \begin{cases} 1 & (k=0), \\ \lambda(\lambda+1)\cdots(\lambda+k-1) & (k\in\mathbb{N}:=\{1,2,\cdots\}). \end{cases}$$

Clearly, we know that $\mathcal{Q}^1_{\alpha,\beta} = \mathcal{Q}_{\alpha,\beta}$.

It is readily verified from (1.6) that

$$z(\mathcal{Q}^\lambda_{\alpha,\beta}f)'(z) = \lambda\mathcal{Q}^{\lambda+1}_{\alpha,\beta}f(z) - (\lambda+1)\mathcal{Q}^\lambda_{\alpha,\beta}f(z), \tag{1.7}$$

and

$$z(\mathcal{Q}_{\alpha+1,\beta}^{\lambda}f)'(z) = (\beta+\alpha)\mathcal{Q}_{\alpha,\beta}^{\lambda}f(z) - (\beta+\alpha+1)\mathcal{Q}_{\alpha+1,\beta}^{\lambda}f(z). \tag{1.8}$$

In [21], Wang *et al.* obtained several inclusion relationships and integral-preserving properties associated with some subclasses involving the operator $\mathcal{Q}_{\alpha,\beta}^{\lambda}$. Several subordination and superordination results involving this family of integral operators are also derived. For some other recent sandwich-type results in analytic function theory, one can find in [1, 2, 3, 5, 6, 7, 8, 9, 16, 17, 18, 19, 20, 22] and the references cited therein.

The main purpose of the present paper is to derive some other new subordination and superordination results involving the operator $\mathcal{Q}_{\alpha,\beta}^{\lambda}$.

2 Preliminary Results

In order to establish our main results, we need the following lemmas.

Lemma 1. (See [15]) *Let q be convex univalent in $\mathbb{U}$ and $\psi,\ \gamma \in \mathbb{C}$ with*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\Re\left(\frac{\psi}{\gamma}\right)\right\}.$$

If p is analytic in $\mathbb{U}$ and

$$\psi p(z) + \gamma z p'(z) \prec \psi q(z) + \gamma z q'(z),$$

then $p \prec q$, and q is the best dominant.

Lemma 2. (See [12]) *Let q be univalent in $\mathbb{U}$, and let θ and ϕ be analytic in the domain $\mathbb{D}$ containing $q(\mathbb{U})$ with $\phi(\omega) \neq 0$ when $\omega \in q(\mathbb{U})$. Setting*

$$Q(z) = zq'(z)\phi(q(z)) \quad \textit{and} \quad h(z) = \theta(q(z)) + Q(z).$$

Suppose also that

1. *Q is starlike univalent in $\mathbb{U}$;*
2. *$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left(\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right) > 0 \quad (z \in \mathbb{U})$.*

If p is analytic in $\mathbb{U}$ with $p(0) = q(0)$, $p(\mathbb{U}) \in \mathbb{D}$, and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then $p \prec q$, and q is the best dominant.

Lemma 3. (See [13]) *Let q be convex univalent in $\mathbb{U}$ and $\zeta \in \mathbb{C}$. Further assume that $\Re(\zeta) > 0$. If $p \in \mathcal{H}[q(0), 1] \cap Q$ and $p(z) + \zeta z p'(z)$ is univalent in $\mathbb{U}$, then*

$$q(z) + \zeta z q'(z) \prec p(z) + \zeta z p'(z),$$

which implies that $q \prec p$ and q is the best subordinant.

Lemma 4. (See [4]) *Let q be convex univalent in $\mathbb{U}$, and let ϑ and φ be analytic in the domain $\mathbb{D}$ containing $q(\mathbb{U})$. Suppose that*

1. $\Re\left(\frac{\vartheta'(q(z))}{\varphi(q(z))}\right) > 0$ *for* $z \in \mathbb{U}$;
2. $zq'(z)\varphi(q(z))$ *is starlike univalent in* $\mathbb{U}$.

If $p \in \mathcal{H}[q(0), 1] \cap Q$ with $p(\mathbb{U}) \subseteq \mathbb{D}$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in $\mathbb{U}$, and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)),$$

then $q \prec p$, and q is the best subordinant.

Lemma 5. (See [14]) *The function*

$$(1-z)^{\nu} \equiv e^{\nu \log(1-z)} \qquad (\nu \neq 0)$$

is univalent in $\mathbb{U}$ if and only if ν is either in the closed disk $|\nu - 1| \leqq 1$ or in the closed disk $|\nu + 1| \leqq 1$.

3 Main Results

Firstly, we derive some subordination results involving the integral operator $\mathcal{Q}^{\lambda}_{\alpha,\beta}$.

Throughout this section, without otherwise mentioned, we assume that the parameters $\gamma,\ \mu,\ \sigma,\ \delta,\ a$ and b satisfy the conditions:

$$\gamma \neq 0;\ \mu \neq 0;\ \sigma,\ \delta,\ a,\ b \in \mathbb{C} \quad \text{with} \quad a + b \neq 0.$$

Theorem 1. *Let q be convex univalent in $\mathbb{U}$ with*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\Re\left(\frac{\lambda}{\eta}\right)\right\} \qquad (\eta \neq 0). \tag{3.1}$$

If $f \in \Sigma$ satisfies the subordination

$$\eta z \mathcal{Q}^{\lambda+1}_{\alpha,\beta} f(z) + (1-\eta) z \mathcal{Q}^{\lambda}_{\alpha,\beta} f(z) \prec q(z) + \frac{\eta z q'(z)}{\lambda}, \tag{3.2}$$

then

$$z\mathcal{Q}^{\lambda}_{\alpha,\beta} f(z) \prec q(z), \tag{3.3}$$

and q is the best dominant.

Proof. Define the function $\mathfrak{h}$ by

$$\mathfrak{h}(z) := z\mathcal{Q}^{\lambda}_{\alpha,\beta} f(z). \tag{3.4}$$

Differentiating both sides of (3.4) with respect to z logarithmically, we have

$$\frac{z\mathfrak{h}'(z)}{\mathfrak{h}(z)} = 1 + \frac{z(\mathcal{Q}^{\lambda}_{\alpha,\beta} f)'(z)}{\mathcal{Q}^{\lambda}_{\alpha,\beta} f(z)}. \tag{3.5}$$

It now follows from (1.7), (3.2) and (3.5) that

$$\mathfrak{h}(z) + \frac{\eta z \mathfrak{h}'(z)}{\lambda} \prec q(z) + \frac{\eta z q'(z)}{\lambda}.$$

An application of Lemma 1, with $\gamma = \frac{\eta}{\lambda}$ and $\psi = 1$, leads to (3.3). $\square$

Taking $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 1, we get the following result.

Corollary 1. *Let* $-1 \leqq B < A \leqq 1$ *and*

$$\Re\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -\Re\left(\frac{\lambda}{\eta}\right)\right\} \qquad (\eta \neq 0).$$

If $f \in \Sigma$ *satisfies the subordination*

$$\eta z \mathcal{Q}_{\alpha,\beta}^{\lambda+1} f(z) + (1-\eta) z \mathcal{Q}_{\alpha,\beta}^{\lambda} f(z) \prec \frac{1+Az}{1+Bz} + \frac{\eta}{\lambda} \frac{(A-B)z}{(1+Bz)^2},$$

then

$$z \mathcal{Q}_{\alpha,\beta}^{\lambda} f(z) \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

In view of (1.8) and Lemma 1, and by similarly applying the method of proof of Theorem 1, we easily get the following results.

Corollary 2. *Let* q *be convex univalent in* $\mathbb{U}$ *with*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\Re\left(\frac{\beta+\alpha}{\eta}\right)\right\} \qquad (\eta \neq 0). \tag{3.6}$$

If $f \in \Sigma$ *satisfies the subordination*

$$\eta z \mathcal{Q}_{\alpha,\beta}^{\lambda} f(z) + (1-\eta) z \mathcal{Q}_{\alpha+1,\beta}^{\lambda} f(z) \prec q(z) + \frac{\eta z q'(z)}{\beta+\alpha},$$

then

$$z \mathcal{Q}_{\alpha+1,\beta}^{\lambda} f(z) \prec q(z)$$

and $q(z)$ *is the best dominant.*

Corollary 3. *Let* $-1 \leqq B < A \leqq 1$ *and*

$$\Re\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -\Re\left(\frac{\beta+\alpha}{\eta}\right)\right\} \qquad (\eta \neq 0).$$

If $f \in \Sigma$ *satisfies the subordination*

$$\eta z \mathcal{Q}_{\alpha,\beta}^{\lambda} f(z) + (1-\eta) z \mathcal{Q}_{\alpha+1,\beta}^{\lambda} f(z) \prec \frac{1+Az}{1+Bz} + \frac{\eta}{\beta+\alpha} \frac{(A-B)z}{(1+Bz)^2},$$

then

$$z \mathcal{Q}_{\alpha+1,\beta}^{\lambda} f(z) \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Theorem 2. *Let q be univalent in $\mathbb{U}$. Suppose that q satisfies*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right)>0. \tag{3.7}$$

Let

$$\varrho(z)=1+\gamma\mu\left(1+\frac{az(\mathcal{Q}^{\lambda+1}_{\alpha,\beta}f)'(z)+bz(\mathcal{Q}^{\lambda}_{\alpha,\beta}f)'(z)}{a\mathcal{Q}^{\lambda+1}_{\alpha,\beta}f(z)+b\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)}\right). \tag{3.8}$$

If

$$\varrho(z)\prec 1+\gamma\frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{az\mathcal{Q}^{\lambda+1}_{\alpha,\beta}f(z)+bz\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)}{a+b}\right)^{\mu}\prec q(z), \tag{3.9}$$

and q is the best dominant.

Proof. Let us consider a function $\mathfrak{p}$ defined by

$$\mathfrak{p}(z):=\left(\frac{az\mathcal{Q}^{\lambda+1}_{\alpha,\beta}f(z)+bz\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)}{a+b}\right)^{\mu}\qquad(\mu\neq 0;\ a+b\neq 0). \tag{3.10}$$

Now, Differentiating (3.10) logarithmically, we get

$$\frac{z\mathfrak{p}'(z)}{\mathfrak{p}(z)}=\mu\left(1+\frac{az(\mathcal{Q}^{\lambda+1}_{\alpha,\beta}f)'(z)+bz(\mathcal{Q}^{\lambda}_{\alpha,\beta}f)'(z)}{a\mathcal{Q}^{\lambda+1}_{\alpha,\beta}f(z)+b\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)}\right).$$

Setting

$$\theta(\omega)=1\quad\text{and}\quad\phi(\omega)=\frac{\gamma}{\omega},$$

by observing that $\theta(\omega)$ is analytic in $\mathbb{C}$ and that $\phi(\omega)\neq 0$ is analytic in $\mathbb{C}\setminus\{0\}$. Furthermore, we let

$$Q(z):=zq'(z)\phi(q(z))=\gamma\frac{zq'(z)}{q(z)},$$

and

$$h(z):=\theta(q(z))+Q(z)=1+\gamma\frac{zq'(z)}{q(z)}.$$

From (3.7), we see that $Q(z)$ is starlike univalent in $\mathbb{U}$, and

$$\Re\left(\frac{zh'(z)}{Q(z)}\right)=\Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right)>0.$$

Thus, an application of Lemma 2 to (3.8) yields the desired result. $\square$

Putting $a=0,\ b=1,\ \gamma=1$ and $q(z)=\frac{1+Az}{1+Bz}$ in Theorem 2, we obtain the following corollary.

Corollary 4. *Let* $-1 \leqq B < A \leqq 1$, $\mu \neq 0$. *If* $f \in \Sigma$, *and*

$$1+\mu\left(1+\frac{z(\mathcal{Q}^{\lambda}_{\alpha,\beta}f)'(z)}{\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)}\right) \prec 1+\frac{(A-B)z}{(1+Az)(1+Bz)},$$

then

$$\left(z\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)\right)^{\mu} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

By similarly applying the method of proof of Theorem 2, we easily get the following result.

Corollary 5. *Let* q *be univalent in* $\mathbb{U}$. *Suppose that* q *satisfies* (3.7). *Let*

$$\chi(z) = 1+\gamma\mu\left(1+\frac{az(\mathcal{Q}^{\lambda}_{\alpha,\beta}f)'(z)+bz(\mathcal{Q}^{\lambda}_{\alpha+1,\beta}f)'(z)}{a\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)+b\mathcal{Q}^{\lambda}_{\alpha+1,\beta}f(z)}\right). \tag{3.11}$$

If

$$\chi(z) \prec 1+\gamma\frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{az\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)+bz\mathcal{Q}^{\lambda}_{\alpha+1,\beta}f(z)}{a+b}\right)^{\mu} \prec q(z)$$

and q *is the best dominant.*

Theorem 3. *Let* q *be univalent in* $\mathbb{U}$. *Suppose that* q *satisfies*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\Re\left(\frac{\sigma}{\gamma}\right)\right\}. \tag{3.12}$$

Let

$$\psi(z) = \left(\frac{az\mathcal{Q}^{\lambda+1}_{\alpha,\beta}f(z)+bz\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)}{a+b}\right)^{\mu}\cdot \tag{3.13}$$

$$\cdot\left[\sigma+\gamma\mu\left(1+\frac{az(\mathcal{Q}^{\lambda+1}_{\alpha,\beta}f)'(z)+bz(\mathcal{Q}^{\lambda}_{\alpha,\beta}f)'(z)}{a\mathcal{Q}^{\lambda+1}_{\alpha,\beta}f(z)+b\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)}\right)\right]+\delta \tag{3.14}$$

If

$$\psi(z) \prec \sigma q(z)+\delta+\gamma zq'(z),$$

then

$$\left(\frac{az\mathcal{Q}^{\lambda+1}_{\alpha,\beta}f(z)+bz\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)}{a+b}\right)^{\mu} \prec q(z),$$

and q *is the best dominant.*

Proof. Define the function $\mathfrak{m}$ by

$$\mathfrak{m}(z) := \left(\frac{az\mathcal{Q}_{\alpha,\beta}^{\lambda+1} f(z) + bz\mathcal{Q}_{\alpha,\beta}^{\lambda} f(z)}{a+b} \right)^{\mu} \qquad (\mu \neq 0;\ a+b \neq 0). \tag{3.15}$$

Taking the logarithmical differentiation on both sides of (3.15), we get

$$\frac{z\mathfrak{m}'(z)}{\mathfrak{m}(z)} = \mu \left(1 + \frac{az(\mathcal{Q}_{\alpha,\beta}^{\lambda+1} f)'(z) + bz(\mathcal{Q}_{\alpha,\beta}^{\lambda} f)'(z)}{a\mathcal{Q}_{\alpha,\beta}^{\lambda+1} f(z) + b\mathcal{Q}_{\alpha,\beta}^{\lambda} f(z)} \right),$$

and hence

$$z\mathfrak{m}'(z) = \mu\mathfrak{m}(z) \left(1 + \frac{az(\mathcal{Q}_{\alpha,\beta}^{\lambda+1} f)'(z) + bz(\mathcal{Q}_{\alpha,\beta}^{\lambda} f)'(z)}{a\mathcal{Q}_{\alpha,\beta}^{\lambda+1} f(z) + b\mathcal{Q}_{\alpha,\beta}^{\lambda} f(z)} \right).$$

Suppose that

$$\theta(\omega) = \sigma\omega + \delta \quad \text{and} \quad \phi(\omega) = \gamma.$$

Also let

$$Q(z) = zq'(z)\phi(q(z)) = \gamma z q'(z),$$

and

$$h(z) = \theta(q(z)) + Q(z) = \sigma q(z) + \delta + \gamma z q'(z).$$

From (3.12), we see that $Q(z)$ is starlike in $\mathbb{U}$, and

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left(\frac{\sigma}{\gamma} + 1 + \frac{zq''(z)}{q'(z)}\right) > 0.$$

Thus, by Lemma 2, we get the assertion of Theorem 3. □

Taking $a = 0,\ b = \gamma = 1$ and $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 3, we obtain the following corollary.

Corollary 6. *Let*

$$\Re\left(\frac{1+Az}{1+Bz}\right) > \max\left\{0, -\Re(\sigma)\right\}.$$

If

$$\left(z\mathcal{Q}_{\alpha,\beta}^{\lambda} f(z)\right)^{\mu} \left[\sigma + \mu\left(1 + \frac{z(\mathcal{Q}_{\alpha,\beta}^{\lambda} f)'(z)}{\mathcal{Q}_{\alpha,\beta}^{\lambda} f(z)}\right)\right] + \delta \prec \sigma\frac{1+Az}{1+Bz} + \delta + \frac{(A-B)z}{(1+Bz)^2},$$

then

$$\left(z\mathcal{Q}_{\alpha,\beta}^{\lambda} f(z)\right)^{\mu} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

By similarly applying the method of proof of Theorem 3, we easily get the following result.

Corollary 7. *Let q be univalent in $\mathbb{U}$. Suppose that q satisfies* (3.12) *and*

$$\varphi(z) = \left(\frac{azQ^{\lambda}_{\alpha,\beta}f(z)+bzQ^{\lambda}_{\alpha+1,\beta}f(z)}{a+b}\right)^{\mu} \cdot \tag{3.16}$$

$$\cdot\left[\sigma+\gamma\mu\left(1+\frac{az(Q^{\lambda}_{\alpha,\beta}f)'(z)+bz(Q^{\lambda}_{\alpha+1,\beta}f)'(z)}{aQ^{\lambda}_{\alpha,\beta}f(z)+bQ^{\lambda}_{\alpha+1,\beta}f(z)}\right)\right]+\delta \tag{3.17}$$

If

$$\varphi(z) \prec \sigma q(z)+\delta+\gamma z q'(z),$$

then

$$\left(\frac{azQ^{\lambda}_{\alpha,\beta}f(z)+bzQ^{\lambda}_{\alpha+1,\beta}f(z)}{a+b}\right)^{\mu} \prec q(z),$$

and q is the best dominant.

With the aid of Lemma 2 and Lemma 5, we can obtain the following results.

Theorem 4. *Let $0 \leqq \rho < 1$. Suppose that $\gamma \in \mathbb{C}$ with $\gamma \neq 0$ and satisfy either $|2\lambda\gamma(1-\rho)+1| \leqq 1$ or $|2\lambda\gamma(1-\rho)-1| \leqq 1$. If f satisfies*

$$\Re\left(\frac{Q^{\lambda+1}_{\alpha,\beta}f(z)}{Q^{\lambda}_{\alpha,\beta}f(z)}\right) > \rho, \tag{3.18}$$

then

$$\left(zQ^{\lambda}_{\alpha,\beta}f(z)\right)^{\gamma} \prec \frac{1}{(1-z)^{2\lambda\gamma(1-\rho)}} = q(z),$$

and q is the best dominant.

Proof. Let

$$\mathbb{H}(z) = \left(zQ^{\lambda}_{\alpha,\beta}f(z)\right)^{\gamma} \qquad (z \in \mathbb{U}). \tag{3.19}$$

Combining (1.7), (3.18) and (3.19), we have

$$1+\frac{z\mathbb{H}'(z)}{\lambda\gamma\mathbb{H}(z)} \prec \frac{1+(1-2\rho)z}{1-z} \qquad (z \in \mathbb{U}). \tag{3.20}$$

If we take

$$q(z) = \frac{1}{(1-z)^{2\lambda\gamma(1-\rho)}}, \qquad \theta(\omega)=1 \quad \text{and} \quad \phi(\omega)=\frac{1}{\lambda\gamma\omega},$$

then q is univalent by the condition of the theorem and Lemma 5. Further, it is easy to show that q, $\theta(\omega)$ and $\phi(\omega)$ satisfy the conditions of Lemma 2. Since

$$Q(z) = zq'(z)\phi\left(q(z)\right) = \frac{2(1-\rho)z}{1-z}$$

is univalent starlike in $\mathbb{U}$ and

$$h(z) = \theta(q(z)) + Q(z) = \frac{1 + (1 - 2\rho)z}{1 - z}$$

satisfy the conditions of Lemma 2. Thus the result follows from (3.20) immediately. The proof is complete. □

Corollary 8. *Let* $0 \leqq \rho < 1$ *and* $\gamma \geqq 1$. *If* $f \in \Sigma$ *satisfies the condition* (3.18), *then*

$$\Re\left(z\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)\right)^{2\lambda\gamma(1-\rho)} > 2^{-1/\gamma},$$

and the bound $2^{-1/\gamma}$ *is the best possible.*

By similarly applying the method of proof of Theorem 4, we easily get the following results.

Corollary 9. *Let* $0 \leqq \rho < 1$. *Suppose that* $\gamma \in \mathbb{C}$ *with* $\gamma \neq 0$ *and satisfy either* $|2\gamma(\alpha + \beta)(1 - \rho) + 1| \leqq 1$ *or* $|2\gamma(\alpha + \beta)(1 - \rho) - 1| \leqq 1$. *If* f *satisfies*

$$\Re\left(\frac{\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)}{\mathcal{Q}^{\lambda}_{\alpha+1,\beta}f(z)}\right) > \rho, \tag{3.21}$$

then

$$\left(z\mathcal{Q}^{\lambda}_{\alpha+1,\beta}f(z)\right)^{\gamma} \prec \frac{1}{(1 - z)^{2\gamma(\alpha+\beta)(1-\rho)}} = q(z),$$

and q *is the best dominant.*

Corollary 10. *Let* $0 \leqq \rho < 1$ *and* $\gamma \geqq 1$. *If* $f \in \Sigma$ *satisfies the condition* (3.21), *then*

$$\Re\left(z\mathcal{Q}^{\lambda}_{\alpha+1,\beta}f(z)\right)^{2\gamma(\alpha+\beta)(1-\rho)} > 2^{-1/\gamma},$$

and the bound $2^{-1/\gamma}$ *is the best possible.*

In the following, we provide some superordination results involving the integral operator $\mathcal{Q}^{\lambda}_{\alpha,\beta}$.

Theorem 5. *Let* q *be convex univalent in* $\mathbb{U}$ *and* $\Re(\eta) > 0$. *Also let*

$$z\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z) \in \mathcal{H}[q(0), 1] \cap Q$$

and

$$\eta z\mathcal{Q}^{\lambda+1}_{\alpha,\beta}f(z) + (1 - \eta)z\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)$$

is univalent in $\mathbb{U}$. *If*

$$q(z) + \frac{\eta z q'(z)}{\lambda} \prec \eta z\mathcal{Q}^{\lambda+1}_{\alpha,\beta}f(z) + (1 - \eta)z\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z), \tag{3.22}$$

then

$$q(z) \prec z\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z) \tag{3.23}$$

and q *is the best subordinant.*

Proof. Let $f \in \Sigma$ and suppose that

$$\varpi(z) = z\mathcal{Q}_{\alpha,\beta}^{\lambda} f(z).$$

We easily find that

$$\varpi(z) + \frac{\eta z\varpi'(z)}{\lambda} = \eta z\mathcal{Q}_{\alpha,\beta}^{\lambda+1} f(z) + (1-\eta) z\mathcal{Q}_{\alpha,\beta}^{\lambda} f(z). \tag{3.24}$$

Next, by means of (3.22), (3.24) and Lemma 3, we readily arrive at the assertion (3.23) of Theorem 5. □

In view of (1.8) and Lemma 3, and by similarly applying the method of proof of Theorem 5, we can get the following result.

Corollary 11. *Let q be convex univalent in $\mathbb{U}$ and $\Re(\eta) > 0$. Also let*

$$z\mathcal{Q}_{\alpha+1,\beta}^{\lambda} f(z) \in \mathcal{H}[q(0), 1] \cap Q$$

and

$$\eta z\mathcal{Q}_{\alpha,\beta}^{\lambda} f(z) + (1-\eta) z\mathcal{Q}_{\alpha+1,\beta}^{\lambda} f(z)$$

is univalent in $\mathbb{U}$. If

$$q(z) + \frac{\eta z q'(z)}{\beta+\alpha} \prec \eta z\mathcal{Q}_{\alpha,\beta}^{\lambda} f(z) + (1-\eta) z\mathcal{Q}_{\alpha+1,\beta}^{\lambda} f(z),$$

then

$$q(z) \prec z\mathcal{Q}_{\alpha+1,\beta}^{\lambda} f(z)$$

and q is the best subordinant.

In view of Lemma 4, and by similarly applying the method of proof of Theorem 5, we get the following results.

Corollary 12. *Let q be convex univalent in $\mathbb{U}$. Also let*

$$\left(\frac{az\mathcal{Q}_{\alpha,\beta}^{\lambda+1} f(z) + bz\mathcal{Q}_{\alpha,\beta}^{\lambda} f(z)}{a+b} \right)^{\mu} \in \mathcal{H}[q(0), 1] \cap Q$$

and ϱ be defined by (3.8) *is univalent in $\mathbb{U}$. If*

$$1 + \gamma \frac{zq'(z)}{q(z)} \prec \varrho(z),$$

then

$$q(z) \prec \left(\frac{az\mathcal{Q}_{\alpha,\beta}^{\lambda+1} f(z) + bz\mathcal{Q}_{\alpha,\beta}^{\lambda} f(z)}{a+b} \right)^{\mu},$$

and q is the best subordinant.

Corollary 13. *Let q be convex univalent in $\mathbb{U}$. Also let*

$$\left(\frac{az\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)+bz\mathcal{Q}^{\lambda}_{\alpha+1,\beta}f(z)}{a+b}\right)^{\mu}\in\mathcal{H}[q(0),1]\cap Q$$

and χ be defined by (3.11) *is univalent in $\mathbb{U}$. If*

$$1+\gamma\frac{zq'(z)}{q(z)}\prec\chi(z),$$

then

$$q(z)\prec\left(\frac{az\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)+bz\mathcal{Q}^{\lambda}_{\alpha+1,\beta}f(z)}{a+b}\right)^{\mu},$$

and q is the best subordinant.

Corollary 14. *Let q be convex univalent in $\mathbb{U}$. Also let*

$$z\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)\in\mathcal{H}[q(0),1]\cap Q$$

and ψ be defined by (3.13) *is univalent in $\mathbb{U}$. If q satisfies*

$$\Re\left(\frac{\sigma q'(z)}{\gamma}\right)>0, \tag{3.25}$$

and

$$\sigma q(z)+\delta+\gamma zq'(z)\prec\psi(z),$$

then

$$q(z)\prec\left(\frac{az\mathcal{Q}^{\lambda+1}_{\alpha,\beta}f(z)+bz\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)}{a+b}\right)^{\mu},$$

and q is the best subordinant.

Corollary 15. *Let q be convex univalent in $\mathbb{U}$. Also let*

$$z\mathcal{Q}^{\lambda}_{\alpha+1,\beta}f(z)\in\mathcal{H}[q(0),1]\cap Q$$

and φ be defined by (3.16) *is univalent in $\mathbb{U}$. If q satisfies* (3.25) *and*

$$\sigma q(z)+\delta+\gamma zq'(z)\prec\varphi(z),$$

then

$$q(z)\prec\left(\frac{az\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)+bz\mathcal{Q}^{\lambda}_{\alpha+1,\beta}f(z)}{a+b}\right)^{\mu},$$

and q is the best subordinant.

Finally, combining the above mentioned subordination and superordination results, we get the following sandwich-type results.

Corollary 16. *Let q_1 and q_2 be convex univalent in $\mathbb{U}$, and $\Re(\eta) > 0$. Suppose that q_2 satisfies* (3.1) *and $z\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z) \in \mathcal{H}[q(0), 1] \cap Q$. Let*

$$\eta z\mathcal{Q}^{\lambda+1}_{\alpha,\beta}f(z) + (1-\eta)z\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)$$

is univalent in $\mathbb{U}$. If

$$q_1(z) + \frac{\eta z q_1'(z)}{\lambda} \prec \eta z\mathcal{Q}^{\lambda+1}_{\alpha,\beta}f(z) + (1-\eta)z\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z) \prec q_2(z) + \frac{\eta z q_2'(z)}{\lambda},$$

then

$$q_1(z) \prec z\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z) \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Corollary 17. *Let q_3 and q_4 be convex univalent in $\mathbb{U}$, and $\Re(\eta) > 0$. Suppose that q_4 satisfies* (3.6) *and $z\mathcal{Q}^{\lambda}_{\alpha+1,\beta}f(z) \in \mathcal{H}[q(0), 1] \cap Q$. Let*

$$\eta z\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z) + (1-\eta)z\mathcal{Q}^{\lambda}_{\alpha+1,\beta}f(z)$$

is univalent in $\mathbb{U}$. If

$$q_3(z) + \frac{\eta z q_3'(z)}{\beta+\alpha} \prec \eta z\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z) + (1-\eta)z\mathcal{Q}^{\lambda}_{\alpha+1,\beta}f(z) \prec q_4(z) + \frac{\eta z q_4'(z)}{\beta+\alpha},$$

then

$$q_3(z) \prec z\mathcal{Q}^{\lambda}_{\alpha+1,\beta}f(z) \prec q_4(z)$$

and q_3 and q_4 are, respectively, the best subordinant and the best dominant.

Corollary 18. *Let q_5 be convex univalent and q_6 be univalent in $\mathbb{U}$. Suppose that q_6 satisfies* (3.7)*, and*

$$\left(\frac{az\mathcal{Q}^{\lambda+1}_{\alpha,\beta}f(z) + bz\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)}{a+b}\right)^{\mu} \in \mathcal{H}[q(0), 1] \cap Q.$$

Let ϱ be defined by (3.8) *is univalent in $\mathbb{U}$. If*

$$1 + \gamma\frac{zq_5'(z)}{q_5(z)} \prec \varrho(z) \prec 1 + \gamma\frac{zq_6'(z)}{q_6(z)},$$

then

$$q_5(z) \prec \left(\frac{az\mathcal{Q}^{\lambda+1}_{\alpha,\beta}f(z) + bz\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)}{a+b}\right)^{\mu} \prec q_6(z),$$

and q_5 and q_6 are, respectively, the best subordinant and the best dominant.

Corollary 19. *Let q_7 be convex univalent and q_8 be univalent in $\mathbb{U}$. Suppose that q_8 satisfies* (3.7)*, and*

$$\left(\frac{az\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)+bz\mathcal{Q}^{\lambda}_{\alpha+1,\beta}f(z)}{a+b}\right)^{\mu}\in\mathcal{H}[q(0),1]\cap Q.$$

Let χ be defined by (3.11) *is univalent in $\mathbb{U}$. If*

$$1+\gamma\frac{zq_7'(z)}{q_7(z)}\prec\chi(z)\prec1+\gamma\frac{zq_8'(z)}{q_8(z)},$$

then

$$q_7(z)\prec\left(\frac{az\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)+bz\mathcal{Q}^{\lambda}_{\alpha+1,\beta}f(z)}{a+b}\right)^{\mu}\prec q_8(z),$$

and q_7 and q_8 are, respectively, the best subordinant and the best dominant.

Corollary 20. *Let q_9 be convex univalent and q_{10} be univalent in $\mathbb{U}$. Suppose that q_9 satisfies* (3.25)*, q_{10} satisfies* (3.12)*, and*

$$z\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)\in\mathcal{H}[q(0),1]\cap Q.$$

Let ψ be defined by (3.13) *is univalent in $\mathbb{U}$. If*

$$\sigma q_9(z)+\delta+\gamma zq_9'(z)\prec\psi(z)\prec\sigma q_{10}(z)+\delta+\gamma zq_{10}'(z),$$

then

$$q_9(z)\prec\left(\frac{az\mathcal{Q}^{\lambda+1}_{\alpha,\beta}f(z)+bz\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)}{a+b}\right)^{\mu}\prec q_{10}(z),$$

and q_9 and q_{10} are, respectively, the best subordinant and the best dominant.

Corollary 21. *Let q_{11} be convex univalent and q_{12} be univalent in $\mathbb{U}$. Suppose that q_{11} satisfies* (3.25)*, q_{12} satisfies* (3.12)*, and*

$$z\mathcal{Q}^{\lambda}_{\alpha+1,\beta}f(z)\in\mathcal{H}[q(0),1]\cap Q.$$

Let φ be defined by (3.16) *is univalent in $\mathbb{U}$. If*

$$\sigma q_{11}(z)+\delta+\gamma zq_{11}'(z)\prec\varphi(z)\prec\sigma q_{12}(z)+\delta+\gamma zq_{12}'(z),$$

then

$$q_{11}(z)\prec\left(\frac{az\mathcal{Q}^{\lambda}_{\alpha,\beta}f(z)+bz\mathcal{Q}^{\lambda}_{\alpha+1,\beta}f(z)}{a+b}\right)^{\mu}\prec q_{12}(z),$$

and q_{11} and q_{12} are, respectively, the best subordinant and the best dominant.

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