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β-OPEN AND β-CLOSED SETS IN DITOPOLOGICAL TEXTURE SPACES

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Abstract

The authors define β-open and β-closed sets in a ditopological texture space and go on to study β-compactness and β-cocompactness, β-stability and β-costability, and β-dicompactness.

1 Introduction

One productive area of research in general topology, which has applications to several branches of science, is the investigation of various types of generalized open set and generalized continuous function, and the study of their structural properties. Early concepts in this area include semi-open sets and semi-continuity introduced by Levine [20], and the preopen sets and (weak) precontinuity of Mashhour *et al.* [21]. A topological space in which every preopen set is semi-open is called a PS-space [3]. A fairly recent application of this concept to digital topology is the result of R. Devi *et al.* [13] that the digital plane [18] is a PS-space.

Abd El Monsef *et al.* [2] introduced the notion of β-open set in topology, and the equivalent notion of semi-preopen set was given independently by Andrijević in [4], and further investigated by Ganster and Andrijević [14]. These and related notions have since been studied by many authors. The reader is referred to Caldas and Jafari [12], and the references therein, for further background and applications of β-open sets.

Textures and ditopological texture spaces were first introduced by the second author as a point-based setting for the study fuzzy sets, and this line of investigation continues, see for example [5, 6, 8, 9, 10], and more recently [23]. On the other hand, textures offer a convenient setting for the investigation of complement-free concepts in general, so much of the recent work has proceeded independently of the fuzzy setting. In particular, the notions of diuniformity and dimetric have been introduced

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in [22], while a textural analogue of the notion of proximity, called a diextremity, is given [26].

The study of compactness in ditopological texture spaces was begun in [5], continued in [11, 24] and extended to real compactness in [25]. In this paper we place β-compactness in a ditopological setting. All the arguments for studying properties related to β-openness and β-closedness in the topological setting apply equally well to this case, and since bitopologies and $\mathbb{L}$-topologies, for $\mathbb{L}$ a Hutton algebra, are special cases of ditopologies, new concepts such as β-stability and β-costability introduced here, may easily be specialized to these settings also.

To complete this introduction we recall various concepts from [8, 9] that will be needed later on in this paper.

Ditopological Texture Spaces: If S is a set, a *texturing* $\mathcal{S}$ of S is a subset of $\mathcal{P}(S)$ which is a point-separating, complete, completely distributive lattice containing S and $\emptyset$, and for which meet coincides with intersection and finite joins with union. The pair $(S, \mathcal{S})$ is then called a *texture*.

For a texture $(S, \mathcal{S})$, most properties are conveniently defined in terms of the *p-sets*

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}$$

and the *q-sets*,

$$Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\}.$$

The following are some basic examples of textures.

Examples 1.1. (1) If X is a set and $\mathcal{P}(X)$ the powerset of X, then $(X, \mathcal{P}(X))$ is the *discrete texture* on X. For $x \in X$, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$.

(2) Setting $\mathbb{I} = [0, 1]$, $\mathcal{I} = \{[0, r), [0, r] \mid r \in \mathbb{I}\}$ gives the *unit interval texture* $(\mathbb{I}, \mathcal{I})$. For $r \in \mathbb{I}$, $P_r = [0, r]$ and $Q_r = [0, r)$.

(3) The texture $(L, \mathcal{L})$ is defined by $L = (0, 1]$, $\mathcal{L} = \{(0, r] \mid r \in \mathbb{I}\}$. For $r \in L$, $P_r = (0, r] = Q_r$.

(4) If $(S, \mathcal{S})$, $(T, \mathcal{T})$ are textures, the *product texturing* $\mathcal{S} \otimes \mathcal{T}$ of $S \times T$ consists of arbitrary intersections of sets of the form $(A \times T) \cup (S \times B)$, $A \in \mathcal{S}, B \in \mathcal{T}$, and $(S \times T, \mathcal{S} \otimes \mathcal{T})$ is called the *product* of $(S, \mathcal{S})$ and $(T, \mathcal{T})$. For $s \in S, t \in T$ we clearly have $P_{(s,t)} = P_s \times P_t$ and $Q_{(s,t)} = (Q_s \times T) \cup (S \times Q_t)$.

Since a texturing $\mathcal{S}$ need not be closed under the operation of taking the set complement, the notion of topology is replaced by that of *dichotomous topology* or *ditopology*, namely a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of *open sets* τ satisfies

1. $S, \emptyset \in \tau$,
2. $G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau$ and
3. $G_i \in \tau, i \in I \implies \bigvee_i G_i \in \tau$,

and the set of *closed sets* κ satisfies

1. $S, \emptyset \in \kappa$,
2. $K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa$ and
3. $K_i \in \kappa, i \in I \implies \bigcap K_i \in \kappa$.

Hence a ditopology is essentially a "topology" for which there is no *a priori* relation between the open and closed sets.

For $A \in \mathcal{S}$ we define the *closure* $[A]$ and the *interior* $]A[$ of A under (τ, κ) by the equalities

$$[A] = \bigcap \{K \in \kappa \mid A \subseteq K\} \text{ and }]A[= \bigvee \{G \in \tau \mid G \subseteq A\}.$$

On the other hand, suppose that $(S, \mathcal{S})$ has a complementation σ, that is an involution $\sigma : \mathcal{S} \to \mathcal{S}$ satisfying $A, B \in \mathcal{S}, A \subseteq B \implies \sigma(B) \subseteq \sigma(A)$. Then if τ and κ are related by $\kappa = \sigma[\tau]$ we say that (τ, κ) is a *complemented ditopology* on $(S, \mathcal{S}, \sigma)$. In this case we have $\sigma([A]) =]\sigma(A)[$ and $\sigma(]A[) = [\sigma(A)]$.

Let $(S, \mathcal{S}), (T, \mathcal{T})$ be textures. In the following definition we consider the product texture $\mathcal{P}(S) \otimes \mathcal{T}$, and denote by $\overline{P}_{(s,t)}$, $\overline{Q}_{(s,t)}$, respectively the p-sets and q-sets for the product texture $(S \times T, \mathcal{P}(S) \otimes \mathcal{T})$ (c.f. Examples 1.1 (4)).

Direlation: Let $(S, \mathcal{S})$, $(T, \mathcal{T})$ be textures. Then

1. $r \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *relation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies

   R1 $r \not\subseteq \overline{Q}_{(s,t)}, P_{s'} \not\subseteq Q_s \implies r \not\subseteq \overline{Q}_{(s',t)}$.

   R2 $r \not\subseteq \overline{Q}_{(s,t)} \implies \exists s' \in S$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq \overline{Q}_{(s',t)}$.

2. $R \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *corelation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies

   CR1 $\overline{P}_{(s,t)} \not\subseteq R, P_s \not\subseteq Q_{s'} \implies \overline{P}_{(s',t)} \not\subseteq R$.

   CR2 $\overline{P}_{(s,t)} \not\subseteq R \implies \exists s' \in S$ such that $P_{s'} \not\subseteq Q_s$ and $\overline{P}_{(s',t)} \not\subseteq R$.

3. A pair (r, R), where r is a relation and R a corelation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$, is called a *direlation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$.

One of the most useful notions of (ditopological) texture spaces is that of difunction. A difunction is a special type of direlation.

Difunctions: Let (f, F) be a direlation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Then (f, F) is called a *difunction from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies the following two conditions.

DF1 For $s, s' \in S$, $P_s \not\subseteq Q_{s'} \implies \exists\, t \in T$ with $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F$.

DF2 For $t, t' \in T$ and $s \in S$, $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t')} \not\subseteq F \implies P_{t'} \not\subseteq Q_t$.

Image and Inverse Image: Let $(f, F) : (S, \mathcal{S}) \to (T, \mathcal{T})$ be a difunction.

1. For $A \in \mathcal{S}$, the *image* $f^{\to}A$ and the *co-image* $F^{\to}A$ are defined by

$$f^{\to}A = \bigcap\{Q_t \mid \forall s,\ f \not\subseteq \overline{Q}_{(s,t)} \implies A \subseteq Q_s\},$$
$$F^{\to}A = \bigvee\{P_t \mid \forall s,\ \overline{P}_{(s,t)} \not\subseteq F \implies P_s \subseteq A\}.$$

2. For $B \in \mathcal{T}$, the *inverse image* $f^{\leftarrow}B$ and the *inverse co-image* $F^{\leftarrow}B$ are defined by

$$f^{\leftarrow}B = \bigvee\{P_s \mid \forall t,\ f \not\subseteq \overline{Q}_{(s,t)} \implies P_t \subseteq B\},$$
$$F^{\leftarrow}B = \bigcap\{Q_s \mid \forall t,\ \overline{P}_{(s,t)} \not\subseteq F \implies B \subseteq Q_t\}.$$

For a difunction, the inverse image and the inverse co-image are equal, but the image and co-image are usually not.

Bicontinuity: The difunction (f, F) is called *continuous* if $B \in \tau_T \implies F^{\leftarrow}B \in \tau_S$, *cocontinuous* if $B \in \kappa_T \implies f^{\leftarrow}B \in \kappa_S$, and *bicontinuous* if it is both continuous and cocontinuous.

On the other hand (f, F) is *open* (*co-open*) if $A \in \tau_S \implies f^{\to}A \in \tau_T$ ($F^{\to}A \in \tau_T$). Also, (f, F) is *closed* (*coclosed*) if $A \in \kappa_S \implies f^{\to}A \in \kappa_T$ ($F^{\to}A \in \kappa_T$).

Injective- surjective difunction: Let $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ be a difunction. Then (f, F) is called *surjective* if it satisfies the condition

SUR. For $t, t' \in T$, $P_t \not\subseteq Q_{t'} \implies \exists\, s \in S$ with $f \not\subseteq \overline{Q}_{(s,t')}$ and $\overline{P}_{(s,t)} \not\subseteq F$.

(f, F) is called *injective* if it satisfies the condition

INJ. For $s, s' \in S$ and $t \in T$, $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F \implies P_s \not\subseteq Q_{s'}$.

If (f, F) is both injective and surjective, then it is called *bijective*.

For terms from lattice theory not defined here the reader is referred to [15]. Also [1] is our general reference for category theory.

2 β-open and β-closed sets

We begin by recalling [2] that a subset A of a topological space X is called β-open if $A \subseteq \operatorname{cl}\operatorname{int}\operatorname{cl} A$. Dually, A is β-closed if $X \setminus A$ is β-open, equivalently if it satisfies $\operatorname{int}\operatorname{cl}\operatorname{int} A \subseteq A$. This leads to the following analogous concepts in a ditopological texture space.

Definition 2.1. Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $A \in \mathcal{S}$.

1. A is *β-open* if $A \subseteq [\,]\,[A]\,[\,]$.
2. A is *β-closed* if $]\,[\,]A[\,]\,[\subseteq A$.

We denote by $\beta O(S, \mathcal{S}, \tau, \kappa)$, or when there can be no confusion by $\beta O(S)$ or even just βO, the set of β-open sets in $\mathcal{S}$. Likewise, $\beta C(S, \mathcal{S}, \tau, \kappa)$, $\beta C(S)$ or βC will denote the set of β-closed sets.

We also recall from [16, 21] the notions of preopen and preclosed sets:

Definition 2.2. Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $A \in \mathcal{S}$.

1. A is *preopen* if $A \subseteq \big]\,[A]\,\big[$.
2. A is *preclosed* if $\big[\,]A[\,\big] \subseteq A$.

We denote by $PO(S, \mathcal{S}, \tau, \kappa)$, more simply by $PO(S)$ or even just PO, the set of preopen sets in $\mathcal{S}$. Likewise, $PC(S, \mathcal{S}, \tau, \kappa)$, $PC(S)$ or PC will denote the set of preclosed sets.

We note for future reference the following elementary facts.

Lemma 2.3. *For a given ditopological texture space:*

1. $\tau \subseteq PO \subseteq \beta O$ *and* $\kappa \subseteq PC \subseteq \beta C$.
2. PO *and* βO *are closed under arbitrary joins.*
3. PC *and* βC *are closed under arbitrary intersections.*

Proof. The results for PO and PC are proved in [16], the proofs for βO and βC are similar and are omitted. □

More generally we note that:

$$\begin{aligned} A \in PO,\ A \subseteq B \subseteq [A] &\implies B \in \beta O, \\ A \in PC,\ ]A[\ \subseteq B \subseteq A &\implies B \in \beta C. \end{aligned} \tag{2.1}$$

Again, the proofs are elementary and are omitted. These results say that a semi-preopen set in the sense of Andrijevic [4] is β-open, while a semi-preclosed set is β-closed.

Generally there is no relation between the β-open and β-closed sets, but for a complemented ditopological space we have the following result.

Proposition 2.4. *Let* $(S, \mathcal{S}, \sigma, \tau, \kappa)$ *be a complemented ditopological texture space. Then*

$$A \in \beta C \iff \sigma(A) \in \beta O.$$

Proof. Immediate on applying $\sigma([B]) =]\sigma(B)[$ and $\sigma(]B[) = [\sigma(B)]$ for $B \in \mathcal{S}$. □

Examples 2.5. (1) If $(X, \mathcal{T})$ is a topological space then $(X, \mathcal{P}(X), \pi_X, \mathcal{T}, \mathcal{T}^c)$ is a complemented ditopological texture space. Here $\pi_X(Y) = X \setminus Y$ for $Y \subseteq X$ is the standard complementation on $(X, \mathcal{P}(X))$ and $\mathcal{T}^c = \{\pi_X(G) \mid G \in \mathcal{T}\}$. Clearly the β-open, β-closed (preopen, preclosed) sets in $(X, \mathcal{T})$ correspond precisely to the β-open, β-closed (preopen, preclosed) sets, respectively, in $(X, \mathcal{P}(X), \pi_X, \mathcal{T}, \mathcal{T}^c)$.

(2) For the unit interval texture $(\mathbb{I}, \mathcal{I})$ of Examples 1.1 (2), let ι be the complementation $\iota([0, r)) = [0, 1 - r]$, $\iota([0, r]) = [0, 1 - r)$, and $(\tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ the standard complemented ditopology given by

$$\tau_{\mathbb{I}} = \{[0, r) \mid r \in \mathbb{I}\} \cup \{\mathbb{I}\}, \quad \kappa_{\mathbb{I}} = \{[0, r] \mid r \in \mathbb{I}\} \cup \{\emptyset\}.$$

For this space we clearly have $PO = \tau_{\mathbb{I}}$, $PC = \kappa_{\mathbb{I}}$ and $\beta O = \beta C = \mathcal{I}$.

We recall that a function between topological spaces is called β-continuous [2] if the inverse image of each open set is β-open, and $M\beta$-continuous [17] if the inverse image of each β-open set is β-open. This leads to the following concepts for a difunction between ditopological texture spaces.

Definition 2.6. The difunction $(f,F):(S_1,\mathcal{S}_1,\tau_1,\kappa_1)\to(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ is called:

1. *β-continuous* (*$M\beta$-continuous*) if $f^{\leftarrow}B\in\beta O(S_1)$ for all $B\in\tau_2$ ($B\in\beta O(S_2)$).
2. *β-cocontinuous* (*$M\beta$-cocontinuous*) if $F^{\leftarrow}B\in\beta C(S_1)$ for all $B\in\kappa_2$ ($B\in\beta C(S_2)$).
3. *β-bicontinuous* (*$M\beta$-bicontinuous*) if it is both β-continuous and β-cocontinuous ($M\beta$-continuous and $M\beta$-cocontinuous).

Clearly $M\beta$-continuity (-cocontinuity, -bicontinuity) is stronger than β-continuity (respectively, -cocontinuity, -bicontinuity). Since $M\beta$-bicontinuity is clearly preserved by composition of difunctions and possessed by the identity difunctions, ditopological texture spaces and $M\beta$-bicontinuous difunctions form a category that we will denote by **M*β*dfDitop**. Note that Examples 2.5 (1) leads to a functor $\mathfrak{F}$ from the category **M*β*Top** of topological spaces and $M\beta$-continuous functions to **M*β*dfDitop** defined by

$$\mathfrak{F}((X,\mathcal{T})\xrightarrow{f}(Y,\mathcal{V}))=(X,\mathcal{P}(X),\pi_X,\mathcal{T},\mathcal{T}^c)\xrightarrow{(f,f^c)}(Y,\mathcal{P}(Y),\pi_Y,\mathcal{V},\mathcal{V}^c).$$

Similar functors may be defined for bitopological spaces, and for Hutton spaces (c.f. [6, 8]), but we omit the details.

In order to give useful characterizations of the above continuity properties we need the following definition.

Definition 2.7. Let $(S,\mathcal{S},\tau,\kappa)$ be a ditopological texture space and $M\in\mathcal{S}$.

(a) $[M]_\beta=\bigcap\{A\in\beta C\mid M\subseteq A\}$.

(b) $]M[_\beta=\bigvee\{A\in\beta O\mid A\subseteq M\}$.

Where it is necessary to indicate the space involved we may write $[M]^S_\beta$, $]M[^S_\beta$, respectively.

By Lemma 2.3 we have $]M[_\beta\in\beta O$, $[M]_\beta\in\beta C$, while $M\in\beta O\iff M=]M[_\beta$ and $M\in\beta C\iff M=[M]_\beta$.

The following characterizations should be compared with those given for continuity and cocontinuity in [11].

Proposition 2.8. *Let $(f,F):(S_1,\mathcal{S}_1,\tau_1,\kappa_1)\to(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ be a difunction.*

1. *The following are equivalent:*

   (a) *(f,F) is $M\beta$-continuous.*

(b) *For each* $A \in \mathcal{S}_1$ *we have* $]F^{\to}A[^{S_2}_{\beta} \subseteq F^{\to}\,]A[^{S_1}_{\beta}$.

(c) *For each* $B \in \mathcal{S}_2$ *we have* $f^{\leftarrow}\,]B[^{S_2}_{\beta} \subseteq \,]f^{\leftarrow}B[^{S_1}_{\beta}$.

2. *The following are equivalent:*

(a) (f,F) *is* $M\beta$*-cocontinuous.*

(b) *For each* $A \in \mathcal{S}_1$ *we have* $f^{\to}[A]^{S_1}_{\beta} \subseteq [f^{\to}A]^{S_2}_{\beta}$.

(c) *For each* $B \in \mathcal{S}_2$ *we have* $[F^{\leftarrow}B]^{S_1}_{\beta} \subseteq F^{\leftarrow}[B]^{S_2}_{\beta}$.

Proof. We prove (1), leaving the dual proof of (2) to the interested reader.

(a) $\Longrightarrow$ (b). Take $A \in \mathcal{S}_1$. Then

$$f^{\leftarrow}]F^{\to}A[^{S_2}_{\beta} \subseteq f^{\leftarrow}(F^{\to}A) \subseteq A$$

by [8, Theorem 2.24 (2 a)]. Now $f^{\leftarrow}]F^{\to}A[^{S_2}_{\beta} = F^{\leftarrow}]F^{\to}A[^{S_2}_{\beta} \in \beta O(S_1)$ by $M\beta$-continuity, so $f^{\leftarrow}]F^{\to}A[^{S_2}_{\beta} \subseteq \,]A[^{S_1}_{\beta}$ and applying [8, Theorem 2.4 (2 b)] gives

$$]F^{\to}A[^{S_2}_{\beta} \subseteq F^{\to}\big(f^{\leftarrow}]F^{\to}A[^{S_2}_{\beta}\big) \subseteq F^{\to}]A[^{S_1}_{\beta},$$

which is the required inclusion.

(b) $\Longrightarrow$ (c). Take $B \in \mathcal{S}_2$. Applying inclusion (b) to $A = f^{\leftarrow}B$ and using [8, Theorem 2.4 (2 b)] gives

$$]B[^{S_2}_{\beta} \subseteq]F^{\to}(f^{\leftarrow}B)[^{S_2}_{\beta} \subseteq F^{\to}]f^{\leftarrow}B[^{S_1}_{\beta}.$$

Hence, $f^{\leftarrow}]B[^{S_2}_{\beta} \subseteq f^{\leftarrow}\big(F^{\to}\,]f^{\leftarrow}B[^{S_1}_{\beta}\,\big) \subseteq \,]f^{\leftarrow}B[^{S_2}_{\beta}$ by [8, Theorem 2.24 (2 a)].

(c) $\Longrightarrow$ (a). Applying (c) for $B \in \beta O(S_2)$ gives

$$f^{\leftarrow}B = f^{\leftarrow}]B[^{S_2}_{\beta} \subseteq \,]f^{\leftarrow}B[^{S_1}_{\beta},$$

so $F^{\leftarrow}B = f^{\leftarrow}B = \,]f^{\leftarrow}B[^{S_1}_{\beta} \in \beta O(S_1)$. Hence, (f,F) is continuous. □

The following proposition gives corresponding characterizations for β-continuity and β-cocontinuity. We omit the proof which follows the same lines as that of Proposition 2.8.

Proposition 2.9. *Let* $(f,F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ *be a difunction.*

1. *The following are equivalent:*

(a) (f,F) *is* β*-continuous.*

(b) *For each* $A \in \mathcal{S}_1$ *we have* $]F^{\to}A[^{S_2} \subseteq F^{\to}\,]A[^{S_1}_{\beta}$.

(c) *For each* $B \in \mathcal{S}_2$ *we have* $f^{\leftarrow}\,]B[^{S_2} \subseteq \,]f^{\leftarrow}B[^{S_1}_{\beta}$.

2. *The following are equivalent:*

(a) (f,F) *is* $M\beta$*-cocontinuous.*

(b) *For each* $A \in \mathcal{S}_1$ *we have* $f^{\to}[A]^{S_1}_{\beta} \subseteq [f^{\to}A]^{S_2}$.

(c) *For each* $B \in \mathcal{S}_2$ *we have* $[F^{\leftarrow}B]^{S_1}_{\beta} \subseteq F^{\leftarrow}[B]^{S_2}$.

3 β-compactness and β-cocompactness

We begin by recalling the definition of a β-compact topological space.

Definition 3.1. [17] Let $(X, \mathcal{T})$ be a topological space. If every cover $\{A_j \mid j \in J\}$ of X by β-open sets A_j of X has a finite subcover then $(X, \mathcal{T})$ is called *β-compact.*

We now give an analogous definition of β-compactness in ditopological texture spaces. As expected, there is also the dual notion of β-cocompactness.

Definition 3.2. A ditopology (τ, κ) on $(S, \mathcal{S})$ is called:

1. *β-compact* if every cover of S by β-open sets has a finite subcover.

2. *β-cocompact* if every cocover of $\emptyset$ by β-closed sets has a finite sub-cocover.

Here we recall that $\mathcal{C} = \{A_j \mid j \in J\}$, $A_j \in \mathcal{S}$ is a cover of S (a cocover of $\emptyset$) if $\bigvee \mathcal{C} = S$ $\left(\bigcap \mathcal{C} = \emptyset\right)$. Since strong compactness (strong cocompactness) [16] is defined in the same way using preopen (preclosed) sets, we have:

Proposition 3.3. *For a ditopological texture space:*

1. β-compact $\implies$ strongly compact $\implies$ compact.

2. β-cocompact $\implies$ strongly cocompact $\implies$ cocompact.

Proof. Immediate from Lemma 2.3 (1). □

It is known from [16, Example 3.4] that compact $\not\Rightarrow$ strongly compact and cocompact $\not\Rightarrow$ strongly cocompact, while the following example establishes that the remaining implications also cannot be reversed in general.

Example 3.4. The unit interval texture $(\mathbb{I}, \mathcal{I})$ with the natural ditopology $(\tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ described in Examples 2.5 (2) is easily seen to be β-compact, but we may modify this space to produce a strongly compact space that is not β-compact as follows.

Consider the product of the texture $(\{a, b\}, \{\emptyset, \{a\}, \{a, b\}\})$ and the principle subtexture [7] of $(\mathbb{I}, \mathcal{I})$ on the set $[0, 1)$. The resulting plain texturing of $S = \{a, b\} \times [0, 1)$ is easily seen to consist of unions of sets of the form

$$\begin{aligned} &\{a, b\} \times [0, r],\ 0 \le r < 1, \quad \{a\} \times [0, s],\ 0 \le s < 1, \\ &\{a, b\} \times [0, r),\ 0 < r < 1, \quad \{a\} \times [0, s),\ 0 < s < 1, \end{aligned}$$

together with $\emptyset$ and S. We define a ditopology on this texture by setting

$$\begin{aligned} \tau &= \{\emptyset\} \cup \{\{a\} \times [0, s) \mid 0 < s < 1\} \cup \{S\}, \\ \kappa &= \{\emptyset\} \cup \{\{a, b\} \times [0, r] \mid 0 \le r < 1\} \cup \{S\}. \end{aligned}$$

It is easy to see that the elements of κ are β-open, whence $\{\{a, b\} \times [0, 1 - \frac{1}{n}] \mid n = 2, 3, \ldots\}$, is a cover of S by β-open sets which has no finite subcover. Hence this space is not β-compact. On the other hand it is clear that $PO(S) = \tau$, so the only way of obtaining a covering of S by preopen sets is to include S, whence $\{S\}$ is a finite subcover and the space is strongly compact.

It is left to the interested reader to produce a modification of this example that is strongly cocompact but not β-cocompact.

The following examples show that in general β-compactness and β-cocompactness are independent.

Examples 3.5. Consider the texture $(L, \mathcal{L})$ of Examples 1.1 (3).

(1) Define the ditopology (τ, κ) by $\tau = \{\emptyset, L\}$ and $\kappa = \mathcal{L}$. Since the only β-open sets are $\emptyset$ and L we see that (τ, κ) is β-compact. However, it is not β-cocompact since it is not cocompact by [11, Examples 2.2 (1)].

(2) Dually, let $\tau = \mathcal{L}$ and $\kappa = \{\emptyset, L\}$. Then the ditopology (τ, κ) is β-cocompact but not β-compact.

On the other hand, for complemented ditopological texture spaces we do have the equivalence of these two properties.

Proposition 3.6. *Let (τ, κ) be a complemented ditopology on $(S, \mathcal{S}, \sigma)$. Then $(S, \mathcal{S}, \sigma, \tau, \kappa)$ is β-compact if and only if it is β-cocompact.*

Proof. Suppose that (τ, κ) is β-compact and let $\mathcal{F} = \{F_j \mid j \in J\}$ be a family of β-closed sets with $\bigcap \mathcal{F} = \emptyset$. Clearly $\mathcal{G} = \{\sigma(F_j) \mid j \in J\}$ is a family of β-open sets. Moreover,

$$\bigvee \mathcal{G} = \bigvee \{\sigma(F_j) \mid j \in J\} = \sigma\Big(\bigcap \{F_j \mid j \in J\}\Big) = \sigma(\emptyset) = S,$$

and so we have $J' \subseteq J$ finite with $\bigvee \{\sigma(F_j) \mid j \in J'\} = S$. Hence $\bigcap \{F_j \mid j \in J'\} = \emptyset$, and we see that (τ, κ) is β-cocompact.

Likewise, if (τ, κ) is β-cocompact then it is β-compact. □

Theorem 3.7. *Let $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be an $M\beta$-continuous difunction. If $A \in \mathcal{S}_1$ is β-compact then $f^{\to}A \in \mathcal{S}_2$ is β-compact.*

Proof. Take $f^{\to}A \subseteq \bigvee_{j \in J} G_j$, where $G_j \in \beta O(S_2)$, $j \in J$. Now by [8, Theorem 2.24 (2 a) and Corollary 2.12 (2)] we have

$$A \subseteq F^{\leftarrow}(f^{\to}A) \subseteq F^{\leftarrow}\Big(\bigvee_{j \in J} G_j\Big) = \bigvee_{j \in J} F^{\leftarrow}G_j.$$

Also, $F^{\leftarrow}G_j \in \beta O(S_1)$ since (f, F) is $M\beta$-continuous, so by the β-compactness of A there exists $J' \subseteq J$ finite such that $A \subseteq \bigcup_{j \in J'} F^{\leftarrow}G_j$. Hence

$$f^{\to}A \subseteq f^{\to}\Big(\bigcup_{j \in J'} F^{\leftarrow}G_j\Big) = \bigcup_{j \in J'} f^{\to}(F^{\leftarrow}G_j) \subseteq \bigcup_{j \in J'} G_j$$

by [8, Corollary 2.12 (2) and Theorem 2.24 (2 b)]. This establishes that $f^{\to}A$ is β-compact. □

Proposition 3.8. *Let $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be a surjective $M\beta$-continuous difunction. Then if $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ is β-compact so is $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$.*

Proof. This follows by taking $A = S_1$ in Theorem 2.4 and noting that $f^{\to}S_1 = f^{\to}(F^{\leftarrow}S_2) = S_2$ by [8, Proposition 2.28 (1 c) and Corollary 2.33 (1)]. □

As expected, we have dual results for cocompactness. We omit the proofs.

Theorem 3.9. *Let $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be an $M\beta$-cocontinuous difunction. If $A \in \mathcal{S}_1$ is β-cocompact then $F^{\to}A \in \mathcal{S}_2$ is β-cocompact.* □

Proposition 3.10. *Let $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be an a surjective $M\beta$-cocontinuous difunction. Then if $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ is β-cocompact so is $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$.* □

4 β-stability and β-costability

The notion of stability for bitopological spaces was introduced by Ralph Kopperman [19]. The analogous notion, and its dual, were given for ditopologies in [5], and studied in greater detail in [11]. We now wish to generalize these concepts for β-open and β-closed sets. The following definition would seem to be appropriate.

Definition 4.1. Let (τ, κ) be a ditopology on the texture space $(S, \mathcal{S})$.

1. (τ, κ) will be called *β-stable* if every β-closed set $F \in \mathcal{S} \setminus \{S\}$ is β-compact in S. That is, whenever G_j, $j \in J$, are β-open sets in $(S, \mathcal{S}, \tau, \kappa)$ satisfying $F \subseteq \bigvee_{j \in J} G_j$, there exists a finite subset J' of J for which $F \subseteq \bigcup_{j \in J'} G_j$.

2. (τ, κ) will be called *β-costable* if every β-open set $G \in \mathcal{S} \setminus \emptyset$ is β-cocompact in S. That is, whenever F_j, $j \in J$, are β-closed sets in $(S, \mathcal{S}, \tau, \kappa)$ satisfying $\bigcap_{j \in J} F_j \subseteq G$, there exists a finite subset J' of J for which $\bigcap_{j \in J'} F_j \subseteq G$.

The following examples show that in general β-stability (β-costability) are unrelated to β-compactness (β-cocompactness), respectively.

Examples 4.2. Consider the texture $(L, \mathcal{L})$ of Examples 1.1 (3).

(1) Let $\tau = \{(0, r] \mid 0 \le r \le 1/2\} \cup \{L\}$ and $\kappa = \{(0, r] \mid 1/2 \le r \le 1\} \cup \{\emptyset\}$. If we take $r \in L$ with $1/2 < r < 1$ and set $A = (0, r]$, then $[A] = A$ and so $][A][=]A[= (0, 1/2] \in \kappa$, from which we see $A \not\subseteq [\,]\![A]\![\,]$, that is A is not β-open. It follows that the only β-open sets are the open sets, so (τ, κ) is β-compact because any open cover of L must contain L. On the other hand the set $F = (0, 1/2]$ is closed, and hence β-closed, and it is clearly not compact so not β-compact. It follows that (τ, κ) is not β-stable.

A dual argument shows that this space is also β-cocompact but not β-costable.

(2) Let $\tau = \mathcal{L}$ and $\kappa = \{\emptyset, L\}$. The ditopology (τ, κ) is not β-compact because it is not compact. On the other hand (τ, κ) is β-stable because in this space every

β-closed set is closed, and the only closed set different from L is $\emptyset$, which is trivially β-compact.

(3) Dually, let $\tau = \{\emptyset, L\}$ and $\kappa = \mathcal{L}$. This ditopology is β-costable but not β-cocompact.

Recalling from [16] that strong stability and strong costability are defined analogously using preopen and preclosed sets, we see from Lemma 2.3 (1) that:

Proposition 4.3. *For a ditopological texture space,*

1. *β-stable $\implies$ strongly stable $\implies$ stable.*
2. *β-costable $\implies$ strongly costable $\implies$ costable.* □

That strong stability (strong costability) is in general strictly more powerful than stability (costability) is known from [16], and the following example establishes that the remaining implications also cannot be reversed in general.

Example 4.4. Consider the unit interval texture $(\mathbb{I}, \mathcal{I}, \tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ with its standard ditopology. Since $PO(\mathbb{I}) = \tau_{\mathbb{I}}$, $PC(\mathbb{I}) = \kappa_{\mathbb{I}}$ this space is strongly stable and strongly costable because it is stable and costable [11]. On the other hand $\beta O(\mathbb{I}) = \beta C(\mathbb{I}) = \mathcal{I}$ so, for example $[0, 1)$ is a β-closed set that is clearly not β-compact. Hence this space is not β-stable, and similarly it is not β-costable.

The following examples show that in general β-stability and β-costability are independent of one another.

Examples 4.5. Consider again the texture $(L, \mathcal{L})$.

(1) Let $\tau = \mathcal{L}$ and $\kappa = \{\emptyset, (0, 1/2], L\}$. Then clearly, $\beta O(L) = \mathcal{L}$ and $\beta C(L) = \kappa$. Since $\mathcal{C} = \{(0, 1/2 - 1/n] \mid n = 3, 4, 5, \ldots\}$ is a β-open cover of the β-closed set $[0, 1/2]$ with no finite subcover, we see that (τ, κ) is not β-stable. On the other hand it is β-costable because $\beta C(L) = \kappa$ is finite.

(2) Dually, let $\tau = \{\emptyset, (0, 1/2], L\}$, $\kappa = \mathcal{L}$. Then $(L, \mathcal{L}, \tau, \kappa)$ is β-stable but not β-costable.

However, for complemented ditopological texture spaces these concepts are equivalent, as we now show.

Proposition 4.6. *Let $(S, \mathcal{S}, \sigma)$ be a texture with complement σ and let (τ, κ) be a complemented ditopology on $(S, \mathcal{S}, \sigma)$. Then (τ, κ) is β-stable if and only if it is β-costable.*

Proof. Let (τ, κ) be β-stable, let G be a β-open set with $G \neq \emptyset$ and $\mathcal{D}$ a β-closed cocover of G. Set $K = \sigma(G)$. Then K is β-closed and satisfies $K \neq S$. Hence K is strongly compact. Let $\mathcal{C} = \{\sigma(F) \mid F \in \mathcal{D}\}$. Since $\bigcap \mathcal{D} \subseteq G$ we have $K \subseteq \bigvee \mathcal{C}$, i.e. $\mathcal{C}$ is an β-open cover of K. Hence there exists $F_1, F_2, \ldots, F_n \in \mathcal{D}$ so that

$$K \subseteq \sigma(F_1) \cup \sigma(F_2) \cup \ldots \cup \sigma(F_n) = \sigma(F_1 \cap F_2 \cap \ldots \cap F_n).$$

This gives $F_1 \cap F_2 \cap \ldots \cap F_n \subseteq \sigma(K) = G$, so G is β-cocompact in S. Hence (τ, κ) is β-costable.

The proof that β-costable implies β-stable is the dual of the above, and is omitted. □

Next let us investigate the preservation of β-stability and β-costability under surjective difunctions.

Theorem 4.7. *Let $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$, $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be ditopological texture spaces with $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ β-stable, and $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ an $M\beta$-bicontinuous surjective difunction. Then $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is β-stable.*

Proof. Take $K \in \beta C(S_2)$ with $K \neq S_2$. Since (f, F) is β-cocontinuous, $f^{\leftarrow}K \in \beta C(S_1)$. Let us prove that $f^{\leftarrow}K \neq S_1$. Assume the contrary. Since $f^{\leftarrow}S_2 = S_1$, by [8, Lemma 2.28 (1 c)] we have $f^{\leftarrow}S_2 \subseteq f^{\leftarrow}K$, whence $S_2 \subseteq K$ by [8, Corollary 2.33 (1 ii)] as (f, F) is surjective. This is a contradiction, so $f^{\leftarrow}(K) \neq S_1$. Hence $f^{\leftarrow}(K)$ is β-compact in $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ by β-stability. As (f, F) is $M\beta$-continuous, $f^{\rightarrow}(f^{\leftarrow}K)$ is β-compact for the ditopology (τ_2, κ_2) by Theorem 3.7, and by [8, Corollary 2.33 (1)] this set is equal to K. This establishes that $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is β-stable. □

Theorem 4.8. *Let $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$, $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be ditopological texture spaces with $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ β-costable, and $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ an $M\beta$-bicontinuous surjective difunction. Then $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is β-costable.*

Proof. This is dual to the proof of Theorem 4.7, and we omit the details. □

5 β-dicompactness

We end by generalizing the notion of β-dicompact space.

Definition 5.1. A ditopological texture space will be called *β-dicompact* if it is β-compact, β-cocompact, β-stable and β-costable.

As a consequence of Propositions 3.8, 3.10 and Theorems 4.7, 4.8 we may state the following:

Theorem 5.2. *β-dicompactness is preserved under a surjective $M\beta$-bicontinuous difunction.*

To give non-trivial characterizations of β-dicompactness, we adapt the following definitions from [5].

Definition 5.3. Let (τ, κ) be a ditopology on $(S, \mathcal{S})$.

1. A set $\mathcal{D} \subseteq \mathcal{S} \times \mathcal{S}$ is called a *difamily* on $(S, \mathcal{S})$. A difamily $\mathcal{D}$ satisfying $\mathcal{D} \subseteq \beta O \times \beta C$ is *β-open, co-β-closed*, one satisfying $\mathcal{D} \subseteq \beta C \times \beta O$ is *β-closed, co-β-open.*

2. A difamily $\mathcal{D}$ has the *finite exclusion property* (fep) if whenever $(F_j, G_j) \in \mathcal{D}$, $j = 1, 2, \ldots, n$ we have $\bigcap_{j=1}^{n} F_j \not\subseteq \bigcup_{j=1}^{n} G_j$.

3. A β-closed, co-β-open difamily $\mathcal{D}$ with $\bigcap\{F \mid F \in \operatorname{dom}\mathcal{D}\} \not\subseteq \bigvee\{G \mid G \in \operatorname{ran}\mathcal{D}\}$ is said to be *bound* in $(S, \mathcal{S}, \tau, \kappa)$.

4. A difamily $\mathcal{D} = \{(G_j, F_j) \mid j \in J\}$ is called a *dicover* of $(S, \mathcal{S})$ if for all partitions J_1, J_2 of J (including the trivial partitions) we have

$$\bigcap_{j \in J_1} F_j \subseteq \bigvee_{j \in J_2} G_j.$$

Theorem 5.4. *For a ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ the following are equivalent:*

1. *$(S, \mathcal{S}, \tau, \kappa)$ is β-dicompact.*

2. *Every β-closed, co-β-open difamily with the finite exclusion property is bound.*

3. *Every β-open, co-β-closed dicover has a sub-dicover which is finite and co-finite.*

Proof. (1) $\Longrightarrow$ (2) Suppose that (1) holds, but that we have a β-closed, co-β-open difamily $\mathcal{B} = \{(F_j, G_j) \mid j \in J\}$ with the fep, which is not bound in $(S, \mathcal{S}, \tau, \kappa)$. Let $F = \bigcap_{i \in I} F_i$. Then F is β-closed by Lemma 2.3 (3), and $F \subseteq \bigvee_{i \in I} G_i$ since $\mathcal{B}$ is not bound. According as $F \neq S$ or $F = S$ we may use β-stability or β-compactness, respectively, to show the existence of a finite subset J_1 of J with $F \subseteq \bigcup_{j \in J_1} G_j$. Now let $G = \bigcup_{j \in J_1} G_j$. By Lemma 2.3 (2), G is a β-open set. Also, $\bigcap_{j \in J} F_j \subseteq G$. Hence, according as $G \neq \emptyset$ or $G = \emptyset$, we may use β-costability or β-cocompactness, respectively, to show that $\bigcap_{j \in J_2} F_j \subseteq G$ for some finite subset J_2 of J. Since now $\bigcap_{j \in J_1 \cup J_2} F_j \subseteq \bigcup_{j \in J_1 \cup J_2} G_j$ we have a contradiction to the fact that $\mathcal{B}$ has the fep.

(2) $\Longrightarrow$ (3) Suppose that $\mathcal{C} = \{(G_i, F_i) \mid i \in I\}$ is a β-open, co-β-closed dicover with no finite, co-finite sub-dicover. As in the proof of [5, Theorem 3.5] we consider the set $\mathcal{F}$ of functions f satisfying

(a) $\operatorname{dom} f$ is a set of finite subsets of I.

(b) $\forall\, J \in \operatorname{dom} f,\ f(J) = (f_1(J), f_2(J)) \in \mathcal{P}_J^{\star\star}$.

(c) $J_1, \ldots, J_n \in \operatorname{dom} f \implies J_1 \cup \ldots \cup J_n \in \operatorname{dom} f$.

(d) $J, K \in \operatorname{dom} f,\ J \subseteq K \implies f_l(J) = J \cap f_l(K),\ l = 1, 2$.

Here

$$\mathcal{P}_J^{\star\star} = \{(J_1, J_2) \in \mathcal{P}_J^{\star} \mid \forall\, K \text{ finite}, J \subseteq K \subseteq I,\ \exists\, (K_1, K_2) \in \mathcal{P}_K^{\star} \text{ with } J \cap K_l = J_l,\ l = 1, 2\}.$$

where

$$\mathcal{P}_J = \{(J_1, J_2) \mid J = J_1 \cup J_2,\ J_1 \cap J_2 = \emptyset\}, \text{ and}$$
$$\mathcal{P}^\star_J = \{(J_1, J_2) \in \mathcal{P}_J \mid \bigcap_{j\in J_1} F_j \not\subseteq \bigvee_{j\in J_2} G_j\}.$$

Exactly as in the proof of [5, Theorem 3.5] it may be verified that $\mathcal{F}$ contains an element g satisfying $\bigcup \operatorname{dom} g = I$.

Now consider the family $\mathcal{B} = \{(\bigcap_{j\in g_1(J)} F_j, \bigcup_{j\in g_2(J)} G_j) \mid J \in \operatorname{dom} g\}$. It is easy to show that $\mathcal{B}$ has the fep. Also $\bigcap_{j\in g_1(J)} F_j$ is β-closed by Lemma 2.3 (3) since each F_j is β-closed, and likewise $\bigcup_{j\in g_2(J)} G_j$ is β-open. Hence by (2) we have

$$\bigcap_{J\in\operatorname{dom} g} (\bigcap_{j\in g_1(J)} F_j) \not\subseteq \bigvee_{J\in\operatorname{dom} g} (\bigcup_{j\in g_2(J)} G_j).$$

Let $I_1 = \bigcup\{g_1(J) \mid J \in \operatorname{dom} g\}$, $I_2 = I \setminus I_1$. Then (I_1, I_2) is a partition of I, and $I_2 \subseteq \bigcup\{g_2(J) \mid J \in \operatorname{dom} g\}$. This gives us

$$\bigcap_{J\in\operatorname{dom} g} (\bigcap_{j\in g_1(J)} F_j) = \bigcap_{i\in I_1} F_i \subseteq \bigvee_{i\in I_2} G_i \subseteq \bigvee_{J\in\operatorname{dom} g} (\bigcup_{j\in g_2(J)} G_j),$$

which is a contradiction.

(3) $\Longrightarrow$ (1) First take β-open sets G_i, $i \in I$, with $S = \bigvee_{i\in I} G_i$. For $i \in I$ let $F_i = \emptyset$. Then $\mathcal{C} = \{(G_i, F_i) \mid i \in I\}$ is a β-open, co-β-closed dicover, so has a finite, co-finite sub-dicover $\{(G_j, F_j) \mid j \in J\}$. For the partition $J_1 = \emptyset$, $J_2 = J$ of J,

$$S = \bigcap_{j\in J_1} F_j \subseteq \bigcup_{j\in J_2} G_j,$$

whence $S = \bigcup_{j\in J} G_j$, and $(S, \mathcal{S}, \tau, \kappa)$ is β-compact. That $(S, \mathcal{S}, \tau, \kappa)$ is β-cocompact is proved in an analogous way.

To establish β-stability, let $F \neq S$ be β-closed and G_i, $i \in I$, be β-open sets with $F \subseteq \bigvee_{i\in I} G_i$. Define $\mathcal{C} = \{(S, F)\} \cup \{(G_i, \emptyset) \mid i \in I\}$. It is clear that $\mathcal{C}$ is a β-open, co-β-closed dicover, and hence has a finite, co-finite sub-dicover $\mathcal{C}_1$. If $\mathcal{C}_1 = \{(G_j, \emptyset) \mid j \in J\}$, J finite, then the fact that $\mathcal{C}_1$ is a dicover implies $\bigcup_{j\in J} G_j = S$, whence $F \subseteq \bigcup_{j\in J} G_j$. On the other hand, if $(S, F) \in \mathcal{C}_1$ then we again obtain $F \subseteq \bigcup_{j\in J} G_j$, as required. That $(S, \mathcal{S}, \tau, \kappa)$ is β-costable can be proved in a similar way.

Hence $(S, \mathcal{S}, \tau, \kappa)$ is β-dicompact. □

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References

[1] Adámek, J. ,Herrlich, H. and Strecker, G. E. *Abstract and Concrete Categories* (John Wiley & Sons, Inc., 1990).

[2] Abd El-Monsef, M. E., El-Deeb, S. N. and Mahmoud, R. A. *β-open sets and β-continuous mappings*, Bull. Fac. Sci., Assiut Univ. **12** (1983), 77–90.

[3] Aho, T. and Nieminen, T. *Spaces in which preopen subsets are semiopen*, Ricerche di Matematica **43** (1) (1994), 45–59.

[4] Andrijević, D. *Semi-preopen sets*, Mat. Vesnik **38** (1986), 24–32.

[5] Brown, L. M. and Diker, M. *Ditopological texture spaces and intuitionistic sets*, Fuzzy Sets and Systems **98** (1998), 217–224.

[6] Brown, L. M. and Ertürk, R. *Fuzzy Sets as Texture Spaces, I. Representation Theorems*, Fuzzy Sets and Systems **110** (2) (2000), 227–236.

[7] Brown, L. M. and Ertürk, R. *Fuzzy sets as texture spaces, II. Subtextures and quotient textures*, Fuzzy Sets and Systems **110** (2) (2000), 237–245.

[8] Brown, L. M., Ertürk, R. and Dost, Ş. *Ditopological texture spaces and fuzzy topology, I. Basic Concepts*, Fuzzy Sets and Systems **147** (2) (2004), 171–199.

[9] Brown, L. M., Ertürk, R. and Dost, Ş. *Ditopological texture spaces and fuzzy topology, II. Topological Considerations*, Fuzzy Sets and Systems **147** (2) (2004), 201–231.

[10] Brown, L. M., Ertürk, R. and Dost, Ş. *Ditopological texture spaces and fuzzy topology, III. Separation Axioms*, Fuzzy Sets and Systems **157** (14) (2006), 1886–1912.

[11] Brown, L. M. and Gohar, M. *Compactness in ditopological texture spaces*, Hacettepe J. Math. Stat. **38** (1) (2009), 21-43.

[12] Caldas, M. and Jafari, S. *A new decomposition of β-open functions*, Chaos Solitons & Fractals **40** (2009), 10–12.

[13] Devi, R., Bhuvaneswari, R. and Maki, H. *Weak forms of $g\rho$-closed sets, where $\rho \in \{\alpha, \alpha^*, \alpha^{**}\}$, and digital planes*, Mem. Fac. Sci. Kochi Univ. (Math) **25** (2004), 37–54.

[14] Ganster, M. and Andrijević, D. *On some questions concerning semi-preopen sets*, J. Inst. Math. & Comp. Sci. (Math. Ser.) **1** (2) (1988), 65–75.

[15] Gierz, G., Hofmann, K. H., Keimel, D, Lawson, J., Mislove, M. and Scott, D. *A compendium of continuous lattices* (Springer–Verlag, Berlin, Heidelberg, New York, 1980).

[16] Gohar, M. and Brown, L. M. *Strong compactness of ditopological texture spaces*, Submitted.

[17] Hanafy, I. M. *Fuzzy β-compactness and fuzzy β-closed spaces*, Turk. J. Math. **28** (2004), 281–294.

[18] Khalimski, E. D., Kopperman, R. and Meyer, P. R. *Computer graphics and connected topologies in finite ordered sets*, Topolology and its Applications **36** (1990), 1–17.

[19] Kopperman, R. D. *Asymmetry and duality in topology*, Topology and its Applications **66** (1995), 1–39.

[20] Levine, N. *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41.

[21] Mashour, A. S., Abd El-Monsef, M. E. and El-Deeb. *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt **53** (1982), 47–53.

[22] Özçağ, S. and Brown, L. M. *Di-uniform texture spaces*, Appl. Gen. Top. **4** (1) (2003), 157–192.

[23] Tirkaki, İ. U. *Fuzzy sets over the poset* $\mathbb{I}$, Hacettepe J. Math. Stat. **37** (2) (2008), 143-166.

[24] Yıldız, F. and Brown, L. M. *Categories of dicompact bi-*T_2 *texture spaces and a Banach-Stone theorem*, Quaestions Mathematicae **30** (2007), 167–192.

[25] Yıldız, F. and Brown, L. M. *Real dicompact textures*, Topology and its Applications **156** (11), (2009), 1970–1984.

[26] Yıldız, G. and Ertürk, R. *Di-extremities on textures*, Hacettepe J. Math. Stat. **38** (3) (2009), 243-257.

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