Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filomat **24:2** (2010), 41–51 DOI: 10.2298/FIL1002041D

ON COMPACT OPERATORS ON SOME SPACES RELATED TO MATRIX $B(r,s)$

Ivana Djolović

Abstract

Many sequence spaces arise from different concepts of summability. Recent results obtained by Altay, Başar and Malkowsky [2] are related to strong Cesàro summability and boundedness. They determined $\beta-$duals of the new sequence spaces and characterized some classes of matrix transformations on them. Here, we will present new results supplementing their research with the characterization of classes of compact operators on those spaces.

1 Introduction

Denote by ω and ϕ the set of all complex and finite sequences $x = (x_k)_{k=0}^{\infty}$, respectively. A Banach space $X \subset \omega$ is a BK *space* if each projection $x \mapsto x_n$ on the n–th coordinate is continuous. A BK space $X \supset \phi$ is said to have AK if $x^{[m]} = \sum_{k=0}^{m} x_k e^{(k)} \to x$ $(m \to \infty)$ for every sequence $x = (x_k)_{k=0}^{\infty} \in X$. As usual, let e and $e^{(n)}$ $(n = 0, 1, \dots)$ be the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ $(k \neq n)$.

Let $1 \leq p < \infty$.The sets of strongly C_1-summable to zero, strongly C_1-summable and strongly C_1-bounded sequences, denoted by w_0^p, w^p and w_∞^p, respectively, are defined and studied by Maddox [6]:

$$w_0^p = \left\{ x \in \omega \mid \lim_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right) = 0 \right\},$$
$$w^p = \{ x \in \omega \mid x - \ell \cdot e \in w_0^p \text{ for some complex number } \ell \}$$

2010 *Mathematics Subject Classifications.* 46B15, 46B45, 46B50.
Key words and Phrases. Strongly bounded and summable sequences, matrix transformations, compact operators, Cesàro method.
Received: March 7, 2010
Communicated by Dragan S. Djordjević
Supported by the research project ♯144003 of the Serbian Ministry of Science, Technology and Development

and

$$w_\infty^p = \left\{ x \in \omega \mid \sup_n \left(\frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right) < \infty \right\}$$

For such defined spaces he obtained that all these spaces are BK with the norm

$$\|x\| = \sup_{\nu \geq 0} \left(\frac{1}{2^\nu} \sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k|^p \right)^{1/p},$$

w_0^p has AK, and every sequence $x \in w^p$ has a unique representation

$$x = \ell \cdot e + \sum_{k=1}^{\infty} (x_k - \xi) e^{(k)}$$

where ℓ is the strong limit of the sequence x.

For our work we need some additional well-known results and notations. So, let us first recall that.

By (X, Y) we denote the set of all matrices that map X into Y. $B(X, Y)$ denotes the set of all bounded linear operators $L : X \to Y$. If we denote by $A = (a_{nk})_{n,k=0}^\infty$ an infinite matrix with complex entries and by A_n its n-th row, we write

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \text{ and } A(x) = (A_n(x))_{n=0}^\infty;$$

then

$A \in (X, Y)$ if and only if $A_n(x)$ converges for all $x \in X$ and all n and $A(x) \in Y$.

Furthermore,

$$X^\beta = \{a \in \omega \mid \sum_k a_k x_k \text{ converges for all } x \in X\}$$

denotes the β–dual of X. The set

$$X_A = \{a \in \omega \mid A(x) \in X\}.$$

is called the matrix domain of A in X. Specially, we are interested in matrix domains of triangle. We say that $T = (t_{nk})_{n,k=0}^\infty$ is a triangle if $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$ $(n = 0, 1, \ldots)$. Such matrix has inverse ([11, 1.4.8, p. 9], [1, Remark 22 (a), p. 22]). Throughout, where it is necessary, we will write T for triangle, S for its inverse and R transpose of S.

If $X \supset \phi$ is a BK space and $a \in \omega$ we write

$$\|a\|_X^* = \sup\{|\sum_{k=0}^{\infty} a_k x_k| \mid \|x\| = 1\}.$$

2 Auxiliary results and motivation

In [3] authors studied matrix domains of triangles on mentioned spaces. Further, in [2] they dealt with the special case, when the triangle is matrix $\triangle = (\triangle_{nk})_{n,k=1}^{\infty}$ with such entries: $\triangle_{nn} = 1$, $\triangle_{n,n-1} = -1$ and $\triangle_{nk} = 0$ otherwise. Actually, in such way they obtained and studied the spaces $w_0^p(\triangle)$, $w^p(\triangle)$and $w_\infty^p(\triangle)$which are matrix domains of $\triangle$ in w_0^p, w^p and w_∞^p respectively.

This motivated us to extend this research supplementing their research with the characterization of classes of compact operators on those spaces. But not only this. The idea is to generalize this replacing mentioned matrix $\triangle$ with new one, $B(r,s)$ $(r \neq 0)$:

$$B(r,s) = \begin{bmatrix} r & 0 & 0 & \dots \\ s & r & 0 & \dots \\ 0 & s & r & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

It is obvious, that $\triangle$ can be obtained as special case of $B(r,s)$ for $r = 1$ and $s = -1$. Matrix $B(r,s)$ is triangle, hence it has inverse, denote it by $S = (s_{nk})_{n,k=1}^{\infty}$. It is easy to obtain that entries of this matrix are defined in this way:

$$s_{nk} = \begin{cases} \frac{(-s)^{n-k}}{r^{n-k+1}} & ,1 \leq k \leq n \\ 0 & ,k > n. \end{cases}$$

Now, let us consider the sequence spaces $w_0^p(r,s)$, $w^p(r,s)$ and $w_\infty^p(r,s)$ obtained as matrix domain of $B(r,s)$ in w_0^p, w^p and w_∞^p respectively, that is, $w_0^p(r,s) = (w_0^p)_{B(r,s)}$, $w^p(r,s) = (w^p)_{B(r,s)}$ and $w_\infty^p(r,s) = (w_\infty^p)_{B(r,s)}$.

The following result is important for the characterization of the classes (X_T, Y) where X is one of the strongly C_1-summable or bounded sequences. Before we give it, let us mention that we will write, as usual, l_∞, c and c_0 for the sets of all bounded, convergent and null sequences.

Lemma 2.1. *[3, Lemma 4.1] (a) Let $X = w_0^p$ or $X = w_\infty^p$, and Y be an arbitrary subset of ω. Then we have $A \in (X_T, Y)$ if and only if $\hat{A} \in (X, Y)$ and $W^{(n)} \in (X, c_0)$ for all $n = 1, 2, \dots,$ where the matrix $\hat{A} = (\hat{a}_{nk})_{n,k=1}^{\infty}$ and the triangles $W^{(n)} = (w_{mk}^{(n)})_{m,k=1}^{\infty}$ are defined by*

$$\hat{a}_{nk} = \sum_{j=k}^{\infty} a_{nj} s_{jk} \text{ for all } n, k \in I\!N$$

and

$$w_{mk}^{(n)} = \sum_{j=m}^{\infty} a_{nj} s_{jk} \text{ for } 1 \leq k \leq m;$$

moreover, if $A \in (X_T, Y)$ then we have

$$Az = \hat{A}(Tz) \text{ for all } z \in Z = X_T.$$

(b) Let Y be an arbitrary linear subspace of ω. Then we have $A \in (w^p(T), Y)$ if and only if

$$\hat{A} \in (w_0^p, Y),\ W^{(n)} \in (w^p, c) \text{ for all } n$$

and

$$\hat{A}e - (\rho_n)_{n=1}^{\infty} \in Y \text{ where } \rho_n = \lim_{m\to\infty} \sum_{k=1}^{m} w_{mk}^{(n)} \text{ for all } n \in \mathbb{N};$$

moreover, if $A \in (w^p(T), Y)$ then we have

$$Az = \hat{A}(Tz) - \xi\,(\rho_n)_{n=1}^{\infty} \text{ for all } z \in w^p(T), \tag{2.1}$$

where $\xi \in C$ is the strong limit of z in $w^p(T)$, that is

$$\frac{1}{n}\sum_{k=1}^{n} |T_k z - \xi|^p = 0. \tag{2.2}$$

Here, we will omit the part with the characterization of appropriate classes. One can achieve that very easy using the previous theorem and result from [7]. The conditions will be obtained putting $T = B(r,s)$ and hence the following will be used:

$$\hat{a}_{nk} = \sum_{j=k}^{\infty} a_{nj} \cdot \frac{(-s)^{j-k}}{r^{j-k+1}} \text{ for all } n,k \in \mathbb{N}; \tag{2.3}$$

$$w_{mk}^{(n)} = \sum_{j=m}^{\infty} a_{nj} \cdot \frac{(-s)^{j-k}}{r^{j-k+1}} \text{ for } 1 \le k \le m; \tag{2.4}$$

So, throughout, we will suppose that necessary and sufficient conditions are obtained, that is, we have the characterizations of appropriate classes of matrix transformations. It is well-known that if X and Y are BK spaces, then $(X,Y) \subset B(X,Y)$, that is, every $A \in (X,Y)$ defines a linear operator $L_A \in B(X,Y)$ where $L_A(x) = Ax$ $(x \in X)$ *([8, Theorem 1.23]; [11, Theorem 4.2.8])*. Also, very important result for the characterizations of matrix transformations between sequence spaces is the following one.

Lemma 2.2. *Let X be a BK space and Y be any of the spaces c_0, c or ℓ_∞. If $A \in (X,Y)$ then*

$$\|L_A\| = \|A\|_{(X,\infty)} = \sup_n \|A_n\|_X^* < \infty \text{ ([8, Theorem 1.23])}. \tag{2.5}$$

Considering "the nature" of the sequence spaces which are the subject of our paper, the next result will be of great importance.

Proposition 2.3. ([3, Theorem 3.2 (d)]) *We write* $\max_\nu = \max_{2^\nu \le k \le 2^{\nu+1}-1}$ *and* $\sum_\nu = \sum_{k=2^\nu}^{2^{\nu+1}-1}$ *for* $\nu = 0, 1, \dots,$ *and put* $\mathcal{M}_p = \{a \in \omega \mid \|a\|_{\mathcal{M}_p} < \infty\}$, *where*

$$\|a\|_{\mathcal{M}_p} = \begin{cases} \sum_{\nu=0}^{\infty} 2^\nu \max_\nu |a_k| & (p = 1) \\ \sum_{\nu=0}^{\infty} 2^{\frac{\nu}{p}} \left(\sum_\nu |a_k|^q\right)^{\frac{1}{q}} & (1 < p < \infty;\ q = p/(p-1)). \end{cases}$$

Let $X = w_0^p$ *or* $X = w_\infty^p$. *If* $a \in (X_T)^\beta$ *then we have*

$$\|a\|^*_{X_T} = \|Ra\|_{\mathcal{M}_p}. \tag{2.6}$$

If $a \in (w^p(T))^\beta$ *then*

$$\|a\|^*_{w^p(T)} = \|Ra\|_{\mathcal{M}_p} + |\eta| \text{ where } \eta = \lim_{m\to\infty} \sum_{k=1}^{m} \sum_{j=m}^{\infty} a_j s_{jk}. \tag{2.7}$$

In our case, for $T = B(r,s)$, we obtain:

$$Ra = (R_k a)_{k=0}^\infty = \left(\sum_{j=k}^{\infty} a_j \frac{(-s)^{j-k}}{r^{j-k+1}} \right)_{k=0}^{\infty}$$

3 Compact operators and Hausdorff measure of noncompactness

The final goal we want to achieve in this paper is characterization of some subclasses of compact operators in terms of conditions for the entries of appropriate infinite matrix. That can be achieved applying the Hausdorff measure of noncompactness.

Here, we will recall some basic definitions and results. More results about measures of noncompactness can be found in [8, 10].

Let X and Y be Banach spaces. A linear operator $L : X \to Y$ is called compact if its domain is all of X and for every bounded sequence $(x_n)_{n=0}^\infty$ in X, the sequence $(L(x_n))_{n=0}^\infty$ has a convergent subsequence in Y. We denote the class of such operators by $K(X,Y)$.

Definition 3.1. Let (X,d) be a metric space, $Q \in \mathcal{M}_X$ and $B(x,r) = \{y \in X \mid d(x,y) < r\}$. Then the Hausdorff measure of noncompactness of Q, denoted by $\chi(Q)$, is defined by

$$\chi(Q) = \inf\{\epsilon > 0 \mid Q \subset \bigcup_{i=1}^{n} B(x_i, r_i), x_i \in X,\ r_i < \epsilon\ (i = 1, \dots, n),\ n \in \mathbb{N}\};$$

the function χ is called the Hausdorff measure of noncompactness.

If Q, Q_1 and Q_2 are bounded subsets of the metric space (X, d), then we have

$$\chi(Q) = 0 \text{ if and only if } Q \text{ is a totally bounded set,}$$

$$\chi(Q) = \chi(\overline{Q}),$$

$$Q_1 \subset Q_2 \text{ implies } \chi(Q_1) \leq \chi(Q_2),$$

$$\chi(Q_1 \cup Q_2) = \max\{\chi(Q_1), \chi(Q_2)\}$$

and

$$\chi(Q_1 \cap Q_2) \leq \min\{\chi(Q_1), \chi(Q_2)\}.$$

If Q, Q_1 and Q_2 are bounded subsets of the normed space X, then we have

$$\chi(Q_1 + Q_2) \leq \chi(Q_1) + \chi(Q_2),$$

$$\chi(Q + x) = \chi(Q) \ (x \in X)$$

and

$$\chi(\lambda Q) = |\lambda|\chi(Q) \text{ for all } \lambda \in \mathbb{C}.$$

Definition 3.2. Let X and Y be Banach spaces and χ_1 and χ_2 be Hausdorff measures on X and Y. Then the operator $L : X \to Y$ is called $(\chi_1, \chi_2) -$ *bounded* if $L(Q)$ is bounded subset of Y for every bounded subset Q of X and there exists a positive constant K such that $\chi_2(L(Q)) \leq K\chi_1(Q)$ for every bounded subset Q of X. If an operator L is $(\chi_1, \chi_2) -$ *bounded* then the number$\|L\|_{(\chi_1,\chi_2)} = \inf\{K > 0 \mid \chi_2(L(Q)) \leq K\chi_1(Q)$ for all bounded $Q \subset X\}$ is called (χ_1, χ_2)- measure of noncompactness of L. In particular, if $\chi_1 = \chi_2 = \chi$, then we write $\|L\|_{(\chi,\chi)} = \|L\|_\chi$.

Lemma 3.3. *Let X and Y be Banach spaces and $L \in B(X, Y)$. Then we have*

$$\|L\|_\chi = \chi(L(\bar{B}_X)) = \chi(L(S_X)) \quad \text{([8, Theorem 2.25]);} \tag{3.1}$$

$$L \in K(X, Y) \quad \textit{if anf only if} \quad \|L\|_\chi = 0 \quad \text{([8, Corollary 2.26 (2.58)]);} \tag{3.2}$$

$$\|L\|_\chi \leq \|L\| \quad \text{([8, Corollary 2.26 (2.59)]).} \tag{3.3}$$

Lemma 3.4 (Goldenštein, Gohberg, Markus)**.** ([8, Theorem 2.23]) *Let X be a Banach space with Schauder basis $(b_n)_{n=0}^\infty$, $Q \in \mathcal{M}_X$, and $P_n : X \to X$ be the projector onto the linear span of $\{b_1, b_2, ..., b_n\}$. Then we have*

$$\frac{1}{a} \limsup_{n\to\infty} \left(\sup_{x\in Q} \|(I - P_n)(x)\| \right) \leq \chi(Q) \leq \limsup_{n\to\infty} \left(\sup_{x\in Q} \|(I - P_n)(x)\| \right), \tag{3.4}$$

where $a = \limsup_{n\to\infty} \|I - P_n\|$.

Lemma 3.5. ([10, Theorem 2.8.]) *Let Q be a bounded subset of the normed space X, where X is l_p for $1 \leq p < \infty$ or c_0. If $P_n : X \to X$ is the operator defined by $P_n(x) = x^{[n]}$ for $x = (x_k)_{k=0}^\infty \in X$, then we have $\chi(Q) = \lim_{n\to\infty}(\sup_{x\in Q} \|(I - P_n)(x)\|)$. ($x^{[n]} = \sum_{k=1}^n x_k e^{(k)}$)*

Lemma 3.6. ([4, Theorem 3.4]) *Let X be a BK space with AK. Then every operator $L \in B(X,c)$ can be represented by an infinite complex matrix $A = (a_{nk})_{n,k=1}^{\infty}$ such that $(L(x))_n = A_n x = \sum_{k=1}^{\infty} a_{nk}x_k$ for all n and all $x \in X$. The Hausdorff measure of noncompactness of L satisfies*

$$\frac{1}{2} \cdot \limsup_{r\to\infty} \left(\sup_{n\geq r} \|A_n - \alpha\|_X^* \right) \leq \|L\|_\chi \leq \limsup_{r\to\infty} \left(\sup_{n\geq r} \|A_n - \alpha\|_X^* \right) \tag{3.5}$$

where

$$\alpha_k = \lim_{k\to\infty} a_{nk} \text{ for every } k \text{ and } \alpha = (\alpha_k)_{k=1}^{\infty}. \tag{3.6}$$

4 Main results

Finally, our main goal is characterization of certain subclasses of compact operator. We will consider the class (X,Y) where X is one of the spaces $w_0^p(r,s)$, $w_\infty^p(r,s)$ or $w^p(r,s)$, and Y is one of the classical sequence spaces c_0, ℓ_∞ or c. According to the space X and the fact that $w_0^p(r,s)$ and $w_\infty^p(r,s)$ have the same $\beta-$duals (Proposotion 2.3, (2.6)), we will distinguish two cases and attempt to define the class $K(X,Y)$.

Theorem 4.1. *Let X be one of the spaces $w_0^p(r,s)$ or $w_\infty^p(r,s)$ and set for $m = 1,2,\ldots$*

$$\|A^{<m>}\| = \begin{cases} \sup_{n>m} \left(\sum_{\nu=0}^{\infty} 2^\nu \max_\nu \left| \sum_{j=k}^{\infty} a_{nj} \frac{(-s)^{j-k}}{r^{j-k+1}} \right| \right) & (p=1) \\ \sup_{n>m} \left(\sum_{\nu=0}^{\infty} 2^{\frac{\nu}{p}} \left(\sum_\nu \left| \sum_{j=k}^{\infty} a_{nj} \frac{(-s)^{j-k}}{r^{j-k+1}} \right|^q \right)^{\frac{1}{q}} \right) & (1<p<\infty) \end{cases} \tag{4.1}$$

and

$$\|A_c^{<m>}\| = \begin{cases} \sup_{n>m} \left(\sum_{\nu=0}^{\infty} 2^\nu \max_\nu \left| \sum_{j=k}^{\infty} a_{nj} \cdot \frac{(-s)^{j-k}}{r^{j-k+1}} - \hat{\alpha}_k \right| \right) & (p=1) \\ \sup_{n>m} \left(\sum_{\nu=0}^{\infty} 2^{\frac{\nu}{p}} \left(\sum_\nu \left| \sum_{j=k}^{\infty} a_{nj} \cdot \frac{(-s)^{j-k}}{r^{j-k+1}} - \hat{\alpha}_k \right|^q \right)^{\frac{1}{q}} \right) & (1<p<\infty), \end{cases} \tag{4.2}$$

where $\hat{\alpha}_k = \lim_{n\to\infty} \sum_{j=k}^{\infty} a_{nj} \cdot \frac{(-s)^{j-k}}{r^{j-k+1}}$ $(k = 1,2,\ldots)$.

(a) If $A \in (X, c_0)$ then we have

$$\|L_A\|_\chi = \lim_{m\to\infty} \|A^{<m>}\|. \tag{4.3}$$

(b) If $A \in (X, \ell_\infty)$ then we have

$$0 \leq \|L_A\|_\chi \leq \lim_{m\to\infty} \|A^{<m>}\|. \tag{4.4}$$

(c) If $A \in (X, c)$ then we have

$$\frac{1}{2}\lim_{m\to\infty}\left(\|A_c^{<m>}\|\right) \le \|L_A\|_\chi \le \lim_{m\to\infty}\left(\|A_c^{<m>}\|\right) \tag{4.5}$$

Proof. (a) Applying Lemmas 3.3 and 3.5, we have

$$\|L_A\|_\chi = \chi(L_A(\bar{B}_X)) = \lim_{m\to\infty}\left[\sup_{x\in\bar{B}_X}\|(I-P_m)(Ax)\|\right] \tag{4.6}$$

where $P_m : c_0 \to c_0$ $(m = 0, 1, \ldots)$ is the projector such that $P_m(x) = x^{[m]}$ for $x = (x_k)_{k=0}^\infty \in c_0$. Let $A^{[m]} = (\bar{a}_{nk})_{n,k=0}^\infty$ be the infinite matrix with

$$\bar{a}_{nk} = \begin{cases} 0 & (0 \le n \le m) \\ a_{nk} & (n > m) \end{cases}.$$

Since $A^{[m]} \in (X, c_0)$, hence $A_n^{[m]} \in X^\beta$, we obtain by Lemmas 2.2 and Proposition 2.3

$$\|A_n^{[m]}\|_X^* = \|RA_n^{[m]}\|_{\mathcal{M}^p} = \begin{cases} \sum_{\nu=0}^\infty 2^\nu max_\nu \left|\sum_{j=k}^\infty a_{nj}\frac{(-s)^{j-k}}{r^{j-k+1}}\right| & (p=1) \\ \sum_{\nu=0}^\infty 2^{\frac{\nu}{p}}\left(\sum_\nu\left|\sum_{j=k}^\infty a_{nj}\frac{(-s)^{j-k}}{r^{j-k+1}}\right|^q\right)^{\frac{1}{q}} & (1<p<\infty) \end{cases} \tag{4.7}$$

Hence, we conclude

$$\sup_{x\in\bar{B}_X}\|(I-P_m)(Ax)\| = \|L_{A^{[m]}}\| = \sup_{n>m}\|A_n^{[m]}\|_X^* = \|A^{<m>}\|. \tag{4.8}$$

Now (4.3) follows from (4.6) and (4.8).

(b) Lemma 3.6 is of great importance for this part. Let $A \in (w_0^p(r,s), c)$. Then it follows by Lemma 2.1 that $\hat{A} \in (w_0^p, c)$. Now, knowing that w_0^p is BK space with AK, by (Lemma 4.1, [9]) we have that $\|L_A\|_\chi = \|L_{\hat{A}}\|_\chi$. Now, applying Theorem 3.6, we obtain

$$\frac{1}{2}\cdot\limsup_{m\to\infty}\left(\sup_{n\ge m}\|\hat{A}_n - \hat{\alpha}\|_{w_0^p}^*\right) \le \|L_A\|_\chi = \|L_{\hat{A}}\|_\chi \le \limsup_{m\to\infty}\left(\sup_{n\ge m}\|\hat{A}_n - \hat{\alpha}\|_{w_0^p}^*\right). \tag{4.9}$$

Further, applying (Lemma 1, [7]), we obtain the following:

$$\|\hat{A}_n - \hat{\alpha}\|_{w_0^p}^* = \begin{cases} \sum_{\nu=0}^\infty 2^\nu \max_\nu\left|\sum_{j=k}^\infty a_{nj}\cdot\frac{(-s)^{j-k}}{r^{j-k+1}} - \hat{\alpha}_k\right| & (p=1) \\ \sum_{\nu=0}^\infty 2^{\frac{\nu}{p}}\left(\sum_\nu\left|\sum_{j=k}^\infty a_{nj}\cdot\frac{(-s)^{j-k}}{r^{j-k+1}} - \hat{\alpha}_k\right|^q\right)^{\frac{1}{q}} & (1<p<\infty), \end{cases} \tag{4.10}$$

This implies (4.5). Let us consider the case $A \in (w_\infty^p(r,s), c)$. We know that $w_0^p(r,s) \subset w_\infty^p(r,s)$, so the fact that $A \in (w_\infty^p(r,s), c)$ implies $A \in (w_0^p(r,s), c)$ and the inequalities in (4.5) follows immediately.

(c) Now, let us suppose that $A \in (X, \ell_\infty)$ and define the projector $P_m : \ell_\infty \to \ell_\infty$ $(m = 0, 1, \ldots)$ by $P_m(x) = x^{[m]}$ for $x = (x_k)_{k=0}^\infty \in \ell_\infty$. Also, let $A^{[m]} = (\bar{a}_{nk})_{n,k=0}^\infty$ be the infinite matrix defined in a way as above. It is obvious that $A^{[m]} \in (X, \ell_\infty)$ if X is one of the spaces . Since $L_A(\bar{B}_X) \subset P_m(L_A(\bar{B}_X)) + (I - P_m)(L_A(\bar{B}_X))$, it follows that

$$\begin{aligned}\chi(L_A(\bar{B}_X)) &\le \chi(P_m(L_A(\bar{B}_X))) + \chi((I-P_m)(L_A(\bar{B}_X))) = \chi((I-P_m)(L_A(\bar{B}_X))) \\ &\le \sup_{x \in \bar{B}_X} \|(I-P_m)(Ax)\| = \|L_{A^{[m]}}\| = \sup_{n>m} \|A_n^{[m]}\|_X^* = \sup_{n>m} \|RA_n^{[m]}\|_{\mathcal{M}^p}.\end{aligned}$$

Since $0 \le \|L_A\|_\chi = \chi(L_A(\bar{B}_X)) \le \sup_{n>m} \|RA_n^{[m]}\|_{\mathcal{M}^p}$, the proof is completed. □

Corollary 4.2. *(a) If $A \in (w_0^p(r,s), c_0)$ or $A \in (w_\infty^p(r,s), c_0)$, then L_A is compact if and only if*

$$\lim_{m\to\infty} \|A^{<m>}\| = 0 \tag{4.11}$$

with $\|A^{<m>}\|$ defined in (4.1).
(b) If $A \in (w_0^p(r,s), \ell_\infty)$ or $A \in (w_\infty^p(r,s), \ell_\infty)$, then the condition in (4.11) is sufficient for L_A to be compact.
(c)If $A \in (w_0^p(r,s), c)$ or $A \in (w_\infty^p(r,s), c)$, then L_A is compact if and only if

$$\lim_{m\to\infty} \|A_c^{<m>}\| = 0 \tag{4.12}$$

with $\|A_c^{<m>}\|$ defined in (4.2).

Proof. This is an immediate consequence of Theorem 4.1 and (3.2). □

Theorem 4.3. *Set for $r = 1, 2, \ldots$*

$$\|B^{<r>}\| = \begin{cases} \sup\limits_{n>r} \left(\sum\limits_{\nu=0}^\infty 2^\nu \max_\nu \left| \sum\limits_{j=k}^\infty a_{nj} \frac{(-s)^{j-k}}{r^{j-k+1}} \right| + |\eta_n| \right) & (p=1) \\ \sup\limits_{n>r} \left(\sum\limits_{\nu=0}^\infty 2^{\frac{\nu}{p}} \left(\sum_\nu \left| \sum\limits_{j=k}^\infty a_{nj} \frac{(-s)^{j-k}}{r^{j-k+1}} \right|^q \right)^{\frac{1}{q}} + |\eta_n| \right) & (1<p<\infty) \end{cases} \tag{4.13}$$

where

$$\eta_n = \lim_{m\to\infty} \sum_{k=1}^m \sum_{j=m}^\infty a_{nj} \frac{(-s)^{j-k}}{r^{j-k+1}} \text{ for } n = 1, 2, \ldots. \tag{4.14}$$

(a) If $A \in (w^p(r,s), c_0)$ then we have

$$\|L_A\|_\chi = \lim_{r\to\infty} \|B^{<r>}\|. \tag{4.15}$$

(b) If $A \in (w^p(r,s), \ell_\infty)$ then we have

$$0 \le \|L_A\|_\chi \le \lim_{r\to\infty} \|B^{<r>}\|. \tag{4.16}$$

Proof. The proof can be done exactly in the same way as in Theorem 4.1 . The only difference which will be used through the proof is: $\|A_n^{[r]}\|^*_{w^p(r,s)} = \sum_{\nu=0}^{\infty} 2^\nu \max_\nu \left| \sum_{j=k}^{\infty} a_{nj} \frac{(-s)^{j-k}}{r^{j-k+1}} \right| + |\eta_n|$ for $p=1$ or $\|A_n^{[r]}\|^*_{w^p(r,s)} = \sum_{\nu=0}^{\infty} 2^{\frac{\nu}{p}} \left(\sum_\nu \left| \sum_{j=k}^{\infty} a_{nj} \frac{(-s)^{j-k}}{r^{j-k+1}} \right|^q \right)^{\frac{1}{q}} + |\eta_n|$ for $1 < p < \infty$ where $\eta_n = \lim_{m\to\infty} \sum_{k=1}^{m} \sum_{j=m}^{\infty} a_{nj} \frac{(-s)^{j-k}}{r^{j-k+1}}$ for $n = 1, 2, \ldots$. □

Corollary 4.4. *(a) If $A \in (w^p(r,s), c_0)$ then L_A is compact if and only if*

$$\lim_{r\to\infty} \|B^{<r>}\| = 0 \tag{4.17}$$

with $\|B^{<r>}\|$ defined in (4.13).
(b) If $A \in (w^p(r,s), \ell_\infty)$, then the condition in (4.17) is sufficient for L_A to be compact.

Proof. This is an immediate consequence of Theorem 4.3 and (3.2). □

Remark 4.5. *It is obvious that it is remains to consider the case $A \in (w^p(r,s), c)$. This technique is not as simply as the previous which are represented. One can try to prove this in the same way as in [4, Theorem 3.7] or use the result [5, Theorem 2.8].*

References

[1] R. C. Cooke, *Infinite Matrices and Sequence Spaces*, MacMillan and Co. Ltd, London, 1950.

[2] B.Altay, F.Başar, E. Malkowsky, Matrix transformations on Some Sequence Spaces related to Strong Cesaro Summability and Boundedness, *Applied Mathematics and Computation*, 2009, **211(2)**, 255–264.

[3] F.Başar, E. Malkowsky, B.Altay, Matrix transformations on the matrix domains of triangles in the spaces of strongly C_1- summable and bounded sequences, *Publ.Math.Debrecen*, 2008, **73(1-2)**, 193–213.

[4] I.Djolović, E. Malkowsky, A note on compact operators on matrix domains, *Journal of Mathematical Analysis and Applications*, 2008, **340(1)**, 291–303.

[5] I.Djolović, E. Malkowsky, The Hausdorff Measure of Noncompactness of Operators on the Matrix Domains of Triangles in the Spaces of Strongly C_1-Summable and Bounded Sequences, *Applied Mathematics and Computation*, 2010, **216**, 1122–1130.

[6] I.J.Maddox, On Kuttner's theorem, *J. London Math.Soc.*, 1968, **43**, 285–290

[7] E. Malkowsky, V. Rakočević, The measure of noncompactness of linear operators between spaces of strongly C_1-summable and bounded sequences, *Acta Math.Hungar.*, 2000, **89(1-2)**, 29–45.

[8] E. Malkowsky, V. Rakočević, An Introduction into the Theory of Sequence Spaces and Measures of Noncompactness, *Zbornik radova* , Matematički institut SANU, Belgrade, 2000, **9(17)**, 143–234.

[9] E. Malkowsky, V. Rakočević, On matrix domains of triangles, *Applied Mathematics and Computations*, 2007, **189(2)**, 1146–1163.

[10] V. Rakočević, Measures of noncompactness and some applications, *Filomat*, 1998, **12** , 87–120.

[11] A. Wilansky, *Summability Through Functional Analysis*, North-Holland Mathematics Studies 85, Amsterdam, 1984.

Technical Faculty, University of Belgrade, VJ 12, 19210 Bor, Serbia
E-mail: zucko@nadlanu.com