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# A NOTE ON WARPED PRODUCT SUBMANIFOLDS OF COSYMPLECTIC MANIFOLDS

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**Abstract**

In this paper, we study warped product anti-slant submanifolds of cosymplectic manifolds. It is shown that the cosymplectic manifold do not admit non trivial warped product submanifolds in the form $N_\perp \times_f N_\theta$ and then we obtain some results for the existence of warped products of the type $N_\theta \times_f N_\perp$, where $N_\perp$ and $N_\theta$ are anti-invariant and proper slant submanifolds of a cosymplectic manifold $\bar{M}$, respectively.

# 1 Introduction

To study the manifolds with negative curvature, R.L. Bishop and B. O'Neill [1] introduced the notion of warped product manifolds by homothetically warping the product metric of a product manifold $N_1 \times N_2$ on to the fibers $p \times N_2$ for each $p \in N_1$. In fact, the warped products appears in the differential geometric studies in a natural way. A surface of revolution is a warped product with leaves the different positions of the rotated curve and fibers the circles of revolution. Recently, warped product semi-slant submanifolds of Kaehler manifolds studied by B. Sahin [8]. After that we have studied warped product semi-slant submanifolds in cosymplectic manifolds and have shown that there exist no proper warped product semi-slant submanifolds in the form $N_T \times_f N_\theta$ and reversing the two factors in cosymplectic manifolds [5]. In this paper, we study warped products of the type $M = N_\perp \times_f N_\theta$ and $M = N_\theta \times_f N_\perp$ which has not been attempted in [5] and obtain some new results for the existence of warped product anti-slant submanifolds of a cosymplectic manifold $\bar{M}$.

# 2 Preliminaries

Let $\bar{M}$ be a $C^\infty-$manifold with $(1,1)$ tensor field $\phi$ such that

$$\phi^2 = -I + \eta \otimes \xi, \tag{2.1}$$

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where $I$ is the identity transformation, $\xi$ a vector field and $\eta$ a $1-$form on $\bar{M}$ satisfying $\phi\xi = \eta \circ \phi = 0$ and $\eta(\xi) = 1$. Then $\bar{M}$ is said to have an almost contact structure. There always exists a Riemannian metric $g$ on $\bar{M}$ such that [2]

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for all vector fields $X, Y$ on $\bar{M}$. Define the tensor $\Phi$ by $\Phi(X, Y) = g(X, \phi Y)$. Then $\Phi$ is a $2-$form. If $[\phi, \phi] + d\eta \otimes \xi = 0$, where $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$, then the almost contact structure is said to be normal. If $\Phi = d\eta$, the almost contact structure is a contact structure. A normal almost contact structure such that $\Phi$ is closed and $d\eta = 0$ is called *cosymplectic structure*. It is well known [7] that the cosymplectic structure is characterized by

$$\bar{\nabla}_X \phi = 0 \quad \text{and} \quad \bar{\nabla}_X \eta = 0, \tag{2.3}$$

where $\bar{\nabla}$ is the Levi-Civita connection of $g$. From the formula $\bar{\nabla}_X \phi = 0$, it follows that $\bar{\nabla}_X \xi = 0$.

Let $M$ be submanifold of an almost contact metric manifold $\bar{M}$ with induced metric $g$ and if $\nabla$ and $\nabla^{\perp}$ are the induced connections on the tangent bundle $TM$ and the normal bundle $T^{\perp}M$ of $M$, respectively then Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.4}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.5}$$

for each $X,\ Y \in TM$ and $N \in T^{\perp}M$, where $h$ and $A_N$ are the second fundamental form and the shape operator (corresponding to the normal vector field $N$) respectively for the immersion of $M$ into $\bar{M}$. They are related as

$$g(h(X, Y), N) = g(A_N X, Y), \tag{2.6}$$

where $g$ denotes the Riemannian metric on $\bar{M}$ as well as the one induced on $M$ [9].

For any $X \in TM$, we write

$$\phi X = PX + FX, \tag{2.7}$$

where $PX$ is the tangential component and $FX$ is the normal component of $\phi X$. Similarly for any $N \in T^{\perp}M$, we write

$$\phi N = BN + CN, \tag{2.8}$$

where $BN$ is the tangential component and $CN$ is the normal component of $\phi N$. If we denote the orthogonal complementary of $F(TM)$ in $TM$ by $\mu$. Then we have the direct sum

$$T^{\perp}M = F(TM) \oplus \mu. \tag{2.9}$$

We can see that $\mu$ is an invariant subbundle with respect to $\phi$. Furthermore the covariant derivatives of the tensor fields $P$ and $F$ are defined as

$$(\bar{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y, \tag{2.10}$$

$$(\bar{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y. \tag{2.11}$$

for all $X, Y \in TM$.

The submanifold $M$ is said to be *invariant* if $F$ is identically zero, that is, $\phi X \in TM$ for any $X \in TM$. On the other hand $M$ is said to be *anti-invariant* if $P$ is identically zero, that is, $\phi X \in T^{\perp}M$, for any $X \in TM$.

We shall always consider $\xi$ to be tangent to the submanifold $M$. There is another class of submanifolds that is called the slant submanifold. For each non zero vector $X$ tangent to $M$ at $x$, such that $X$ is not proportional to $\xi_x$, we denote by $0 \leq \theta(X) \leq \pi/2$, the angle between $\phi X$ and $T_xM$ is called the *slant angle*. If the slant angle $\theta(X)$ is constant for all $X \in T_xM - \langle \xi_x \rangle$ and $x \in M$ then $M$ is said to be *slant* submanifold [4]. Obviously if $\theta = 0$, $M$ is invariant and if $\theta = \pi/2$, $M$ is an anti-invariant submanifold. A slant submanifold is said to be *proper slant* if it is neither invariant nor anti-invariant submanifold.

We recall the following result for slant submanifold.

**Theorem 2.1** [4] *Let $M$ be a submanifold of an almost contact metric manifold $\bar{M}$, such that $\xi \in TM$. Then $M$ is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$P^2 = \lambda(-I + \eta \otimes \xi). \tag{2.12}$$

*Furthermore, if $\theta$ is slant angle, then $\lambda = \cos^2 \theta$.*

Following relations are straightforward consequence of equation (2.12)

$$g(PX, PY) = \cos^2 \theta[g(X, Y) - \eta(X)\eta(Y)] \tag{2.13}$$

$$g(FX, FY) = \sin^2 \theta[g(X, Y) - \eta(X)\eta(Y)] \tag{2.14}$$

for any $X, Y$ tangent to $M$.

# 3 Warped Product Submanifolds

Let $(N_1, g_1)$ and $(N_2, g_2)$ be two Riemannian manifolds and $f$, a positive differentiable function on $N_1$. The warped product of $N_1$ and $N_2$ is the Riemannian manifold $N_1 \times_f N_2 = (N_1 \times N_2, g)$, where

$$g = g_1 + f^2 g_2. \tag{3.1}$$

A warped product manifold $N_1 \times_f N_2$ is said to be *trivial* if the warping function $f$ is constant.We recall the following general formula on a warped product [2].

$$\nabla_X V = \nabla_V X = (X \ln f)V, \tag{3.2}$$

where $X$ is tangent to $N_1$ and $V$ is tangent to $N_2$.

Let $M = N_1 \times_f N_2$ be a warped product manifold, this means that $N_1$ is totally geodesic and $N_2$ is totally umbilical submanifold of $M$, respectively.

For warped product submanifolds of cosymplectic manifolds, we recall the following lemma.

**Lemma 3.1 [5].** *Let $M = N_1 \times {}_f N_2$ be a proper warped product submanifold of a cosymplectic manifold $\bar{M}$, with $\xi \in TN_1$, where $N_1$ and $N_2$ are any Riemannian submanifolds of $\bar{M}$, then*

*(i)* $\xi \ln f = 0$,

*(ii)* $g(h(X,Y), FZ) = g(h(X,Z), FY)$,

*(iii)* $g(h(X,Z), FW) = g(h(X,W), FZ)$

*for any $X,\ Y \in TN_1$ and $Z,\ W \in TN_2$.*

In the following section we shall investigate warped product submanifolds of a cosymplectic manifold with slant factor.

# 4 Warped Product Submanifolds with Slant Factor

The study of semi-slant submanifolds of almost contact metric manifolds was introduced by J.L. Cabrerizo et.al [3]. A semi-slant submanifold $M$ of an almost contact metric manifold $\bar{M}$ is a submanifold which admits two orthogonal complementary distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ such that $\mathcal{D}_1$ is invariant under $\phi$ and $\mathcal{D}_2$ is slant with slant angle $\theta \neq 0$ i.e., $\phi\mathcal{D}_1 = \mathcal{D}_1$ and $\phi Z$ makes a constant angle $\theta$ with $TM$ for each $Z \in \mathcal{D}_2$. In particular, if $\theta = \frac{\pi}{2}$, then a semi-slant submanifold reduces to a contact CR-submanifold. For a semi-slant submanifold $M$ of an almost contact metric manifold, we have

$$TM = D_1 \oplus D_2 \oplus \{\xi\}.$$

Then he defined anti-slant submanifolds as a particular class of bi-slant submanifolds. The submanifold $M$ is said to be *anti-slant submanifold* of $\bar{M}$ if $\mathcal{D}_1$ is an anti-invariant distribution of $M$ i.e., $\phi\mathcal{D}_1 \subseteq T^{\perp}M$ and $\mathcal{D}_2$ is slant with slant angle $\theta \neq 0$.

In this section we study warped product anti-slant submanifolds of cosymplectic manifolds. If the manifolds $N_\theta$ and $N_\perp$ are slant and anti-invariant submanifolds of a cosymplectic manifold $\bar{M}$, then their warped product anti-slant submanifolds may be given by one of the following forms:

$(i)$ $N_\perp \times {}_f N_\theta$,

$(ii)$ $N_\theta \times {}_f N_\perp$.

For the warped product of the type $(i)$, we have

**Theorem 4.1.** *There do not exist non-trivial warped product submanifolds $M = N_\perp \times {}_f N_\theta$ of a cosymplectic manifold $\bar{M}$ such that $\xi \in TN_\perp$, where $N_\perp$ and $N_\theta$ are* *anti-invariant and proper slant submanifolds of $\bar{M}$, respectively.*

*Proof.* Let $M = N_\perp \times {}_f N_\theta$ be a warped product anti-slant submanifold of a cosymplectic manifold $\bar{M}$ and for any $X \in TN_\theta$ and $Z \in TN_\perp$, we have

$$(\bar{\nabla}_X \phi)Z = \bar{\nabla}_X \phi Z - \phi \bar{\nabla}_X Z.$$

As $\bar{M}$ is cosymplectic, then the left hand side of the above equation is zero, that is

$$\bar{\nabla}_X \phi Z = \phi \bar{\nabla}_X Z.$$

Using (2.4), (2.5), (2.7) and (2.8), we obtain

$$-A_{FZ}X + \nabla^\perp_X FZ = P\nabla_X Z + F\nabla_X Z + Bh(X,Z) + Ch(X,Z).$$

Equating the tangential components and then using (3.2), we get

$$(Z \ln f)PX = -A_{FZ}X - Bh(X,Z). \tag{4.1}$$

Taking the product with $PX$ in (4.1) and making use of formula (2.13), we obtain

$$(Z \ln f)\cos^2\theta \|X\|^2 = -g(A_{FZ}X, PX) - g(Bh(X,Z), PX).$$

Then from (2.2) and (2.6), we get

$$(Z \ln f)\cos^2\theta \|X\|^2 = -g(h(X,PX), FZ) + g(h(X,Z), FPX). \tag{4.2}$$

As $\theta \neq \pi/2$, interchanging $X$ by $PX$ in (4.2) and taking account of equations (2.12) and (2.13), we deduce that

$$(Z \ln f)\cos^4\theta \|X\|^2 = \cos^2\theta g(h(X,PX), FZ) - \cos^2\theta g(h(PX,Z), FX),$$

or,

$$(Z \ln f)\cos^2\theta \|X\|^2 = g(h(X,PX), FZ) - g(h(PX,Z), FX). \tag{4.3}$$

Adding equations (4.2) and (4.3), we get

$$2(Z \ln f)\cos^2\theta \|X\|^2 = g(h(X,Z), FPX) - g(h(PX,Z), FX). \tag{4.4}$$

The right hand side of the above equation is zero by Lemma 3.1 (iii), then

$$(Z \ln f)\cos^2\theta \|X\|^2 = 0. \tag{4.5}$$

As $N_\theta \neq \{0\}$ proper slant, it follows from equation (4.5) that $Z \ln f = 0$. Also, as $\xi \in TN_\perp$ then by Lemma 3.1 $(i)$, we have $\xi \ln f = 0$. This means that $f$ is constant on $N_\perp$. This completes the proof. □

Now, the other case, i.e., the warped product of the type $N_\theta \times {}_f N_\perp$ with $\xi \in TN_\theta$ is dealt in the following.

**Theorem 4.2.** *Let $M = N_\theta \times {}_f N_\perp$ be a warped product submanifold of a cosymplectic manifold $\bar{M}$ such that $\xi \in TN_\theta$. Then*

$$g((\bar{\nabla}_X F)Z, FX) + g((\bar{\nabla}_{PX} F)Z, FPX) = \sin^2\theta g(h(X, PX), FZ),$$

*for all $X \in TN_\theta$ and $Z \in TN_\perp$.*

*Proof.* For any $X \in TN_\theta$ and $Z \in TN_\perp$, we have

$$g(\phi\bar{\nabla}_X Z, \phi X) = g(\bar{\nabla}_X Z, X) - \eta(X)\eta(\bar{\nabla}_X Z).$$

Using (2.4), (3.2) and the fact that $\xi$ is tangent to $N_\theta$, we obtain by orthogonality of two distributions that

$$g(\phi\bar{\nabla}_X Z, \phi X) = (X \ln f)g(Z, X) = 0.$$

As $\bar{M}$ is cosymplectic, then the above equation takes the form

$$g(\bar{\nabla}_X \phi Z, \phi X) = 0.$$

Then from (2.5) and (2.7), we get

$$-g(A_{FZ}X, PX) + g(\nabla^\perp_X FZ, FX) = 0,$$

i.e., $$g(\nabla^\perp_X FZ, FX) = g(A_{FZ}X, PX).$$

By (2.6), we obtain

$$g(\nabla^\perp_X FZ, FX) = g(h(X, PX), FZ). \tag{4.6}$$

On the other hand, we have

$$\nabla^\perp_X FZ = (\bar{\nabla}_X F)Z + F\nabla_X Z. \tag{4.7}$$

Taking the product with $FX$ in (4.7) and using (3.2), we obtain that

$$g(\nabla^\perp_X FZ, FX) = g((\bar{\nabla}_X F)Z, FX) + (X \ln f)g(FZ, FX).$$

Thus from (2.2) and orthogonality of two distributions, we get

$$g(\nabla^\perp_X FZ, FX) = g((\bar{\nabla}_X F)Z, FX). \tag{4.8}$$

Then by (4.6) and (4.8), we obtain

$$g((\bar{\nabla}_X F)Z, FX) = g(h(X, PX), FZ). \tag{4.9}$$

As $\theta \neq \pi/2$, then substituting $X$ by $PX$ in (4.9) and taking account of equation (2.12), we deduce that

$$g((\bar{\nabla}_{PX} F)Z, FPX) = -\cos^2\theta g(h(X, PX), FZ). \tag{4.10}$$

Thus the result follows from equations (4.9) and (4.10) on their addition. □

From the above theorem, we have the following consequences:

(i) If $\theta = 0$, i.e. $M$ is a CR-warped product submanifold of a cosymplectic manifold $\bar{M}$, then

$$g((\bar{\nabla}_X F)Z, FX) = -g((\bar{\nabla}_{PX} F)Z, FPX)$$

for all $X \in TN_T$ and $Z \in TN_\perp$.

(ii) If $\theta = \pi/2$, i.e. $M$ is anti-invariant submanifold of a cosymplectic manifold $\bar{M}$, then from Theorem 4.2, we obtain

$$g((\bar{\nabla}_U F)V, FU) = 0$$

for all $U, V \in TM$.

**Theorem 4.3.** *Let $M = N_\theta \times {}_f N_\perp$ be a proper warped product submanifold of a cosymplectic manifold $\bar{M}$ such that $\xi$ is tangent to $N_\theta$. Then $(\bar{\nabla}_X F)Z$ lies in the invariant normal subbundle for each $X \in TN_\theta$ and $Z \in TN_\perp$.*

*Proof.* For a cosymplectic manifold $\bar{M}$, we have

$$\bar{\nabla}_U \phi V = \phi \bar{\nabla}_U V,$$

for all $U, V \in T\bar{M}$. Now, for any $X \in TN_\theta$ and $Z \in TN_\perp$ by above relation we get

$$\bar{\nabla}_X \phi Z = \phi \bar{\nabla}_X Z.$$

Using (2.4), (2.5), (2.7) and (2.8) we obtain that

$$-A_{FZ}X + \nabla^\perp_X FZ = P\nabla_X Z + F\nabla_X Z + Bh(X,Z) + Ch(X,Z).$$

From the normal components of the above equation and formula (3.2) gives

$$\nabla^\perp_X FZ = (X \ln f)FZ + Ch(X,Z). \tag{4.11}$$

Taking the product in (4.11) with $FW$ for any $W \in TN_\perp$, we get

$$\begin{aligned} g(\nabla^\perp_X FZ, FW) &= (X \ln f)g(FZ, FW) + g(Ch(X,Z), FW), \\ &= (X \ln f)g(\phi Z, \phi W) + g(\phi h(X,Z), \phi W). \end{aligned}$$

Then from (2.2) we obtain

$$g(\nabla^\perp_X FZ, FW) = (X \ln f)g(Z, W). \tag{4.12}$$

Also, on taking the product in (4.7) with $FW$ for any $W \in TN_\perp$ and using (3.2), we deduce that

$$g(\nabla^\perp_X FZ, FW) = g((\bar{\nabla}_X F)Z, FW) + (X \ln f)g(Z, W). \tag{4.13}$$

From equations (4.12) and (4.13), it follows that

$$g((\bar{\nabla}_X F)Z, FW) = 0, \tag{4.14}$$

for any $X \in TN_\theta$ and $Z,\ W \in TN_\perp$. Since $N_\perp \neq \{0\}$ is a Riemannian and anti-invariant submanifold, then (4.14) implies that $(\bar{\nabla}_X F)Z \in \mu$, for all $X \in TN_\theta$ and $Z \in TN_\perp$. This proves the theorem completely. $\square$

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