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# COMMUTATOR AND SELF-COMMUTATOR APPROXIMANTS II

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**Abstract**

We minimize the quantities (i) $\|T-(AX-XA)\|$, (ii) $\|T-(X^*X-XX^*)\|$ and (iii) $\|T-(AX-XB)\|$ where $T$ is isometric and where in (i) $A$ is paranormal and commutes with $T$, in (ii) $X^*$ (or $X$) is paranormal and commutes with $T$, and in (iii) $A$ and $B$ are paranormal and $AT=TB$ and $TA=BT$. The upshot is that these quantities are minimized when $0=AX-XA=X^*X-XX^*=AX-XB$. To prove these results we obtain the power norm equality for paranormal operators: if $A$ is paranormal then $\|A^n\|=\|A\|^n$ if $n\in\mathbb{N}$.

# 1 Introduction

As in [11] and [12] we approximate an operator by a commutator $AX-XA$ of operators, by a self-commutator $X^*X-XX^*$ and, as in [4], by a "generalized commutator" $AX-XB$. There, in [4], [11] and [12] the approximation is in the von Neumann-Schatten norm $\|\cdot\|_p$, where $1\le p<\infty$, on the von Neumann-Schatten classes $C_p$; here, the approximation is in the sup norm on $L(H)$ (For operators on Banach space, see the recent paper by Duggal [6]).

The pertinent concept is that of paranormality which, as is well known from [9], is a strong generalization of hyponormality.

For self-commutator approximation with paranormal $X$ we have to restrict ourselves to the sup norm. Consider the more transparent hyponormal special case of this: that is, approximation by a self-commutator $X^*X-XX^*$ for hyponormal $X$; that is, approximation by a positive self-commutator. This topic may be regarded as an obvious extension of [12] since there can be no question of minimizing $\|T-(X^*X-XX^*)\|_p$ where $X^*X-XX^*$ is compact and $X$ is hyponormal. For if $X^*X-XX^*$ is compact then $X^*XP=(X^*X-XX^*)P$ is compact where $P$ is the orthogonal projection onto $\operatorname{Ker} XX^*$ (that is, $I-P$ is the orthogonal projection onto $(\operatorname{Ker} XX^*)^\perp=(\operatorname{Ker} X^*)^\perp=\operatorname{Ran} X$); hence $X$ is compact and therefore,

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being hyponormal, is normal [10, Problem 206]. Thus, if $X^*X - XX^*$ is compact for hyponormal $X$ then $X^*X - XX^* = 0$. The same result holds if, more generally, $X$ is paranormal; for, as is proved in Theorem 2.2 below, a compact paranormal operator is normal.

Another property the paranormal operators share with the hyponormal ones is the power norm equality: if $A$ is paranormal then $\|A^n\| = \|A\|^n$ if $n \in \mathbb{N}$ as is proved in Theorem 2.1.

We use the power norm equality to obtain the approximation results here (Theorems 3.1, 3.2 and 3.3). Theorem 3.1 says that if $A$ is a paranormal operator commuting with the isometry $T$ then $\|T - (AX - XA)\| \geq T$. Theorem 3.2 gives a similar result about minimizing $\|T - (X^*X - XX^*)\|$ for (a) paranormal $X^*$ commmuting with the isometry $T$ and for (b) paranormal $X$ commuting with $T$; Example 3.1 shows that this commutativity assumption is necessary. From Theorem 3.1 we obtain - via operator matrices - Theorem 3.3, a result about minimizing $\|T - (AX - XB)\|$ for paranormal $A$ and $B$. Those minimization results are interpreted geometrically in Corollaries 3.1 and 3.4.

# 2 Paranormality

**Definition 2.1.** *An operator $A$ in $L(H)$ is paranormal if*

$$(A^*)^*A^2 - 2\lambda A^*A + \lambda^2 \geq 0$$

*for all real $\lambda \geq 0$.*

With the definition of positivity in $L(H)$ and the discriminant criterion for the quadratic in $\lambda$, Definition 2.1 is easily proved to be equivalent to Definition 2.2 [3, Theorem 4].

**Definition 2.2.** *An operator $A$ in $L(H)$ is paranormal if*

$$\|Af\|^2 \leq \|A^2f\|\|f\|$$

*for all $f$ in $H$.*

The class or paranormal operators strictly contains many other classes of operators including the hyponormal operators [5, §3], [7, §1], [9]. Paranormal operators share with the hyponormal ones the following properties given in Theorems 2.1 and 2.2 below.

**Theorem 2.1** (Power norm equality)**.** *If $A$ is paranormal, $\|A^n\| = \|A\|^n$ where $n \in \mathbb{N}$.*

*Proof.* Equality is trivial for $n = 1$. Proceed by induction. Now, by Definition 2.2

$$\begin{aligned}
\|A^nf\|^2 &= \|A(A^{n-1}f)\|^2 = \|Ag\|^2, \text{ say} \\
&\leq \|A^2g\|\,\|g\| \\
&= \|A^{n+1}f\|\|A^{n-1}f\| \\
&\leq \|A^{n+1}\|\|A^{n-1}\|\|f\|
\end{aligned}$$

for all $f$ in $H$. Therefore, using the induction hypothesis $\|A^k\| = \|A\|^k$ for $1 \leq k \leq n$, we get

$$\|A^n\|^2 = \|A\|^{2n} \leq \|A^{n+1}\|\|A\|^{n-1}$$

whence $\|A\|^{n+1} \leq \|A^{n+1}\|$. Since $\|A^{n+1}\| \leq \|A\|^{n+1}$ automatically the inductive step follows. □

**Theorem 2.2.** *A compact paranormal operator is normal.*

*Proof.* Since the paranormal operator $A$, say, is compact it has a countable spectrum (cf. [13, Theorem 1.8.2]). Isolated points of the spectrum are poles, hence eigenvalues; further, the eigenspaces corresponding to these eigenvalues are mutually orthogonal. So if one generates the space corresponding to these eigenvalues one obtains a diagonal operator. What is left is at best the limit point which is the limit point of the diagonal entries. Conclusion: $A$ is normal. □

# 3 Approximation results

The proofs of the approximation results below use Theorem 2.1 and hinge on the following identity: if $AT = TA$ then

$$nTA^{n-1} = A^nB - BA^n + \sum_{i=0}^{n-1} A^{n-i-1}(T - (AB - BA))A^i \tag{3.1}$$

for all $B$ in $L(H)$.

The next result is a variant of the well-known result of Anderson [1, Theorem 1.7] on minimizing $\|T - (AX - XA)\|$ for normal $A$.

**Theorem 3.1.** *If $A$ is paranormal and $T$ is an isometry such that $AT = TA$ then*

$$\|T - (AX - XA)\| \geq \|T\|$$

*for all $X$ in $L(H)$.*

*Proof.* Let "$B$" $= X$ in (3.1). Take norms:

$$n\|TA^{n-1}\| \leq 2\|A^n\|\|X\| + \|T - (AX - XA)\| \sum_{i=0}^{n-1} \|A^{n-i-1}\|\|A^i\|.$$

Since $A$ is paranormal then, by Theorem 2.1, $\|A^k\| = \|A\|^k$ for all $k$ in $\mathbb{N}$ and so the summation above equals $n\|A\|^{n-1}$; and, further, since $T$ is isometric then $n\|TA^{n-1}\| = n\|A^{n-1}\| = n\|A\|^{n-1}$. Dividing through by $n\|A\|^{n-1}$ gives

$$1 \leq \frac{2}{n}\|A\|\|X\| + \|T - (AX - XA)\|.$$

Since this holds for all $n$ we have $\|T - (AX - XA)\| \geq 1 = \|T\|$. □

This result may be expressed geometrically. Consider the linear map $\Delta_A : L(H) \to L(H)$ given by

$$\Delta_A = AX - XA$$

for fixed $A$ and varying $X$. Let the linear subsets $\operatorname{Ran}\Delta_A$ and $\operatorname{Ker}\Delta_A$ be given by

$$\operatorname{Ran}\Delta_A = \{Y \in L(H) : Y = \Delta_A(X) \text{ for varying } X \text{ in } L(H)\},$$
$$\operatorname{Ker}\Delta_A = \{X \in L(H) : \Delta_A(X) = 0\}.$$

Let $\mathcal{M}$ and $\mathcal{N}$ be linear subsets of a normed space $\mathcal{L}$, say. We say $\mathcal{M}$ is orthogonal to $\mathcal{N}$, denoted $\mathcal{M}\perp\mathcal{N}$, if for all $m$ in $\mathcal{M}$ and $n$ in $\mathcal{N}$

$$\|m + n\| \geq \|m\|.$$

For historical remarks on this asymmetric definition of orthogonality see [8, p. 93].

**Corollary 3.1.** *If $A$ is paranormal and $T$ is an isometry in $Ker\,\Delta_A$ then*

$$Ker\,\Delta_A \perp Ran\,\Delta_A$$

.

The next result generalizes a well-known result for hyponormal $A$ [10, Problem 233].

**Corollary 3.2.** *If $A$ is paranormal then*

$$\|I - (AX - XA)\| \geq \|I\|$$

*for all $X$ in $L(H)$.*

**Theorem 3.2.** *(a) If $X^*$ is paranormal and $T$ is an isometry such that $X^*T = TX^*$ then*

$$\|T - (X^*X - XX^*)\| \geq \|T\|;$$

*(b) The same conclusion holds if, instead, $X$ is paranormal and $T$ is an isometry such that $XT = TX$.*

*Proof.* (a) In the identity (3.1) take "$A$" $= X^*$ and "$B$" $= X$ and proceed as in the proof of Theorem 3.1: then

$$1 \leq \frac{2}{n}\|X^*\|\|X\| + \|T - (X^*X - XX^*)\| \tag{3.2}$$

which gives the result.

(b) In (3.1) take "$A$" $= -X$ and "$B$" $= X^*$. Then, because $X$ is paranormal, $\|A^n\| = \|(-X)^n\| = \|X^n\| = \|X\|^n = \|A\|^n$ and because $\|T - ((-X)X^* - X^*(-X))\| = \|T - (X^*X - XX^*)\|$ we get, as in the proof of Theorem 3.1, the inequality 3.2 above, giving the result. □

The following example shows that Theorem 3.2 (b) fails if $XT \neq TX$.

**Example 3.1.** *Let $H = l_2$, the space of square-summable sequences of complex numbers, and let $X = S$, the simple unilateral shift. Then $S$ is hyponormal and hence paranormal: for, with $f = (x_n)_1^\infty$,*

$$\langle (S^*S - SS^*)f, f\rangle = |x_1|^2 \geq 0$$

Let $T$ be given by $T(x_1, x_2, x_3, ...) = (x_2, x_1, x_3, ...)$. Then $\|Tf\| = \|f\|$ and $ST \neq TS$. Now,

$$\begin{aligned}\|T - (S^*S - SS^*)\| &= \sup_{\|f\|=1} |\langle Tf, f\rangle - \langle (S^*S - SS^*)f, f\rangle| \\ &= \sup_{\|f\|=1} |\langle (x_2, x_1, x_3, ...)(\bar{x}_1, \bar{x}_2, \bar{x}_3, ...)\rangle - |x_1|^2| \\ &= \sup_{\|f\|=1} |x_2\bar{x}_1 + x_1\bar{x}_2 + \sum_{n=3}^{\infty} |x_n|^2 - |x_1|^2|. \qquad (3.3)\end{aligned}$$

Without loss of generality suppose that $\mathcal{R}x_1 \geq \mathcal{R}x_2$, $\mathcal{J}x_1 \geq \mathcal{J}x_2$ and $x_2 \neq 0$ and $\|f\| = 1$. Then one can check that

$$\begin{aligned}(3.3) &\leq \sup_{\|f\|=1} |2|x_1|^2 + \sum_{n=3}^{\infty} |x_n|^2 - |x_1|^2| \\ &= \sup_{\|f\|=1} ||\|f\|^2 - |x_2|^2| < \|f\|^2 = 1 = \|T\|.\end{aligned}$$

The next result deals with minimizing $\|T - (AX - XB)\| \geq \|T\|$. It reduces to Theorem 3.1 if $A = B$.

**Theorem 3.3.** *If $A$ and $B$ are paranormal and $T$ is an isometry such that $AT = TB$ and $TA = BT$ then*

$$\|T - (AX - XB)\| \geq \|T\|.$$

*Proof.* On $H \oplus H$, let $\mathcal{T} = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}$, $\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $\mathcal{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$. Then $\mathcal{T}$ is isometric on $H \oplus H$ (since $T$ is isometric on $H$) and $\mathcal{A}\mathcal{T} = \mathcal{T}\mathcal{A}$ (since $AT = TB$ and $TA = BT$). Further, $\mathcal{A}$ is paranormal on $H \oplus H$: for, with $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, on using the paranormality of $A$ and $B$ we get

$$\begin{aligned}\|\mathcal{A}f\|^4 &= \| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \|^4 = (\|Af_1\|^2 + \|Bf_2\|^2)^2 \\ &\leq (\|A^2 f_1\|\|f_1\| + \|B^2 f_2\|\|f_2\|)^2 \\ &= \|A^2 f_1\|^2\|f_1\|^2 + \|B^2 f_2\|^2\|f_2\|^2 + 2\|A^2 f_1\|\|f_1\|\|B^2 f_2\|\|f_2\| \\ &\leq (\|A^2 f_1\|^2 + \|B^2 f_2\|^2)(\|f_1\|^2 + \|f_2\|^2) \qquad (3.4) \\ &= \| \begin{bmatrix} A^2 & 0 \\ 0 & B^2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \|^2 \| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \|^2 = \|\mathcal{A}^2 f\|^2 \|f\|^2\end{aligned}$$

so that $\|\mathcal{A}f\|^2 \leq \|\mathcal{A}^2 f\|\|f\|$ as desired (The inequality (3.4) comes from $0 \leq (\|A^2 f_1\|\|f_2\| - \|B^2 f_2\|\|f_1\|)^2$). Therefore, $\mathcal{A}$ and $\mathcal{T}$ satisfy Theorem 3.1 and so $\|\mathcal{T} - (\mathcal{A}\mathcal{X} - \mathcal{X}\mathcal{A})\| \geq \|\mathcal{T}\|$ whence

$$\|T - (AX - XA)\| \geq \|T\|.$$

□

Let the linear map $\Delta_{A,B} : L(H) \to L(H)$ and the linear subsets $\operatorname{Ran}\Delta_{A,B}$ and $\operatorname{Ker}\Delta_{A,B}$ be given by, for fixed $A$ and $B$,

$$\begin{aligned}
&\Delta_{A,B}(X) = AX - XB,\\
&\operatorname{Ran}\Delta_{A,B} = \{Y \in L(H) : Y = \Delta_{A,B}(X) \text{ for varying } X \text{ in } L(H)\},\\
&\operatorname{Ker}\Delta_{A,B} = \{X \in L(H) : \Delta_{A,B}(X) = 0\}.
\end{aligned}$$

Theorem 3.3 can be expressed geometrically as follows.

**Corollary 3.3.** *If $A$ and $B$ are paranormal and $T$ is an isometry such that $T \in Ker\,\Delta_{A,B} \cap Ker\,\Delta_{B,A}$ then $Ker\,\Delta_{A,B} \cap Ker\,\Delta_{B,A} \perp Ran\,\Delta_{A,B}$.*

It is proved in [2, Theorem 1.5] that if $A$ and $B$ are normal and $T$ is such that $AT = TB$ then

$$||T - (AX - XB)|| \geq ||T||. \qquad (A - F)$$

Geometrically: if $A$ and $B$ are normal and if $\operatorname{Ker}\Delta_{A,B} \neq \{0\}$ then

$$\operatorname{Ker}\Delta_{A,B} \perp \operatorname{Ran}\Delta_{A,B}.$$

This last result, (A-F), together with the rest of this paper, prompts the following questions.

**Question 1**. Can the condition in Theorems 3.1, 3.2 and 3.3 (and in Corollaries 3.1 and 3.3) that $T$ is isometric be dropped?

**Question 2**. More generally, can the condition of normality in (A-F) be weakened to that of paranormality?

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