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Boundary value problems associated with first order rectangular Kronecker product systems

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Abstract

In this paper first, we establish a general solution of the non-linear Kronecker product system $(\mathrm{P}\otimes\mathrm{Q})(\mathrm{t})\mathrm{y}'(\mathrm{t})+(\mathrm{R}\otimes\mathrm{S})(\mathrm{t})\mathrm{y}(\mathrm{t}) = \mathrm{f}(\mathrm{t},\mathrm{y}(\mathrm{t}))$ with the help of variation of parameters formula. Finally, we prove existence and uniqueness results for the non-linear Kronecker product system satisfying general boundary condition $\mathrm{Uy} = \alpha$, by using Schauder-Tychonov's and Brouwer's fixed point theorems.

1 Introduction

The importance and applications of the Kronecker products in various fields of science and technology are well known. These Kronecker products are also used in studying the behavior of various functions of stochastic matrices in the field of statistics [7]. In finding solutions to non-linear as well as non-homogeneous matrix systems, the construction of Green's matrix is vital. However, the theory involving rectangular matrices have many difficulties due to non-existence of usual inverse of a matrix. Here by a suitable transformation, the rectangular matrices are transformed into non-singular square matrices and the solutions are finally expressed in terms of rectangular matrices [5].

In this paper, we focus our attention to the following Kronecker product boundary value problem associated with the first order non-linear system

$$(\mathrm{P}\otimes\mathrm{Q})(\mathrm{t})\mathrm{y}'(\mathrm{t}) + (\mathrm{R}\otimes\mathrm{S})(\mathrm{t})\mathrm{y}(\mathrm{t}) = \mathrm{f}(\mathrm{t},\mathrm{y}(\mathrm{t})) \tag{1}$$

satisfying the general boundary conditions

$$\mathrm{Uy} = \alpha \tag{2}$$

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where P(t), Q(t), R(t) and S(t) are rectangular matrices of order $\mathrm{m} \times \mathrm{n}$ and y(t) is of order $n^2 \times 1$, $\mathrm{f} \in \mathrm{C}[[0, \mathrm{b}] \times \mathrm{R}^{\mathrm{n}^2 \times 1}, \mathrm{R}^{\mathrm{m}^2 \times 1}]$, the components of P(t), Q(t), R(t) and S(t) are continuous on [0, b]. We assume through out this paper that the rows of P(t), Q(t) are linearly independent on [0, b] and the system (1) is consistent. We assume for the sake of convenience that $\mathrm{f}(\mathrm{t}, 0) \equiv 0$, so that the system (1) admits trivial solution. The operator $\mathrm{U} : \mathrm{C}[0, \mathrm{b}] \to R^{\mathrm{m}^2 \times 1}$, where C[0, b] is the space of all n^2-vector valued continuous bounded functions on [0, b]. Let $R^{\mathrm{m} \times \mathrm{n}}$ denote the space of $\mathrm{m} \times \mathrm{n}$ matrices, whose elements are real numbers.

In section 2, we develop general solution of the homogeneous Kronecker product system and also establish the general solution of the non-linear Kronecker product system (1) with the help of variation of parameters formula.

In section 3, we formulate the integral equation corresponding to the boundary value problem (1) and (2) and obtain existence, existence and uniqueness theorems by using Schauder-Tychonov's theorem and Brouwer's fixed point theorem respectively.

2 Preliminaries

In this section we obtain the general solution to the Kronecker product system (1) by using variation of parameters method.

Lemma 1. ([6]) *If* A *is an* $\mathrm{m} \times \mathrm{n}$ *matrix whose rows are linearly independent and system of equations*

$$\mathrm{AX} = \mathrm{b} \tag{3}$$

is consistent, then there exists a unique solution to (3) *given by* $\mathrm{x} = \mathrm{A}^{\mathrm{T}}(\mathrm{AA}^{\mathrm{T}})^{-1}\mathrm{b}$.

By using the transformation $\mathrm{y}(\mathrm{t}) = (\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t})\mathrm{z}(\mathrm{t})$, the equation

$$(\mathrm{P} \otimes \mathrm{Q})(\mathrm{t})\mathrm{y}'(\mathrm{t}) + (\mathrm{R} \otimes \mathrm{S})(\mathrm{t})\mathrm{y}(\mathrm{t}) = 0 \tag{4}$$

can be transformed into

$$\mathrm{z}'(\mathrm{t}) = -\mathrm{A}^{-1}(\mathrm{t})\mathrm{B}(\mathrm{t})\mathrm{z}(\mathrm{t}) \tag{5}$$

and the equation (1) is transformed into

$$\mathrm{z}'(\mathrm{t}) = -\mathrm{A}^{-1}(\mathrm{t})\mathrm{B}(\mathrm{t})\mathrm{z}(\mathrm{t}) + \mathrm{A}^{-1}(\mathrm{t})\mathrm{f}(\mathrm{t}, (\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t})\mathrm{z}(\mathrm{t})) \tag{6}$$

where $\mathrm{A}(\mathrm{t}) = (\mathrm{P}(\mathrm{t})\mathrm{P}^{\mathrm{T}}(\mathrm{t}) \otimes \mathrm{Q}(\mathrm{t})\mathrm{Q}^{\mathrm{T}}(\mathrm{t}))$ and

$$\mathrm{B}(\mathrm{t}) = (\mathrm{P}(\mathrm{t}) \otimes \mathrm{Q}(\mathrm{t}))(\mathrm{P}^{\mathrm{T}}(\mathrm{t}) \otimes \mathrm{Q}^{\mathrm{T}}(\mathrm{t}))' + (\mathrm{R}(\mathrm{t})\mathrm{P}^{\mathrm{T}}(\mathrm{t}) \otimes \mathrm{S}(\mathrm{t})\mathrm{Q}^{\mathrm{T}}(\mathrm{t})).$$

Definition 1. Any set of n linearly independent solutions of (5) is a *fundamental set of solutions.* The matrix with these elements as columns is called a *fundamental matrix* for the given equation.

Theorem 1. *If the system* (4) *is consistent, then any solution of* (4) *is of the form* $(\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t})\Phi(\mathrm{t})\mathrm{c}$, *where* $\Phi(t)$ *is a fundamental matrix of* (5) *and c is a constant vector of order* $\mathrm{m}^2 \times 1$.

Proof. The transformation $\mathrm{y(t)} = (\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t})\mathrm{z(t)}$ transforms the equation (4) into (5). Since $\Phi(t)$ is a fundamental matrix of (5), it follows that any solution of z(t) is of the form $\mathrm{z(t)} = \Phi(t)\mathrm{c}$, where c is a constant vector of order $\mathrm{m}^2 \times 1$. Hence $\mathrm{y(t)} = (\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t})\Phi(\mathrm{t})\mathrm{c}$. $\square$

Theorem 2. *Any solution y(t) of the non-homogeneous matrix differential equation* (1) *is of the form*

$$y(t) = (\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t})\Phi(\mathrm{t})\mathrm{c} + \bar{\mathrm{y}}(\mathrm{t}),$$

where $\bar{\mathrm{y}}(\mathrm{t})$ *is a particular solution of* (1).

Proof. It can easily verified that for any constant $\mathrm{m}^2 \times 1$ vector c,

$$(\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t})\Phi(\mathrm{t})\mathrm{c} + \bar{\mathrm{y}}(\mathrm{t})$$

is a solution of (1). Now to show that every solution is of this form. Let y(t) be any solution of (1) and $\bar{\mathrm{y}}(\mathrm{t})$ be a particular solution of (1). Then it can be easily verified that $y(t) - \bar{\mathrm{y}}(\mathrm{t})$ is a solution of (4). Hence by Theorem 1, we have $\mathrm{y(t)} - \bar{\mathrm{y}}(\mathrm{t}) = (\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t})\Phi(\mathrm{t})\mathrm{c}$. Therefore $\mathrm{y(t)} = (\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t})\Phi(\mathrm{t})\mathrm{c} + \bar{\mathrm{y}}(\mathrm{t})$. $\square$

Theorem 3. ([5]) *A particular solution* $\bar{\mathrm{y}}(\mathrm{t})$ *of* (1) *is of the form*

$$\bar{\mathrm{y}}(\mathrm{t}) = (\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t})\Phi(\mathrm{t}) \int_0^{\mathrm{t}} \Phi^{-1}(\mathrm{s})(\mathrm{P}\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}\mathrm{Q}^{\mathrm{T}})^{-1}(\mathrm{s})\mathrm{f}(\mathrm{s}, \mathrm{y}(\mathrm{s}))\mathrm{ds}.$$

3 Main Results

In this section we first, obtain the existence theorem for the boundary value problem (1) and (2) by using Schauder-Tychonov's fixed point theorem. Further, we establish existence and uniqueness theorem for the boundary value problem (1), (2) with the help of Brouwer's fixed point theorem. In this section, we use the notation $\|\mathrm{f}\|_\infty = \sup_{\mathrm{t}\in[0,\mathrm{b}]} \|\mathrm{f(t)}\|$.

The general solution $\mathrm{y(t)} = (\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t})\Phi(\mathrm{t})\mathrm{c}$ of the homogeneous equation (4) satisfies the general boundary condition (2) if and only if

$$\mathrm{U}[(\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(.)\Phi(.)\mathrm{c}] = \mathrm{Wc}$$

where W is the square matrix whose columns are values of U on the corresponding columns of

$$(\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(.)\Phi(.).$$

Therefore the general solution of (1) can be written as

$$\mathrm{y(t)} = (\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t})\Phi(\mathrm{t})\mathrm{c} + \psi(\mathrm{t}, \mathrm{y}), \tag{7}$$

where

$$\psi(\mathrm{t}, \mathrm{y}) = (\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t})\Phi(\mathrm{t}) \int\limits_0^{\mathrm{t}} \Phi^{-1}(\mathrm{s})(\mathrm{PP}^{\mathrm{T}} \otimes \mathrm{QQ}^{\mathrm{T}})^{-1}(\mathrm{s})\mathrm{f}(\mathrm{s}, \mathrm{y(s)})\mathrm{ds}.$$

This solution satisfies the boundary condition matrix (2) if and only if

$$\mathrm{Uy} = \alpha = \mathrm{Wc} + \mathrm{U}\psi(., \mathrm{y}).$$

This equation in 'c' has a unique solution for some $\alpha \in R^{\mathrm{m}^2 \times 1}$ if and only if

$$\mathrm{c} = \mathrm{W}^{-1}[\alpha - \mathrm{U}\psi(., \mathrm{y})]. \tag{8}$$

From the above discussion, it follows that the boundary value problem (1) and (2) possesses a solution on [0, b], if a function y(t) can be found that satisfies the integral equation

$$\mathrm{y(t)} = (\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t})\Phi(\mathrm{t})\mathrm{W}^{-1}[\alpha - \mathrm{U}\psi(., \mathrm{y})] + \psi(\mathrm{t}, \mathrm{y}) \tag{9}$$

Now we prove the following existence theorem.

Theorem 4. *Define* $\mathrm{V} : [0, \mathrm{b}] \to R^{\mathrm{m}^2 \times 1}$ *by*

$$\mathrm{V(t)} = \max_{\|\mathrm{y(t)}\| \le \beta} \{\Phi^{-1}(\mathrm{t})(\mathrm{PP}^{\mathrm{T}} \otimes \mathrm{QQ}^{\mathrm{T}})^{-1}(\mathrm{t})\mathrm{f}(\mathrm{t}, \mathrm{y(t)})\}$$

and the operator K by

$$\mathrm{Kh} = \mathrm{W}^{-1}[\alpha - \mathrm{U}\psi(., \mathrm{y})]$$

for every $\mathrm{h} \in \mathrm{B}^{\beta}$*, where* B^{β} *is the closed ball of C[0, b] centered at origin with radius* $\beta > 0$*. Then the boundary value problem* (1) *and* (2) *possesses at least one solution defined on the interval [0, b]. If* $\mathrm{L}_1\mathrm{L}_2(\mathrm{M} + \mathrm{N}) \le \beta$*, where*

$$\mathrm{L}_1 = \max_{\mathrm{t} \in [0, \mathrm{b}]} \left\|(\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t})\right\|, \mathrm{L}_2 = \max_{\mathrm{t} \in [0, \mathrm{b}]} \|\Phi(\mathrm{t})\|, \mathrm{M} = \sup_{\mathrm{h} \in \mathrm{B}^{\beta}} \|\mathrm{Kh}\|$$

and

$$\mathrm{N} = \max_{\mathrm{t} \in [0, \mathrm{b}]} \int\limits_0^{\mathrm{t}} \mathrm{V(s)ds}.$$

Proof. We first, show that the operator $\mathrm{T} : \mathrm{B}^{\beta} \to \mathrm{C}[0, \mathrm{b}]$ defined by

$$\mathrm{Ty(t)} = (\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t})\Phi(\mathrm{t})[\mathrm{Kh} + \int\limits_0^{\mathrm{t}} \Phi^{-1}(\mathrm{s})(\mathrm{PP}^{\mathrm{T}} \otimes \mathrm{QQ}^{\mathrm{T}})^{-1}(\mathrm{s})\mathrm{f}(\mathrm{s}, \mathrm{y(s)})\mathrm{ds}]. \tag{10}$$

has a fixed point in B^{β}. For $\mathrm{t}, \mathrm{t}_1 \in [0, \mathrm{b}]$, consider

$$\|\mathrm{Ty}(\mathrm{t}) - \mathrm{Ty}(\mathrm{t}_1)\|$$

$$= \left\| (\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t})\Phi(\mathrm{t})[\mathrm{Kh} + \int_0^{\mathrm{t}} \Phi^{-1}(\mathrm{s})(\mathrm{PP}^{\mathrm{T}} \otimes \mathrm{QQ}^{\mathrm{T}})^{-1}(\mathrm{s})\mathrm{f}(\mathrm{s}, \mathrm{y}(\mathrm{s}))\mathrm{ds}] \right.$$

$$\left. -(\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t}_1)\Phi(\mathrm{t}_1)[\mathrm{Kh} + \int_0^{\mathrm{t}_1} \Phi^{-1}(\mathrm{s})(\mathrm{PP}^{\mathrm{T}} \otimes \mathrm{QQ}^{\mathrm{T}})^{-1}(\mathrm{s})\mathrm{f}(\mathrm{s}, \mathrm{y}(\mathrm{s}))\mathrm{ds}] \right\|$$

$$\leq \mathrm{L}_1(\mathrm{M}+\mathrm{N}) \|\Phi(\mathrm{t}) - \Phi(\mathrm{t}_1)\| + \mathrm{L}_2(\mathrm{M}+\mathrm{N}) \left\| (\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t}) - (\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t}_1) \right\|$$

$$+\mathrm{L} \left\| \int_{\mathrm{t}}^{\mathrm{t}_1} \mathrm{V}(\mathrm{s})\mathrm{ds} \right\|$$

where $L = L_1 L_2$.

Given $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ satisfying

$$\|\Phi(\mathrm{t}) - \Phi(\mathrm{t}_1)\| < \frac{\varepsilon}{3\mathrm{L}_1(\mathrm{M}+\mathrm{N})}, \tag{11}$$

$$\left\| (\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t}) - (\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t}_1) \right\| < \frac{\varepsilon}{3\mathrm{L}_2(\mathrm{M}+\mathrm{N})}, \tag{12}$$

and

$$\left\| \int_{\mathrm{t}}^{\mathrm{t}_1} \mathrm{V}(\mathrm{s})\mathrm{ds} \right\| < \frac{\varepsilon}{3\mathrm{L}} \quad , \tag{13}$$

for every $\mathrm{t}, \mathrm{t}_1 \in [0, \mathrm{b}]$ with $|\mathrm{t} - \mathrm{t}_1| < \delta(\varepsilon)$.

This is clear, because $\Phi(\mathrm{t})$ and $(\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t})$ are uniformly continuous on [0, b] and the function $\mathrm{H}(\mathrm{t}) = \int_0^{\mathrm{t}} \mathrm{V}(\mathrm{s})\mathrm{ds}$ is uniformly continuous on the interval [0, b].

Inequalities (11),(12) and (13) imply that the set TB^{β} is equi-continuous. Now, taking the norm on both sides of (10), we have

$$\|\mathrm{Ty}(\mathrm{t})\| = \left\| (\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t})\Phi(\mathrm{t})[\mathrm{Kh} + \int_0^{\mathrm{t}} \Phi^{-1}(\mathrm{s})(\mathrm{PP}^{\mathrm{T}} \otimes \mathrm{QQ}^{\mathrm{T}})^{-1}(\mathrm{s})\mathrm{f}(\mathrm{s}, \mathrm{y}(\mathrm{s}))\mathrm{ds}] \right\|$$

$$\leq \left\| (\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{t}) \right\| \|\Phi(\mathrm{t})\| \left[\|\mathrm{Kh}\| + \left\| \int_0^{\mathrm{t}} \mathrm{V}(\mathrm{s})\mathrm{ds} \right\| \right]$$

$$\leq \mathrm{L}(\mathrm{M}+\mathrm{N}) \leq \beta.$$

Thus $TB^{\beta} \subset B^{\beta}$ and hence TB^{β} is relatively compact.

Next, we show that T is continuous on B^{β}. Let $\{\mathrm{y_m(t)}\}_{\mathrm{m=1}}^{\infty} \subset \mathrm{B}^{\beta}$ and $y \in B^{\beta}$ be such that $\|y_m - y\| \to 0$ as $m \to \infty$. It follows that

$$\|Ty_m - Ty\| = \Big\|(\mathrm{P^T} \otimes \mathrm{Q^T})(\mathrm{t})\Phi(\mathrm{t})\{\mathrm{W}^{-1}[\alpha - \mathrm{U}\psi(.,\mathrm{y_m}] + \int_0^{\mathrm{t}} \Phi^{-1}(\mathrm{s})(\mathrm{PP^T} \otimes \mathrm{QQ^T})^{-1}(\mathrm{s})\mathrm{f}(\mathrm{s},\mathrm{y_m}(\mathrm{s}))\mathrm{ds}\} - (\mathrm{P^T} \otimes \mathrm{Q^T})(\mathrm{t})\Phi(\mathrm{t})\{\mathrm{W}^{-1}[\alpha - \mathrm{U}\psi(.,\mathrm{y})] + \int_0^{\mathrm{t}} \Phi^{-1}(\mathrm{s})(\mathrm{PP^T} \otimes \mathrm{QQ^T})^{-1}(\mathrm{s})\mathrm{f}(\mathrm{s},\mathrm{y}(\mathrm{s}))\mathrm{ds}]\Big\|$$

$$\leq \mathrm{L}\{\mathrm{L}\left\|\mathrm{W}^{-1}\right\| \|\mathrm{U}\| + 1\} \int_0^{\mathrm{t}} \left\|\Phi^{-1}(\mathrm{s})\right\| \left\|(\mathrm{PP^T} \otimes \mathrm{QQ^T})^{-1}(\mathrm{s})\right\| \times \|\mathrm{f}(\mathrm{s},\mathrm{y_m}(\mathrm{s})) - \mathrm{f}(\mathrm{s},\mathrm{y}(\mathrm{s}))\| \,\mathrm{ds}.$$

The above integral tends to zero as $m \to \infty$. By the application of Schauder-Tychonov fixed point theorem, there exists a fixed point $y \in B^{\beta}$which is a solution of the boundary value problem (1) and (2).

The following theorem establishes the existence and uniqueness criteria for the boundary value problem (1) and (2). □

Theorem 5. *Let B^{β} be the closed ball of C [0, b] centered at the origin with radius $\beta > 0$. Assume that there exists a constant $\lambda > 0$ such that for every $\gamma \in B^{\beta}$, the solution y(t, 0, γ) of* (1) *with y(0) = γ exists on [0, b], is unique and satisfies*

$$\sup_{\mathrm{t}\in[0,\mathrm{b}]} \|\mathrm{y}(\mathrm{t},0,\gamma)\| \leq \lambda.$$

Let V(t), N and $L = L_1 L_2$ be as in Theorem 4 and assume that

$$\left\|\mathrm{W}^{-1}\right\| [\, \|\alpha\| + \|\mathrm{U}\| \,\mathrm{LN}] \leq \beta.$$

Then the boundary value problem (1) *and* (2) *has a unique solution on any interval [0, b].*

Proof. For every $\gamma \in B^{\beta}$, consider the operator T defined by

$$\mathrm{T}\gamma = \mathrm{W}^{-1}[\alpha - \mathrm{U}\psi_1(.,\mathrm{y}(.,0,\gamma))], \tag{14}$$

where

$$\psi_1(\mathrm{t},\mathrm{y}(\mathrm{t},0,\gamma)) = (\mathrm{P^T} \otimes \mathrm{Q^T})(\mathrm{t})\Phi(\mathrm{t}) \int_0^{\mathrm{t}} \Phi^{-1}(\mathrm{s})(\mathrm{PP^T} \otimes \mathrm{QQ^T})^{-1}(\mathrm{s})\mathrm{f}(\mathrm{s},\mathrm{y}(\mathrm{s},0,\gamma))\mathrm{ds}.$$

By taking norm on both sides of the equation (14), we have

$$\begin{aligned}\|T\gamma\| &= \left\|\mathrm{W}^{-1}[\alpha - \mathrm{U}\psi_1(.,\mathrm{y}(.,0,\gamma))]\right\| \\ &\leq \left\|\mathrm{W}^{-1}\right\| [\, \|\alpha\| + \|\mathrm{U}\| \,\mathrm{LN}] \leq \beta.\end{aligned}$$

Thus $TB^{\beta} \subset B^{\beta}$. Now, we prove the continuity of T, we first show the continuity of y(t, 0, γ) with respect to γ.

Let y_m and y be solutions of the system of equations

$$(\mathrm{P}\otimes\mathrm{Q})(\mathrm{t})\mathrm{y}'(\mathrm{t}) + (\mathrm{R}\otimes\mathrm{S})(\mathrm{t})\mathrm{y}(\mathrm{t}) = \mathrm{f}(\mathrm{t},\mathrm{y}(\mathrm{t})), \qquad \mathrm{y}(0) = \gamma_{\mathrm{m}}$$

$$(\mathrm{P}\otimes\mathrm{Q})(\mathrm{t})\mathrm{y}'(\mathrm{t}) + (\mathrm{R}\otimes\mathrm{S})(\mathrm{t})\mathrm{y}(\mathrm{t}) = \mathrm{f}(\mathrm{t},\mathrm{y}(\mathrm{t})), \qquad \mathrm{y}(0) = \gamma$$

respectively.

Let z_m and z denote the solutions of the system of equations:

$$\mathrm{z}'(\mathrm{t}) = -\mathrm{A}^{-1}(\mathrm{t})\mathrm{B}(\mathrm{t})\mathrm{z}(\mathrm{t}) + \mathrm{A}^{-1}(\mathrm{t})\mathrm{f}(\mathrm{t},(\mathrm{P}^{\mathrm{T}}\otimes\mathrm{Q}^{\mathrm{T}})(\mathrm{t})\mathrm{z}(\mathrm{t})), \mathrm{z}(0) = \mathrm{k}_{\mathrm{m}},$$

$$\mathrm{z}'(\mathrm{t}) = -\mathrm{A}^{-1}(\mathrm{t})\mathrm{B}(\mathrm{t})\mathrm{z}(\mathrm{t}) + \mathrm{A}^{-1}(\mathrm{t})\mathrm{f}(\mathrm{t},(\mathrm{P}^{\mathrm{T}}\otimes\mathrm{Q}^{\mathrm{T}})(\mathrm{t})\mathrm{z}(\mathrm{t})), \mathrm{z}(0) = \mathrm{k},$$

respectively, obtained by using the transformation $\mathrm{y}(\mathrm{t}) = (\mathrm{P}^{\mathrm{T}}\otimes\mathrm{Q}^{\mathrm{T}})(\mathrm{t})\mathrm{z}(\mathrm{t})$.

Let $\{\mathrm{k_m}\}_{\mathrm{m}=1}^{\infty} \subset B^{\beta}$ and $k \in B^{\beta}$ be such that $\|\mathrm{k_m} - \mathrm{k}\| \to 0$ as $\mathrm{m} \to \infty$. Our assumptions implies that there exists a constant $\lambda > 0$ such that

$$\begin{aligned} &\|\mathrm{y_m}(\mathrm{t})\|_{\infty} \leq \lambda \\ \Rightarrow \left\|(\mathrm{P}^{\mathrm{T}}\otimes\mathrm{Q}^{\mathrm{T}})(\mathrm{t})\mathrm{z_m}(\mathrm{t})\right\|_{\infty} &\leq \left\|(\mathrm{P}^{\mathrm{T}}\otimes\mathrm{Q}^{\mathrm{T}})(\mathrm{t})\right\|_{\infty} \|\mathrm{z_m}(\mathrm{t})\|_{\infty} \\ &\leq \mathrm{L}_1 \|\mathrm{z_m}(\mathrm{t})\|_{\infty} \leq \lambda. \end{aligned}$$

Therefore

$$\|\mathrm{z_m}(\mathrm{t})\|_{\infty} \leq \frac{\lambda}{\mathrm{L}_1} (\mathrm{m} = 1,2,3,\dots) \text{and} \, \|\mathrm{z}(\mathrm{t})\|_{\infty} \leq \frac{\lambda}{\mathrm{L}_1},$$

for all $t \in [0,\mathrm{b}]$. Then

$$\|\mathrm{z'}_{\mathrm{m}}(\mathrm{t})\|_{\infty} \leq \frac{\lambda}{\mathrm{L}_1} \sup_{\mathrm{t}\in[0,\mathrm{b}]} \left\|\mathrm{A}^{-1}(\mathrm{t})\mathrm{B}(\mathrm{t})\right\| +$$

$$\sup_{\left\|(\mathrm{P}^{\mathrm{T}}\otimes\mathrm{Q}^{\mathrm{T}})(\mathrm{t})\mathrm{z_m}(\mathrm{t})\right\|_{\infty} \leq \lambda, \mathrm{t}\in[0,\mathrm{b}]} \left\|\mathrm{A}^{-1}(\mathrm{t})\mathrm{f}(\mathrm{t},(\mathrm{P}^{\mathrm{T}}\otimes\mathrm{Q}^{\mathrm{T}})(\mathrm{t})\mathrm{z_m}(\mathrm{t}))\right\|.$$

The above inequality proves the sequence of functions $\{z_m(t)\}$ is equi-continuous and uniformly bounded and hence by Ascoli's Lemma, there exists a subsequence $\{\mathrm{z_{m_j}}(\mathrm{t})\}_{\mathrm{j}=1}^{\infty}$ of $\{\mathrm{z_m}(\mathrm{t})\}_{\mathrm{m}=1}^{\infty}$ such that $\mathrm{z_{m_j}}(\mathrm{t}) \to \bar{\mathrm{z}}(\mathrm{t})$ as $\mathrm{j} \to \infty$ uniformly on [0,b], where $\bar{\mathrm{z}}(\mathrm{t}) \in \mathrm{C}[0,\mathrm{b}]$.

Taking the limit as $j \to \infty$ in

$$\mathrm{z_{m_j}}(\mathrm{t}) = \mathrm{z_{m_j}}(0) - \int_0^{\mathrm{t}} \mathrm{A}^{-1}(\mathrm{s})\mathrm{B}(\mathrm{s})\mathrm{z_{m_j}}(\mathrm{s})\mathrm{ds} + \int_0^{\mathrm{t}} \mathrm{A}^{-1}(\mathrm{s})\mathrm{f}(\mathrm{s},(\mathrm{P}^{\mathrm{T}}\otimes\mathrm{Q}^{\mathrm{T}})(\mathrm{s})\mathrm{z_{m_j}}(\mathrm{s}))\mathrm{ds},$$

we obtain

$$\bar{\mathrm{z}}(\mathrm{t}) = \mathrm{k} - \int_0^{\mathrm{t}} \mathrm{A}^{-1}(\mathrm{s})\mathrm{B}(\mathrm{s})\bar{\mathrm{z}}(\mathrm{s})\mathrm{ds} + \int_0^{\mathrm{t}} \mathrm{A}^{-1}(\mathrm{s})\mathrm{f}(\mathrm{s}, (\mathrm{P}^{\mathrm{T}} \otimes \mathrm{Q}^{\mathrm{T}})(\mathrm{s})\bar{\mathrm{z}}(\mathrm{s}))\mathrm{ds}.$$

Hence from uniqueness of solutions of initial value problems, it follows that

$$\bar{\mathrm{y}}(\mathrm{t}) = \mathrm{y}(\mathrm{t})$$

and every sequence of $\{y_m(t)\}$ contains a subsequence which is convergent uniformly to y(t) on [0, b]. Thus $\{y_m(t)\}$ converges to y(t) uniformly on [0, b]. Note that if $\mathrm{k_m} \in B^\beta(\mathrm{m} = 1, 2, 3, \ldots)$ and $k \in \mathrm{B}^\beta$ such that $\|\mathrm{k_m} - \mathrm{k}\| \to 0$ as $m \to \infty$. Then $\|\mathrm{y}(\mathrm{t}, 0, \gamma_{\mathrm{m}}) - \mathrm{y}(\mathrm{t}, 0, \gamma)\| \to 0$ as $m \to \infty$. Thus the function y(t, 0, γ) is continuous with respect to γ and uniformly continuous with respect to 't' on [a, b] . Consider

$$\begin{aligned} &\|\mathrm{T}\gamma_{\mathrm{m}} - \mathrm{T}\gamma\| \\ &\quad \leq \left\|\mathrm{W}^{-1}\right\| \|\mathrm{U}\| \, \mathrm{L} \int_0^{\mathrm{b}} \left\|\Phi^{-1}(\mathrm{s})\right\| \left\|(\mathrm{PP}^{\mathrm{T}} \otimes \mathrm{QQ}^{\mathrm{T}})^{-1}(\mathrm{s})\right\| \\ &\quad \|[\mathrm{f}(\mathrm{s}, \mathrm{y}(\mathrm{s}, 0, \gamma_{\mathrm{m}})) - \mathrm{f}(\mathrm{s}, \mathrm{y}(\mathrm{s}, 0, \gamma))]\| \, \mathrm{ds} \to 0 \ \text{ as } \ \mathrm{m} \to \infty. \end{aligned}$$

This proves the continuity of T on B^β. From Brouwer's fixed point theorem, there exists a fixed point $\gamma_0 \in B^\beta$ such that $T\gamma_0 = \gamma_0$ which results a solution to the boundary value problem. Hence there exists a unique solution to the boundary value problem (1) and (2). $\square$

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