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Ascoli-type theorems and ideal (α)-convergence

E. Athanassiadou[a], A. Boccuto[b], X. Dimitriou[b], N. Papanastassiou[b]

[a]*Department of Mathematics, University of Athens, Panepistimiopolis, Athens 15784, Greece*
[b]*Department of Mathematics and Computer Sciences, University of Perugia, via Vanvitelli 1, I-06123 Perugia, Italy*

Abstract. We investigate fundamental properties of $\mathcal{I}$-exhaustiveness and $\mathcal{I}$-convergence of real-valued function sequences, giving some characterizations. Furthermore, we establish new versions of Ascoli and Helly theorems, giving also applications to measure theory. Finally, we pose an open problem.

1. Introduction

The concept of *α-convergence* or *continuous convergence* or *stetige Konvergenz* of real-valued function sequences has been known in the literature since the beginning of the last century (see for example [7, 15, 16, 20]). This notion was formulated in the case of an ordered structure by E. Wolk in 1975 ([21]).

In [15] there are some comparisons between the notions and main properties of α-convergence, equicontinuity and exhaustiveness of function sequences. These results have been extended in [9] for the statistical convergence and in [19] for the $\mathcal{I}$-convergence introduced in [18].

In this paper we prove some properties of $(\mathcal{I}\alpha)$-convergence, and in particular its relation to α-convergence. Furthermore, we continue the investigation of $\mathcal{I}$-compactness started in [5], giving a Heine-type lemma which relates pointwise and uniform $\mathcal{I}$-exhaustiveness and proving some versions of Ascoli and Helly theorems, which extend earlier results of [12, 15]. Recently some versions of these theorems were proved in a different setting (see [14]). Also relations with the Alexandroff convergence ([1, 8, 13]) and strong uniform convergence on finite sets ([4, 8]) and their ideal versions are investigated.

Moreover, some applications of exhaustiveness to measure theory are presented, and in particular we give the example of a measure sequence, ideal exhaustive but not weakly exhaustive in the classical sense. Note that in general, in the context of ideal pointwise convergence, the Brooks-Jewett, Nikodým convergence and Vitali-Hahn-Saks theorems do not hold (see [6]). However, under suitable additional hypotheses, it is possible to prove some versions of limit theorems even if we require the simple $\mathcal{I}$-pointwise convergence of measures. Some results along this line were just proved in [5, 6]. In the last section of this paper we continue such an investigation, obtaining new limit theorems as applications of the results presented previously.

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Email addresses: eathan@math.uoa.gr (E. Athanassiadou), boccuto@dmi.unipg.it (A. Boccuto), dxenof@math.uoa.gr (X. Dimitriou), npapanas@math.uoa.gr, Corresponding author (N. Papanastassiou)

2. Modes of ideal convergence

We begin with some basic notions about ideals of $\mathbb{N}$.

Definition 2.1. a) A family of sets $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is called an *ideal* of $\mathbb{N}$ iff $A \cup B \in \mathcal{I}$ whenever $A, B \in \mathcal{I}$ and for each $A \in \mathcal{I}$ and $B \subset A$ we get $B \in \mathcal{I}$.

b) An ideal $\mathcal{I}$ is said to be *non-trivial* iff $\mathcal{I} \neq \emptyset$ and $\mathbb{N} \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is *admissible* iff it contains all singletons of $\mathbb{N}$.

c) Given an ideal $\mathcal{I}$ of $\mathbb{N}$, we call the *dual filter* associated with $\mathcal{I}$ the set $\mathcal{F} = \mathcal{F}(\mathcal{I}) := \{\mathbb{N} \setminus A : A \in \mathcal{I}\}$. A strictly increasing sequence $(n_t)_t$ in $\mathbb{N}$ is $\mathcal{I}$*-thick* iff the set $\{n_t : t \in \mathbb{N}\}$ belongs to the dual filter $\mathcal{F} = \mathcal{F}(\mathcal{I})$.

d) An admissible ideal $\mathcal{I}$ of $\mathbb{N}$ is a *P-ideal* iff for any sequence $(A_j)_j$ in $\mathcal{I}$ there are sets $B_j \subset \mathbb{N}$, $j \in \mathbb{N}$, such that the symmetric difference $A_j \Delta B_j$ is finite for all $j \in \mathbb{N}$ and $\bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$ (see also [3, 17, 18]).

Remark 2.2. a) Observe that the ideal $\mathcal{I}_{\text{fin}}$ of all finite subsets of $\mathbb{N}$ is a P-ideal. Another example of P-ideal of $\mathbb{N}$ is the ideal $\mathcal{I}_d$ of all subsets of $\mathbb{N}$ having zero asymptotic density, where the asymptotic density of a set $A \subset \mathbb{N}$ is defined as

$$d(A) = \lim_n \frac{\text{card}(A \cap \{1, \dots, n\})}{n},$$

(provided that the limit exists). Here the symbol card denotes the cardinality of the set into brackets.

Some further examples of P-ideals can be found, e.g., in [5, 18].

b) When $\mathcal{I} = \mathcal{I}_d$, the concept of $\mathcal{I}$-thick coincides with that of *statistically dense* mentioned in [9, 10].

We now define some ideal versions of different type of convergence.

From now on, (X, d) is a metric space, and for all $x \in X$ and $\delta > 0$ let us denote by $B(x, \delta)$ the set $\{z \in X : d(z, x) < \delta\}$. For $\delta > 0$, the *δ-enlargement* of a set $A \subseteq X$ is $A^\delta := \bigcup_{x \in A} B(x, \delta)$. We denote by $\mathbb{R}^X$, $C_{\mathbb{R}}(X)$, $Bd(X)$ the spaces of all functions $f : X \to \mathbb{R}$, of all continuous functions of $\mathbb{R}^X$, and all bounded functions of $\mathbb{R}^X$ (endowed with the sup norm $\|\cdot\|_\infty$ respectively).

Definition 2.3. a) A sequence $(x_n)_n$ in X is called $\mathcal{I}$*-convergent to* $x \in X$ iff $\{n \in \mathbb{N} : d(x_n, x) > \varepsilon\} \in \mathcal{I}$ for any $\varepsilon > 0$. In this case we write $\mathcal{I} - \lim_n x_n = x$ (see also [17, 18]).

b) A sequence $(x_n)_n$ in X $\mathcal{I}^*$*-converges to* $x \in X$ iff there is an $\mathcal{I}$-thick sequence $(n_t)_t \in \mathbb{N}$ such that $\lim_t x_{n_t} = x$.

Note that $\mathcal{I}^*$-convergence of a sequence $(x_n)_n$ to x always implies $\mathcal{I}$-convergence of $(x_n)_n$ to x and the converse is true if and only if $\mathcal{I}$ is a P-ideal (see [18]).

c) Let $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$, $(f_n)_n \subseteq \mathbb{R}^X$ be a function sequence and $x \in X$. Then $(f_n)_n$ is $\mathcal{I}$*-exhaustive at* x iff for every $\varepsilon > 0$ there exist a $\delta > 0$ and a set $A \in \mathcal{I}$ (depending on ε and x) such that $|f_n(x) - f_n(z)| < \varepsilon$, whenever $n \in \mathbb{N} \setminus A$ and $z \in B(x, \delta)$. The sequence $(f_n)_n$ is said to be $\mathcal{I}$*-exhaustive on* X iff $(f_n)_n$ is $\mathcal{I}$-exhaustive at every $x \in X$ (see [5, 19]).

When $\mathcal{I} = \mathcal{I}_{\text{fin}}$, the above concept coincides with that of exhaustiveness of a function sequence given in [15]. If $\mathcal{I} = \mathcal{I}_d$, then the notion of $\mathcal{I}$-exhaustiveness coincides with that of *statistical exhaustiveness* given in [10].

d) A sequence $(f_n)_n \subseteq \mathbb{R}^X$ is *uniformly* $\mathcal{I}$*-exhaustive on* X iff for each $\varepsilon > 0$ there are a $\delta > 0$ and a set $A \in \mathcal{I}$ (depending on ε) such that $|f_n(x) - f_n(z)| < \varepsilon$ for all $n \in \mathbb{N} \setminus A$ and $d(x, z) < \delta$.

e) We say that an $(f_n)_n \subseteq \mathbb{R}^X$ is *weakly-*$\mathcal{I}$*-exhaustive at* $x \in X$ iff for every $\varepsilon > 0$ there is a $\delta > 0$ such that for each $z \in B(x, \delta)$ there is a set $A \in \mathcal{I}$ (depending on ε, x and z) with $|f_n(x) - f_n(z)| < \varepsilon$ for all $n \in \mathbb{N} \setminus A$. The sequence $(f_n)_n$ is called *weakly-*$\mathcal{I}$*-exhaustive on* X iff it is weakly-$\mathcal{I}$-exhaustive at every $x \in X$ (see [19]).

When $\mathcal{I} = \mathcal{I}_{\text{fin}}$ the above notion coincides with that of weak-exhaustiveness of a function sequence introduced in [15]. If $\mathcal{I} = \mathcal{I}_d$, then the concept of weak $\mathcal{I}$-exhaustiveness coincides with that of *st-weak exhaustiveness* given in [10].

f) A sequence $(x_n)_n$ in $\mathbb{R}$ is $\mathcal{I}$*-bounded* iff there exists a $K > 0$ such that $\{n \in \mathbb{N} : |x_n| > K\} \in \mathcal{I}$.

A function sequence $(f_n)_n \subseteq Bd(X)$ is $\mathcal{I}$*-bounded* iff there exists a $K > 0$ such that $\{n \in \mathbb{N} : \|f_n\|_\infty > K\} \in \mathcal{I}$.

g) A sequence $(f_n)_n \subseteq \mathbb{R}^X$ is called *$(\mathcal{I}\alpha)$-convergent to* $f \in \mathbb{R}^X$ (shortly $f_n \stackrel{\mathcal{I}\alpha}{\to} f$) iff for each $x \in X$ and for every sequence $(x_n)_n$ in X with $\mathcal{I} - \lim_n x_n = x$ we have $\mathcal{I} - \lim_n f_n(x_n) = f(x)$ (see [19]).

When $\mathcal{I} = \mathcal{I}_{\text{fin}}$ the above concept coincides with that of (α)-convergence (see [7, 13, 16, 20, 21]). Moreover, the $(\mathcal{I}_d\alpha)$-convergence coincides with the *statistically α-convergence* given in [10].

h) A sequence $(f_n)_n \subseteq C_{\mathbb{R}}(X)$ is *$\mathcal{I}$-Alexandroff convergent to* $f \in \mathbb{R}^X$ (shortly $f_n \stackrel{\mathcal{I}-Al.}{\to} f$) iff $(f_n)_n$ $\mathcal{I}$-converges pointwise to f ($f_n \stackrel{\mathcal{I}-p.w.}{\to} f$) and for every $\varepsilon > 0$ and any set $A \in \mathcal{P}(\mathbb{N}) \setminus \mathcal{I}$ there exist an infinite set $M_A = \{n_1 < n_2 < \ldots < n_k < \ldots\} \subseteq A$ and an open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of X such that for every $k \in \mathbb{N}$ and $x \in U_k$ we have $|f_{n_k}(x) - f(x)| < \varepsilon$.

When $\mathcal{I} = \mathcal{I}_{\text{fin}}$ the above concept coincides with that of Alexandroff convergence (denoted by $f_n \stackrel{Al.}{\to} f$) introduced in 1948 by P. S. Alexandroff (see also [1, 8, 13]). When $I = I_d$ this notion coincides with that of statistical Alexandroff convergence introduced in [9, 10]

i) Let $\mathcal{F}$ be the family of all finite subsets of X, $(f_n)_n \subseteq \mathbb{R}^X$, $f \in \mathbb{R}^X$ and $\mathcal{I}$ be a fixed admissible ideal of $\mathbb{N}$. We say that $(f_n)_n$ *converges $\mathcal{I}$-strongly uniformly to f on $\mathcal{F}$* and we write $f_n \stackrel{\mathcal{I}-\mathcal{T}^s_{\mathcal{F}}}{\to} f$, iff for every $\varepsilon > 0$ and for each $B \in \mathcal{F}$ there exist a $\delta > 0$ and a set $A \in \mathcal{I}$ such that for every $z \in B^\delta$ and $n \in \mathbb{N} \setminus A$ we have $|f_n(z) - f(z)| < \varepsilon$.

When $\mathcal{I} = \mathcal{I}_{\text{fin}}$ the above definition coincides with that of strong uniform convergence of a function sequence on finite sets (or $\mathcal{T}^s_{\mathcal{F}}$ convergence) introduced in [4] (see also [9, 10] when $\mathcal{I} = \mathcal{I}_d$).

Proposition 2.4. *With the same notations as above, let $\mathcal{I}$ be any fixed admissible ideal, $\mathcal{I} \neq \mathcal{I}_{fin}$, $(f_n)_n \subseteq \mathbb{R}^X$, and let us consider the following statements:*

(i) $(f_n)_n \stackrel{\mathcal{I}\alpha}{\to} f$.
(ii) $(f_n)_n \stackrel{\alpha}{\to} f$.
(iii) $(f_n)_n \stackrel{\mathcal{I}-p.w.}{\to} f$ and $(f_n)_n$ is $\mathcal{I}$-exhaustive on X.
(iv) $(f_n)_n \stackrel{p.w.}{\to} f$ and $(f_n)_n$ is exhaustive on X.
Then (ii) $\Longleftrightarrow$ (iv), (iv) $\Longrightarrow$ (iii) $\Longrightarrow$(i), (ii) $\Longrightarrow$ (i) and (i) $\not\Rightarrow$ (ii).

Proof. (i) $\not\Rightarrow$ (ii): Let $y_1 \neq y_2 \in \mathbb{R}$ and $H \in \mathcal{I}$ be an infinite set. Since $\mathcal{I} \neq \mathcal{I}_{\text{fin}}$, such a H does exist. Set $f_n(x) = y_1$ for all $x \in X$ and $n \in \mathbb{N} \setminus H$, and $f_n(x) = y_2$ for each $x \in X$ and $n \in H$. Set $f(x) = y_1$ for any $x \in X$. It is not hard too see that (i) is fulfilled, but for any sequence $(x_n)_n$ in X we get that $\lim_n f_n(x_n)$ does not exist in the usual sense.

(ii) $\Longleftrightarrow$ (iv): See [15, Theorem 2.6].

(iv) $\Longrightarrow$ (iii): It is an immediate consequence of definitions of $\mathcal{I}$-pointwise convergence and $\mathcal{I}$-exhaustiveness of a function sequence.

(iii) $\Longrightarrow$ (i): See [19, Theorem 2.5].

(ii) $\Longrightarrow$ (i): Immediate by definition of $(\mathcal{I}\alpha)$-convergence. □

Remark 2.5. The implication (i) $\Longrightarrow$ (iii) was previously proved in [19, Theorem 2.7] for ideals consisting of all subsets of $\mathbb{N}$ which intersect only a finite number of elements of a given infinite partition of $\mathbb{N}$.

The next proposition is an extension of [15, Corollary 3.2.8] to the context of metric spaces and ideal convergence.

Proposition 2.6. *Let $\mathcal{I}$ be any admissible ideal, let $(f_n)_n \stackrel{\mathcal{I}-p.w.}{\to} f$ and $(f_n)_n$ be $\mathcal{I}$-exhaustive on X. Then f is continuous and $(f_n)_n$ $\mathcal{I}$-converges uniformly to f on every compact subset of X.*

Proof. The continuity of f follows from [19, Proposition 2.3]. Let $C \subset X$ be any compact set and fix arbitrarily $\varepsilon > 0$ and $x \in C$. Since $(f_n)_n$ is $\mathcal{I}$-exhaustive at x and f is continuous at x, in correspondence with ε and x there exist $\Lambda_x \in \mathcal{I}$ and an open ball B_x centered at x, with

$$|f_n(z) - f_n(x)| \leq \varepsilon/3 \quad \text{and } |f(z) - f(x)| \leq \varepsilon/3 \tag{1}$$

for each $n \in \mathbb{N} \setminus \Lambda_x$ and $z \in B_x$. Let us consider the family $\{B_x : x \in C\}$. Since C is compact, there exists a finite subfamily $\{B_{x_1}, B_{x_2}, \ldots, B_{x_p}\}$, covering C. Since $(f_n)_n \stackrel{\mathcal{I}-p.w.}{\to} f$, then in correspondence with ε and x_1, x_2, $\ldots$, x_p there exists $\Lambda_0 \in \mathcal{I}$ with

$$|f_n(x_j) - f(x_j)| \leq \varepsilon/3, \quad j = 1, \ldots, p \tag{2}$$

whenever $n \in \mathbb{N} \setminus \Lambda_0$. Set $\Lambda := \Lambda_0 \bigcup\left(\bigcup_{j=1}^{p} \Lambda_{x_j}\right)$. Then $\Lambda \in \mathcal{I}$.

Now, choose arbitrarily $z \in C$: there exists $j \in \{1, 2, \ldots, p\}$ such that $z \in B_{x_j}$. Then, from (1) and (2), for every $n \in \mathbb{N} \setminus \Lambda$ we get

$$|f_n(z) - f(z)| \leq |f_n(z) - f_n(x_j)| + |f_n(x_j) - f(x_j)| + |f(x_j) - f(z)| \leq \varepsilon.$$

This ends the proof. □

Proposition 2.7. *Let (X, d) be a metric space, $(f_n)_n \subseteq \mathbb{R}^X$, $f \in \mathbb{R}^X$ and $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$ such that $f_n \stackrel{\mathcal{I}-p.w.}{\to} f$.*
Then the following are equivalent:

(i) $(f_n)_n$ is weakly $\mathcal{I}$-exhaustive on X.

(ii) f is continuous on X.

Proof. (i)⇒ (ii): See [19, Proposition 2.14].
(ii)⇒ (i): Let $x \in X$ and $\varepsilon > 0$. Since f is continuous at x there is a $\delta > 0$ such that for every $z \in B(x, \delta)$,

$$|f(z) - f(x)| < \frac{\varepsilon}{3}. \tag{3}$$

But $f_n \stackrel{\mathcal{I}-p.w.}{\to} f$, which means that

$$f_n(z) \stackrel{\mathcal{I}}{\to} f(z), \tag{4}$$
$$f_n(x) \stackrel{\mathcal{I}}{\to} f(x). \tag{5}$$

By (4) there exists a set $A_1 \in \mathcal{I}$ with

$$|f_n(z) - f(z)| < \frac{\varepsilon}{3}, \text{ for all } n \notin A_1. \tag{6}$$

By (5) there exists a set $A_2 \in \mathcal{I}$ with

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}, \text{ for all } n \notin A_2. \tag{7}$$

Let $A = A_1 \cup A_2 \in \mathcal{I}$. Then for every $n \notin A_1 \cup A_2$ by (3), (6), (7) and the triangle inequality we get that:

$$\begin{aligned} |f_n(z) - f_n(x)| &\leq |f_n(z) - f(z)| + |f_n(x) - f(x)| + |f(z) - f(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned} \tag{8}$$

By (8) we get that to every $z \in B(x, \delta)$ there corresponds a set $A \in \mathcal{I}$ such that $|f_n(z) - f_n(x)| < \varepsilon$, for all $n \notin A$, and the definition of weak-$\mathcal{I}$-exhaustiveness at x is satisfied. Since $x \in X$ was chosen arbitrarily the result follows. □

Remark 2.8. The notion of weak $\mathcal{I}$-exhaustiveness is strictly weaker than that of $\mathcal{I}$-exhaustiveness. Indeed, let $\mathcal{I}$ be any admissible ideal of $\mathbb{N}$, $\mathbb{R}$ be endowed with the usual metric and fix an arbitrary point $x_0 \in \mathbb{R}$. We consider the sequence $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, defined as follows:

$$f_n(x) = \begin{cases} \frac{1}{2n}, & x \leq x_0 \\ n, & x = x_0 + \frac{1}{n} \\ \frac{1}{n}, & x > x_0,\ x \neq x_0 + \frac{1}{n}. \end{cases}$$

Then obviously $(f_n)_n$ converges $\mathcal{I}$-pointwise to 0, and thus by Proposition 2.7 $(f_n)_n$ is weakly $\mathcal{I}$-exhaustive at x_0. But taking $\varepsilon = 1/4$ it is not hard to see that $(f_n)_n$ is not $\mathcal{I}$-exhaustive at x_0.

Remark 2.9. Propositions 2.6 and 2.7 were obtained independently in [10] for the statistical case. We thank Prof. Lj.D.R. Kočinac who gave us the preprint of [10]. Taking it into account we formulate the following Propositions 2.10, 2.11, 2.12 and 2.13.

The next result is a consequence of Proposition 2.7 and [8, Theorem 2.9] and strengthens [10, Theorem 4.3].

Proposition 2.10. *Let (X, d) be a metric space, $f \in \mathbb{R}^X$, $(f_n)_n \subseteq C_{\mathbb{R}}(X)$ pointwise convergent to f and $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$. Then the following are equivalent:*

(i) $(f_n)_n$ *is weakly-$\mathcal{I}$-exhaustive on X*

(ii) f *is continuous on X*

(iii) $f_n \overset{Al.}{\to} f$

(iv) $f_n \overset{\mathcal{T}^s_{\mathcal{F}}}{\to} f$.

The following result is a strengthening of [10, Proposition 4.4] to the general ideal context.

Proposition 2.11. *Let (X, d), $(f_n)_n$, f be as above such that $f_n \overset{\mathcal{I}-p.w.}{\to} f$ and $(f_n)_n$ is $\mathcal{I}$-exhaustive on X. Then $f_n \overset{\mathcal{I}-\mathcal{T}^s_{\mathcal{F}}}{\to} f$.*

The following result is a generalization of [10, Propositions 4.5 and 4.7].

Proposition 2.12. *Let $(f_n)_n \subseteq C_{\mathbb{R}}(X)$ and $f \in \mathbb{R}^X$ be such that $f_n \overset{\mathcal{I}-\mathcal{T}^s_{\mathcal{F}}}{\to} f$. Then $f_n \overset{\mathcal{I}-p.w.}{\to} f$ and $(f_n)_n$ is weakly-$\mathcal{I}$-exhaustive on X.*

Another kind of ideal convergence which preserves continuity of the limit function is the ideal Alexandroff convergence (see Def. 2.3 h)). The following proposition strengthen [10, Theorem 4.8], which was proved in the particular case of the statistical Alexandroff convergence.

Proposition 2.13. *Let $(f_n)_n \subseteq C_{\mathbb{R}}(X)$, $f \in \mathbb{R}^X$ and $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$. If $f_n \overset{\mathcal{I}-Al.}{\to} f$, then f is continuous.*

3. Compactness and Ascoli-type theorems

We now give the notions of ideal closure and (sequential) compactness with respect to the ideal convergence (see also [5, Definitions 3.1]).

Definition 3.1. Let (X, d) be a metric space.

a) For $F \subset X$ and $u \in X$, we say that u is *in the $\mathcal{I}$-closure of F* iff there is a sequence $(x_n)_n$ in F such that $\mathcal{I} - \lim_n x_n = u$.

b) A subset $F \subset X$ is said to be *$\mathcal{I}$-sequentially compact* iff every sequence $(x_n)_n$ in F contains an $\mathcal{I}$-convergent subsequence $(x_{n_k})_k$ with $\mathcal{I} - \lim_k x_{n_k} \in F$.

We now recall the following proposition ([5, Proposition 2.5 b)]).

Proposition 3.2. *Let $\mathcal{I}$ be any admissible ideal and $(x_n)_n$ be a sequence in $\mathbb{R}$ with $\mathcal{I} - \lim_n x_n = x \in \mathbb{R}$. Then there exists a subsequence $(x_{n_q})_q$ of $(x_n)_n$, such that $\lim_q x_{n_q} = x$ in the usual sense.*

Remark 3.3. An immediate consequence of Proposition 3.2 is that the $\mathcal{I}$-closure of a set coincides with its ordinary closure.

The following result was given in [5, Proposition 3.2].

Proposition 3.4. *Let $\mathcal{I}$ be any admissible ideal of $\mathbb{N}$. Then a subset K of X is sequentially compact if and only if it is $\mathcal{I}$-sequentially compact.*

Corollary 3.5. *A subset K of X is relatively $\mathcal{I}$-sequentially compact (that is its $\mathcal{I}$-closure is $\mathcal{I}$-sequentially compact) iff it is relatively sequentially compact.*

Proof. It is an immediate consequence of Remark 3.3 and Proposition 3.4. □

The following lemma for exhaustive function sequences will be useful to prove our version of the Ascoli theorem.

Lemma 3.6. *Suppose that X is a compact metric space, $\mathcal{I}$ is a P-ideal and $(f_n)_n \subseteq \mathbb{R}^X$ is $\mathcal{I}$-exhaustive on X. Then $(f_n)_n$ is uniformly $\mathcal{I}$-exhaustive on X.*

Moreover, if $\mathcal{I}$ is a P-ideal, then there exists an $\mathcal{I}$-thick sequence $(n_t)_t$ in $\mathbb{N}$ such that $(f_{n_t})_t$ is uniformly exhaustive on X.

Proof. Fix arbitrarily $\varepsilon > 0$. Since $(f_n)_n$ is $\mathcal{I}$-exhaustive on X, by hypothesis we know that for every $x \in X$ and $\varepsilon > 0$ there are $D = D(x,\varepsilon) \in \mathcal{I}$ and $\eta = \eta(x,\varepsilon) > 0$ with $|f_n(x) - f_n(z)| < \varepsilon$ for each $z \in X$ with $d(x,z) < \eta(x,\varepsilon)$ and $n \in \mathbb{N} \setminus D(x,\varepsilon)$. For any $x \in X$ set

$$C(x) := \{z \in X : d(x,z) < \eta(x,\varepsilon/2)\};$$

$$B(x) := \{z \in X : d(x,z) < \frac{1}{2}\eta(x,\varepsilon/2)\}.$$

The set $\mathcal{B} := \{B(x) : x \in X\}$ is an open covering of X. Since X is a compact metric space, $\mathcal{B}$ contains a finite sub-covering of X, say $\mathcal{B}' := \{B(x_1), \dots, B(x_q)\}$, where $q \in \mathbb{N}$.

Let now $\delta(\varepsilon) := \frac{1}{2} \min_{j \in \{1,\dots,q\}} \eta(x_j, \varepsilon/2)$, $A(\varepsilon) := \bigcup_{j=1}^{q} D(x_j, \varepsilon/2)$, and fix arbitrarily $x, z \in X$ with $d(x,z) < \delta(\varepsilon)$. There exists $j \in \{1,2,\dots,q\}$ with $x \in B(x_j) \subset C(x_j)$. It is not difficult to check that $z \in C(x_j)$ and that $A(\varepsilon) \in \mathcal{I}$, since $\mathcal{I}$ is closed under finite unions.

Let now $n \in \mathbb{N} \setminus A(\varepsilon)$: then $n \in \mathbb{N} \setminus D(x_j, \varepsilon/2)$ for all $j \in \{1,\dots,q\}$. Hence,

$$|f_n(x) - f_n(x_j)| < \varepsilon/2, \quad |f_n(z) - f_n(x_j)| < \varepsilon/2,$$

and thus

$$|f_n(x) - f_n(z)| \le |f_n(x) - f_n(x_j)| + |f_n(z) - f_n(x_j)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

by the above inequalities. So we get uniform $\mathcal{I}$-exhaustiveness of $(f_n)_n$.

We now turn to the final part. By uniform $\mathcal{I}$-exhaustiveness of $(f_n)_n$, in correspondence with $j \in \mathbb{N}$ there exist $A_j \in \mathcal{I}$ and $\delta_j > 0$ with $|f_n(x) - f_n(z)| < 1/j$ whenever $d(x,z) < \delta_j$ and $n \in \mathbb{N} \setminus A_j$. Since $\mathcal{I}$ is a P-ideal, by [3, Proposition 1] there is a set $A_\infty \in \mathcal{I}$ such that $A_j \setminus A_\infty$ is finite for all $j \in \mathbb{N}$.

Arguing similarly as in [3, Proposition 3], it is possible to prove that there exists an $\mathcal{I}$-thick sequence $(n_t)_t$ in $\mathbb{N}$ such that for every $\varepsilon > 0$ there are $\bar{t} \in \mathbb{N}$ and $\delta > 0$ with the property that $|f_{n_t}(x) - f_{n_t}(z)| < \varepsilon$ for any $x, z \in X$ with $d(x,z) < \delta$ and for all $t > \bar{t}$. Thus the assertion of Lemma 3.6 follows. □

We now prove our version of the Ascoli theorem (see [2, 7, 13, 15, 16]) in the context of P-ideals, which is an extension of [15, Theorem 3.1.1] to the ideal setting.

Theorem 3.7. *Let X be a compact metric space, $\mathcal{I}$ be a P-ideal, and $(f_n)_n \subseteq Bd(X)$. If the set $\{f_n : n \in \mathbb{N}\}$ is $\mathcal{I}$-bounded and the sequence $(f_n)_n$ is $\mathcal{I}$-exhaustive on X, then $(f_n)_n$ contains a uniformly convergent subsequence. Moreover, the set $\{f_n : n \in \mathbb{N}\}$ is sequentially compact in $Bd(X)$.*

Proof. Since $(f_n)_n$ is $\mathcal{I}$-exhaustive on X, by Lemma 3.6 there exists an $\mathcal{I}$-thick sequence $(n_t)_t$ in $\mathbb{N}$, such that $(f_{n_t})_t$ is uniformly exhaustive on X. Let $B^* := \{n_t : t \in \mathbb{N}\}$. From $\mathcal{I}$-thickness it follows that B^* belongs to the dual filter $\mathcal{F}(\mathcal{I})$. Since $(f_n)_n$ is $\mathcal{I}$-bounded, then there exists a $b > 0$ such that $A_* := \{n \in \mathbb{N} : \|f_n\|_\infty > b\} \in \mathcal{I}$. From this it follows that, if $B_0 := B^* \cap (\mathbb{N} \setminus A_*)$, then $B_0 \in \mathcal{F}(\mathcal{I})$ and the sequence $(f_n)_{n\in B_0}$ is bounded.

Since X is compact, then X is separable (see [13]). Let $\{x_h : h \in \mathbb{N}\}$ be a countable dense subset of X. The boundedness of the sequence $(f_n)_{n\in B_0}$ implies that the sequence $(f_n(x_1))_{n\in B_0}$ is bounded in $\mathbb{R}$. Thus there exists a set $K_1 \subset B_0$, $K_1 =: \{k_n^{(1)} : n \in \mathbb{N}\}$, such that the sequence $(f_{k_n^{(1)}}(x_1))_n$ is convergent in the ordinary sense to a real number y_1.

Proceeding by induction, arguing analogously as in the classical case, we obtain that for all $h \in \mathbb{N}$ there exists a set $\{k_n^{(h)} : n \in \mathbb{N}\}$, with the property that

$$\{k_n^{(h+1)} : n \in \mathbb{N}\} \subset \{k_n^{(h)} : n \in \mathbb{N}\} \subset B_0$$

and such that the sequence $(f_{k_n^{(h)}}(x_h))_n$ is convergent (in the usual sense) to a real number y_h.

For all $h \in \mathbb{N}$, set $g_h(x) := f_{k_h^{(h)}}(x)$. Note that

$$\lim_h g_h(x_j) = y_j \quad \text{for all } j \in \mathbb{N}. \tag{9}$$

We claim that the sequence $(g_k)_k$ is uniformly Cauchy in the ordinary sense. By virtue of the classical results, for each $\delta > 0$ there exists a finite number $s = s(\delta)$ of elements of X, say $x_{j_1}, \ldots, x_{j_s}$, such that

$$X = \bigcup_{r=1}^{s} B(x_{j_r}, \delta). \tag{10}$$

Fix arbitrarily $x \in X$ and $\varepsilon > 0$, and let $\delta > 0$ satisfy the condition of uniform exhaustiveness of the sequence $(f_n)_{n\in B_0}$. So there exists $r \in \{1, 2, \ldots, s(\delta)\}$ with

$$d(x, x_{j_r}) < \delta. \tag{11}$$

By (9), the sequence $(g_k(x_{j_r}))_k$ is convergent to y_{j_r} for each $r \in \{1, \ldots, s(\delta)\}$, so there is a positive integer ν such that

$$|g_k(x_{j_r}) - g_h(x_{j_r})| < \varepsilon \tag{12}$$

whenever $h, k \geq \nu$ and $r = 1, \ldots, s(\delta)$. Note that the natural number ν can be chosen independently of the considered point x. Furthermore, thanks to uniform exhaustiveness of the sequence $(f_n)_{n\in B_0}$ and (11), the integer ν can be chosen with

$$|g_k(x) - g_k(x_{j_r})| < \varepsilon \tag{13}$$

for all $k \geq \nu$. By (12) and (13) we obtain:

$$\begin{aligned} |g_k(x) - g_h(x)| \quad &\leq \quad |g_k(x) - g_k(x_{j_r})| + |g_k(x_{j_r}) - g_h(x_{j_r})| + \\ &+ \quad |g_h(x) - g_h(x_{j_r})| \end{aligned} \tag{14}$$

whenever $h, k \geq \nu$. By (14) we get that the sequence $(g_k)_k$ is uniformly Cauchy, and hence $(g_k)_k$ is uniformly convergent. From this it is easy to deduce sequential compactness of the set $\{f_n : n \in \mathbb{N}\}$ in $Bd(X)$. This completes the proof. □

We now prove a Helly-type theorem, which is an extension of [14, Theorem 5.1], where a similar result is proved in a different context.

Theorem 3.8. *Let $\mathcal{I}$ be any admissible ideal on $\mathbb{N}$, $[a,b]$ be a compact subinterval of the real line and $(g_n)_n \subseteq \mathbb{R}^{[a,b]}$ be an equibounded sequence of monotone functions. Assume that the set $(g_n(x))_n$ is relatively $\mathcal{I}$-sequentially compact for any $x \in [a,b]$.*

Then the sequence $(g_n)_n$ admits a subsequence $(h_n)_n$, convergent pointwise to a function h.

Proof. It is an easy consequence of Corollary 3.5 and [12, Helly Theorem]. □

4. Some applications of exhaustiveness to measures

Let G be any infinite set and $\Sigma \subset \mathcal{P}(G)$ be a σ-algebra. We denote by $ba(\Sigma)$ the set of all real-valued finitely additive bounded measures on Σ and by $ca(\Sigma)$ the linear subspace of $ba(\Sigma)$ consisting of all σ-additive measures on Σ.

We now give some preliminary definitions, and present the notions of ideal exhaustiveness and ideal α-convergence in the context of measures.

Definition 4.1. a) For a positive $\lambda \in ba(\Sigma)$ and $A, B \in \Sigma$ the *(pseudo)-λ-distance* between A and B is defined by $d_\lambda(A,B) := \lambda(A\Delta B)$, where Δ denotes the symmetric difference.

A measure $\mu \in ba(\Sigma)$ is *λ-continuous at* $E \in \Sigma$ iff it is continuous at E on (Σ, d_λ). We say that μ is *λ-continuous on* Σ iff μ is λ-continuous at every $E \in \Sigma$. Observe that μ is λ-absolutely continuous iff μ is λ-continuous at $\emptyset$.

b) Let $\mathcal{I}$ be any admissible ideal of $\mathbb{N}$. A sequence $(\mu_n)_n$ in $ba(\Sigma)$ is *$\mathcal{I}$-exhaustive at* $E \in \Sigma$ iff for each $\varepsilon > 0$ there are a $\delta > 0$ and a set $A \in \mathcal{I}$ such that $|\mu_n(E) - \mu_n(F)| < \varepsilon$ for every $F \in \Sigma$ with $d_\lambda(E,F) < \delta$ and for all $n \in \mathbb{N} \setminus A$. We say that $(\mu_n)_n$ is *$\mathcal{I}$-exhaustive on* Σ iff it is $\mathcal{I}$-exhaustive at E, for every $E \in \Sigma$.

c) A sequence $(\mu_n)_n$ in $ba(\Sigma)$ is *weakly-$\mathcal{I}$-exhaustive at* $E \in \Sigma$ iff for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $F \in \Sigma$ with $d_\lambda(E,F) < \delta$ there is a set $A \in \mathcal{I}$ with $|\mu_n(E) - \mu_n(F)| < \varepsilon$ for all $n \in \mathbb{N} \setminus A$. We say that $(\mu_n)_n$ is *weakly-$\mathcal{I}$-exhaustive on* Σ iff it is weakly-$\mathcal{I}$-exhaustive at every $E \in \Sigma$.

d) We say that $(\mu_n)_n$ *$(\mathcal{I}\alpha)$-converges to μ at* $E \in \Sigma$ iff for every sequence $(E_n)_n$ in Σ with $\mathcal{I} - \lim_n d_\lambda(E_n, E) = 0$ we get $\mathcal{I} - \lim_n \mu_n(E_n) = \mu(E)$. The sequence $(\mu_n)_n$ *$(\mathcal{I}\alpha)$-converges to μ on* Σ iff it $(\mathcal{I}\alpha)$-converges to μ at every $E \in \Sigma$.

Remark 4.2. Observe that Propositions 2.4 and 2.7 can be formulated and proved similarly in the measure setting by using [5, Theorem 4.16].

Proposition 4.3. *Let $\mathcal{I}$ be any fixed admissible ideal, $\mathcal{I} \neq \mathcal{I}_{fin}$, Σ be a σ-algebra, $\lambda \in ba(\Sigma)$ non-negative, $(\mu_n)_n$ be a sequence in $ba(\Sigma)$ and $\mu \in ba(\Sigma)$ (in the sense of Definitions 4.1), and let us consider the following statements:*

(i) $(\mu_n)_n$ is $(\mathcal{I}\alpha)$-convergent to μ

(ii) $(\mu_n)_n$ is (α)-convergent to μ

(iii) $(\mu_n)_n$ $\mathcal{I}$-converges pointwise to μ and $(\mu_n)_n$ is $\mathcal{I}$-exhaustive on Σ

(iv) $(\mu_n)_n$ converges pointwise to μ and $(\mu_n)_n$ is exhaustive on Σ.

Then (ii) $\Longleftrightarrow$ (iv), (iv) $\Longrightarrow$ (iii) $\Longrightarrow$(i), (ii) $\Longrightarrow$(i) and (i) $\not\Rightarrow$ (ii).

Proof. (i) $\not\Rightarrow$ (ii): Let $\Sigma = \mathcal{P}(\mathbb{N})$, $\lambda(A) := \sum_{n\in A} \frac{1}{2^n}$, $A \in \Sigma$, and $H := \{q_1, \ldots, q_n, \ldots\} \in \mathcal{I}$ be an infinite set. Since $\mathcal{I} \neq \mathcal{I}_{\text{fin}}$, such a H does exist. Set $\mu(A) = 0$ for all $A \in \Sigma$, and $\mu_i(A) = 0$ whenever $A \in \Sigma$ and $i \in \mathbb{N} \setminus H$. For each $n \in \mathbb{N}$, let δ_n be the Dirac measure defined by setting $\delta_n(A) = 1$ if $A \in \Sigma$ and $n \in A$, and $\delta_n(A) = 0$ if $A \in \Sigma$ and $n \notin A$. Let $E = \emptyset$ and $E_n = \{n\}$, $n \in \mathbb{N}$. We get $\lim_n \lambda(E_n) = 0$, but $\delta_n(E_n) = 1$ for all $n \in \mathbb{N}$. Set $\mu_{q_n}(A) = \delta_n(A)$ for every $n \in \mathbb{N}$ and $A \in \Sigma$. It is not hard to see that (i) is fulfilled, but $\lim_n \mu_n(E_n)$ does not exist in the ordinary sense.

The other parts can be proved similarly as in Proposition 2.4. □

Remark 4.4. Implication (i) $\Rightarrow$ (iii) was proved in [5, Theorem 4.16] for ideals consisting of all subsets of $\mathbb{N}$ which intersect a finite number of elements of a given infinite partition of $\mathbb{N}$.

Note that, when the ideal involved $\mathcal{I} \neq \mathcal{I}_{\text{fin}}$ satisfies this property, the sequence μ_n, $n \in \mathbb{N}$, defined in Proposition 4.3, (i) $\not\Rightarrow$ (ii), is an $\mathcal{I}$-exhaustive measure sequence, by virtue of [5, Theorem 4.16]. However, this sequence is not $\mathcal{I}_{\text{fin}}$-exhaustive, and it is even not weakly-$\mathcal{I}_{\text{fin}}$-exhaustive at $\emptyset$. Indeed, observe that for every $\vartheta > 0$ there is a cofinite set $E \subset \mathbb{N}$, with the property that $\lambda(E\Delta\emptyset) = \lambda(E) < \vartheta$. Note that for every cofinite subset $M \subset \mathbb{N}$ it is possible to find an integer $\overline{n}$ large enough with $q_{\overline{n}} \in M \cap E \cap H$, so that $1 = \delta_{\overline{n}}(E) = \mu_{q_{\overline{n}}}(E)$.

Similarly as in Proposition 2.7 we get the following

Proposition 4.5. *Let $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$, $(\mu_n)_n$, μ and λ be as above such that $(\mu_n)_n$ $\mathcal{I}$-converges setwise to μ on Σ. Then the following are equivalent:*

(i) $(\mu_n)_n$ is weakly-$\mathcal{I}$-exhaustive on Σ

(ii) μ is λ-continuous on Σ.

Problem 4.6. *Find a pseudometric space (Σ, d_λ), a sequence $(\mu_n)_n$ in $ba(\Sigma)$, an admissible ideal $\mathcal{I}$ of $\mathbb{N}$ and a set $E \in \Sigma$ such that $(\mu_n)_n$ is weakly-$\mathcal{I}$-exhaustive at E but $(\mu_n)_n$ is not $\mathcal{I}$-exhaustive at E.*

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