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Polynomials of the Laguerre type

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Abstract. In this note we shall study a class of polynomials $\{f_{n,m}^{c,r}(x)\}$, where c is some real number, $r \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$. These polynomials are defined by the generating function. Also, for these polynomials we find an explicit representation in the form of the hypergeometric function; some identities of the convolution type are presented; some special cases are shown. The special cases of these polynomials are: Panda's polynomials [2], [4]; the generalized Laguerre polynomials [1], [6]; the Celine Fasenmyer polynomials [3].

1. Polynomials $f_{n,m}^{c,r}(x)$

Let $\phi(u)$ be a formal power–series expansion

$$\phi(u) = \sum_{n=0}^{\infty} \gamma_n u^n, \quad \gamma_n = \frac{(-r^r)^n}{n!}. \tag{1.1}$$

We define the polynomials $f_{n,m}^{c,r}(x)$ as

$$(1-t^m)^{-c}\phi\left(\frac{-4xt}{(1-t^m)^r}\right) = \sum_{n=0}^{\infty} f_{n,m}^{c,r}(x)t^n. \tag{1.2}$$

We prove the following result.

Theorem 1.1. *The polynomials $f_{n,m}^{c,r}(x)$ satisfy the following relations:*

$$f_{n,m}^{c,r}(x) = \frac{(4r^r x)^n}{n!}\, {}_{(r+1)m}F_{rm-1} \times A, \tag{1.3}$$

where

$$A = \begin{bmatrix} -\frac{n}{m}, \dots, \frac{m-1-n}{m}; \frac{1-c-rn}{rm}, \dots, \frac{rm-c-rn}{rm}; \frac{(-1)^{m-1}(4r^r x)^{-m}(rm)^{rm}}{(rm-1)^{rm-1}} \\ \frac{1-c-rn}{rm-1}, \frac{2-c-rn}{rm-1}, \dots, \frac{rm-1-c-rn}{rm-1}; \end{bmatrix} \tag{1.4}$$

$$x^n = \frac{n!}{4^n (r^r)^n} \sum_{k=0}^{[n/m]} (-1)^k \binom{c+rn-rmk}{k} f_{n-mk,m}^{c,r}(x). \tag{1.5}$$

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Proof. Using (1.1) and (1.2), we find:

$$
\begin{aligned}
(1-t^m)^{-c}\phi\left(\frac{-4xt}{(1-t^m)^r}\right) &= (1-t^m)^{-c}\sum_{n=0}^{\infty}\frac{(-r^r)^n}{n!}\frac{(-4x)^n t^n}{(1-t^m)^{rn}} = \sum_{n=0}^{\infty}\frac{(4r^r x)^n t^n}{n!}(1-t^m)^{-c-rn}\\
&= \left(\sum_{n=0}^{\infty}\frac{(4r^r x)^n t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\binom{-c-rn}{k}(-t^m)^k\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(4r^r x)^{n-k}t^{n-k}}{(n-k)!}\binom{-c-r(n-k)}{k}(-1)^k t^{mk}\\
&\qquad (n-k-mk := n,\ \ n-k := n-mk,\ \ k\le [n/m])\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{(-1)^k(4r^r)^{n-mk}x^{n-mk}}{(n-mk)!}\binom{-c-r(n-mk)}{k}t^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[n/m]}\frac{(4r^r)^{n-mk}x^{n-mk}(c+r(n-mk))_k}{k!(n-mk)!}\right)t^n.
\end{aligned}
$$

Using the well–known equalities ([5])

$$
(\alpha)_{n+k} = (\alpha)_n(\alpha+n)_k,\qquad \frac{(-1)^k}{(n-k)!} = \frac{(-n)_k}{n!},\qquad (\alpha)_{n-k} = \frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k},
$$

we find that

$$
\begin{aligned}
\frac{(c+r(n-mk))_k}{(n-mk)!} &= \frac{(c)_{r(n-mk)+k}}{(c)_{r(n-mk)}}\cdot\frac{(-1)^{mk}(-1)^{mk}}{(n-mk)!} = \frac{(-1)^{(rm-1)k}(c)_{rn}}{(1-c-rn)_{(rm-1)k}}\cdot\frac{(1-c-rn)_{rmk}}{(-1)^{rmk}(c)_{rn}}\cdot\frac{(-1)^{mk}(-n)_{mk}}{n!}\\
&= \frac{(-1)^{(m-1)k}(1-c-rn)_{rmk}(-n)_{mk}}{(1-c-rn)_{(rm-1)k}\ n!} = \frac{(-1)^{(m-1)k}(rm)^{rmk}m^{mk}\cdot A\cdot B}{(rm-1)^{(rm-1)k}\ n!\cdot C}
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \left(\frac{1-c-rn}{rm}\right)_k\cdot\left(\frac{2-c-rn}{rm}\right)_k\cdot\cdots\cdot\left(\frac{rm-c-rn}{rm}\right)_k,\\
B &= \left(\frac{-n}{m}\right)_k\cdot\left(\frac{1-n}{m}\right)_k\cdot\cdots\cdot\left(\frac{m-1-n}{m}\right)_k,\\
C &= \left(\frac{1-c-rn}{rm-1}\right)_k\cdot\left(\frac{2-c-rn}{rm-1}\right)_k\cdot\cdots\cdot\left(\frac{rm-1-c-rn}{rm-1}\right)_k.
\end{aligned}
$$

From the other side, because of the next equality,

$$
(1-t^m)^{-c}\,\phi\left(\frac{-4xt}{(1-t^m)^r}\right) = \sum_{n=0}^{\infty}f_{n,m}^{c,r}(x)t^n
$$

it follows

$$
f_{n,m}^{c,r}(x) = \frac{(4r^r x)^n}{n!}\,{}_{(r+1)m}F_{rm-1}\left[\begin{matrix}\frac{1-c-rn}{rm},\dots,\frac{rm-c-rn}{rm};\frac{-n}{m},\dots,\frac{m-1-n}{m};\ \frac{(-1)^{m-1}(4r^r x)^{-m}(rm)^{rm}}{(rm-1)^{rm-1}}\\ \frac{1-c-rn}{rm-1},\frac{2-c-rn}{rm-1},\dots,\frac{rm-1-c-rn}{rm-1};\end{matrix}\right].
$$

These are the required equalities (1.3) and (1.4).

Again, from (1.1) and (1.2), we get:

$$
\begin{aligned}
\sum_{n=0}^{\infty}(-4x)^n t^n\frac{(-r^r)^n}{n!} &= (1-t^m)^{c+rn}\sum_{n=0}^{\infty}f_{n,m}^{c,r}(x)t^n = \left(\sum_{k=0}^{\infty}\binom{c+rn}{k}(-1)^k t^{mk}\right)\left(\sum_{n=0}^{\infty}f_{n,m}^{c,r}(x)t^n\right)\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}(-1)^k\binom{c+r(n-mk)}{k}f_{n-mk,m}^{c,r}(x)t^n.
\end{aligned}
$$

Hence, we get

$$x^n = \frac{n!}{4^n (r^r)^n} \sum_{k=0}^{[n/m]} (-1)^k \binom{c + r(n - mk)}{k} f^{c,r}_{n-mk,m}(x),$$

which yields the equality (1.5). □

2. Some special cases of $f^{c,r}_{n,m}(x)$

If $r = 1$ and $m > 1$, then (1.3)–(1.4) become

$$f^{c,1}_{n,m}(x) = \frac{(-4x)^n}{n!} {}_{2m}F_{m-1} \left[\begin{matrix} \frac{1-c-n}{m}, \frac{2-c-n}{m}, \dots, \frac{m-c-n}{m}; \frac{-n}{m}, \frac{1-n}{m}, \dots, \frac{m-1-n}{m}; \frac{\left(\frac{m}{4x}\right)^m}{(1-m)^{m-1}} \\ \frac{1-c-n}{m-1}, \frac{2-c-n}{m-1}, \dots, \frac{m-1-c-n}{m-1}; \end{matrix} \right].$$

If $m = 1$ and $r > 1$, then (1.3)–(1.4) yield

$$f^{c,r}_{n,1}(x) = \frac{(4r^r x)^n}{n!} {}_{r+1}F_{r-1} \left[\begin{matrix} \frac{1-c-rn}{r}, \dots, \frac{r-c-rn}{r}; -n; \frac{(4r^r x)^{-1}}{(r-1)^{r-1}} \\ \frac{1-c-rn}{r-1}, \dots, \frac{r-1-c-rn}{r-1}; \end{matrix} \right].$$

For $r = 0$ in (1.2), and $\phi(u) = e^u$, we have $(1 - t^m)^{-c} \phi(-4xt) = \sum_{n=0}^{\infty} f^{c,0}_{n,m}(x) t^n$, and hence we get the following equalities:

$$e^{-4xt} = (1 - t^m)^c \sum_{n=0}^{\infty} f^{c,0}_{n,m}(x) t^n,$$

and also

$$\sum_{n=0}^{\infty} \frac{(-4x)^n t^n}{n!} = \left(\sum_{k=0}^{\infty} \binom{c}{k} (-t^m)^k \right) \left(\sum_{n=0}^{\infty} f^{c,0}_{n,m}(x) t^n \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} (-1)^k \binom{c}{k} f^{c,0}_{n-mk,m}(x) t^n.$$

So, we get

$$x^n = \frac{n!}{(-4)^n} \sum_{k=0}^{[n/m]} (-1)^k \binom{c}{k} f^{c,0}_{n-mk,m}(x). \tag{2.1}$$

For $c = 0$ in (1.3) and (1.4), we get the following formula

$$f^{0,r}_{n,m}(x) = \frac{(4r^r x)^n}{n!} {}_{(r+1)m}F_{rm-1} \left[\begin{matrix} \frac{1-rn}{rm}, \dots, \frac{rm-rn}{rm}; \frac{-n}{m}, \dots, \frac{m-1-n}{m}; \frac{(-1)^{m-1}(4r^r x)^{-m} m^m}{(rm-1)^{rm-1}} \\ \frac{1-rn}{rm-1}, \dots, \frac{rm-1-rn}{rm-1}; \end{matrix} \right].$$

For $c = 1$, $m = 2$ and $r = 2$, then, $\gamma_n = \dfrac{(-4)^n}{n!}$ and by (1.3) and (1.4), we get the following formula

$$f^{1,2}_{n,2}(x) = \frac{(4x)^n}{n!} {}_6F_3 \left[\begin{matrix} \frac{-2n}{4}, \frac{1-2n}{4}, \frac{2-2n}{4}, \frac{3-2n}{4}; \frac{-n}{2}, \frac{1-n}{2}; \frac{-1}{12^3 x^2} \\ \frac{-2n}{3}, \frac{1-2n}{3}, \frac{2-2n}{3}; \end{matrix} \right]$$

Note that the generalized Laguerre polynomials are the special case of the polynomials $f_{n,m}^{c,r}(x)$, that is, $L_{n,m}^{c}(x) = f_{n,m}^{c-1,1}(x/4)$. So, we get the following representation

$$L_{n,m}^{c} = \frac{x^n}{n!}\, {}_{2m}F_{m-1}\left[\begin{matrix} \frac{2-c-n}{m}, \ldots, \frac{m+1-c-n}{m}; \frac{-n}{m}, \ldots, \frac{m-1-n}{m}; \frac{x^{-m}m^m}{(1-m)^{m-1}} \\ \frac{2-c-n}{m-1}, \ldots, \frac{m-c-n}{m-1}; \end{matrix}\right].$$

For the Laguerre polynomials $L_{n,1}^{c}(x) \equiv L_n^c(x)$, where

$$(1-t)^{-c} \exp\left\{\frac{-xt}{1-t}\right\} = \sum_{n=0}^{\infty} L_n^c(x)\frac{t^n}{n!},$$

the following statement holds.

Theorem 2.1. *Let* $D = \dfrac{d}{dx}$ *and* $g \in C^\infty(-\infty, +\infty)$ *and* $g(x) \neq 0$, *then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}D^{n-k-j}\{g^{-1}\}D^j\{g\} = (-1)^n n!. \tag{2.2}$$

Proof. Using the known formula ([5])

$$DL_n^c(x) = DL_{n-1}^c(x) - L_{n-1}^c(x), \quad n \geq 1,$$

we get the following equalities:

$$DL_n^c(x) = (D-1)L_{n-1}^c(x), \quad D^2L_n^c(x) = (D-1)^2L_{n-2}^c(x), \quad \ldots, \quad D^sL_n^c(x) = (D-1)^sL_{n-s}^c(x),\ n \geq s.$$

Hence, for $s = n$, we obtain that

$$D^nL_n^c(x) = \left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}D^{n-k}\right)\{1\} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}D^{n-k-j}\{g^{-1}\}D^j\{g\}.$$

Since $D^nL_n^c(x) = (-1)^n n!$, we get

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}D^{n-k-j}\{g^{-1}\}D^j\{g\} = (-1)^n n!,$$

which leads to (2.2). □

Depending on the chosen functions $g(x)$ and from (2.2), we get some interesting relations.

1° For $g(x) = e^{ax}$, a is any rial number, and $g^{-1}(x) = e^{-ax}$, we get

$$a^n\sum_{k=0}^{n}\frac{a^{-k}}{k!}\sum_{j=0}^{n-k}\frac{(-1)^j}{j!(n-k-j)!} = 1.$$

2° If $g(x) = (1+x)^\alpha$, for $x > -1$, $\alpha \neq 0$, then we get

$$(\alpha)_n \Gamma(\alpha) \sum_{k=0}^{n} \frac{(1+x)^k}{k!} \sum_{j=0}^{n-k} \frac{1}{(1-\alpha-n)_{k+j-2}\Gamma(\alpha-j+2)} = \frac{(1+x)^n}{\alpha n!},$$

or

$$\sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{(-1)^j (1+x)^k \left(\prod_{i=0}^{j}(\alpha+1-i)\right)\left(\prod_{s=0}^{n-k-j-1}(\alpha+s)\right)}{k!\,j!(n-k-j)!} = \frac{(1+x)^n}{\alpha}.$$

3° For $g(x) = a^x$, $a > 0$ and $a \neq 1$, we obtain

$$\sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{(-1)^j (\ln a)^{n-k}}{j!k!(n-k-j)!} = 1.$$

4° For $g(x) = x^\alpha e^x$, we have the following formula

$$\sum_{k=0}^{n} \sum_{j=0}^{n-k} \sum_{i=0}^{n-k-j} \sum_{l=0}^{j} \frac{(-1)^j (\alpha)_j (\alpha+1-l)_l x^{-i-l}}{k!i!l!(n-k-j)!(j-l)!} = 1,$$

or in the form

$$\sum_{k=0}^{n} \sum_{j=0}^{n-k} \sum_{i=0}^{n-k-j} \sum_{l=0}^{j} \frac{(-1)^{k+i} (-n)_{k+j+i} (\alpha)_j}{x^{i+l} k!i!l!(j-l)!} = \frac{n!}{\Gamma(\alpha+1)}.$$

Hence, for $\alpha = n$, $n \in \mathbb{N}$, we get

$$\sum_{k=0}^{n} \sum_{j=0}^{n-k} \sum_{i=0}^{n-k-j} \sum_{l=0}^{j} \frac{(-1)^{k+i} (-n)_{k+j+i} j!}{x^{i+l} i!l!(j-l)!} = 1.$$

5° For $g(x) = x^\alpha$, $x \geq 0$ and $x \neq 1$, $\alpha \neq 0$, we obtain

$$\sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{(-1)^j \prod_{i=0}^{n-k-j-1}(\alpha+j) \prod_{s=0}^{j-1}(\alpha-s) x^k}{k!\,j!} = x^n.$$

3. Some identities of the convolution type

Using the following equality

$$(1-t^m)^{-c/2}\phi\left(\frac{-4xt}{(1-t^m)^r}\right)=(1-t^m)^{c/2}\sum_{n=0}^{\infty} f^{c/2,r}(x)t^n, \tag{3.1}$$

and by (1.1) and (1.2), we get

$$\sum_{n=0}^{\infty} f_{n,m}^{c/2,r}(x)t^n=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{c/2}{k}(-1)^k f_{n-k,m}^{c/2,r}(x)t^{n+mk-k}=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}\binom{c/2}{k}(-1)^k f_{n-mk,m}^{c/2,r}(x)t^n.$$

Hence, we get

$$f_{n,m}^{c/2,r}(x)=\sum_{k=0}^{[n/m]}(-1)^k\binom{c/2}{k}f_{n-mk,m}^{c/2,r}(x). \tag{3.2}$$

Again, by (1.1) and (1.2), for $\phi(u)=e^u$ we find:

$$(1-t^m)^{-c}\phi\left(\frac{-4(x_1+x_2+\cdots+x_s)t}{(1-t^m)^r}\right)=\sum_{n=0}^{\infty} f_{n,m}^{c,r}(x_1+x_2+\cdots+x_s)t^n,$$

that is,

$$(1-t^m)^{-c/s}e^{\frac{-4x_1t}{(1-t^m)^r}}\cdot\cdots\cdot(1-t^m)^{-c/s}e^{\frac{-4x_st}{(1-t^m)^r}}=\sum_{n=0}^{\infty} f_{n,m}^{c,r}(x_1+\cdots+x_s)t^n,$$

hence

$$\left(\sum_{n=0}^{\infty} f_{n,m}^{c/s,r}(x_1)t^n\right)\left(\sum_{n=0}^{\infty} f_{n,m}^{c/s,r}(x_2)t^n\right)\cdots\left(\sum_{n=0}^{\infty} f_{n,m}^{c/s,r}(x_s)t^n\right)=\sum_{n=0}^{\infty} f_{n,m}^{c,r}(x_1+\cdots+x_s)t^n.$$

So, we get

$$\sum_{i_1+\cdots+i_s=n} f_{i_1,m}^{c/s,r}(x_1)f_{i_2,m}^{c/s,r}(x_2)\ldots f_{i_s,m}^{c/s,r}(x_s)=f_{n,m}^{c,r}(x_1+x_2+\cdots+x_s). \tag{3.3}$$

Let $c=c_1+\cdots+c_k$ and $x=x_1+\cdots+x_k$, then we have, at, on the one side

$$(1-t^m)^{-c_1-\cdots-c_k}\phi\left(\frac{-4(x_1+\cdots+x_k)t}{(1-t^m)^r}\right)=\sum_{n=0}^{\infty} f_{n,m}^{c_1+\cdots+c_k,r}(x_1+\cdots+x_k)t^n,$$

and on the other hand

$$\begin{aligned}(1-t^m)^{-c_1-\cdots-c_k}\phi\left(\frac{-4(x_1+\cdots+x_k)t}{(1-t^m)^r}\right)&=\left(\sum_{n=0}^{\infty} f_{n,m}^{c_1,r}(x_1)t^n\right)\cdot\cdots\cdot\left(\sum_{n=0}^{\infty} f_{n,m}^{c_k,r}(x_k)t^n\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{i_1+\cdots+i_k=n} f_{i_1,m}^{c_1,r}(x_1)\ldots f_{i_k,m}^{c_k,r}(x_k)\right)t^n.\end{aligned}$$

Hence we get the following formula

$$\sum_{i_1+\cdots+i_k=n} f_{i_1,m}^{c_1,r}(x_1)\cdot\cdots\cdot f_{i_k,m}^{c_k,r}(x_k) = f_{n,m}^{c_1+\cdots+c_k,r}(x_1+\cdots+x_k). \tag{3.4}$$

If $x_1 = x_2 = \cdots = x_s = \dfrac{x}{s}$, then (3.3) becomes

$$\sum_{i_1+\cdots+i_s=n} f_{i_1,m}^{c/s,r}(x/s)\ldots f_{i_s,m}^{c/s,r}(x/s) = f_{n,m}^{c,r}(x), \tag{3.5}$$

where s is a natural number.

For $r = 0$ in (1.2), and $\phi(u) = e^u$, we have $(1-t^m)^{-c}\phi(-4xt) = \sum_{n=0}^{\infty} f_{n,m}^{c,0}(x)t^n$, whence we get

$$e^{-4xt} = (1-t^m)^c \sum_{n=0}^{\infty} f_{n,m}^{c,0}(x)t^n,$$

and also

$$\sum_{n=0}^{\infty}\frac{(-4x)^n t^n}{n!} = \left(\sum_{k=0}^{\infty}\binom{c}{k}(-t^m)^k\right)\left(\sum_{n=0}^{\infty} f_{n,m}^{c,0}(x)t^n\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}(-1)^k\binom{c}{k} f_{n-mk,m}^{c,0}(x)t^n.$$

So, we get

$$x^n = \frac{n!}{(-4)^n}\sum_{k=0}^{[n/m]}(-1)^k\binom{c}{k} f_{n-mk,m}^{c,0}(x). \tag{3.6}$$

For $c = 0$ in (1.3) and (1.4), we get the following formula

$$f_{n,m}^{0,r}(x) = \frac{(4r^r x)^n}{n!}\,{}_{(r+1)m}F_{rm-1}\left[\begin{matrix}\frac{1-rn}{rm},\ldots,\frac{rm-rn}{rm};\frac{-n}{m},\ldots,\frac{m-1-n}{m}; & \frac{(-1)^{m-1}(4r^r x)^{-m} m^m}{(rm-1)^{rm-1}}\\ \frac{1-rn}{rm-1},\ldots,\frac{rm-1-rn}{rm-1}; & \end{matrix}\right].$$

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