

Well posedness for a class of ultra-parabolic equations with discontinuous flux

Jela Susić^a

^aUniversity of Montenegro, Cetinjski put bb, 81000 Podgorica, Montenegro

Abstract. We prove existence and uniqueness of a weak solution to an ultra-parabolic equation with discontinuous convection term. Due to degeneracy in the parabolic term, the equation does not admit the classical solution. Equations of this type describe processes where transport is negligible in some directions.

1. Introduction

The subject of the paper is the well posedness for the following Cauchy problem

$$u_t + \operatorname{div} f(x, u) = \sum_{i,j=1}^k \partial_{x_i} (a_{ij} \partial_{x_j} (u)), \quad k \leq d \quad (1)$$

$$u|_{t=0} = u_0 \in L^2 \cap L^\infty(\mathbb{R}^d), \quad (2)$$

for a constants a and b . The matrix $(a_{ij})_{i,j=1,\dots,k}$ is a constant matrix such that there exists $\lambda > 0$ satisfying

$$\sum_{i,j=1}^k a_{ij} \xi_i \xi_j > \lambda |\xi|^2, \quad \xi \in \mathbb{R}^k, \quad (3)$$

and $f = (f_1, \dots, f_k, 0, \dots, 0)$ is such that there exists constants $C_i, i = 1, \dots, 4$:

$$\max_{1 \leq i \leq k} \|\partial_\xi f_i(x, \xi)\|_\infty < C_1;$$

$$\int_{\mathbb{R}^d} \sum_{i=1}^k \sup_{\xi \in \mathbb{R}} |f_i(x, \xi)|^2 dx < C_2, \quad \int_{\mathbb{R}^d} \sum_{i=1}^k \sup_{\xi \in \mathbb{R}} |\partial_\xi f_i(x, \xi)|^2 dx < C_3;$$

$$\int_{\mathbb{R}^d} \sup_{\xi \in \mathbb{R}} \frac{|f_i(x + \Delta x, \xi) - f_i(x, \xi)|^2}{\Delta x} dx < C_4, \quad \Delta x \in \mathbb{R}^d.$$

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Email address: jela@rc.pmf.ac.me (Jela Susić)

The latter assumptions clearly include cases of discontinuous flux. For instance, the third assumption is fulfilled if $f \in C^1(\mathbb{R}_\xi; BV(\mathbb{R}_x^d))^d$ and $\sup_{\xi \in \mathbb{R}} |f(\cdot, \xi)| \in BV(\mathbb{R}_x^d) \cap L^2(\mathbb{R}^d)$.

If $k = d$ then (1) is parabolic and the corresponding Cauchy problem admits a unique smooth solution with the initial data satisfied in the sense of strong traces of the solution u on $t = 0$ (see [8]). If $k < d$, then (1) is ultraparabolic. Remark that the condition $u_0 \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is not substantial (it is enough to assume $u_0 \in L^\infty(\mathbb{R}^d)$) but to avoid certain technical moments we shall assume exactly (2).

Specific situation modeled by (1) is the one when transport can be neglected in the directions x_{k+1}, \dots, x_d which is expected in certain physical situations (see Figure 1). In general, ultraparabolic equations were firstly considered by Graetz [4], and Nusselt [9] in their investigations concerning the heat transfer. Besides the heat transfer, equations of type (1) describe processes in porous media (cf. [12]) such as oil extraction or CO2 sequestration which typically occur in highly heterogeneous surroundings (again Figure 1). One can also find applications in sedimentation processes, traffic flow, radar shape-from-shading problems, blood flow, gas flow in a variable duct and so on.

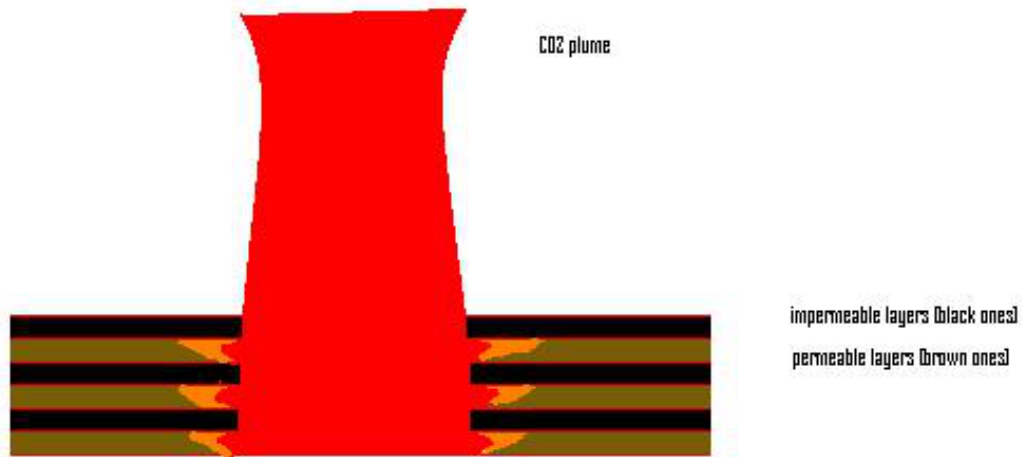


Figure 1: CO2-plume expansion in a highly stratified surrounding. Transport in the vertical direction can be obviously neglected. Diffusion term in the equations modeling the transport is usually linear (e.g. [11]).

Concerning the technical moments, if the flux is regular enough, then Cauchy problem (1), (2) can be solved using either the kinetic approach [2] or the Kruzhkov [7] method of doubling of variables [3]. Precompactness properties of families of solutions to equations of type (1) are considered in [10, 13] using techniques of Tartar’s H -measures [14] (see also [6] for a more general situation).

Usual situation for Cauchy problem (1), (2) is that it does not admit the classical solution, but it admits several weak solutions. Physically relevant weak solution is then singled out by using entropy inequalities [2, 3, 7]. In the case of (1), (2), we are not able to prove existence of the classical solution, but we do not need the entropy type inequalities to prove uniqueness of the weak solution defined as follows.

Definition 1.1. We say that a function $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ to (1), (2) is a weak solution to (1), (2) if

- $\partial_{x_j} u \in L^2(\mathbb{R}^+ \times \mathbb{R}^d)$, $j = 1, \dots, k$;
- for every $T > 0$ and every $\varphi \in C_c^2([0, T] \times \mathbb{R}^d)$

$$\int_{[0, T] \times \mathbb{R}^d} (u \partial_t \varphi + \langle f(x, u), \nabla \varphi \rangle) dx dt - \int_{\mathbb{R}^d} u_0 \varphi dx = \int_{[0, T] \times \mathbb{R}^d} \sum_{i, j}^k a_{ij} \partial_{x_j} u \partial_{x_i} \varphi dx dt.$$

The main theorem of the paper is:

Theorem 1.2. *There exists a unique weak solution to (1), (2).*

Moreover, if u and v are two weak solutions to (1) with initial conditions u_0 and v_0 , respectively, then it holds

$$\int_0^T \int_{\|x\| < R} |u(t, x) - v(t, x)| dx dt \leq C(R, T) \int_{R^d} |u_0(x) - v_0(x)| dx,$$

for a constant $C(T, R)$ depending on T and R .

The next section will be dedicated to the proof of the theorem. Existence proof will be given via the method of shifting of variables, while the uniqueness will be provided by deriving the corresponding Kato inequality.

2. Proof of Theorem 1.2

In order to prove existence, we shall use the vanishing viscosity and flux regularization of (1), (2), and prove that the family of solutions to the regularized problems is strongly L^1_{loc} precompact. More precisely, we shall prove that the family admits the L^1_{loc} -limit along a subsequence, and that limit will obviously be wanted solution to (1), (2).

The uniqueness will follow by a simple adaptation of the existence proof.

2.1. The existence proof

We first regularize functions $f_i, i = 1, 2, \dots, k$ in x .

$$f_{i\varepsilon}(x, \lambda) = f_i(x, \lambda) * \rho_\varepsilon(x) = \int f_i(y, \lambda) \cdot \rho_\varepsilon(x - y) dy,$$

where

$$\rho_\varepsilon(z) = \frac{1}{\varepsilon^d} \rho\left(\frac{z_1}{\varepsilon}\right) \cdot \rho\left(\frac{z_2}{\varepsilon}\right) \cdot \dots \cdot \rho\left(\frac{z_d}{\varepsilon}\right), \quad \rho \geq 0, \quad \varepsilon > 0.$$

$$\rho \in C_c^\infty(\mathbb{R}), \quad \text{supp} \rho \subset (-1, 1), \quad \int \rho(z) dz = 1$$

Consider now a viscous regularization of the equation(1):

$$\partial_t u_\varepsilon + \text{div} f_\varepsilon(x, u_\varepsilon) = \sum_{i,j=1}^k \partial_{x_i} (a_{ij} \partial_{x_j} u_\varepsilon) + \varepsilon \sum_{i=k+1}^d \partial_{x_i x_i} u_\varepsilon \tag{4}$$

It is well known that, since (4) is strictly parabolic, Cauchy problem (4), (2) admits a unique solution belonging to $H^s(\mathbb{R}^+ \times \mathbb{R}^d)$ for any $s > 0$. If u_ε converges in L^1_{loc} as $\varepsilon \rightarrow 0$, then it converges to u , where u is wanted solution of (1), (2). In order to show that u_ε converges, we will apply the method of shifting of variables and the Kolmogorov-Riesz compactness criterion (see e.g. [5]).

First, we need the following apriori estimate.

Lemma 2.1. *Let u be a weak solution to (1). Then, there exists a constant C_5 such that it holds for any $T > 0$*

$$\int_{[0,T] \times \mathbb{R}^d} |\partial_{x_r} u_\varepsilon|^2 dx dt \leq C_5, \quad r = 1, \dots, k.$$

Proof. By multiplying (4) by u_ε and integrating over $R^+ \times R^d$, we obtain after elementary partial integration and using ellipticity condition (3)

$$\int_{R^d} u_\varepsilon^2(T, x) dx - \int_{R^d} u_0^2(x) dx + \lambda \sum_{r=1}^k \int_{[0, T] \times R^d} |\partial_{x_r} u_\varepsilon|^2 \leq \int_{[0, T] \times R^d} \sum_{r=1}^k |f_{r\varepsilon}(x, u_\varepsilon) \partial_{x_r} u_\varepsilon| dx dt.$$

Applying the Young inequality on the right-hand side of the latter expression, we obtain:

$$\int_{R^d} u_\varepsilon^2(T, x) dx + \frac{\lambda}{2} \sum_{r=1}^k \int_{[0, T] \times R^d} |\partial_{x_r} u_\varepsilon|^2 \leq \frac{1}{2\lambda} \int_{[0, T] \times R^d} \sum_{r=1}^k |f_{r\varepsilon}(x, u_\varepsilon)|^2 dx dt + \int_{R^d} u_0^2(x) dx.$$

This concludes the proof. \square

Consider equation (4) with the variable x shifted by Δx (i.e. we introduce the change of variables $x \mapsto x + \Delta x$). It holds:

$$\begin{aligned} \partial_t \tilde{u}_\varepsilon + \operatorname{div} f_\varepsilon(x + \Delta x, \tilde{u}_\varepsilon) &= \sum_{i,j=1}^k \partial_{x_i} (a_{ij} \partial_{x_j} \tilde{u}_\varepsilon) + \varepsilon \sum_{i=k+1}^d \partial_{x_i x_i} \tilde{u}_\varepsilon \\ \tilde{u}_\varepsilon|_{t=0} &= u_0(x + \Delta x), \end{aligned} \tag{5}$$

where $\tilde{u}_\varepsilon(t, x) = u_\varepsilon(t, x + \Delta x)$, and u_ε is the solution to

$$\begin{aligned} \partial_t u_\varepsilon + \operatorname{div} f_\varepsilon(x, u_\varepsilon) &= \sum_{i,j=1}^k \partial_{x_i} (a_{ij} \partial_{x_j} u_\varepsilon) + \varepsilon \sum_{i=k+1}^d \partial_{x_i x_i} u_\varepsilon, \\ u_\varepsilon|_{t=0} &= u_0(x). \end{aligned} \tag{6}$$

Subtracting the equations (5) and (6), we obtain:

$$\partial_t (\tilde{u}_\varepsilon - u_\varepsilon) + \operatorname{div} (f_\varepsilon(x + \Delta x, \tilde{u}_\varepsilon) - f_\varepsilon(x, u_\varepsilon)) = \sum_{i,j=1}^k \partial_{x_i} [a_{ij} \partial_{x_j} (\tilde{u}_\varepsilon - u_\varepsilon)] + \varepsilon \sum_{i=k+1}^d \partial_{x_i x_i} (\tilde{u}_\varepsilon - u_\varepsilon).$$

Multiplying the previous equality by $\eta'(\tilde{u}_\varepsilon - u_\varepsilon)$, we obtain:

$$\begin{aligned} \partial_t \eta(\tilde{u}_\varepsilon - u_\varepsilon) + \operatorname{div} (\eta'(\tilde{u}_\varepsilon - u_\varepsilon) (f_\varepsilon(x + \Delta x, \tilde{u}_\varepsilon) - f_\varepsilon(x, u_\varepsilon))) & \\ - \sum_{i=1}^k \eta''(\tilde{u}_\varepsilon - u_\varepsilon) (f_{i\varepsilon}(x + \Delta x, \tilde{u}_\varepsilon) - f_{i\varepsilon}(x, u_\varepsilon)) \partial_{x_i} (\tilde{u}_\varepsilon - u_\varepsilon) & \\ = \sum_{i,j=1}^k \partial_{x_i} (\eta'(\tilde{u}_\varepsilon - u_\varepsilon) \partial_{x_j} (a_{ij} (\tilde{u}_\varepsilon - u_\varepsilon))) & \\ + \varepsilon \sum_{i=k+1}^d \partial_{x_i x_i} (\eta(\tilde{u}_\varepsilon - u_\varepsilon)) - \varepsilon \sum_{i=k+1}^d \eta''(\tilde{u}_\varepsilon - u_\varepsilon) (\partial_{x_i} (\tilde{u}_\varepsilon - u_\varepsilon))^2 & \\ - \sum_{i,j=1}^k \eta''(\tilde{u}_\varepsilon - u_\varepsilon) a_{ij} \partial_{x_j} (\tilde{u}_\varepsilon - u_\varepsilon) \partial_{x_i} (\tilde{u}_\varepsilon - u_\varepsilon) & \end{aligned} \tag{7}$$

Observe that $u_\varepsilon(x_1, \dots, x_{i-1}, \pm\infty, x_{i+1}, \dots, x_d) = 0$ for almost every $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d) \in R^{d-1}$ because of u_ε is a solution of strongly parabolic equation with the initial

condition that vanish at infinity. Similarly, there holds $\tilde{u}_\varepsilon(x_1, \dots, x_{i-1}, \pm\infty, x_{i+1}, \dots, x_d) = 0$. This yields:

$$\left\{ \begin{aligned} &\int_{\mathbb{R}^d} \operatorname{div}(\eta'(\tilde{u}_\varepsilon - u_\varepsilon)(f_\varepsilon(x + \Delta x, \tilde{u}_\varepsilon) - f_\varepsilon(x, u_\varepsilon))) \, dx = 0 \\ &\int_{\mathbb{R}^d} \partial_{x_i x_i}(\eta(\tilde{u}_\varepsilon - u_\varepsilon)) \, dx = 0 \\ &\int_{\mathbb{R}^d} \sum_{i,j=1}^d \partial_{x_i}(\eta'(\tilde{u}_\varepsilon - u_\varepsilon) \cdot \partial_{x_i}(a_{ij}(\tilde{u}_\varepsilon) \partial_{x_j} \tilde{u}_\varepsilon - a_{ij}(u_\varepsilon) \partial_{x_j} u_\varepsilon)) \, dx = 0 \end{aligned} \right. \tag{8}$$

Assume now that $\eta \geq 0$ and $\eta'' \geq 0$. By (8) and from the ellipticity condition (3), it follows by integrating equality (7) over the set $[0, t] \times \mathbb{R}^d$:

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^d} \partial_t \eta(\tilde{u}_\varepsilon - u_\varepsilon) \\ &+ \int_0^T \int_{\mathbb{R}^d} \sum_{i=1}^d \eta''(\tilde{u}_\varepsilon - u_\varepsilon)(f_{i\varepsilon}(x + \Delta x, u) - f_{i\varepsilon}(x, u)) \partial_{x_i}(\tilde{u}_\varepsilon - u_\varepsilon) \, dx \, dt \\ &= \int_{\mathbb{R}^d} [\eta(\tilde{u}_\varepsilon(T, x) - u_\varepsilon(T, x)) - \eta(\tilde{u}_\varepsilon(0, x + \Delta x) - u_\varepsilon(0, x))] \, dx \\ &- \int_0^T \int_{\mathbb{R}^d} \sum_{i=1}^d \eta''(\tilde{u}_\varepsilon - u_\varepsilon)(f_{i\varepsilon}(x + \Delta x, \tilde{u}_\varepsilon) - f_{i\varepsilon}(x, u_\varepsilon)) \partial_{x_i}(\tilde{u}_\varepsilon - u_\varepsilon) \, dx \, dt \\ &\leq 0. \end{aligned} \tag{9}$$

Now, we can specify the entropy η . Let $\eta(\tilde{u}_\varepsilon - u_\varepsilon) = |z|_\delta$, where $|z|_\delta$ is a C^2 -function such that

$$|z|'_\delta = \begin{cases} 1, & z > \delta \\ -1, & z < -\delta \end{cases}, \quad 0 \leq |z|''_\delta < \frac{1}{\delta}.$$

Then (9) becomes:

$$\begin{aligned} &\int_{\mathbb{R}^d} [|u_\varepsilon(T, x + \Delta x) - u_\varepsilon(T, x)|_\delta - |u_0(x + \Delta x) - u_0(x)|_\delta] \, dx \\ &\leq \int_0^T \int_{\mathbb{R}^d} \sum_{i=1}^k |\tilde{u}_\varepsilon - u_\varepsilon|''_\delta (f_{i\varepsilon}(x + \Delta x, \tilde{u}_\varepsilon) - f_{i\varepsilon}(x + \Delta x, u_\varepsilon)) \partial_{x_i}(\tilde{u}_\varepsilon - u_\varepsilon) \, dx \, dt \\ &+ \int_0^T \int_{\mathbb{R}^d} \sum_{i=1}^k |\tilde{u}_\varepsilon - u_\varepsilon|''_\delta (f_{i\varepsilon}(x + \Delta x, u_\varepsilon) - f_{i\varepsilon}(x, u_\varepsilon)) \partial_{x_i}(\tilde{u}_\varepsilon - u_\varepsilon) \, dx \, dt. \end{aligned} \tag{10}$$

Denote the first summand on the right hand side by (a) and the second by (b). Furthermore, denote by $G_r(x) = \sup_{\xi \in \mathbb{R}} \sup_{\varepsilon \in (0,1)} |f'_{r\varepsilon, \xi}(x, \xi)|$, $r = 1, \dots, k$. It holds according to Lemma 2.1 and definition of $|\cdot|_\delta$:

$$(a) \quad \int_0^T \int_{\mathbb{R}^d} \sum_{i=1}^k |\tilde{u}_\varepsilon - u_\varepsilon|''_\delta (f_{i\varepsilon}(x + \Delta x, \tilde{u}_\varepsilon) - f_{i\varepsilon}(x + \Delta x, u_\varepsilon)) \partial_{x_i}(\tilde{u}_\varepsilon - u_\varepsilon) \, dx \, dt$$

$$\begin{aligned}
 &= \sum_{i=1}^k \int_0^T \int_{\mathbb{R}^d} |\tilde{u}_\varepsilon - u_\varepsilon|'_\delta f'_{i\varepsilon, \xi}(x + \Delta x, \xi(t, x)) (\tilde{u}_\varepsilon - u_\varepsilon) \partial_{x_i} (\tilde{u}_\varepsilon - u_\varepsilon) dx dt \\
 &\leq \sum_{i=1}^k \int_0^T \int_{\mathbb{R}^d} |\tilde{u}_\varepsilon - u_\varepsilon|'_\delta (\tilde{u}_\varepsilon - u_\varepsilon) \cdot G_i(x) \cdot \partial_{x_i} (\tilde{u}_\varepsilon - u_\varepsilon) dx dt \\
 &\leq \sum_{i=1}^k \left(\int_0^T \int_{\mathbb{R}^d} (|\tilde{u}_\varepsilon - u_\varepsilon|'_\delta (\tilde{u}_\varepsilon - u_\varepsilon))^2 G_i^2(x) dx dt \right)^{1/2} \cdot \left(\int_0^T \int_{\mathbb{R}^d} (\partial_{x_i} (\tilde{u}_\varepsilon - u_\varepsilon))^2 dx dt \right)^{1/2} \\
 &= o(1), \quad \delta \rightarrow 0,
 \end{aligned}$$

according to the Lebesgue dominated convergence theorem since $|\tilde{u}_\varepsilon - u_\varepsilon|'_\delta (\tilde{u}_\varepsilon - u_\varepsilon) \rightarrow 0$ pointwisely (keep in mind that $G_i \in L^2(\mathbb{R}^d)$, $i = 1, \dots, k$). Similarly as in (a), and since $f = f(x, \lambda)$ is of bounded variation with respect to $x \in \mathbb{R}^d$ (see conditions on f), we conclude

$$\begin{aligned}
 (b) \quad &\int_0^T \int_{\mathbb{R}^d} \sum_{i=1}^k |\tilde{u}_\varepsilon - u_\varepsilon|'_\delta (f_{i\varepsilon}(x + \Delta x, u_\varepsilon) - f_{i\varepsilon}(x, u_\varepsilon)) \partial_{x_i} (\tilde{u}_\varepsilon - u_\varepsilon) dx dt \\
 &= \sum_{i=1}^k \int_0^T \int_{\mathbb{R}^d} |\tilde{u}_\varepsilon - u_\varepsilon|'_\delta \frac{f_{i\varepsilon}(x + \Delta x, u_\varepsilon) - f_{i\varepsilon}(x, u_\varepsilon)}{(\Delta x)^{1/2}} \cdot (\Delta x)^{1/2} \cdot \partial_{x_i} (\tilde{u}_\varepsilon - u_\varepsilon) dx dt \\
 &\leq \sum_{i=1}^k \left(\int_0^T \int_{\mathbb{R}^d} \left(\frac{f_{i\varepsilon}(x + \Delta x, u_\varepsilon) - f_{i\varepsilon}(x, u_\varepsilon)}{(\Delta x)^{1/2}} \right)^2 dx dt \right)^{1/2} \\
 &\quad \times \left(\int_0^T \int_{\mathbb{R}^d} \Delta x \cdot (|\tilde{u}_\varepsilon - u_\varepsilon|'_\delta)^2 \cdot (\partial_{x_i} (\tilde{u}_\varepsilon - u_\varepsilon))^2 dx dt \right)^{1/2} \\
 &\leq C_3 \cdot \Delta x \cdot (|\tilde{u}_\varepsilon - u_\varepsilon|'_\delta)^2 \cdot C_5 \leq C_6 \frac{\Delta x}{\delta^2}.
 \end{aligned}$$

Now, fix $\sigma > 0$ and a compact $K \subset \subset \mathbb{R}^d$. Choose Δx and δ so that $(o(1)$ below is from (a)):

$$\begin{aligned}
 &C_6 \frac{\Delta x}{\delta^2} < \sigma, \quad o(1) < \sigma, \\
 &\int_{\mathbb{R}^d} |u_0(x + \Delta x) - u_0(x)|_\delta dx < \sigma, \\
 &\int_0^T \int_K (|\tilde{u}_\varepsilon - u_\varepsilon|_\delta - |\tilde{u}_\varepsilon - u_\varepsilon|) dx < \sigma.
 \end{aligned} \tag{11}$$

Since $\int_0^T \int_K |\tilde{u}_\varepsilon - u_\varepsilon|_\delta dx dt \leq \int_0^T \int_{\mathbb{R}^d} |\tilde{u}_\varepsilon - u_\varepsilon|_\delta dx dt$, from (11), (a), (b), and (10) it follows that for any fixed $T > 0$:

$$\begin{aligned}
 \int_K |\tilde{u}_\varepsilon(T, x) - u_\varepsilon(T, x)| dx &\leq \int_K (|\tilde{u}_\varepsilon(T, x) - u_\varepsilon(T, x)| - |\tilde{u}_\varepsilon(T, x) - u_\varepsilon(T, x)|_\delta) dx \\
 &\quad + \int_{\mathbb{R}^d} |u_0(x + \Delta x) - u_0(x)| dx + 2\sigma \leq 4\sigma.
 \end{aligned}$$

Thus, we have proved L^1_{loc} equicontinuity of the family u_ε with respect to $x \in \mathbb{R}^d$. Equicontinuity with respect to $t \in \mathbb{R}^+$ follows from [7, Section 4].

Therefore, by using the Kolmogorov-Riesz criterion, it follows that (u_ε) is strongly precompact in L^1_{loc} . This concludes the existence proof.

2.2. The uniqueness proof

First, notice that the following lemma holds.

Lemma 2.2. For any weak solution u to (1), (2) it holds

$$\partial_{x_s} u \in L^2(\mathbb{R}^+ \times \mathbb{R}^d), \quad s = 1, \dots, k.$$

Proof. We take the convolution kernel ρ_ε defined on the beginning of Section 2.1 and apply it on (1). Denoting $u_\varepsilon = u \star \rho_\varepsilon$ and multiplying the convoluted (1) by u_ε , we obtain as in the proof of Lemma 2.1:

$$\int_{\mathbb{R}^d} u_\varepsilon^2(T, x) dx - \int_{\mathbb{R}^d} u_0^2(x) dx + \frac{\lambda}{2} \sum_{r=1}^k \int_{[0, T] \times \mathbb{R}^d} |\partial_{x_r} u_\varepsilon|^2 < \infty$$

uniformly in ε .

By letting $\varepsilon \rightarrow 0$ here, we immediately reach to the statement of the lemma. \square

Next, take two arbitrary weak solutions to (1), denote them by u and v , corresponding to the initial data $u|_{t=0} = u_0, v|_{t=0} = v_0$.

Take the regularizing kernel $\rho_\varepsilon(t)$ given at the beginning of subsection 2.1. By convoluting (1) (first as it is and then with $u = v$) by $\rho_\varepsilon(t)$, we obtain the equations:

$$u_t^\varepsilon + \operatorname{div} f_{\varepsilon u}(x, u) = \sum_{i,j=1}^k \partial_{x_i} (a_{ij} \partial_{x_j} (u_\varepsilon)),$$

$$v_t^\varepsilon + \operatorname{div} f_{\varepsilon v}(x, v) = \sum_{i,j=1}^k \partial_{x_i} (a_{ij} \partial_{x_j} (v_\varepsilon)),$$

where $f_{\varepsilon u}(x, u) = f(x, u) \star \rho_\varepsilon(t)$ and $f_{\varepsilon v}(x, v) = f(x, v) \star \rho_\varepsilon(t)$. By subtracting the latter two equations and multiplying by $\eta'(u_\varepsilon - v_\varepsilon)$ for a convex function $\eta \in C^2(\mathbb{R})$, we obtain

$$\begin{aligned} & \partial_t \eta(u_\varepsilon - v_\varepsilon) + \eta'(u_\varepsilon - v_\varepsilon) \operatorname{div} (f_{\varepsilon u}(x, u) - f_{\varepsilon v}(x, v)) \\ &= \sum_{i,j=1}^k \partial_{x_i} (a_{ij} \partial_{x_j} \eta(u_\varepsilon - v_\varepsilon)) - \sum_{i,j=1}^k \eta''(u_\varepsilon - v_\varepsilon) a_{ij} \partial_{x_i} (u_\varepsilon - v_\varepsilon) \partial_{x_j} (u_\varepsilon - v_\varepsilon). \end{aligned}$$

Letting here $\varepsilon \rightarrow 0$, we get in the weak sense (keep in mind Lemma 2.2 when dealing with the second term on the right-hand side)

$$\begin{aligned} & \partial_t \eta(u - v) + \eta'(u - v) \operatorname{div} (f(x, u) - f(x, v)) \\ &= \sum_{i,j=1}^k \partial_{x_i} (a_{ij} \partial_{x_j} \eta(u - v)) - \sum_{i,j=1}^k \eta''(u - v) a_{ij} \partial_{x_i} (u - v) \partial_{x_j} (u - v). \end{aligned}$$

We rewrite the latter in the form

$$\partial_t \eta(u - v) + \operatorname{div} \eta'(u - v) (f(x, u) - f(x, v)) - \sum_{i=1}^k \eta''(u - v) (f(x, u) - f(x, v)) \partial_{x_i} (u - v)$$

$$= \sum_{i,j=1}^k \partial_{x_i} (a_{ij} \partial_{x_j} \eta(u-v)) - \sum_{i,j=1}^k \eta''(u-v) a_{ij} \partial_{x_i} (u-v) \partial_{x_j} (u-v).$$

By putting $\eta(z) = |z|$ and having in mind $\eta''(u-v)(f(x,u) - f(x,v)) = \delta(u-v)(f(x,u) - f(x,v)) = 0$, we reach to the so called Kato inequality:

$$\begin{aligned} & \int_{R^+ \times R^d} (|u-v| \partial_t \varphi + \langle \text{sgn}(u-v)(f(x,v) - f(x,u)), \nabla \varphi \rangle) dx dt \\ & \leq \sum_{i,j=1}^k \int_{R^+ \times R^d} (a_{ij} \text{sgn}(u-v) \partial_{x_j} (u-v)) \partial_{x_i} \varphi dx dt. \end{aligned}$$

The L^1 -stability follows from here by using the standard procedure which can be found in e.g. [3]. The proof is over.

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