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New criteria for generalized weighted composition operators from mixed norm spaces into Zygmund-type spaces

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Abstract. New criteria for the boundedness and the compactness of the generalized weighted composition operators from mixed norm spaces into Zygmund-type spaces are given in this paper.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. Let $0 < p, q < \infty$, $\gamma > -1$. If an $f \in H(\mathbb{D})$ such that (see, e.g., [12, 13])

$$\|f\|^q_{H_{p,q,\gamma}} = \int_0^1 \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{q/p} (1-r)^\gamma dr < \infty,$$

we say that f belongs to the mixed norm space, which denoted by $H_{p,q,\gamma} = H_{p,q,\gamma}(\mathbb{D})$.

Let $\beta > 0$. The Zygmund-type space, denoted by $\mathscr{Z}^\beta$, consists of all $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathscr{Z}^\beta} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |f''(z)|.$$

$\mathscr{Z}^\beta$ becomes a Banach space under the above norm $\|\cdot\|_{\mathscr{Z}^\beta}$. Let $\beta = 1$. $\mathscr{Z}^1 = \mathscr{Z}$ is the classical Zygmund space. For more information on the Zygmund space on the unit disk, see, e.g., [4].

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map. The composition operator C_φ is the linear operator on $H(\mathbb{D})$ defined by

$$C_\varphi(f) = f \circ \varphi, \quad f \in H(\mathbb{D}).$$

Let u be a fixed analytic function on $\mathbb{D}$. The weighted composition operator uC_φ, which induced by φ and u, is defined as follows.

$$(uC_\varphi f)(z) = u(z) f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

We refer [3, 11] for the theory of the composition operator on function spaces.

The generalized weighted composition operator $D^n_{\varphi,u}$, which induced by Zhu (see [21–23]), is defined as follows.

$$(D^n_{\varphi,u} f)(z) = u(z)\cdot f^{(n)}(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

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Here $f^{(n)}(z)$ denote the n-th derivative of f. This operator includes many known operators. If $n = 0$, then we get the weighted composition operator uC_φ. If $n = 0$ and $u(z) \equiv 1$, then we obtain the composition operator C_φ. If $n = 1$, $u(z) = \varphi'(z)$, then $D^n_{\varphi,u} = DC_\varphi$, which was studied in [5, 8–10, 14]. When $n = 1$ and $u(z) = 1$, then $D^n_{\varphi,u} = C_\varphi D$, which was studied in [5, 9, 14].

Composition operators and weighted composition operators between Zygmund-type spaces and some other spaces were studied, for example, in [1, 2, 6, 7]. See [15–23] for the study of the generalized weighted composition operator on various function spaces.

In [15], the author studied the generalized weighted composition operators $D^n_{\varphi,u}$ from $H_{p,q,\gamma}$ into weighted-type spaces. In [16], the author studied the generalized weighted composition operators $D^n_{\varphi,u}$ from $H_{p,q,\gamma}$ into the mth weighted-type space. Among others, he obtained the following result.

Theorem A Let $u \in H(\mathbb{D})$, φ be an analytic self-map of $\mathbb{D}$ and n be a nonnegative integer. Assume that $0 < p, q < \infty$, $\gamma > -1$ and $0 < \beta < \infty$. Then the following propositions hold:
(a) The operator $D^n_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{Z}^\beta$ is bounded if and only if

$$M_1 := \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |u''(z)|}{(1-|\varphi(z)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n}} < \infty, \quad M_2 := \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n+1}} < \infty \tag{1}$$

and

$$M_3 := \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n+2}} < \infty. \tag{2}$$

(b) The operator $D^n_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{Z}^\beta$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |u''(z)|}{(1-|\varphi(z)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n}} = 0, \quad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n+1}} = 0 \tag{3}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n+2}} = 0. \tag{4}$$

In this paper, motivated by [16, 20, 24], we give a new criteria for the boundedness and compactness of the generalized weighted composition operators $D^n_{\varphi,u}$ from $H_{p,q,\gamma}$ into $\mathcal{Z}^\beta$.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the next. The notation $a \asymp b$ means that there is a positive constant C such that $C^{-1}b \le a \le Cb$.

2. Main results and proofs

In this section we give our main results and proofs. For this purpose, we need two lemmas as follows.

Lemma 2.1. [16] *Assume that $0 < p, q < \infty$ and $\gamma > -1$. Let $f \in H_{p,q,\gamma}$. Then there is a positive constant C independent of f such that*

$$|f^{(n)}(z)| \le C \frac{\|f\|_{H_{p,q,\gamma}}}{(1-|z|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n}}.$$

The following criterion follows from standard arguments similar, for example, to those outlined in Proposition 3.11 of [3].

Lemma 2.2. *Let $u \in H(\mathbb{D})$, φ be an analytic self-map of $\mathbb{D}$ and n be a nonnegative integer. Assume that $0 < p, q < \infty$, $\gamma > -1$ and $0 < \beta < \infty$. The operator $D^n_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{Z}^\beta$ is compact if and only if $D^n_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{Z}^\beta$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $H_{p,q,\gamma}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\|D^n_{\varphi,u} f_k\|_{\mathcal{Z}^\beta} \to 0$ as $k \to \infty$.*

Fix $0 < p, q < \infty$, $\gamma > -1$. For $a \in \mathbb{D}$ and $b > \frac{\gamma+1}{q}$, set

$$f_{a,j}(z) = \left(\frac{1-|a|^2}{1-\bar{a}z}\right)^j \frac{(1-|a|^2)^{b-\frac{\gamma+1}{q}}}{(1-\bar{a}z)^{\frac{1}{p}+b}}, \quad j = 0,1,2. \tag{5}$$

We will use these three families of functions to characterize the generalized weighted composition operators $D^n_{\varphi,u} : H_{p,q,\gamma} \to \mathscr{Z}^\beta$. We note that since the constant function 1 belongs to $H_{p,q,\gamma}$, the boundedness of uC_φ requires that $u = uC_\varphi 1 \in \mathscr{Z}^\beta$. Thus, we shall assume throughout that $u \in \mathscr{Z}^\beta$.

Theorem 2.3. *Let $u \in \mathscr{Z}^\beta$, φ be an analytic self-map of $\mathbb{D}$ and n be a nonnegative integer. Assume that $0 < p, q < \infty$, $\gamma > -1$ and $0 < \beta < \infty$. Then the following conditions are equivalent:*
(a) *The operator $D^n_{\varphi,u} : H_{p,q,\gamma} \to \mathscr{Z}^\beta$ is bounded;*
(b)

$$N_1 := \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |u(z)||\varphi'(z)|^2 < \infty, \quad N_2 := \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |2u'(z)\varphi'(z) + u(z)\varphi''(z)| < \infty,$$

$$A := \max\left\{\sup_{w\in\mathbb{D}} \|D^n_{\varphi,u} f_{\varphi(w),0}\|_{\mathscr{Z}^\beta},\ \sup_{w\in\mathbb{D}} \|D^n_{\varphi,u} f_{\varphi(w),1}\|_{\mathscr{Z}^\beta},\ \sup_{w\in\mathbb{D}} \|D^n_{\varphi,u} f_{\varphi(w),2}\|_{\mathscr{Z}^\beta}\right\} < \infty.$$

Proof. $(a) \Rightarrow (b)$. Assume that $D^n_{\varphi,u} : H_{p,q,\gamma} \to \mathscr{Z}^\beta$ is bounded. Taking the functions z^n and z^{n+1} and using the boundedness of $D^n_{\varphi,u}$ and the fact that $|\varphi(z)| \le 1$ we see that N_1, N_2 are finite.

For each $a \in \mathbb{D}$, it is easy to check that $f_{a,j} \in H_{p,q,\gamma}$. Moreover $\|f_{a,j}\|_{H_{p,q,\gamma}} (j = 0,1,2)$, are bounded by constants independent of a (see [15]). By the boundedness of $D^n_{\varphi,u} : H_{p,q,\gamma} \to \mathscr{Z}^\beta$, we get

$$\sup_{a\in\mathbb{D}} \|D^n_{\varphi,u} f_{\varphi(a),j}\|_{\mathscr{Z}^\beta} \le \|D^n_{\varphi,u}\| \sup_{a\in\mathbb{D}} \|f_{\varphi(a),j}\|_{H_{p,q,\gamma}} \le C\|D^n_{\varphi,u}\| < \infty, \quad j = 0,1,2,$$

as desired.

$(b) \Rightarrow (a)$. Suppose that N_1, N_2 and A are finite. A calculation shows that

$$f^{(n)}_{a,0}(a) = \frac{\prod_{j=0}^{n-1}(\frac{1}{p}+b+j)\bar{a}^n}{(1-|a|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n}}, \quad f^{(n)}_{a,1}(a) = \frac{\prod_{j=1}^{n}(\frac{1}{p}+b+j)\bar{a}^n}{(1-|a|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n}}, \quad \text{and} \quad f^{(n)}_{a,2}(a) = \frac{\prod_{j=2}^{n+1}(\frac{1}{p}+b+j)\bar{a}^n}{(1-|a|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n}}. \tag{6}$$

Denote $2u'(z)\varphi'(z) + u(z)\varphi''(z)$ by $v(z)$. From (6), for $w \in \mathbb{D}$, we have

$$\begin{aligned}(D^n_{\varphi,u} f_{\varphi(w),0})''(w) &= \frac{\prod_{j=0}^{n-1}(\frac{1}{p}+b+j)u''(w)\overline{\varphi(w)}^n}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n}} + \\ &\quad \frac{\prod_{j=0}^{n}(\frac{1}{p}+b+j)v(w)\overline{\varphi(w)}^{n+1}}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n+1}} + \frac{\prod_{j=0}^{n+1}(\frac{1}{p}+b+j)u(w)(\varphi'(w))^2\overline{\varphi(w)}^{n+2}}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n+2}},\end{aligned} \tag{7}$$

$$\begin{aligned}(D^n_{\varphi,u} f_{\varphi(w),1})''(w) &= \frac{\prod_{j=1}^{n}(\frac{1}{p}+b+j)u''(w)\overline{\varphi(w)}^n}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n}} + \\ &\quad \frac{\prod_{j=1}^{n+1}(\frac{1}{p}+b+j)v(w)\overline{\varphi(w)}^{n+1}}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n+1}} + \frac{\prod_{j=1}^{n+2}(\frac{1}{p}+b+j)u(w)(\varphi'(w))^2\overline{\varphi(w)}^{n+2}}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n+2}},\end{aligned} \tag{8}$$

and

$$(D^n_{\varphi,u}f_{\varphi(w),2})''(w) = \frac{\prod_{j=2}^{n+1}(\frac{1}{p}+b+j)u''(w)\overline{\varphi(w)}^n}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n}} + \frac{\prod_{j=2}^{n+2}(\frac{1}{p}+b+j)v(w)\overline{\varphi(w)}^{n+1}}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n+1}} + \frac{\prod_{j=2}^{n+3}(\frac{1}{p}+b+j)u(w)(\varphi'(w))^2\overline{\varphi(w)}^{n+2}}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n+2}}. \tag{9}$$

Let

$$p_0 = \prod_{j=0}^{n}(\frac{1}{p}+b+j), \quad p_1 = \prod_{j=0}^{n+1}(\frac{1}{p}+b+j), \quad p_2 = \prod_{j=0}^{n+2}(\frac{1}{p}+b+j).$$

Multiplying (7) by $-(\frac{1}{p}+b+n)$ and (8) by $(\frac{1}{p}+b)$ respectively, we get

$$\begin{aligned}&-(\frac{1}{p}+b+n)(D^n_{\varphi,u}f_{\varphi(w),0})''(w) + (\frac{1}{p}+b)(D^n_{\varphi,u}f_{\varphi(w),1})''(w)\\ =\ & p_0\frac{v(w)\overline{\varphi(w)}^{n+1}}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n+1}} + 2p_1\frac{u(w)(\varphi'(w))^2\overline{\varphi(w)}^{n+2}}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n+2}}.\end{aligned} \tag{10}$$

Multiplying (8) by $-(\frac{1}{p}+b+n+1)$, (9) by $(\frac{1}{p}+b+1)$, we obtain

$$\begin{aligned}&(\frac{1}{p}+b)\Big[-(\frac{1}{p}+b+n+1)(D^n_{\varphi,u}f_{\varphi(w),1})''(w) + (\frac{1}{p}+b+1)(D^n_{\varphi,u}f_{\varphi(w),2})''(w)\Big]\\ =\ & p_1\frac{v(w)\overline{\varphi(w)}^{n+1}}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n+1}} + 2p_2\frac{u(w)(\varphi'(w))^2\overline{\varphi(w)}^{n+2}}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n+2}}.\end{aligned} \tag{11}$$

Multiply (10) by $(\frac{1}{p}+b+n+1))$, we get

$$\begin{aligned}&(\frac{1}{p}+b+n+1)\Big[(\frac{1}{p}+b)(D^n_{\varphi,u}f_{\varphi(w),1})''(w) - (\frac{1}{p}+b+n)(D^n_{\varphi,u}f_{\varphi(w),0})''(w)\Big]\\ =\ & p_1\frac{v(w)\overline{\varphi(w)}^{n+1}}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n+1}} + 2(\frac{1}{p}+b+n+1)p_1\frac{u(w)(\varphi'(w))^2\overline{\varphi(w)}^{n+2}}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n+2}}.\end{aligned} \tag{12}$$

Subtracting (12) from (11), we obtain

$$\begin{aligned}&\frac{2p_1u(w)(\varphi'(w))^2\overline{\varphi(w)}^{n+2}}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n+2}} = (\frac{1}{p}+b+n)(\frac{1}{p}+b+n+1)(D^n_{\varphi,u}f_{\varphi(w),0})''(w)\\ &-2(\frac{1}{p}+b)(\frac{1}{p}+b+n+1)(D^n_{\varphi,u}f_{\varphi(w),1})''(w) + (\frac{1}{p}+b)(\frac{1}{p}+b+1)(D^n_{\varphi,u}f_{\varphi(w),2})''(w),\end{aligned} \tag{13}$$

which implies that

$$
\begin{aligned}
&\frac{(1-|w|^2)^\beta |u(w)(\varphi'(w))^2||\varphi(w)|^{n+2}}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n+2}} \\
\leq\ & \frac{1}{2p_1}(\frac{1}{p}+b+n)(\frac{1}{p}+b+n+1)(1-|w|^2)^\beta |(D^n_{\varphi,u} f_{\varphi(w),0})''(w)| \\
&+\frac{1}{p_1}(\frac{1}{p}+b)(\frac{1}{p}+b+n+1)(1-|w|^2)^\beta |(D^n_{\varphi,u} f_{\varphi(w),1})''(w)| + \frac{1}{2p_1}(\frac{1}{p}+b)(\frac{1}{p}+b+1)(1-|w|^2)^\beta |(D^n_{\varphi,u} f_{\varphi(w),2})''(w)| \\
\leq\ & \frac{1}{2p_1}(\frac{1}{p}+b+n)(\frac{1}{p}+b+n+1)\|D^n_{\varphi,u} f_{\varphi(w),0}\|_{\mathscr{Z}^\beta} \\
&+\frac{1}{p_1}(\frac{1}{p}+b)(\frac{1}{p}+b+n+1)\|D^n_{\varphi,u} f_{\varphi(w),1}\|_{\mathscr{Z}^\beta} + \frac{1}{2p_1}(\frac{1}{p}+b)(\frac{1}{p}+b+1)\|D^n_{\varphi,u} f_{\varphi(w),2}\|_{\mathscr{Z}^\beta} \qquad (14) \\
\leq\ & \frac{1}{2p_1}(\frac{1}{p}+b+n)(\frac{1}{p}+b+n+1)A + \frac{1}{p_1}(\frac{1}{p}+b)(\frac{1}{p}+b+n+1)A + \frac{1}{2p_1}(\frac{1}{p}+b)(\frac{1}{p}+b+1)A. \qquad (15)
\end{aligned}
$$

From (12) and (13), we obtain

$$
\begin{aligned}
&p_1 \frac{v(w)\overline{\varphi(w)}^{n+1}}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n+1}} \\
=\ & -(\frac{1}{p}+b+n)(\frac{1}{p}+b+n+1)(\frac{1}{p}+b+n+2)(D^n_{\varphi,u} f_{\varphi(w),0})''(w) \\
&+(\frac{1}{p}+b)(\frac{1}{p}+b+n+1)(\frac{2}{p}+2b+2n+3)(D^n_{\varphi,u} f_{\varphi(w),1})''(w) \\
&-(\frac{1}{p}+b)(\frac{1}{p}+b+1)(\frac{1}{p}+b+n+1)(D^n_{\varphi,u} f_{\varphi(w),2})''(w), \qquad (16)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\frac{(1-|w|^2)^\beta |v(w)||\varphi(w)|^{n+1}}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n+1}} \\
\leq\ & \frac{1}{p_1}(\frac{1}{p}+b+n)(\frac{1}{p}+b+n+1)(\frac{1}{p}+b+n+2)(1-|w|^2)^\beta |(D^n_{\varphi,u} f_{\varphi(w),0})''(w)| \\
&+\frac{1}{p_1}(\frac{1}{p}+b)(\frac{1}{p}+b+n+1)(1+2(\frac{1}{p}+b+n+2))(1-|w|^2)^\beta |(D^n_{\varphi,u} f_{\varphi(w),1})''(w)| \\
&+\frac{1}{p_1}(\frac{1}{p}+b)(\frac{1}{p}+b+1)(\frac{1}{p}+b+n+1)(1-|w|^2)^\beta |(D^n_{\varphi,u} f_{\varphi(w),2})''(w)| \\
\leq\ & \frac{1}{p_1}(\frac{1}{p}+b+n)(\frac{1}{p}+b+n+1)(\frac{1}{p}+b+n+2)\|D^n_{\varphi,u} f_{\varphi(w),0}\|_{\mathscr{Z}^\beta} \\
&+\frac{1}{p_1}(\frac{1}{p}+b)(\frac{1}{p}+b+n+1)(1+2(\frac{1}{p}+b+n+2))\|D^n_{\varphi,u} f_{\varphi(w),1}\|_{\mathscr{Z}^\beta} \\
&+\frac{1}{p_1}(\frac{1}{p}+b)(\frac{1}{p}+b+1)(\frac{1}{p}+b+n+1)\|D^n_{\varphi,u} f_{\varphi(w),2}\|_{\mathscr{Z}^\beta} \qquad (17) \\
\leq\ & \frac{1}{p_1}(\frac{1}{p}+b+n)(\frac{1}{p}+b+n+1)(\frac{1}{p}+b+n+2)A \\
&+\frac{1}{p_1}(\frac{1}{p}+b)(\frac{1}{p}+b+n+1)(1+2(\frac{1}{p}+b+n+2))A + \frac{1}{p_1}(\frac{1}{p}+b)(\frac{1}{p}+b+1)(\frac{1}{p}+b+n+1)A. \qquad (18)
\end{aligned}
$$

By (7), (13) and (16), we have

$$
\begin{aligned}
& p_0 \frac{u''(w)\overline{\varphi(w)}^n}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n}} \\
= & \Big[(\frac{1}{p}+b+n)+(\frac{1}{p}+b+n)^2[\frac{1}{2}(\frac{1}{p}+b+n+1)+1]\Big](D^n_{\varphi,u}f_{\varphi(w),0})''(w) \\
& -(\frac{1}{p}+b)(\frac{1}{p}+b+n)(\frac{1}{p}+b+n+2)(D^n_{\varphi,u}f_{\varphi(w),1})''(w)+\frac{1}{2}(\frac{1}{p}+b)(\frac{1}{p}+b+1)(\frac{1}{p}+b+n)(D^n_{\varphi,u}f_{\varphi(w),2})''(w), \quad (19)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& \frac{(1-|w|^2)^\beta|u''(w)||\varphi(w)|^n}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n}} \\
\le & \frac{\Big[(\frac{1}{p}+b+n)+(\frac{1}{p}+b+n)^2[\frac{1}{2}(\frac{1}{p}+b+n+1)+1]\Big]}{p_0}(1-|w|^2)^\beta|(D^n_{\varphi,u}f_{\varphi(w),0})''(w)| \\
& -\frac{(\frac{1}{p}+b)(\frac{1}{p}+b+n)(\frac{1}{p}+b+n+2)}{p_0}(1-|w|^2)^\beta|(D^n_{\varphi,u}f_{\varphi(w),1})''(w)| \\
& +\frac{(\frac{1}{p}+b)(\frac{1}{p}+b+1)(\frac{1}{p}+b+n)}{2p_0}(1-|w|^2)^\beta|(D^n_{\varphi,u}f_{\varphi(w),2})''(w)| \\
\le & \frac{\Big[(\frac{1}{p}+b+n)+(\frac{1}{p}+b+n)^2[\frac{1}{2}(\frac{1}{p}+b+n+1)+1]\Big]}{p_0}\|D^n_{\varphi,u}f_{\varphi(w),0}\|_{\mathscr{Z}^\beta} \\
& +\frac{(\frac{1}{p}+b)(\frac{1}{p}+b+n)(\frac{1}{p}+b+n+2)}{p_0}\|D^n_{\varphi,u}f_{\varphi(w),1}\|_{\mathscr{Z}^\beta} \\
& +\frac{(\frac{1}{p}+b)(\frac{1}{p}+b+1)(\frac{1}{p}+b+n)}{2p_0}\|D^n_{\varphi,u}f_{\varphi(w),2}\|_{\mathscr{Z}^\beta} \quad (20) \\
\le & \frac{\Big[(\frac{1}{p}+b+n)+(\frac{1}{p}+b+n)^2[\frac{1}{2}(\frac{1}{p}+b+n+1)+1]\Big]}{p_0}A \\
& +\frac{(\frac{1}{p}+b)(\frac{1}{p}+b+n)(\frac{1}{p}+b+n+2)}{p_0}A+\frac{(\frac{1}{p}+b)(\frac{1}{p}+b+1)(\frac{1}{p}+b+n)}{2p_0}A. \quad (21)
\end{aligned}
$$

Fix $r \in (0,1)$. If $|\varphi(w)| > r$, then from (21) we obtain

$$
\begin{aligned}
\frac{(1-|w|^2)^\beta|u''(w)|}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n}} \le & \frac{\Big[(\frac{1}{p}+b+n)+(\frac{1}{p}+b+n)^2[\frac{1}{2}(\frac{1}{p}+b+n+1)+1]\Big]}{p_0r^n}A \\
& +\frac{(\frac{1}{p}+b)(\frac{1}{p}+b+n)(\frac{1}{p}+b+n+2)}{p_0r^n}A+\frac{(\frac{1}{p}+b)(\frac{1}{p}+b+1)(\frac{1}{p}+b+n)}{2p_0r^n}A<\infty. \quad (22)
\end{aligned}
$$

On the other hand, if $|\varphi(w)| \le r$, we get

$$
\frac{(1-|w|^2)^\beta|u''(w)|}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n}} \le \frac{1}{(1-r^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n}}\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u''(z)|<\infty. \quad (23)
$$

From (22) and (23) we see that M_1 is finite. Using similar arguments, (15) and (18) we can obtain that M_2 and M_3 are finite as well. By Theorem A, we complete the proof of this theorem.

Theorem 2.4. *Let $u \in \mathscr{Z}^\beta$, φ be an analytic self-map of $\mathbb{D}$ and n be a nonnegative integer. Assume that $p, q > 0$, $\gamma > -1$ and $0 < \beta < \infty$. Suppose that the operator $D^n_{\varphi,u} : H_{p,q,\gamma} \to \mathscr{Z}^\beta$ is bounded, then the following conditions are equivalent:*

(a) *The operator $D^n_{\varphi,u} : H_{p,q,\gamma} \to \mathscr{Z}^\beta$ is compact;*

(b)

$$\lim_{|\varphi(w)|\to 1} \|D^n_{\varphi,u} f_{\varphi(w),0}\|_{\mathscr{Z}^\beta} = \lim_{|\varphi(w)|\to 1} \|D^n_{\varphi,u} f_{\varphi(w),1}\|_{\mathscr{Z}^\beta} = \lim_{|\varphi(w)|\to 1} \|D^n_{\varphi,u} f_{\varphi(w),2}\|_{\mathscr{Z}^\beta} = 0.$$

Proof. $(a) \Longrightarrow (b)$. Assume that $D^n_{\varphi,u} : H_{p,q,\gamma} \to \mathscr{Z}^\beta$ is compact. Let $\{w_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $\lim_{k\to\infty} |\varphi(w_k)| = 1$. Since the sequences $\{f_{\varphi(w_k),j}\}$, $j = 0, 1, 2$, are bounded in $H_{p,q,\gamma}$ and converge to 0 uniformly on compact subsets of $\mathbb{D}$, by Lemma 2.2, we get $\|D^n_{\varphi,u} f_{\varphi(w_k),j}\|_{\mathscr{Z}^\beta} \to 0$, $j = 0, 1, 2$, as $k \to \infty$, which means that (b) holds.

$(b) \Longrightarrow (a)$. Suppose that the limits in (b) are 0. Using the inequality (20), we get

$$\begin{aligned}
&\frac{(1-|w|^2)^\beta |u''(w)|}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n}} \\
\le\ & \frac{\left[(\frac{1}{p}+b+n) + (\frac{1}{p}+b+n)^2[\frac{1}{2}(\frac{1}{p}+b+n+1)+1]\right]}{p_0|\varphi(w)|^n} \|D^n_{\varphi,u} f_{\varphi(w),0}\|_{\mathscr{Z}^\beta} + \\
& \frac{(\frac{1}{p}+b)(\frac{1}{p}+b+n)(\frac{1}{p}+b+n+2)}{p_0|\varphi(w)|^n} \|D^n_{\varphi,u} f_{\varphi(w),1}\|_{\mathscr{Z}^\beta} + \frac{(\frac{1}{p}+b)(\frac{1}{p}+b+1)(\frac{1}{p}+b+n)}{2p_0|\varphi(w)|^n} \|D^n_{\varphi,u} f_{\varphi(w),2}\|_{\mathscr{Z}^\beta} \to 0
\end{aligned}$$

as $|\varphi(w)| \to 1$. Using the inequality (17), we get

$$\begin{aligned}
&\frac{(1-|w|^2)^\beta |v(w)|}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n+1}} \\
\le\ & \frac{(\frac{1}{p}+b+n)(\frac{1}{p}+b+n+1)(\frac{1}{p}+b+n+2)}{p_1|\varphi(w)|^{n+1}} \|D^n_{\varphi,u} f_{\varphi(w),0}\|_{\mathscr{Z}^\beta} \\
& + \frac{(\frac{1}{p}+b)(\frac{1}{p}+b+n+1)(1+2(\frac{1}{p}+b+n+2))}{p_1|\varphi(w)|^{n+1}} \|D^n_{\varphi,u} f_{\varphi(w),1}\|_{\mathscr{Z}^\beta} \\
& + \frac{(\frac{1}{p}+b)(\frac{1}{p}+b+1)(\frac{1}{p}+b+n+1)}{p_1|\varphi(w)|^{n+1}} \|D^n_{\varphi,u} f_{\varphi(w),2}\|_{\mathscr{Z}^\beta} \to 0
\end{aligned}$$

as $|\varphi(w)| \to 1$. Moreover, using (14), we deduce

$$\begin{aligned}
&\frac{(1-|w|^2)^\beta |u(w)(\varphi'(w))^2|}{(1-|\varphi(w)|^2)^{\frac{\gamma+1}{q}+\frac{1}{p}+n+2}} \\
\le\ & \frac{(\frac{1}{p}+b+n)(\frac{1}{p}+b+n+1)}{2p_1|\varphi(w)|^{n+2}} \|D^n_{\varphi,u} f_{\varphi(w),0}\|_{\mathscr{Z}^\beta} + \frac{(\frac{1}{p}+b)(\frac{1}{p}+b+n+1)}{p_1|\varphi(w)|^{n+2}} \|D^n_{\varphi,u} f_{\varphi(w),1}\|_{\mathscr{Z}^\beta} \\
& + \frac{1}{2p_1|\varphi(w)|^{n+2}} (\frac{1}{p}+b)(\frac{1}{p}+b+1) \|D^n_{\varphi,u} f_{\varphi(w),2}\|_{\mathscr{Z}^\beta} \to 0
\end{aligned}$$

as $|\varphi(w)| \to 1$. By Theorem A, we see that $D^n_{\varphi,u} : H_{p,q,\gamma} \to \mathscr{Z}^\beta$ is compact. The proof of this theorem is complete.

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