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On generalized difference ideal convergence in random 2-normed spaces

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Abstract. An ideal I is a family of subsets of positive integers **N** which is closed under taking finite unions and subsets of its elements. In [17], Kostyrko et. al introduced the concept of ideal convergence as a sequence (x_k) of real numbers is said to be I-convergent to a real number ℓ, if for each $\varepsilon > 0$ the set $\{k \in \mathbf{N} : |x_k - \ell| \geq \varepsilon\}$ belongs to I. In [28], Mursaleen and Alotaibi introduced the concept of I-convergence of sequences in random 2-normed spaces. In this paper, we define and study the notion of Δ^n-ideal convergence and Δ^n-ideal Cauchy sequences in random 2-normed spaces, and prove some interesting theorems.

1. Introduction

The probabilistic metric space was introduced by Menger [25] which is an interesting and important generalization of the notion of a metric space. Karakus [15] studied the concept of statistical convergence in probabilistic normed spaces. The theory of probabilistic normed spaces was initiated and developed in [1, 33–36] and further it was extended to random/probabilistic 2-normed spaces by Goleţ [10] using the concept of 2-norm which is defined by Gähler [9], and Gürdal and Pehlivan [13] studied statistical convergence in 2-Banach spaces.

The notion of I-convergence was initially introduced by Kostyrko, et. al [17] as a generalization of statistical convergence which is based on the structure of the ideal I of subset of natural numbers **N**. Kostyrko, et. al [18] gave some of basic properties of I-convergence and dealt with extremal I-limit points.

Although an ideal is defined as a hereditary and additive family of subsets of a non-empty arbitrary set X, here in our study it suffices to take I as a family of subsets of N, positive integers, i.e. $I \subset 2^{\mathbf{N}}$, such that $A \cup B \in I$ for each $A, B \in I$, and each subset of an element of I is an element of I.

A non-empty family of sets $F \subset 2^{\mathbf{N}}$ is a filter on **N** if and only if $\Phi \notin F$, $A \cap B \in F$ for each $A, B \in F$, and any subset of an element of F is in F. An ideal I is called *non-trivial* if $I \neq \Phi$ and $\mathbf{N} \notin I$. Clearly I is a non-trivial ideal if and only if $F = F(I) = \{\mathbf{N} - A : A \in I\}$ is a filter in **N**, called the filter associated with the ideal I. A non-trivial ideal I is called *admissible* if and only if $\{\{n\} : n \in \mathbf{N}\} \subset I$. A non-trivial ideal I is maximal if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset. Further details on ideals can be found in Kostyrko, et.al (see [17]). Throughout this paper we assume I is a non-trivial admissible ideal in **N**. Recall

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that a sequence $x = (x_k)$ of points in $\mathbf{R}$ is said to be I-convergent to a real number ℓ if $\{k \in \mathbf{N} : |x_k - \ell| \geq \varepsilon\} \in I$ for every $\varepsilon > 0$ ([17]). In this case we write $I - \lim x_k = \ell$.

In 1981, the idea of difference seqoence spaces was introduced by Kizmaz (see [16]). For difference sequences spaces of order m (see [24]). The generalized difference ideal convergence of real sequences was introduced and studied by Hazarika [14] and Gumus and Nuray [11] independently.

The notion of statistical convergence was introduced by Fast [7] and Schoenberg [32] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated by Fridy [8], Šalát [31], Çakalli [2], Caserta, et. al, [3], Di Maio and Kočinac [23], Miller [26], Maddox [22] and many others.

The notion of statistical convergence depends on the density (asymptotic or natural)of subsets of $\mathbf{N}$. A subset of $\mathbf{N}$ is said to have natural density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ exists.}$$

Definition 1.1. A sequence $x = (x_k)$ is said to be *statistically convergent* to ℓ if for every $\varepsilon > 0$

$$\delta(\{k \in \mathbf{N} : |x_k - \ell| \geq \varepsilon\}) = 0.$$

In this case, we write $S - \lim x = \ell$ or $x_k \to \ell(S)$ and S denotes the set of all statistically convergent sequences.

Remark 1.2. If we take $I = I_f = \{A \subseteq \mathbf{N} : A \text{ is a finite subset }\}$. Then I_f is a non-trivial admissible ideal of $\mathbf{N}$ and the corresponding convergence coincides with the usual convergence.

Remark 1.3. If we take $I = I_\delta = \{A \subseteq \mathbf{N} : \delta(A) = 0\}$ where $\delta(A)$ denote the asymptotic density of the set A. Then I_δ is a non-trivial admissible ideal of $\mathbf{N}$ and the corresponding convergence coincides with the statistical convergence.

Definition 1.4. ([6]) A sequence $x = (x_k)$ is said to be Δ^n-*statistically convergent* to ℓ if for every $\varepsilon > 0$ the set $\{k \in \mathbf{N} : |\Delta^n x_k - \ell| \geq \varepsilon\}$ has natural density zero. i.e.

$$\lim_{m\to\infty} \frac{1}{m} |\{k \leq m : |\Delta^n x_k - \ell| \geq \varepsilon\}| = 0,$$

where $n \in \mathbf{N}$ and $\Delta^0 x_k = (x_k)$, $\Delta x_k = (x_k - x_{k+1})$, $\Delta^n x_k = (\Delta^n x_k) = (\Delta^{n-1} x_k - \Delta^{n-1} x_{k+1})$, and also this generalized difference notion has the following binomial representation:

$$\Delta^n x_k = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x_{k+i} \text{ for all } k \in \mathbf{N}.$$

Definition 1.5. ([11, 14]) A sequence $x = (x_k)$ is said to be Δ^n-I-*convergent* to ℓ if for every $\varepsilon > 0$ the set $\{k \in \mathbf{N} : |\Delta^n x_k - \ell| \geq \varepsilon\}$ belong to I, where $n \in \mathbf{N}$.

Definition 1.6. ([11]) Let $I \subset 2^{\mathbf{N}}$ be an ideal in $\mathbf{N}$. If $\{k + 1 : k \in A\} \in I$, for any $A \in I$, then I is said to be a *translation invariant ideal*.

The existing literature on ideal convergence and its generalizations appears to have been restricted to real or complex sequences [14], but in recent years these ideas have been also extended to the sequences of fuzzy real numbers in fuzzy normed spaces [20] and intuitionistic fuzzy normed spaces [19]. Further details on ideal convergence can be found in [18, 21, 29, 30, 37–39].

2. Preliminaries

Definition 2.1. A function $f : \mathbf{R} \to \mathbf{R}_0^+$ is called a *distribution function* if it is a non-decreasing and left continuous with $\inf_{t\in\mathbf{R}} f(t) = 0$ and $\sup_{t\in\mathbf{R}} f(t) = 1$. By D^+, we denote the set of all distribution functions such that $f(0) = 0$. If $a \in \mathbf{R}_0^+$, then $H_a \in D^+$, where

$$H_a(t) = \begin{cases} 1, & \text{if } t > a; \\ 0, & \text{if } t \le a \end{cases}$$

It is obvious that $H_0 \ge f$ for all $f \in D^+$.

A *t-norm* is a continuous mapping $* : [0,1] \times [0,1] \to [0,1]$ such that $([0,1], *)$ is abelian monoid with unit one and $c * d \ge a * b$ if $c \ge a$ and $d \ge b$ for all $a, b, c \in [0,1]$. A triangle function τ is a binary operation on D^+, which is commutative, associative and $\tau(f, H_0) = f$ for every $f \in D^+$.

In [9], Gähler introduced the following concept of 2-normed space.

Definition 2.2. Let X be a linear space of dimension $d > 1$ (d may be infinite). A real-valued function $\|.,.\|$ from X^2 into $\mathbf{R}$ satisfying the following conditions:

(1) $\|x_1, x_2\| = 0$ if and only if x_1, x_2 are linearly dependent,
(2) $\|x_1, x_2\|$ is invariant under permutation,
(3) $\|\alpha x_1, x_2\| = |\alpha| \|x_1, x_2\|$, for any $\alpha \in \mathbf{R}$,
(4) $\|x + \overline{x}, x_2\| \le \|x, x_2\| + \|\overline{x}, x_2\|$
is called an 2-norm on X and the pair $(X, \|.,.\|)$ is called an 2-normed space.

A trivial example of an 2-normed space is $X = \mathbf{R}^2$, equipped with the Euclidean 2-norm $\|x_1, x_2\|_E =$ the volume of the parallelogram spanned by the vectors x_1, x_2 which may be given explicitly by the formula

$$\|x_1, x_2\|_E = |det(x_{ij})| = abs\left(\det(< x_i, x_j >)\right)$$

where $x_i = (x_{i1}, x_{i2}) \in \mathbf{R}^2$ for each $i = 1, 2$.

Recently, Goleţ [10] used the idea of 2-normed space to define the random 2-normed space.

Definition 2.3. Let X be a linear space of dimension $d > 1$ (d may be infinite), τ a triangle, and $\mathcal{F} : X \times X \to D^+$. Then $\mathcal{F}$ is called a *probabilistic 2-norm* and $(X, \mathcal{F}, \tau)$ a *probabilistic 2-normed* space if the following conditions are satisfied:
$(P2N_1)$ $\mathcal{F}(x, y; t) = H_0(t)$ if x and y are linearly dependent, where $\mathcal{F}(x, y; t)$ denotes the value of $\mathcal{F}(x, y)$ at $t \in \mathbf{R}$,
$(P2N_2)$ $\mathcal{F}(x, y; t) \ne H_0(t)$ if x and y are linearly independent,
$(P2N_3)$ $\mathcal{F}(x, y; t) = \mathcal{F}(y, x; t)$, for all $x, y \in X$,
$(P2N_4)$ $\mathcal{F}(\alpha x, y; t) = \mathcal{F}(x, y; \frac{t}{|\alpha|})$, for every $t > 0, \alpha \ne 0$ and $x, y \in X$,
$(P2N_5)$ $\mathcal{F}(x + y, z; t) \ge \tau\left(\mathcal{F}(x, z; t), \mathcal{F}(y, z; t)\right)$, whenever $x, y, z \in X$.
If $(P2N_5)$ is replaced by
$(P2N_6)$ $\mathcal{F}(x + y, z; t_1 + t_2) \ge \mathcal{F}(x, z; t_1) * \mathcal{F}(y, z; t_2)$, for all $x, y, z \in X$ and $t_1, t_2 \in \mathbf{R}_0^+$;
then $(X, \mathcal{F}, *)$ is called a *random 2-normed* space (for short, R2NS).

Remark 2.4. Every 2-normed space $(X, \|.,.\|)$ can be made a random 2-normed space in a natural way, by setting
(i) $\mathcal{F}(x, y; t) = H_0(t - \|x, y\|)$, for every $x, y \in X, t > 0$ and $a * b = \min\{a, b\}, a, b \in [0,1]$;
(ii) $\mathcal{F}(x, y; t) = \frac{t}{t + \|x, y\|}$, for every $x, y \in X, t > 0$ and $a * b = ab, a, b \in [0,1]$.

In [12], Gürdal and Pehlivan studied statistical convergence in 2-normed spaces and in 2-Banach spaces in [13]. In fact, Mursaleen [27] studied the concept of statistical convergence of sequences in random 2-normed spaces. In [5], Esi and Özdemir introduced and studied the concept of generalized Δ^m-statistical convergence of sequences in probabilistic normed spaces. Recently in [28], Mursaleen and Alotaibi introduced the concepts of I-convergence of sequences in random 2-normed spaces.

In this paper we define and study Δ^n-ideal convergence in random 2-normed spaces which is quite a new and interesting idea to work with. We show that some properties of Δ^n-ideal convergence of real numbers also hold for sequences in random 2-normed spaces. We find some relations related to Δ^n-ideal convergent sequences in random 2-normed spaces. Also we find out the relation between Δ^n-ideal convergent and Δ^n-ideal Cauchy sequences in this spaces.

3. Δ^n-ideal convergence

In this section we define Δ^n-ideal convergent sequences in random 2-normed $(X, \mathcal{F}, *)$. Also we obtained some basic properties of this notion in random 2-normed spaces.

Definition 3.1. A sequence $x = (x_k)$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be Δ^n-*convergent* to $\ell \in X$ with respect to $\mathcal{F}$ if for each $\varepsilon > 0$, $\theta \in (0, 1)$ there exists an positive integer n_0 such that $\mathcal{F}(\Delta^n x_k - \ell, z; \varepsilon) > 1 - \theta$, whenever $k \geq n_0$ and non zero $z \in X$. In this case we write $\mathcal{F}\text{-}\lim_k \Delta^n x_k = \ell$, and ℓ is called the $\mathcal{F}_{\Delta^n}$-limit of $x = (x_k)$.

Definition 3.2. A sequence $x = (x_k)$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be Δ^n-*Cauchy* with respect to $\mathcal{F}$ if for each $\varepsilon > 0$, $\theta \in (0, 1)$ and non zero $z \in X$, there exists a positive integer $n_0 = n_0(\varepsilon, z)$ such that $\mathcal{F}(\Delta^n x_k - \Delta^n x_s, z; \varepsilon) > 1 - \theta$, whenever $k, s \geq n_0$.

Now, we define the following definitions.

Definition 3.3. A sequence $x = (x_k)$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be Δ^n-*I-convergent* or I_{Δ^n}-*convergent* to $\ell \in X$ with respect to $\mathcal{F}$ if for every $\varepsilon > 0$, $\theta \in (0, 1)$ and non zero $z \in X$ such that

$$\{k \in \mathbf{N} : \mathcal{F}(\Delta^n x_k - \ell, z; \varepsilon) \leq 1 - \theta\} \in I.$$

or equivalently

$$\{k \in \mathbf{N} : \mathcal{F}(\Delta^n x_k - \ell, z; \varepsilon) > 1 - \theta\} \in F,$$

In this case we write $I^{R2N}\text{-}\lim \Delta^n x = \ell$ or $x_k \to \ell(\Delta^n(I^{R2N}))$ and

$$\Delta^n(I^{R2N}) = \{x = (x_k) : \exists\, \ell \in \mathbf{R}, I^{R2N}\text{-}\lim \Delta^n x = \ell\}.$$

Let $\Delta^n(I^{R2N})$ denotes the set of all Δ^n-ideal convergent sequences in random 2-normed space $(X, \mathcal{F}, *)$.

Definition 3.4. A sequence $x = (x_k)$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be Δ^n-*I-Cauchy* with respect to $\mathcal{F}$ if for every $\varepsilon > 0$, $\theta \in (0, 1)$ and non zero $z \in X$, there exists a positive integer $n_0 = n_0(\varepsilon, z)$ such that for all $k, s \geq n_0$

$$\{k \in \mathbf{N} : \mathcal{F}(\Delta^n x_k - \Delta^n x_s, z; \varepsilon) \leq 1 - \theta\} \in I.$$

or equivalently

$$\{k \in \mathbf{N} : \mathcal{F}(\Delta^n x_k - \Delta^n x_s, z; \varepsilon) > 1 - \theta\} \in F.$$

Definition 3.3, immediately implies the following lemma.

Lemma 3.5. *Let $(X, \mathcal{F}, *)$ be a random 2-normed space. If $x = (x_k)$ is a sequence in X, then for every $\varepsilon > 0$, $\theta \in (0,1)$ and non zero $z \in X$, then the following statements are equivalent:*
(i) $I^{R2N}\text{-}\lim_{k\to\infty} \Delta^n x_k = \ell$.
(ii) $\{k \in \mathbf{N} : \mathcal{F}(\Delta^n x_k - \ell, z; \varepsilon) \leq 1 - \theta\} \in I$.
(iii) $\{k \in \mathbf{N} : \mathcal{F}(\Delta^n x_k - \ell, z; \varepsilon) > 1 - \theta\} \in F$.
(iv) $I\text{-}\lim_{k\to\infty} \mathcal{F}(\Delta^n x_k - \ell, z; \varepsilon) = 1$.

Lemma 3.6. *Let $(X, \mathcal{F}, *)$ be a random 2-normed space. If I is a translation invariant ideal and $I^{R2N}\text{-}\lim_{k\to\infty} \Delta^n x_k = \ell$, then $I^{R2N}\text{-}\lim_{k\to\infty} \Delta^n x_{k+1} = \ell$.*

Proof of the lemma is straightforward, thus omitted.

Proposition 3.7. *Let $(X, \mathcal{F}, *)$ be a random 2-normed space. If I is an admissible translation invariant ideal and $I^{R2N} - \lim_{k\to\infty} \Delta^{n-1} x_k = \ell$, then $I^{R2N}\text{-}\lim_{k\to\infty} \Delta^n x_k = \ell$.*

Proof of the proposition is straightforward, thus omitted.

Theorem 3.8. *Let $(X, \mathcal{F}, *)$ be a random 2-normed space. If $x = (x_k)$ is a sequence in X such that $I^{R2N}\text{-}\lim \Delta^n x_k = \ell$ exists, then it is unique.*

Proof. Suppose that there exist elements ℓ_1, ℓ_2 $(\ell_1 \neq \ell_2)$ in X such that

$$I^{R2N}\text{-}\lim_{k\to\infty} \Delta^n x_k = \ell_1 \text{ and } I^{R2N}\text{-}\lim_{k\to\infty} \Delta^n x_k = \ell_2.$$

Let $\varepsilon > 0$ be given. Choose $r > 0$ such that

$$(1 - r) * (1 - r) > 1 - \varepsilon. \tag{1}$$

Then, for any $t > 0$ and non zero $z \in X$ we define

$$K_1(r,t) = \left\{k \in \mathbf{N} : \mathcal{F}\left(\Delta^n x_k - \ell_1, z; \frac{t}{2}\right) \leq 1 - r\right\};$$

$$K_2(r,t) = \left\{k \in \mathbf{N} : \mathcal{F}\left(\Delta^n x_k - \ell_2, z; \frac{t}{2}\right) \leq 1 - r\right\}.$$

Since $I^{R2N}\text{-}\lim_{k\to\infty} \Delta^n x_k = \ell_1$ and $I^{R2N}\text{-}\lim_{k\to\infty} \Delta^n x_k = \ell_2$, we have

$K_1(r,t) \in I$ and $K_2(r,t) \in I$ for all $t > 0$.

Now let $K(r,t) = K_1(r,t) \cup K_2(r,t)$, then it is easy to observe that $K(r,t) \in I$. But we have $K^c(r,t) \in F$.

Now if $k \in K^c(r,t)$ then we have

$$\mathcal{F}(\ell_1 - \ell_2, z; t) \geq \mathcal{F}\left(\Delta^n x_k - \ell_1, z; \frac{t}{2}\right) * \mathcal{F}\left(\Delta^n x_k - \ell_2, z; \frac{t}{2}\right) > (1 - r) * (1 - r).$$

It follows by (3.1) that

$$\mathcal{F}(\ell_1 - \ell_2, z; t) > (1 - \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, we get $\mathcal{F}(\ell_1 - \ell_2, z; t) = 1$ for all $t > 0$ and non zero $z \in X$. Hence $\ell_1 = \ell_2$. □

Next theorem gives the algebraic characterization of Δ^n-I-convergence on random 2-normed spaces.

Theorem 3.9. *Let $(X, \mathcal{F}, *)$ be a random 2-normed space, and $x = (x_k)$ and $y = (y_k)$ be two sequences in X. (a) If I^{R2N}-$\lim \Delta^n x_k = \ell$ and $c(\neq 0) \in \mathbf{R}$, then I^{R2N}-$\lim c\Delta^n x_k = c\ell$. (b) If I^{R2N}-$\lim \Delta^n x_k = \ell_1$ and I^{R2N}-$\lim \Delta^n y_k = \ell_2$, then I^{R2N}-$\lim \Delta^n (x_k + y_k) = \ell_1 + \ell_2$.*

Proof of the theorem is straightforward, thus omitted.

Theorem 3.10. *Let $(X, \mathcal{F}, *)$ be a random 2-normed space. If $x = (x_k)$ be a sequence in X such that $\mathcal{F}$-$\lim \Delta^n x_k = \ell$, then I^{R2N}-$\lim \Delta^n x_k = \ell$.*

Proof. Let $\mathcal{F}$-$\lim \Delta^n x_k = \ell$. Then for every $0 < \varepsilon < 1, t > 0$ and non zero $z \in X$, there is a positive integer $m = m(\varepsilon, z)$ such that

$$\mathcal{F}(\Delta^n x_k - \ell, z; t) > 1 - \varepsilon$$

for all $k \geq m$. Since the set

$$K(\varepsilon, t) = \{k \in \mathbf{N} : \mathcal{F}(\Delta^n x_k - \ell, z; t) \leq 1 - \varepsilon\} \subset \mathbf{N} - \{m_{k+1}, m_{k+2}, ...\}.$$

Also, since I is admissible, and consequently we have $K(\varepsilon, t) \in I$. This shows that I^{R2N}-$\lim \Delta^n x_k = \ell$. □

Remark 3.11. The converse of the above theorem is not true in general. It follows from the following example.

Example 3.12. Let $X = \mathbf{R}^2$, with the 2-norm $\|x, z\| = |x_1 z_2 - x_2 z_1|$, $x = (x_1, x_2), z = (z_1, z_2)$ and $a * b = ab$ for all $a, b \in [0, 1]$. Let $\mathcal{F}(x, y; t) = \frac{t}{t + \|x,y\|}$, for all $x, z \in X, z_2 \neq 0$, and $t > 0$. Now we define a sequence $x = (x_k)$ by

$$\Delta^n x_k = \begin{cases} (k, 0), & \text{if } k = i^3 \text{ for } i \in \mathbf{N}; \\ (0, 0), & \text{otherwise.} \end{cases}$$

Now for every $0 < \varepsilon < 1$ and $t > 0$, write

$$K(\varepsilon, t) = \{k \in \mathbf{N} : \mathcal{F}(\Delta^n x_k - \ell, z; t) \leq 1 - \varepsilon\}, \ell = (0, 0).$$

We have

$$\mathcal{F}(\Delta^n x_k - \ell, z; t) = \begin{cases} \frac{t}{t + kz_2}, & \text{if } k = i^3 \text{ for } i \in \mathbf{N}; \\ 1, & \text{otherwise.} \end{cases}$$

and hence

$$\lim_k \mathcal{F}(\Delta^n x_k - \ell, z; t) = \begin{cases} 0, & \text{if } k = i^3 \text{ for } i \in \mathbf{N}; \\ 1, & \text{otherwise.} \end{cases}$$

This shows that $x = (x_k)$ is not convergent in $(X, \mathcal{F}, *)$. But if we take $I = I_\delta = \{A \subset \mathbf{N} : \delta(A) = 0\}$ and since $K(\varepsilon, t) \subset \{(1, 0), (8, 0), (27, 0), (64, 0), ...\}$, then $\delta(K(\varepsilon, t)) = 0$. Thus we have I^{R2N}-$\lim \Delta^n x_k = \ell$.

Theorem 3.13. *Let $(X, \mathcal{F}, *)$ be a random 2-normed space. If $x = (x_k)$ be a sequence in X, then I^{R2N}-$\lim \Delta^n x_k = \ell$ if and only if there exists a subset $K \subseteq \mathbf{N}$ such that $K \in F$ and $\mathcal{F}$-$\lim \Delta^n x_k = \ell$.*

Proof. Suppose first that I^{R2N}-$\lim \Delta^n x_k = \ell$. Then for any $t > 0$, $r = 1, 2, 3, ...$ and non zero $z \in X$, let

$$A(r, t) = \left\{k \in \mathbf{N} : \mathcal{F}(\Delta^n x_k - \ell, z; t) > 1 - \frac{1}{r}\right\}$$

and

$$K(r,t) = \left\{k \in \mathbf{N} : \mathcal{F}(\Delta^n x_k - \ell, z; t) \leq 1 - \frac{1}{r}\right\}.$$

Since I^{R2N}-lim $\Delta^n x_k = \ell$ it follows that

$$K(r,t) \in I.$$

Now for $t > 0$ and $r = 1, 2, 3, ...$, we observe that

$$A(r,t) \supset A(r+1,t)$$

and

$$A(r,t) \in F. \tag{2}$$

Now we have to show that, for $k \in A(r,t)$, $\mathcal{F}$-lim $\Delta^n x_k = \ell$. Suppose that for $k \in A(r,t)$, (x_k) not convergent to ℓ with respect to $\mathcal{F}$. Then there exists some $s > 0$ such that

$$\{k \in \mathbf{N} : \mathcal{F}(\Delta^n x_k - \ell, z; t) \leq 1 - s\}.$$

Let

$$A(s,t) = \{k \in \mathbf{N} : \mathcal{F}(\Delta^n x_k - \ell, z; t) > 1 - s\}$$

and

$$s > \frac{1}{r}, r = 1, 2, 3, ...$$

Then we have

$$A(s,t) \in I.$$

Furthermore, $A(r,t) \subset A(s,t)$ implies that $A(r,t) \in I$, which contradicts (3.2) as $A(r,t) \in F$. Hence $\mathcal{F}$-lim $\Delta^n x_k = \ell$.

Conversely, suppose that there exists a subset $K \subseteq \mathbf{N}$ such that $K \in F$ and $\mathcal{F}$-lim $\Delta^n x_k = \ell$.

Then for every $0 < \varepsilon < 1$, $t > 0$ and non zero $z \in X$, we can find out a positive integer $m = m(\varepsilon, z)$ such that

$$\mathcal{F}(\Delta^n x_k - \ell, z; t) > 1 - \varepsilon$$

for all $k \geq m$. If we take

$$K(\varepsilon, t) = \{k \in \mathbf{N} : \mathcal{F}(\Delta^n x_k - \ell, z; t) \leq 1 - \varepsilon\}$$

then it is easy to see that

$$K(\varepsilon, t) \subset \mathbf{N} - \{m_{k+1}, m_{k+2}, ...\}$$

and since I is admissible, consequently

$$K(\varepsilon, t) \in I.$$

Hence I^{R2N}-lim $\Delta^n x_k = \ell$. $\square$

Now, we establish the Cauchy convergence criteria in random 2-normed spaces.

Theorem 3.14. *Let $(X,\mathcal{F},*)$ be a random 2-normed space. Then a sequence (x_k) in X is Δ^n-I-convergent if and only if it is Δ^n-I-Cauchy.*

Proof. Let (x_k) be a Δ^n-I-convergent sequence in X. We assume that I^{R2N}-$\lim \Delta^n x_k = \ell$. Let $\varepsilon > 0$ be given. Choose $r > 0$ such that (3.1) is satisfied. For $t > 0$ and non zero $z \in X$ define

$$A(r,t) = \left\{k \in \mathbf{N} : \mathcal{F}(\Delta^n x_k - \ell, z; \frac{t}{2}) \leq 1 - r\right\}.$$

Then

$$A^c(r,t) = \left\{k \in \mathbf{N} : \mathcal{F}(\Delta^n x_k - \ell, z; \frac{t}{2}) > 1 - r\right\}.$$

Since I^{R2N}-$\lim \Delta^n x_k = \ell$ it follows that $A(r,t) \in I$ and consequently $A^c(r,t) \in F$. Let $p \in A^c(r,t)$. Then

$$\mathcal{F}(\Delta^n x_p - \ell, z; \frac{t}{2}) > 1 - r. \tag{3}$$

If we take

$$B(\varepsilon,t) = \{k \in \mathbf{N} : \mathcal{F}(\Delta^n x_k - \Delta^n x_p, z; t) \leq 1 - \varepsilon\}$$

then to prove the result it is sufficient to prove that $B(\varepsilon,t) \subseteq A(r,t)$. Let $k \in B(\varepsilon,t) \cap A^c(r,t)$, then for non zero $z \in X$

$$\mathcal{F}(\Delta^n x_k - \Delta^n x_p, z; t) \leq 1 - \varepsilon \text{ and } \mathcal{F}(\Delta^n x_k - \ell, z; \frac{t}{2}) > 1 - r. \tag{4}$$

Then by (3.1), (3.3) and (3.4) we get

$$1 - \varepsilon \geq \mathcal{F}(\Delta^n x_n - \Delta^n x_p, z; t) \geq \mathcal{F}(\Delta^n x_n - \ell, z; \frac{t}{2}) * \mathcal{F}(\Delta^n x_p - \ell, z; \frac{t}{2})$$

$$> (1 - r) * (1 - r) > (1 - \varepsilon)$$

which is not possible. Thus $B(\varepsilon,t) \subset A(r,t)$. Since $A(r,t) \in I$, it follows that $B(\varepsilon,t) \in I$. This shows that (x_k) is Δ^n-I-Cauchy.

Conversely, suppose (x_k) is Δ^n-I-Cauchy but not Δ^n-I-convergent. Then there exists positive integer p and non zero $z \in X$ such that

$$A(\varepsilon,t) = \{k \in \mathbf{N} : \mathcal{F}(\Delta^n x_k - \Delta^n x_p, z; t) \leq 1 - \varepsilon\}.$$

then

$$A(\varepsilon,t) \in I$$

and consequently

$$A^c(\varepsilon,t) \in F. \tag{5}$$

For $r > 0$ such that (3.1) is satisfied and we take

$$B(r,t) = \{k \in \mathbf{N} : \mathcal{F}(\Delta^n x_k - \ell, z; \frac{t}{2}) > 1 - r\}.$$

If $p \in B(r,t)$ then

$$\mathcal{F}(\Delta^n x_p - \ell, z; \frac{t}{2}) > 1 - r.$$

Since

$$\mathcal{F}(\Delta^n x_k - \Delta^n x_p, z; t) \geq \mathcal{F}(\Delta^n x_k - \ell, z; \frac{t}{2}) * \mathcal{F}(\Delta^n x_p - \ell, z; \frac{t}{2}) > (1-r) * (1-r) > 1 - \varepsilon,$$

then we have

$$\{k \in \mathbf{N} : \mathcal{F}(\Delta^n x_n - \Delta^n x_p, z; t) > 1 - \varepsilon\} \in I$$

i.e. $A^c(\varepsilon, t) \in I$, which contradicts (3.5) as $A^c(\varepsilon, t) \in F$. Hence (x_k) is Δ^n-I-convergent. □

Combining Theorem 3.13 and Theorem 3.14 we get the following corollary.

Corollary 3.15. *Let $(X, \mathcal{F}, *)$ be a random 2-normed space and and $x = (x_k)$ be a sequence in X. Then the following statements are equivalent:*
(a) x is Δ^n-I-convergent.
(b) x is Δ^n-I-Cauchy.
(c) there exists a subset $K \subseteq \mathbf{N}$ such that $K \in F$ and $\mathcal{F}$-$\lim \Delta^n x_k = \ell$.

Theorem 3.16. *Let $(X, \mathcal{F}, *)$ be a random 2-normed space and $x = (x_k)$ be a sequence in X. Let I be a non-trivial ideal in $\mathbf{N}$. If there is a Δ^n-I-convergent sequence $y = (y_k)$ in X such that $\{k \in \mathbf{N} : \Delta^n y_k \neq \Delta^n x_k\} \in I$, then x is also Δ^n-I-convergent in X.*

Proof. Suppose that $\{k \in \mathbf{N} : \Delta^n y_k \neq \Delta^n x_k\} \in I$ and I^{R2N}-$\lim \Delta^n y_k = \ell$. Let $0 < \varepsilon < 1$ be given. Then for $t > 0$ and non zero $z \in X$ we get

$$\left\{k \in \mathbf{N} : \mathcal{F}(\Delta^n y_k - \ell, z; \frac{t}{2}) \leq 1 - \varepsilon\right\} \in I.$$

For every $0 < \varepsilon < 1$,

$$\left\{k \in \mathbf{N} : \mathcal{F}(\Delta^n x_k - \ell, z; \frac{t}{2}) \leq 1 - \varepsilon\right\} \subseteq \{k \in \mathbf{N} : \Delta^n y_k \neq \Delta^n x_k\} \cup \left\{k \in \mathbf{N} : \mathcal{F}(\Delta^n y_k - \ell, z; \frac{t}{2}) \leq 1 - \varepsilon\right\}. \tag{6}$$

As both the sets of right-hand side of (3.6) is in I, therefore we have that

$$\left\{k \in \mathbf{N} : \mathcal{F}(\Delta^n x_k - \ell, z; \frac{t}{2}) \leq 1 - \varepsilon\right\} \in I.$$

□

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